SURFACE MODELING

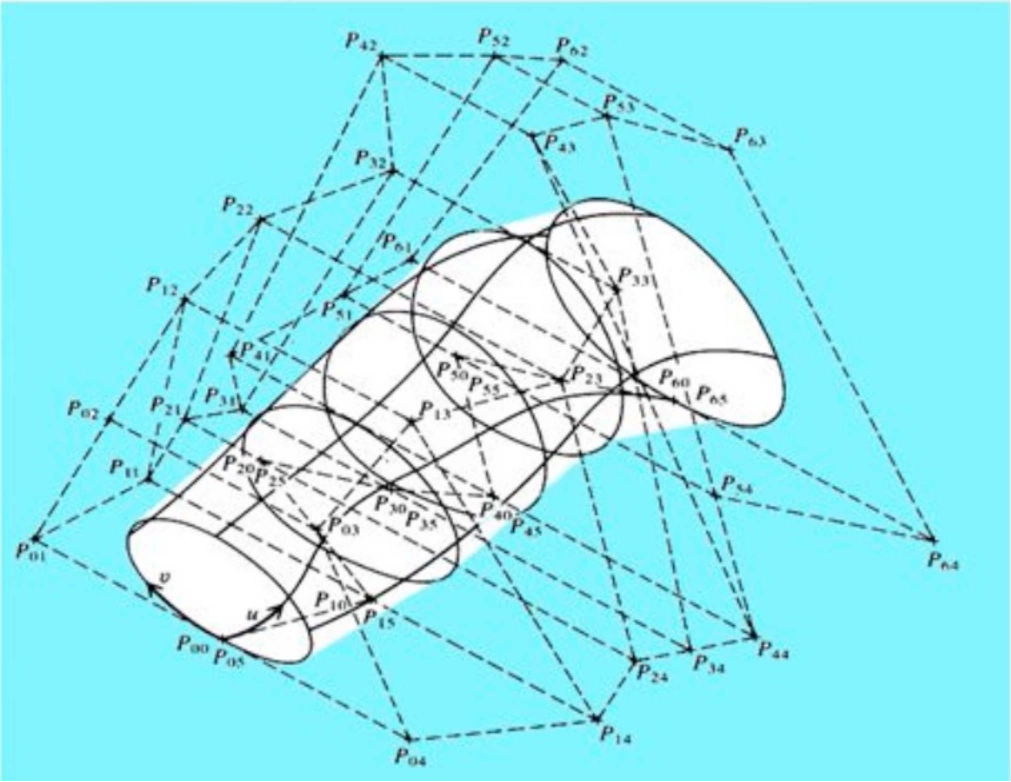
LECTURE #7

MKS 537E – Intro to CAE

Surface modeling

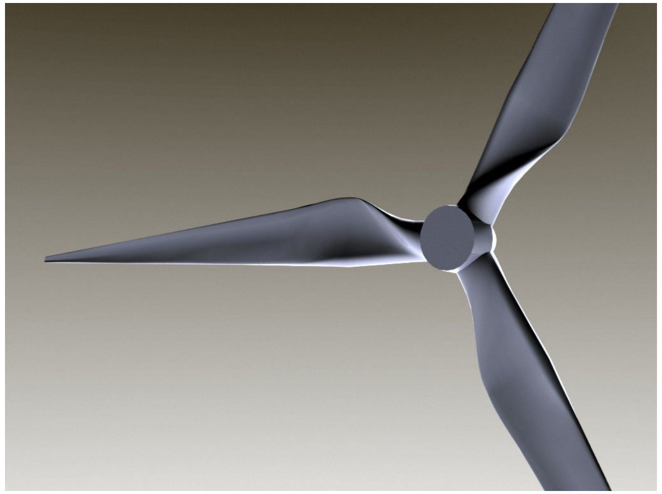
Surface modeling is a mathematical method usually provided in computer-aided design applications for displaying solid-appearing objects.

