Incorporating the Modified Orifice Equation into Pipe Network Solvers for More Realistic Leakage Modeling

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Abstract: It has been well established in several experimental and modeling studies that leak areas are often not fixed, but vary as linear functions of pressure. Replacing this linear equation into the orifice equation results in a two-part modified orifice leakage equation with head exponents of 0.5 and 1.5. The purpose of this study was to incorporate the modified orifice equation into the hydraulic network formulation and evaluate its impact on model performance. The conventional and modified software were applied to 600 instances of stochastic leakage distributions in three different pipe networks. It was found that the conventional power leakage equation results in significant leakage volume and flow-rate errors under certain conditions. In addition, a problem of nonconvergence of the conventional global gradient algorithm for leakage exponents greater than 2 was observed and is discussed. **DOI:** 10.1061/(ASCE)HY.1943-7900.0001410. *© 2017 American Society of Civil Engineers.*

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Introduction

Hydraulic network modeling software, including the commonly used *EPANET* package, (Rossman 2000), uses a power equation to model pressure-dependent outflows such as leakage. Researchers and practitioners in the leakage field have long realized that leakage does not adhere to the theoretical orifice equation (a power equation with a fixed exponent of 0.5) but mostly require higher-power exponents to be simulated realistically (Ogura 1979; Hiki 1981).

Several causes for the power leakage exponent diverging from the theoretical value of 0.5 have been investigated, including leakage hydraulics (van Zyl and Clayton 2007), soil-leak interaction (Walski et al. 2006; van Zyl et al. 2013) and the distribution of leaks in a network (Schwaller and van Zyl 2015). However, the overriding cause of variations in the leakage exponent has been shown to be that leak areas are not fixed, but vary with system pressure (May 1994; van Zyl and Clayton 2007; Cassa et al. 2010; Ferrante et al. 2011; Massari et al. 2012; De Marchis et al. 2016; Fox et al. 2017). In addition, the variations in leak area have been shown to be a linear function of pressure head under both elastic and viscoelastic deformation conditions for different leak types, pipe materials, and loading states (Cassa and van Zyl 2013; van Zyl and Cassa 2014; Malde 2015; Ssozi et al. 2016).

The variations in leak area under plastic deformation and fracture conditions cannot be assumed linear, but van Zyl et al. (2017) argues that these mechanisms are unlikely to dominate the pressure-leakage response of distribution systems because they (1) are irreversible and (2) only occur when pressures are increased and not when they are decreased. Thus, it is assumed in this study that all leak areas vary linearly with pressure head.

van Zyl et al. (2017) investigated the implications of a linear head-area relationship and concluded that the power equation is flawed in that its parameters are not constant, but vary with system pressure, and that the leakage exponent approaches infinity under certain conditions. They concluded that significant modeling errors are possible if the power equation is extrapolated beyond its calibration pressure range and at high exponent values. Finally, they recommended that the linear head-area relationship be explicitly incorporated in a hydraulic simulation software for a more realistic leakage modeling.

The aim of this study was to incorporate and test the inclusion of a modified orifice equation, which explicitly includes a linear head-area relationship in the standard hydraulic network solver model.

The following section develops a modified orifice equation that incorporates a linear pressure-area relationship, describes its relationship to the power formulation and provides a brief description of relevant water-loss benchmarks. The standard hydraulic network model is then modified to incorporate the linear head-area relationship. Finally, the implications of this modification for model convergence and accuracy are investigated on the basis of three example networks with stochastic leak distributions.

Background

Realistic Leakage Modeling

Hydraulically, leaks are orifices that should adhere to the orifice equation

$$Q = C_d A \sqrt{2gh} \tag{1}$$

where Q = orifice flow rate; C_d = discharge coefficient; A = orifice area; g = gravitational acceleration; and h = head differential over the leak.

The areas of real leaks are not constant but can be assumed to vary linearly with pressure, i.e.

$$A = A_0 + mh \tag{2}$$

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where A_0 = initial area (the area of the leak opening when the head differential is zero); and m = head-area slope.

Replacing Eq. (2) into Eq. (1) results in the modified orifice equation, which is known in leakage practice as the fixed and variable area discharges (FAVAD) equation (May 1994)

$$Q = C_d\sqrt{2g}(A_0 h^{0.5} + mh^{1.5}) \tag{3}$$

An important consequence of the linearity assumption of the head-area relationship is that the total leakage area of a system with many leaks will also display a linear head-area relationship. In addition, the total system initial area (A_0) will be the sum of all the individual leak initial areas, and the total system head-area slope (m) will be the sum of all the individual leak head-area slopes (Schwaller et al. 2015). This indicates that Eq. (3) may be applied equally to a system (or pipe) with a single or multiple leaks.

A power formulation is commonly used to model the behavior of leaks, both in hydraulic modeling software (Rossman 2000) and leakage management practice (Lambert 2000; Farley and Trow 2003)

$$Q = Ch^{\alpha} \tag{4}$$

where C = leakage coefficient; and α = leakage exponent. In modeling practice, such as the widely used *EPANET* software, the power equation is called an emitter (Rossman 2000) and is used to model pressure-dependent consumer demands and leakage. In leakage management practice, Eq. (4) is called the power leakage equation, and the symbol $N1$ is used instead of α (Lambert 2000).