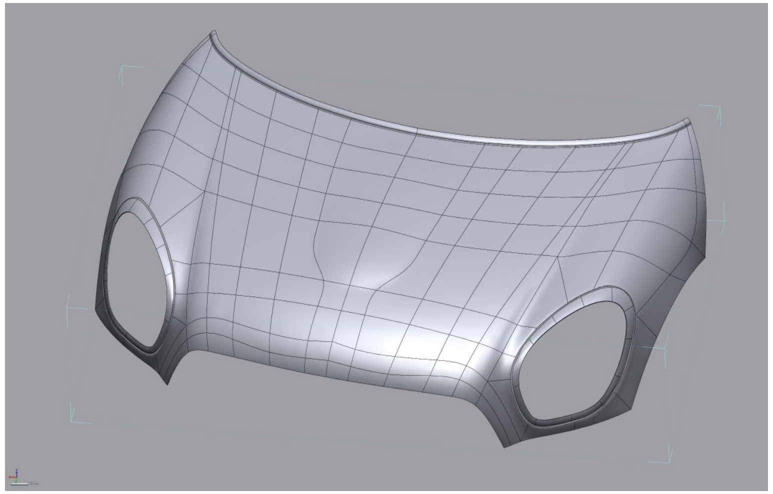
Surface modeling defines a component with greater mathematical integrity as it models the surfaces to give more definitive spatial boundaries to the design.



Surface modeling

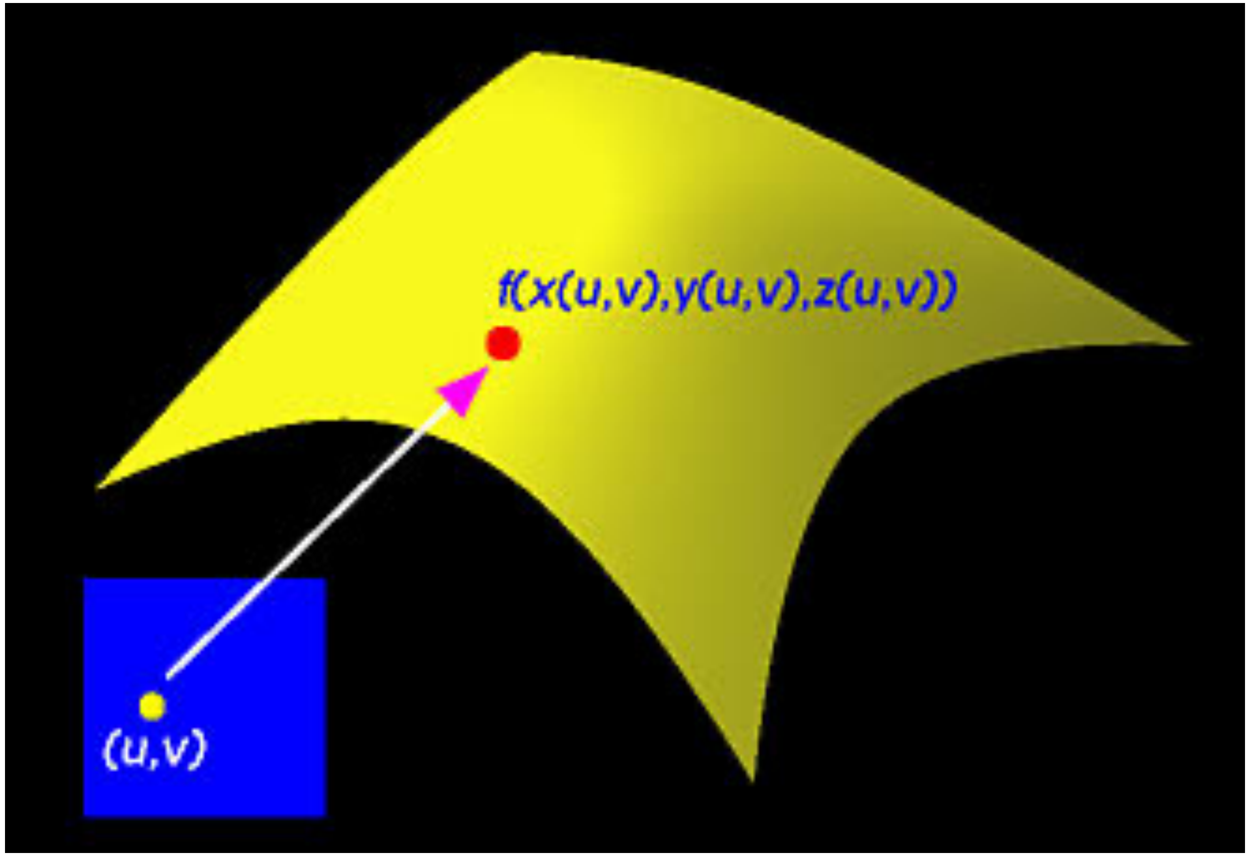
It is particularly useful for modeling objects, which can be modeled as shells, such as car body panels, aircraft fuselages or fan blades.





Surface modeling are used in

- ➢calculating mass properties
- ➢checking for interference
- ➢between mating parts
- ➢generating cross-section views
- ➢generating finite elements meshes
- ➢generating NC tool paths for
- ➢continuous path machining



Surface modeling

Advantages

- Eliminates ambiguity and non-uniqueness present in wireframe models by hiding lines not seen.
- Renders the model for better visualization and presentation, objects appear more realistic.
- Provides the surface geometry for CNC machining.
- Provides the geometry needed for mold and die design.
- Can be used to design and analyze complex freeformed surfaces (ship hulls, airplane fuselages, car bodies, …).
- Surface properties such as roughness, color and reflectivity can be assigned and demonstrated.

Disadvantages

- Surface models provide no information about the inside of an object.
- Complicated computation, depending on the number of surfaces

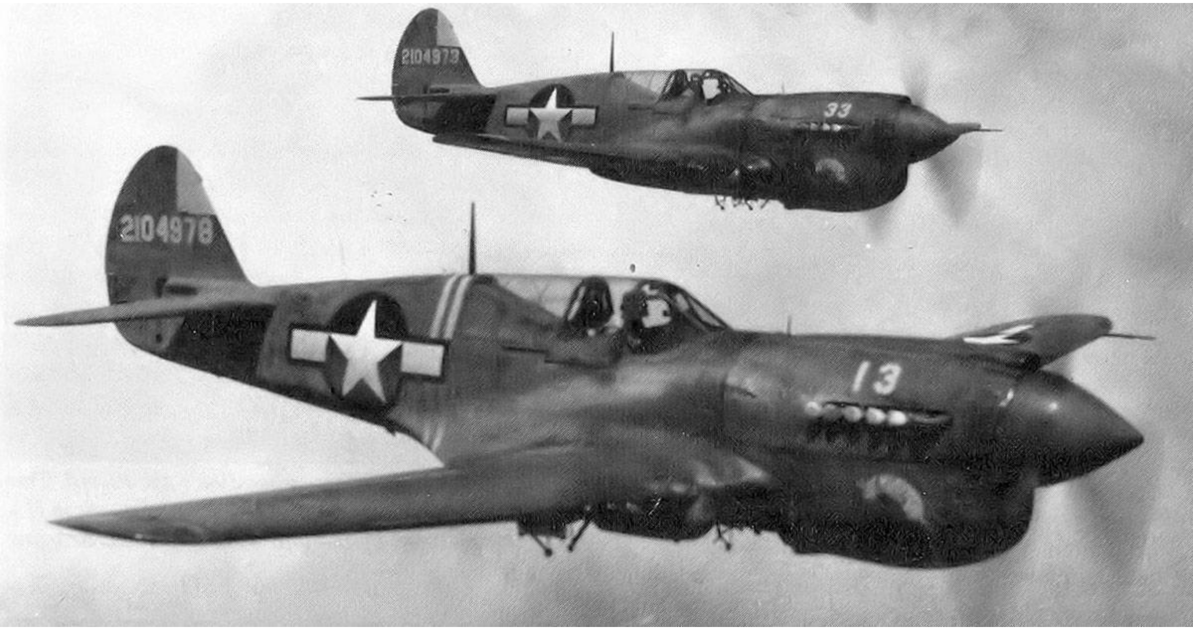
Surface modeling

In 1940s.

Surface modeling was essentially the situation in the early 1940s.

The pressures of wartime production, particularly in the aircraft industry, led to changes in the way the geometry was represented.

A book by Liming [1944] explains many of the techniques used at this time.



Liming, Roy A. Practical Analytic Geometry with Applications to Aircraft, New York: Macmillan Company, 1944.

Surface modeling

In 1950s.

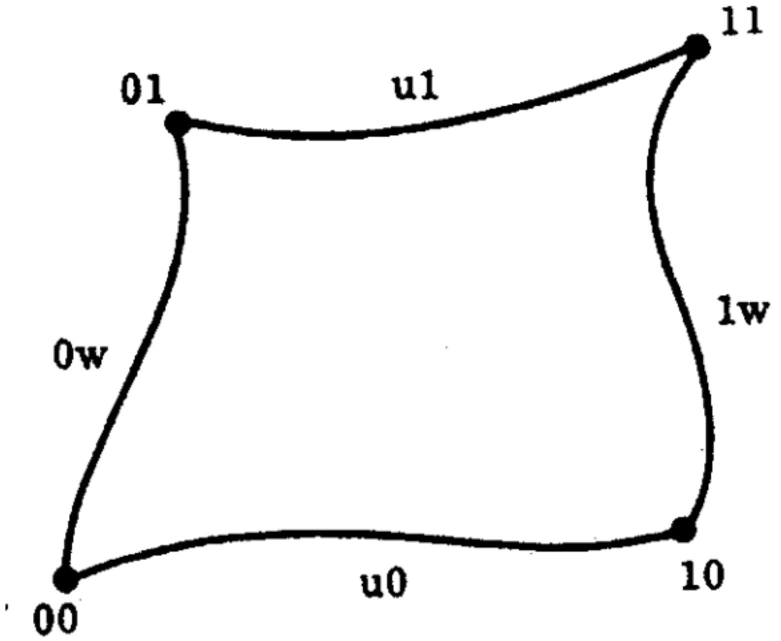
N. Lidbro describes a system used by Saab Scania in Sweden in 1956.