Leakage Benchmarks

A brief overview is provided of internationally accepted water-loss terminology and benchmarks relevant to this study (AWWA 2016). It is desirable to split a large distribution system into smaller discrete zones called district metered areas (DMAs). Ideally, DMAs should be supplied through a single metered supply point where, if desired, the inlet pressure can be controlled.

The level of real losses in a DMA is appraised by using the concept of an infrastructure leakage index (ILI), defined by the current annual real losses (CARL) divided by the unavoidable annual real losses (UARL) in the system

$$\mathrm{ILI} = \frac{\mathrm{CARL}}{\mathrm{UARL}} \tag{5}$$

The values of the CARL is normally estimated by measuring the minimum night flow rate (MNF) entering a DMA (typically between 3 and 5 a.m.) and subtracting the estimated user consumption.

The value of the UARL (L/day) represents the theoretical low limit of leakage that could be achieved in a specific system at a given pressure (AWWA 2016; Lambert 2009). The UARL formula is given by

$$\mathrm{UARL} = (18L_m + 0.8N_c + 25L_s)h_{AZ} \tag{6}$$

where L_m = length of mains (km); N_c = number of service connections; L_s = length of service lines between the property boundary and water meter (km); and h_{AZ} = average zonal pressure head. The length of service lines L_s is only included in Eq. (6) when consumer meters are installed within the property boundary; otherwise, it is set to zero.

Model Development

The aim of this section is to propose a hydraulic model formulation that incorporates the modified orifice equation in Eq. (3) for leakage modeling. The standard model formulation is first presented, and the modifications to implement the modified orifice equation are then discussed.

Network Formulation with Standard Leakage Power Function

The topology of a water-distribution network is represented by a set of nodes and links (a graph). A link corresponds to a collection of pipes, valves, and pumps.

The steady-state system equations are formulated by applying principles of conservation of mass and energy at each node and link, respectively. These equations are solved simultaneously to obtain pressure heads at the junction nodes and flow rates in the links.

User demands are lumped at network nodes and consist of both fixed and pressure-dependent demands. The standard model for pressure-dependent demands and leakage is a power function

$$\mu_i = c_i(H_i - z_i)^{\alpha}, \qquad \text{if } H_i \geq z_i \tag{7}$$

where μ_i = lumped nodal leakage outflow; c_i = leakage coefficient; z_i = elevation of node i; and α = leakage exponent.

The steady-state equations read

$$\begin{aligned} -\sum_{j=1}^{np} A_{ij}Q_j - \mu_i &= D_i, i = 1, \cdots, nu \\ h_j(Q_j) - \sum_{i=1}^{nu} A_{ij}H_i &= \sum_{i=1}^{nf} B_{ij}H_i^f, j = 1, \cdots, np \\ c_i^{-\frac{1}{\alpha}}[\mu_i]^{\frac{1}{\alpha}-1}\mu_i + h_i^{CV}(\mu_i) - H_i &= -z_i, i = 1, \cdots, nu \end{aligned} \tag{8}$$

where nu = number of junction nodes; nf = number of fixed-head nodes; np = number of links; Q_j = flow rate in link j; D_i = fixed demand at node i; A_{ij} = coefficient describing the existence and direction of link j (-1 if link j is directed toward node i, $+1$ if link j is directed away from node i, and 0 if link j is not connected to node i); B_{ij} = similar to A_{ij} but for fixed-head nodes; H_i = head at node i; H_i^f = head at fixed-head node i; h_j = head loss in link j; and h_i^{CV} = check valve head-loss penalty function as proposed in Piller and van Zyl (2014) and given by

$$h_i^{CV}(\mu_i) = -r_{\max}\max(0, -\mu_i)^2 = -h_0 \max\left(0, -\frac{\mu_i}{\Delta Q}\right)^2 \tag{9}$$

where $r_{\max} = \frac{h_0}{\Delta Q^2}$ = large resistance coefficient, which is chosen to produce an equivalent head loss less than or equal to $-h_0$ if the leakage outflow is less than or equal $-\Delta Q$; for practical application, $h_0 = 15$ m, and $\Delta Q = 0.01$ L/s.

In *EPANET*, emitters at nodes are modeled by adding a fictitious pipe between the node and a fictitious reservoir (Rossman 2000). The head at the fictitious reservoir is the elevation of the junction. The frictional pipe head loss is described by using Eq. (7), with resistance $c^{-1/\alpha}$ and exponent $1/\alpha$. The system shown in Eq. (8) describes the behavior of the fictional pipe and reservoir and, unlike the *EPANET* emitter function, includes a check valve to prevent backflow.

The function h_j is differentiable with respect to Q_j. Moreover, h_j is assumed as a strictly increasing function of Q_j, which is nonbounded at infinity. The system shown in Eq. (8) can be rewritten in matrix form as

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$$
\begin{aligned}
-\mathbf{A}\mathbf{Q}-\boldsymbol{\mu}&=\mathbf{D}\\
\mathbf{h}(\mathbf{Q})-\mathbf{A}^T\mathbf{H}&=\mathbf{B}^T\mathbf{H}^f\\
\mathbf{g}(\boldsymbol{\mu})-\mathbf{H}&=-\mathbf{Z}
\end{aligned}\tag{10}
$$

where $\mathbf{Q}$ = vector of link flow rates; $\mathbf{D}$ = vector of fixed nodal demands; $\mathbf{A}=nu\times np$ incidence matrix of the variable-head nodes; $\mathbf{B}=nf\times np$ incidence matrix of fixed-head nodes; $\mathbf{H}$ = vector of variable nodal heads; $\mathbf{H}^f$ = vector of fixed nodal heads; $\mathbf{h}$ = vector of link head losses; $\mathbf{Z}$ = vector of elevations at variable-head nodes; and $\mathbf{g}(\boldsymbol{\mu})$ = vector of link head losses in the fictional pipes.