N. Lidbro, "Modern Aircraft Geometry: A Description of the Mathematical Method used at SAAB, Sweden, for Aircraft Dimensioning and Shape Determination", Aircraft Engineering and Aerospace Technology, Vol. 28-11, (1956) pp. 388 - 394

Surface modeling

In 1960s.

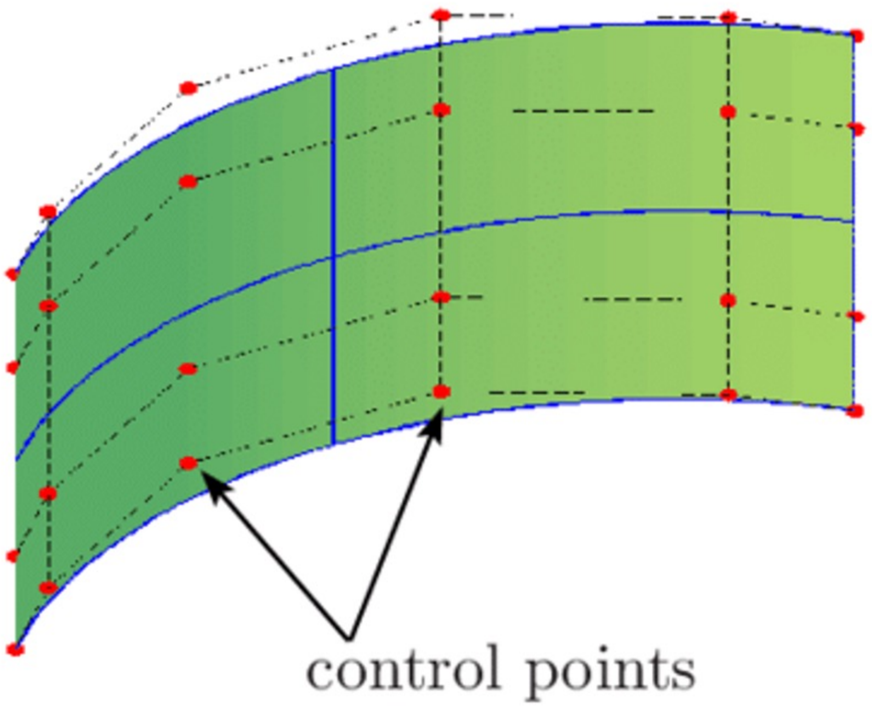
The use of parametric techniques became popular in the 1960s, largely due to the pioneering work of S.A. Coons in 1964.