Eq. (10) has only one solution if matrix $\mathbf{A}$ is full rank in the number of junction nodes (Piller 1995). Indeed, there is an elliptic function (the content function) whose minimization restricted to the mass-balance equation is equivalent to solving Eq. (10). For $\mathbf{A}$ to be full rank, it is sufficient that each connected component or group of nodes possesses at least one fixed-head node.

The method of Newton applied to the system in Eq. (10) is an iterative procedure that consists of sequentially solving the linear system

$$
\begin{pmatrix}
-\mathbf{A} & -\mathbf{I}_{nu,nu} & \mathbf{0}_{nu,nu}\\
\mathbf{J}_n & \mathbf{0}_{np,nu} & -\mathbf{A}^T\\
\mathbf{0}_{nu,np} & \mathbf{K}_n & -\mathbf{I}_{nu,nu}
\end{pmatrix}
\begin{pmatrix}
\mathbf{Q}^{n+1}-\mathbf{Q}^n\\
\boldsymbol{\mu}^{n+1}-\boldsymbol{\mu}^n\\
\mathbf{H}^{n+1}-\mathbf{H}^n
\end{pmatrix}
=\begin{pmatrix}
\mathbf{A}\mathbf{Q}^n+\boldsymbol{\mu}^n+\mathbf{D}\\
-\mathbf{h}(\mathbf{Q}^n)+\mathbf{A}^T\mathbf{H}^n+\mathbf{B}^T\mathbf{H}^f\\
-\mathbf{g}(\boldsymbol{\mu}^n)+\mathbf{H}^n-\mathbf{Z}
\end{pmatrix}\tag{11}
$$

where $\mathbf{J}_n=\nabla_{\mathbf{Q}}\mathbf{h}(\mathbf{Q}^n)$ = matrix Jacobian of the $\mathbf{h}$ function with regard to $\mathbf{Q}$ at $\mathbf{Q}^n$; and $\mathbf{K}_n=\nabla_{\boldsymbol{\mu}}\mathbf{g}(\boldsymbol{\mu}^n)$ = matrix Jacobian of the $\mathbf{g}$ function with regard to $\boldsymbol{\mu}$ at $\boldsymbol{\mu}^n$.

It may be observed that $\mathbf{Q}^{n+1}$ satisfies the mass-balance equation as the first row of Eq. (11) is reduced to $-\mathbf{A}\mathbf{Q}^{n+1}-\boldsymbol{\mu}^{n+1}=\mathbf{D}$.

Eq. (11) may be further reduced by applying a block Gaussian elimination to the system

$$
\begin{aligned}
\mathbf{Q}^{n+1}&=\mathbf{Q}^n-\mathbf{J}_n^{-1}[\mathbf{h}(\mathbf{Q}^n)-\mathbf{A}^T\mathbf{H}^{n+1}-\mathbf{B}^T\mathbf{H}^f]\\
\boldsymbol{\mu}^{n+1}&=\boldsymbol{\mu}^n-\mathbf{K}_n^{-1}[g(\boldsymbol{\mu}^n)-\mathbf{H}^{n+1}+\mathbf{Z}]
\end{aligned}\tag{12}
$$

where $\mathbf{H}^{n+1}$ = solution of the linear system

$$
\begin{gathered}
(\mathbf{A}\mathbf{J}_n^{-1}\mathbf{A}^T+\mathbf{K}_n^{-1})\mathbf{H}^{n+1}=\cdots\\
\mathbf{A}\mathbf{J}_n^{-1}[\mathbf{h}(\mathbf{Q}^n)-\mathbf{B}^T\mathbf{H}^f]+\mathbf{K}_n^{-1}[g(\boldsymbol{\mu}^n)+\mathbf{Z}]-[\mathbf{A}\mathbf{Q}^n+\boldsymbol{\mu}^n+\mathbf{D}]
\end{gathered}\tag{13}
$$

Eqs. (12) and (13) are the foundation for the so-called hybrid methods (Hamam and Brameller 1971; Carpentier et al. 1985; Todini and Pilati 1988). One implementation of these hybrid methods is the global gradient algorithm (GGA) used in *EPANET*, which follows from the application of the Newton method to the system shown in Eq. (10).

Modified Orifice Formulation

The modified orifice formulation has two important differences to the power equation: it requires the addition of two fictional pipe-reservoir systems to each node to simulate the two terms in Eq. (3), and the exponents of the two terms are fixed at 0.5 and 1.5.

The modified orifice equation [Eq. (3)] is written as follows:

$$
\mu_i=\mu_{1i}+\mu_{2i}=c_{1i}(H_i-z_i)^{0.5}+c_{2i}(H_i-z_i)^{1.5},\quad \text{if } H_i\geq z_i \tag{14}
$$

where μ_{1i} = lumped leakage outflow corresponding to exponent 0.5; μ_{2i} = lumped leakage outflow corresponding to exponent 1.5; coefficient c_{1i} = sum of $\sqrt{2g}C_dA_0$; and c_{2i} = sum of $\sqrt{2g}C_dm$ of all the individual leaks lumped to node i.

Including the modified orifice equation in the initial system in Eq. (8) leads to

$$
\begin{aligned}
&-\sum_{j=1}^{np}A_{ij}Q_j-\mu_{1i}-\mu_{2i}=D_i,\ i=1,\cdots,nu\\
&h_j(Q_j)-\sum_{i=1}^{nu}A_{ij}H_i=\sum_{i=1}^{nf}B_{ij}H_i^f,\ j=1,\cdots,np\\
&c_{1i}^{-2}[\mu_{1i}]\mu_{1i}+h_i^{CV}(\mu_{1i})-H_i=-z_i,\ i=1,\cdots,nu\\
&c_{2i}^{-\frac{2}{3}}[\mu_{2i}]^{-\frac{1}{3}}\mu_{2i}+h_i^{CV}(\mu_{2i})-H_i=-z_i,\ i=1,\cdots,nu
\end{aligned}\tag{15}
$$

The equivalent matrix form is

$$
\begin{aligned}
-\mathbf{A}\mathbf{Q}-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2&=\mathbf{D}\\
\mathbf{h}(\mathbf{Q})-\mathbf{A}^T\mathbf{H}&=\mathbf{B}^T\mathbf{H}^f\\
\mathbf{g}^F(\boldsymbol{\mu}_1)-\mathbf{H}&=-\mathbf{Z}\\
\mathbf{g}^V(\boldsymbol{\mu}_2)-\mathbf{H}&=-\mathbf{Z}
\end{aligned}\tag{16}
$$

where $\mathbf{g}^F(\boldsymbol{\mu}_1)$ = vector of link head losses at fixed-area fictional pipes; and $\mathbf{g}^V(\boldsymbol{\mu}_2)$ = vector of link head losses at variable-area fictional pipes.

The method of Newton applied to the system shown in Eq. (16) consists of sequentially solving the linear system

$$
\begin{pmatrix}
-\mathbf{A} & -\mathbf{I}_{nu,nu} & -\mathbf{I}_{nu,nu} & 0_{nu,nu}\\
\mathbf{J}_n & 0_{np,nu} & 0_{np,nu} & -\mathbf{A}^T\\
0_{nu,np} & \mathbf{F}_n & 0_{nu,nu} & -\mathbf{I}_{nu,nu}\\
0_{nu,np} & 0_{nu,nu} & \mathbf{V}_n & -\mathbf{I}_{nu,nu}
\end{pmatrix}
\begin{pmatrix}
\mathbf{Q}^{n+1}-\mathbf{Q}^n\\
\boldsymbol{\mu}_1^{n+1}-\boldsymbol{\mu}_1^n\\
\boldsymbol{\mu}_2^{n+1}-\boldsymbol{\mu}_2^n\\
\mathbf{H}^{n+1}-\mathbf{H}^n
\end{pmatrix}
=\begin{pmatrix}
\mathbf{A}\mathbf{Q}^n+\boldsymbol{\mu}_1^n+\boldsymbol{\mu}_2^n+\mathbf{D}\\
-\mathbf{h}(\mathbf{Q}^n)+\mathbf{A}^T\mathbf{H}^n+\mathbf{B}^T\mathbf{H}^f\\
-\mathbf{g}^F(\boldsymbol{\mu}_1^n)+\mathbf{H}^n-\mathbf{Z}\\
-\mathbf{g}^V(\boldsymbol{\mu}_2^n)+\mathbf{H}^n-\mathbf{Z}
\end{pmatrix}\tag{17}
$$

where $\mathbf{F}_n=\nabla_{\boldsymbol{\eta}_1}\mathbf{g}^F(\boldsymbol{\mu}_1^n)$ = matrix Jacobian of the $\mathbf{g}^F$ function with regard to $\boldsymbol{\mu}_1$ at $\boldsymbol{\mu}_1^n$; and $\mathbf{V}_n=\nabla_{\boldsymbol{\eta}_2}\mathbf{g}^V(\boldsymbol{\mu}_2^n)$ = matrix Jacobian of the $\mathbf{g}^V$ function with regard to $\boldsymbol{\mu}_2$ at $\boldsymbol{\mu}_2^n$.

It reduced to the update formulas for the flow rate on the network graph and the leakage components

$$
\begin{aligned}
\mathbf{Q}^{n+1}&=\mathbf{Q}^n-\mathbf{J}_n^{-1}[\mathbf{h}(\mathbf{Q}^n)-\mathbf{A}^T\mathbf{H}^{n+1}-\mathbf{B}^T\mathbf{H}^f]\\
\boldsymbol{\mu}_1^{n+1}&=\boldsymbol{\mu}_1^n-\mathbf{F}_n^{-1}[\mathbf{g}^F(\boldsymbol{\mu}_1^n)-\mathbf{H}^{n+1}+\mathbf{Z}]\\
\boldsymbol{\mu}_2^{n+1}&=\boldsymbol{\mu}_2^n-\mathbf{V}_n^{-1}[\mathbf{g}^V(\boldsymbol{\mu}_2^n)-\mathbf{H}^{n+1}+\mathbf{Z}]
\end{aligned}\tag{18}
$$

where $\mathbf{H}^{n+1}$ = solution of the linear system

$$
\begin{gathered}
(\mathbf{A}\mathbf{J}_n^{-1}\mathbf{A}^T+\mathbf{F}_n^{-1}+\mathbf{V}_n^{-1})\mathbf{H}^{n+1}=\mathbf{A}\mathbf{J}_n^{-1}[\mathbf{h}(\mathbf{Q}^n)-\mathbf{B}^T\mathbf{H}^f]+\cdots\\
\mathbf{F}_n^{-1}[\mathbf{g}^F(\boldsymbol{\mu}_1^n)+\mathbf{Z}]+\mathbf{V}_n^{-1}[\mathbf{g}^V(\boldsymbol{\mu}_2^n)+\mathbf{Z}]-[\mathbf{A}\mathbf{Q}^n+\boldsymbol{\mu}_1^n+\boldsymbol{\mu}_2^n+\mathbf{D}]
\end{gathered}\tag{19}
$$