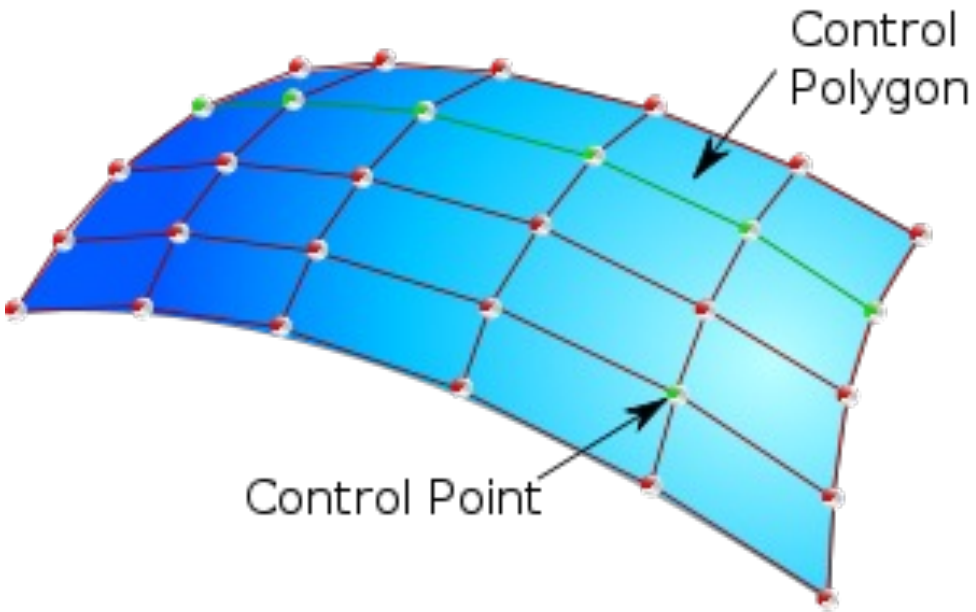


S. Coons, Surfaces for Computer Aided Design of Space Forms, Technical Report, MIT, 1967, Project MAC-TR 41.

Control points and polygon

The poles (sometimes known as *control points*) of a surface define its shape.