Uniqueness and Convergence Properties

The existence, uniqueness of solution, and convergence properties of the power and modified orifice formulations may be investigated by introducing the minimization problem equivalent to the system in Eq. (16)

$$\min C(\mathbf{Q}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) := \sum_{j=1}^{np} C_j(Q_j) + \sum_{i=1}^{nu} C_i^F(\mu_{1i}) + \sum_{i=1}^{nu} C_i^V(\mu_{2i})$$

$$\text{subject to:} -\sum_{j=1}^{np} A_{ij} Q_j - \mu_{1i} - \mu_{2i} = D_i, i = 1, \ldots, nu \tag{20}$$

where C = content function [a generalization of the content function introduced by Piller et al. (2003) for pressure-driven modeling (PDM) and demand-driven modeling (DDM) cases]; $C_j(Q_j) = \int_0^{Q_j} [h_j(u) - \sum_{i=1}^{nf} B_{ij} H_i^f] du$ = contribution of link j; $C_i^F(\boldsymbol{\mu}_{1i}) = \int_0^{\boldsymbol{\mu}_{1i}} [g_i^F(\eta) + z_i] d\eta$ = contribution for the fixed-area fictitious link at node i; and $C_i^V(\boldsymbol{\mu}_{2i}) = \int_0^{\boldsymbol{\mu}_{2i}} [g_i^V(\eta) + z_i] d\eta$ = contribution for the variable-area fictitious link at node i. C has the dimension of power and is expressed with unknown flow rates and leakage components as basic unknowns. The gradient vector of the content function has the following components:

$$\begin{aligned}
\frac{\partial C}{\partial Q_j}(\mathbf{Q}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= h_j(Q_j) - \sum_{i=1}^{nf} B_{ij} H_i^f, j = 1, \cdots, np \\
\frac{\partial C}{\partial \mu_{1i}}(\mathbf{Q}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= g_i^F(\mu_{1i}) + z_i, i = 1, \cdots, nu \\
\frac{\partial C}{\partial \mu_{2i}}(\mathbf{Q}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= g_i^V(\mu_{2i}) + z_i, i = 1, \cdots, nu
\end{aligned} \tag{21}$$

The Hessian matrix is diagonal with positive diagonal elements as follows:

$$\begin{aligned}
\frac{\partial^2 C}{\partial Q_j^2}(\mathbf{Q}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= J_{jj}(Q_j) \geq 0, j = 1, \cdots, np \\
\frac{\partial^2 C}{\partial \mu_{1i}^2}(\mathbf{Q}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= 2c_{1i}^{-2}[\mu_{1i}] + \frac{h_0}{\Delta Q^2} \max(0, -\mu_{1i}) \\
&= F_{ii}(\mu_{1i}) \geq 0, i = 1, \cdots, nu \\
\frac{\partial^2 C}{\partial \mu_{2i}^2}(\mathbf{Q}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= \frac{2}{3} c_{2i}^{-\frac{2}{3}} [\mu_{2i}]^{-\frac{1}{3}} + \frac{h_0}{\Delta Q^2} \max(0, -\mu_{2i}) \\
&= V_{ii}(\mu_{2i}) \geq 0, \mu_{2i} \neq 0, i = 1, \cdots, nu
\end{aligned} \tag{22}$$

The Hessian matrix is not defined for $\mu_{2i} = 0$, but this could be regularized to avoid division by zero and small values (Piller 1995).

C is a strictly convex function because h_j, g_i^F, and g_i^V are strictly increasing functions. This and the fact that the constraint set is convex proves the uniqueness of the solution. Moreover, C is a norm-coercive function (it tends to plus infinity if one of the components tends to plus or minus infinity). This indicates that level sets are bounded. The constraint set is nonempty and closed; C is continuous and norm-coercive, which involves the existence of a solution for problem optimization [Eq. (20)].

Eq. (16) is the sufficient Euler-Lagrange condition for optimality of the content model [Eq. (20)] (the nodal head is the Lagrange multiplier associated to the mass-balance constraint). Because of the convexity of C and full rank of the incidence matrix $\mathbf{A}$, solving the minimization problem [Eq. (20)] and the saddle point problem [Eq. (16)] is equivalent.