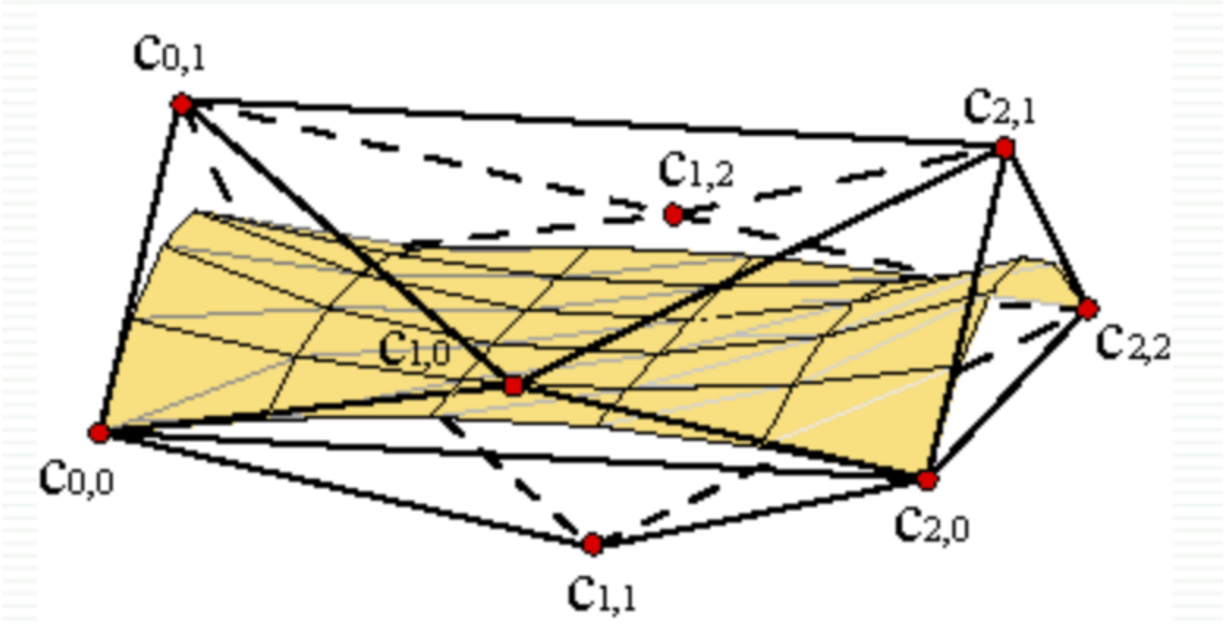




Convex hull

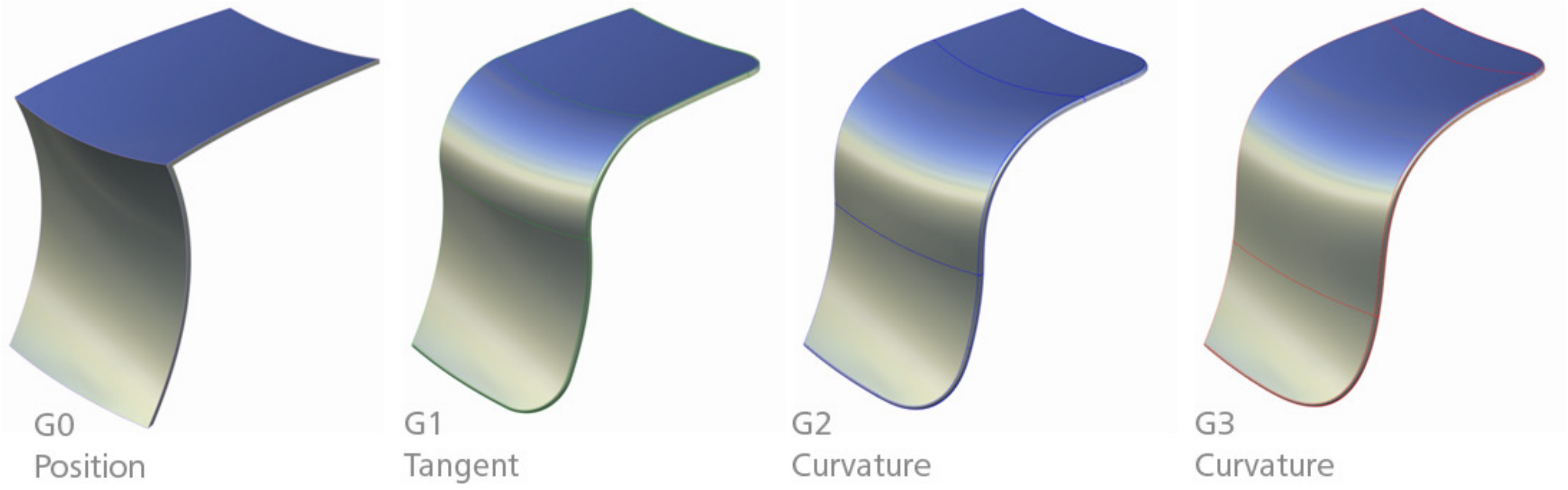
Convex hull property of the control net: the surface is included in the convex hull of the control net.

Example. For piecewise surfaces, every piece is included in the convex hull of their control net.



Geometric continuity

The smoothness of surfaces is a kind of geometric characteristic. Therefore, in constructing smooth piecewise surfaces, people usually consider only geometric continuity, namely, G^n continuity, which is irrelevant to the selected parameters.

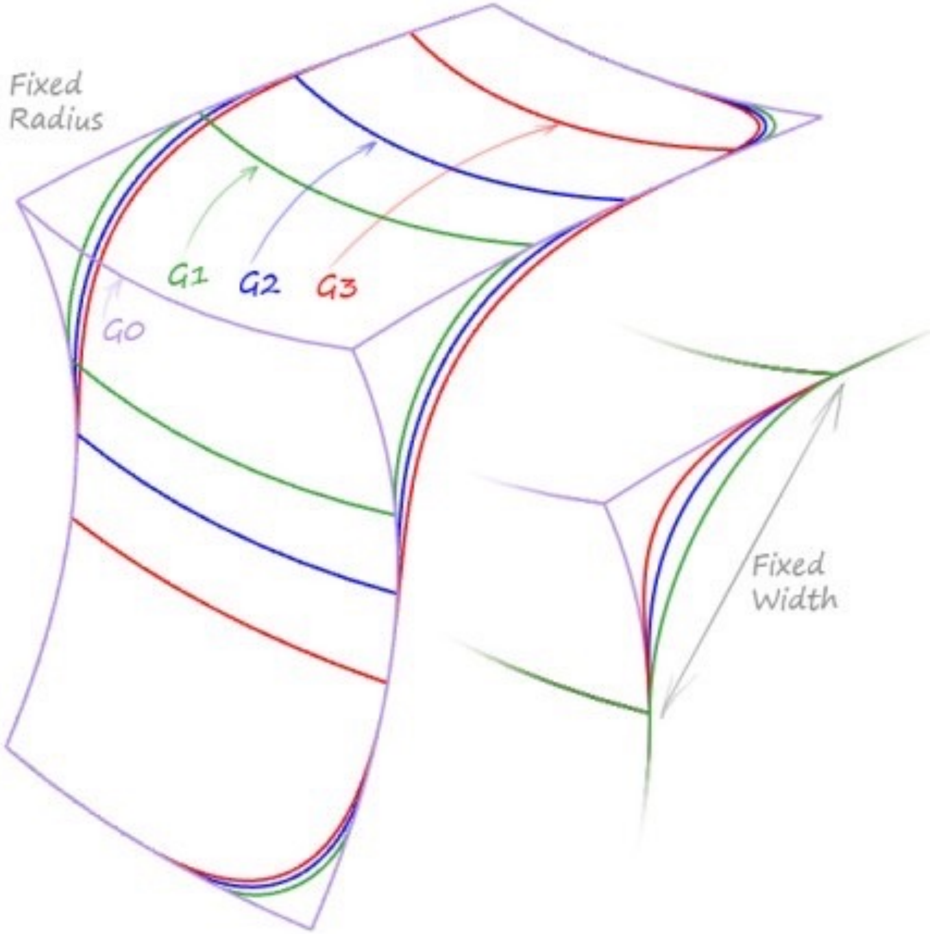


Geometric continuity

The continuity is defined using the terms

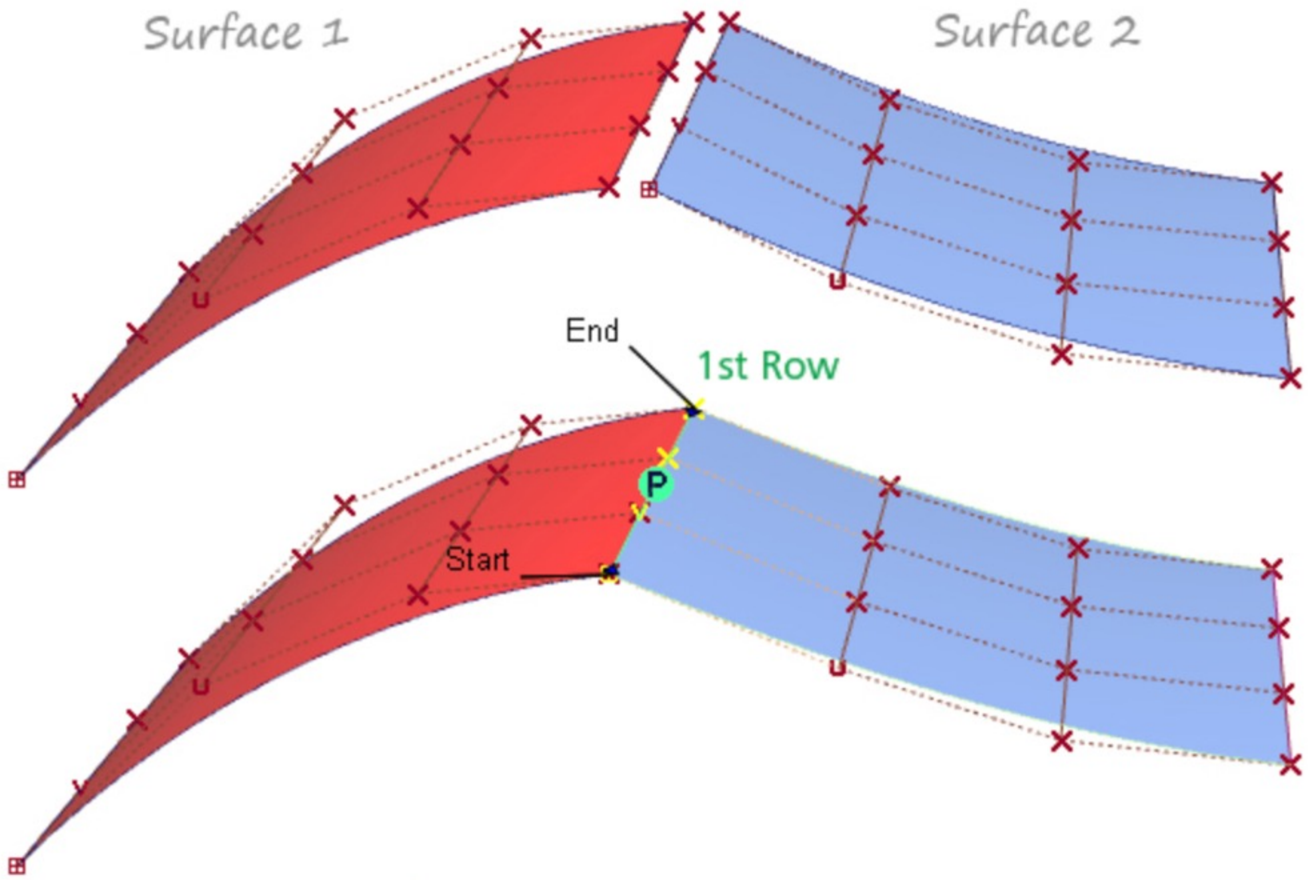
- G^0 – position (touching)
- G^1 – tangent (angle)
- G^2– curvature (radius)
- G^3 – acceleration

(rate of change of curvature)