Limitations of the Formulations

There is no guarantee of global convergence (safe convergence whatever the initial point), even if Eq. (16) is a stationary condition for a convex minimization problem (minimization of the content). The Newton procedure can be unstable, and the introduction of a damping (or under relaxation) factor has been proposed to get the global convergence under some conditions (Ortega and Rheinboldt 1970). Nonconvergence cases may occur in *EPANET* with an emitter exponent greater than or equal to 2: The corresponding head-loss function on the fictitious pipe is sublinear, and the first derivative of the head-loss function may not be defined near the solution. For power equations with an exponent between 1 and 2, the number of iterations generally increases with increasing exponent, but it does converge. Similar reasoning can be done when comparing the GGA that is using head-loss functions with exponents close to 2 and the Newton method applied to nodal equations that are using the inverse head functions with exponents close to 0.5. The GGA is preferable because it produces fewer iterations in general. So, these limitations are common to all hydraulic solvers but depend on the formulation used.

Another known problem in *EPANET* is linked with the presence of zero-flow conditions ($\mathbf{J}_n$ is no longer invertible for zero flows, or the ratio of the smallest to the largest elements on the diagonal of $\mathbf{J}_n$ is very large). Elhay and Simpson (2011) suggested a regularization technique that limits the bounds of the Schur complement eigenvalues. Only a modest extra computational cost is incurred when the technique is applied with a sufficiently large bounded range. Other techniques may also be applied such as damping (plus a positive coefficient) on the head-loss Jacobian by Carpentier et al. (1985) or a modification of the Hazen-Williams friction head-loss function near zero to force the slope to be greater than a fixed suitable number (Piller 1995).

With the iterates in Eqs. (18) and (19), the first limitation is corrected as improved GGA is used with a pipe head loss with an exponent close to 2 and two emitters with exponents of 0.5 and 1.5, producing head loss on fictitious pipes with exponents of 2 and 2/3. For correcting the second problem of zero flow in *EPANET* or other hydraulic solver, one of the techniques described previously may be used (e.g., Piller 1995).

Application to Example Networks

The modified orifice model was implemented in the public-domain *EPANET* hydraulic modeling package and the implications of the modification were tested on three networks of different sizes with stochastic leak distributions. The aim of this part of the study was to compare the proposed modified orifice leakage model with the conventional power leakage model in terms of simulation accuracy and convergence. This was done for the normal diurnal demand patterns, and cases in which the system pressure was significantly reduced, to investigate model performance under pressure management conditions.

The modified orifice formulation was implemented by modifying the *EPANET* source code in such a way that each node can have two emitters with exponents fixed to 0.5 and 1.5.

Example Networks

Three networks of different sizes (small, medium, and large) were used to evaluate the modified orifice formulation. The small and medium networks were adapted from *EPANET's* example networks—Net1 and Net3, respectively—whereas the large network was adapted from the model of the central business district of Durban, which is a coastal city in South Africa.

Modifications were made to the example networks to ensure that they operate under gravity and that each was supplied from a single point. Although the example networks were modified, the formulations can be used to model networks with pumps and multiple sources. The purpose of the modifications was to control the pressures that were used in this study. Each network was simulated at

Table 1. System Properties of the Small, Medium, and Large Water-Distribution Systems Used in This Study

Network	Number of junctions	Total pipe length (km)	Pipe diameter range (mm)	Initial AZP (m)	AZP after pressure management (m)
Small	8	19.3	150–450	81.2	21.8
Medium	85	60.0	200–750	86.7	37.6
Large	747	103.8	25–800	107.8	28.3

two input pressures selected for the purposes of this study: a high initial pressure and a low pressure representing the implementation of pressure management to manage leakage. The difference between the initial and postpressure management pressures were made intentionally large to allow simulation errors to be studied. These pressures are not unfeasible, but they were not selected to represent typical distribution systems.

Table 1 summarizes the properties of the three example networks, including the average diurnal pressures at their average zonal points (AZPs), defined as the pressure at the node with an average static water pressure (AWWA 2016).

Leakage Generation and Distribution

To implement realistic leakage distributions, leaks were generated and allocated to system nodes according to a stochastic model proposed by Schwaller and van Zyl (2015). One leak was generated at a time on the basis of the statistical distributions described by Schwaller and van Zyl (2015), summarized as follows:

- The discharge coefficient C_d was modeled by using a normal distribution.
- The initial leak area A_0 was modeled by using a lognormal distribution for background leaks and a normal distribution for potentially detectable leaks.
- The head-area slope was calculated by using a generalized power function of the initial leakage area based on the work by Cassa and van Zyl (2013).

The generated leak was then allocated to a random pipe by using a uniform distribution weighted by pipe length. Finally, a random number R between 0 and 1 was generated and used to distribute the leakage parameters between the two nodes of the pipe. On the basis of Eq. (3), the terms $RC_dA_0\sqrt{2g}$ and $(1-R)C_dA_0\sqrt{2g}$ were added to the coefficients of the emitter with an exponent of 0.5 of the two pipe nodes. Similarly, the terms $RC_dm\sqrt{2g}$ and $(1-R)C_dm\sqrt{2g}$ were added to the coefficients of the emitter with an exponent of 1.5 of the two pipe nodes.

Background leaks were generated first up to a level of 69% of the system's UARL as per the original definition of the UARL concept (Lambert 2009). To replicate realistic systems in which the leakage exponent of background leakage is often found to be close to 1.5 (Lambert 2000; Lambert et al. 2013), 20% of the background leaks were assigned a head-area slope equivalent to a leakage number of 100, as was done by Schwaller and van Zyl (2015).

Potentially detectable leaks were then generated up to the desired level of leakage for the system. In this study, the level of leakage used was at an ILI of 8, which represents a system with high but not excessive leakage. Leaks were generated until the ILI was within 0.001% of the target value.

For the small network, 100% of the nodes had leakage; for the medium and large networks, approximately 90 and 70% of the nodes, respectively, had leakage.

Once all the leaks were generated, the equivalent leakage parameters for a system modeled with the conventional power leakage approach were estimated. The same system nodes with leakage in the modified orifice approach were maintained in the conventional power leakage approach. To do this, two simulations were performed at different supply pressures (a 5-m head differential was used) under minimum night flow conditions. Two average zonal night pressures and corresponding system leakage flows were then obtained and used to estimate the system leakage exponent. Once the leakage exponent was known, the leakage coefficients for identical nodal leakage flows were calculated for each node under minimum night flow conditions.

The aforementioned process was used to generate 600 random leakage distribution scenarios, whereby each of the three example networks had 200 scenarios, each with an ILI of 8.

Results

The 200 leakage scenarios in each of the three networks were simulated by using both the conventional power equation and the newly implemented modified orifice equation approaches. The results were then analyzed to allow the performance of the two algorithms to be compared in terms of simulation accuracy and convergence speed.

The modified orifice equation is based on a fundamental fluid mechanics theory, and it incorporates the linear head-area slope, which was demonstrated in several studies under elastic and viscoelastic deformation conditions. In contrast, the power equation is purely an empirical equation with no theoretical basis. Thus, when presenting the results, the modified orifice formulation was assumed to provide the true behavior, and the differences between the results of the two formulations were described as errors in the power equation formulation.

Simulation Accuracy

Fig. 1 shows the variation of the medium system's pressure at its average zonal point without leakage and under initial and pressure management conditions for both formulations. The pressure for the medium system without any leakage is also shown, and although the system had a high level of leakage, it had only a small influence on the average zonal pressure.

Because of the small influence of leakage on pressure, no differences in the pressures obtained from the modified orifice and power formulations are evident in the figure.

The volumetric error in the total daily leakage volumes (the fractional difference in total system leakage volume over a 24-h simulation) for the two leakage formulations is negligibly small for all three systems without pressure management as shown in Fig. 2.

However, after pressure management the power formulation error in the daily leakage volume becomes substantial as shown in Fig. 3. The power formulation is an empirical approach that can only be used safely within its pressure calibration range. Beccause the power formulation in the example was calibrated under minimum night flow conditions for the initial system, it is not able to accurately model system leakage at the substantially different

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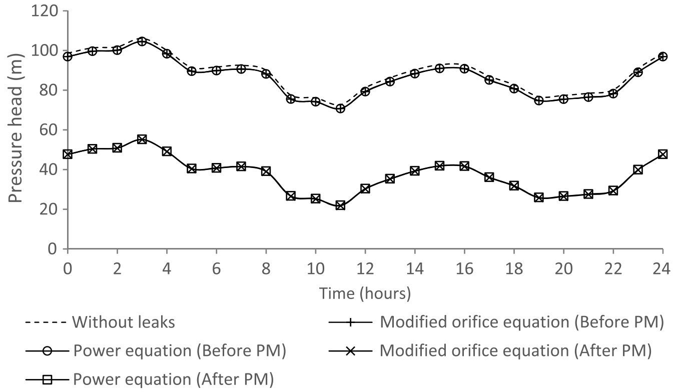

Fig. 1. Pressure head at the medium system's AZP node before and after pressure management (PM)

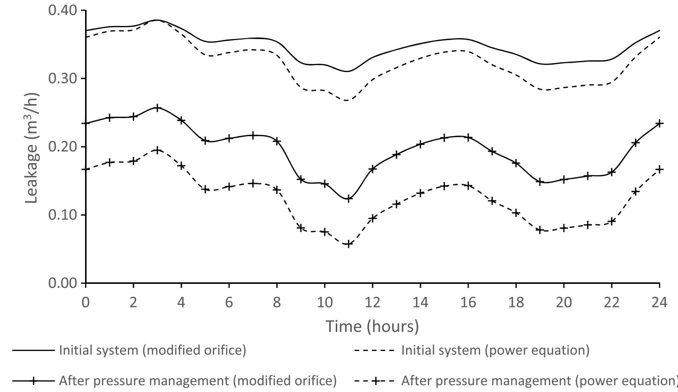

Fig. 4. Typical diurnal leakage pattern at the critical node of the medium system

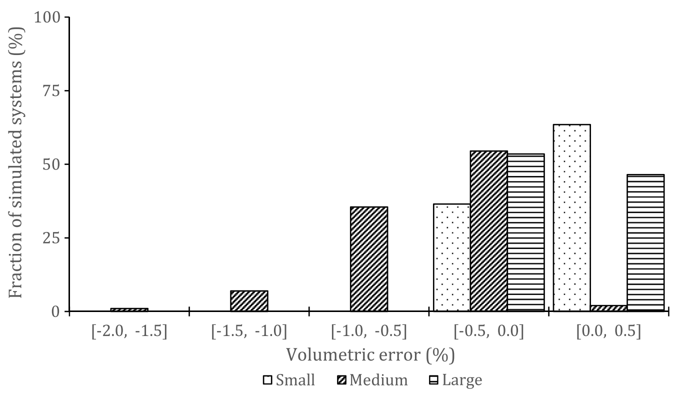

Fig. 2. Power formulation error in the total daily leakage volume for the system without pressure management

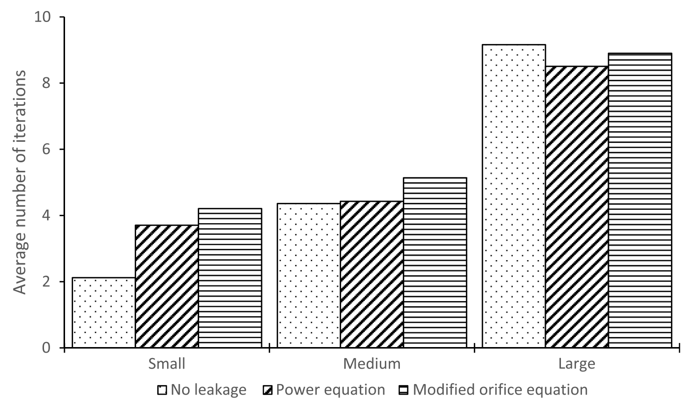

Fig. 5. Average number of iterations to achieve convergence for the different formulations

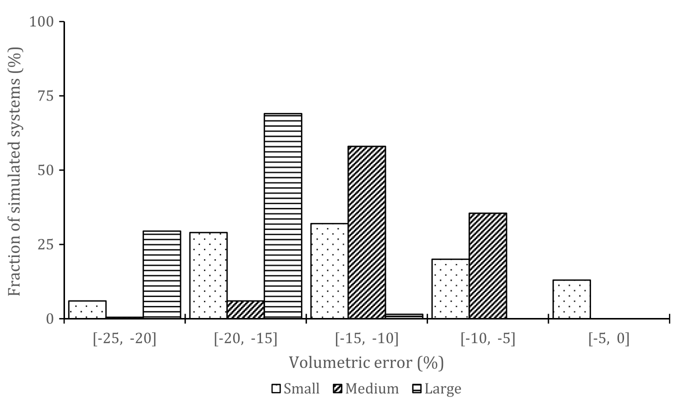

Fig. 3. Power formulation error in the total daily leakage volume for the system with pressure management

pressures resulting from pressure management. This is an important factor to consider when estimating the expected savings in leakage when planning pressure management interventions.

Finally, the results showed that, although the error in total system leakage for the systems without pressure management was negligible (Fig. 2), this was not true for the leakage volumes at individual nodes. Fig. 4 shows the typical leakage pattern at the critical node (system node with the lowest pressure) of the medium system for the two formulations, both with and without pressure management.

The two formulations give identical leakage in the initial system under minimum night flow conditions, as this is where the power formulation was calibrated. However, substantial differences are evident for other times in the initial system and for the whole day during pressure management.

Convergence

The modified orifice formulation required slightly more iterations to converge compared with the power formulation for all three systems, as shown in Fig. 5. This increase is likely caused by the increased number of unknowns in the modified orifice equation as a result of adding a second emitter to each node.

It is known that the global gradient method has convergence problems under certain conditions. Some of these problems will be experienced with both the power and modified orifice formulations, for instance, because of ill-conditioned Jacobian matrices, badly chosen initial solutions, a large range of link resistances, or a lack of smoothness in the head-loss function or its derivatives (inverse of a leakage function is like a head-loss function).

However, it was found for the power formulation that an equivalent leakage exponent greater than 2 sometimes resulted in nonconvergence of the standard global gradient algorithm. Because the modified orifice exponents are fixed at 0.5 and 1.5, this problem cannot be experienced by this formulation.

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For the power equation formulation, leakage exponents greater than 2 sometimes occurred when generating stochastic systems with high ILI values. These large leakage exponents are not unrealistic and have been observed in field and laboratory studies (Greyvenstein and van Zyl 2007) [see Schwaller and van Zyl (2015) for a summary of international field studies], and thus this problem needs to be addressed.

The reason for the nonconvergence was found to be related to conditions under which the Newton-Raphson method overshoots the root and subsequently diverges away from it. In the global gradient algorithm, the emitter behavior is simulated through the headloss function of a fictional pipe, which is a power function of flow rate with an exponent of $1/\alpha$. Nonconvergence of this function for $\alpha > 2$ is well known (e.g., Ortega and Rheinboldt 1970).

Like most other convergence problems in the global gradient algorithm, this problem may be solved by introducing a damping parameter into the algorithm as in Elhay et al. (2016).

Conclusion

This study investigated the implementation of a modified orifice formulation for a more realistic simulation of leakage in water-distribution systems. A modified hydraulic network formulation is proposed that includes a second emitter at each leakage node.

The application of the modified formulation showed that the power and modified orifice leakage formulations produced similar results for total system leakage flows and volumes under normal diurnal pressure variations. However, the leakage flows at individual nodes at elevations different from the average zonal pressure were found to differ significantly.

In addition, it was found that simulating systems at pressures significantly different from the normal diurnal range, for instance, as a result of pressure management, results in substantial errors in both leakage flows and volumes.

The average convergence speed of the modified orifice formulation was found to be slightly larger than that of the power formulation with one more iteration required on average.

The current power formulation used in the global gradient algorithm (and implemented in *EPANET*) resulted in nonconvergence when a leakage exponent greater than 2 was used. Like most other known instabilities of the Newton-Raphson method, this problem can be solved by introducing a damping factor into the convergence process.

Finally, further studies are recommended to investigate the implications of a more realistic modified orifice leakage formulation on the results of modeling studies that include leakage, such as calibration, leak detection, and operational optimization studies.

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