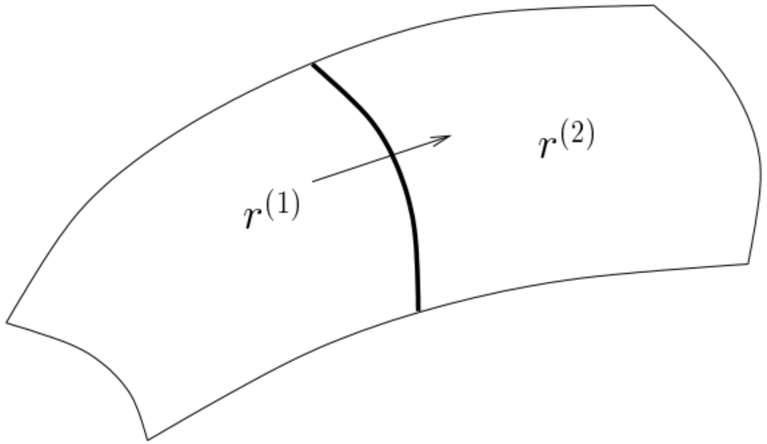
G^0 Geometric continuity

The lowest order of regularity is G^0 , which means that the surface is connected. (sharp edges)



G^1 Geometric continuity

A continuous surface is said to be tangent plane continuous, denoted by G^1 , if every point on the surface has a unique tangential plan, which varies continuously on the surface. Such a surface is also said to be geometrically continuous of order one.



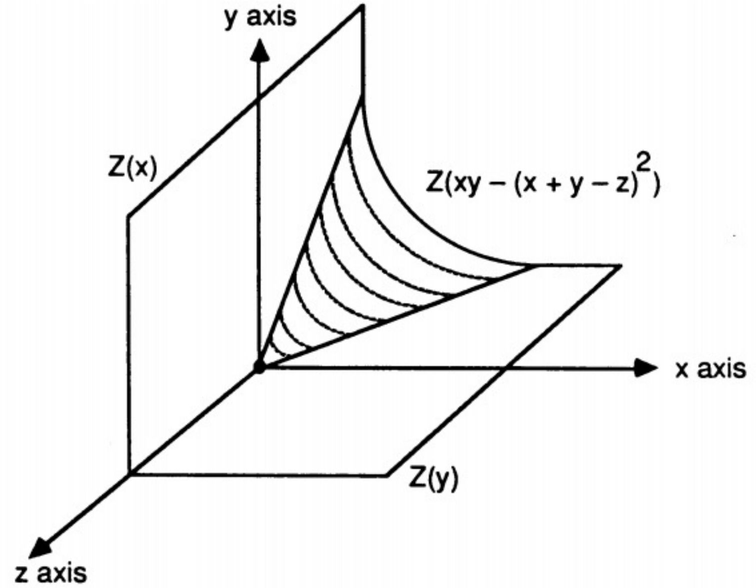
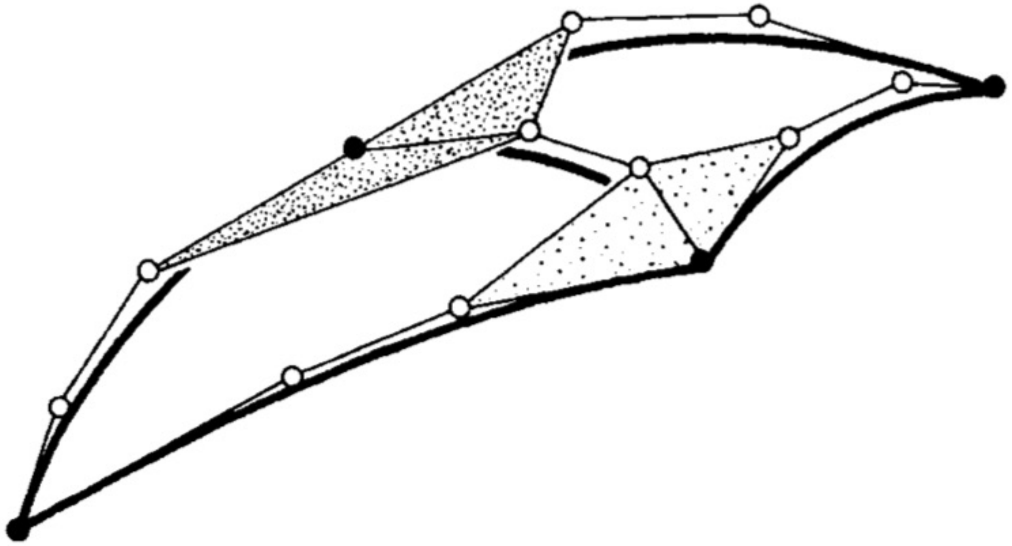
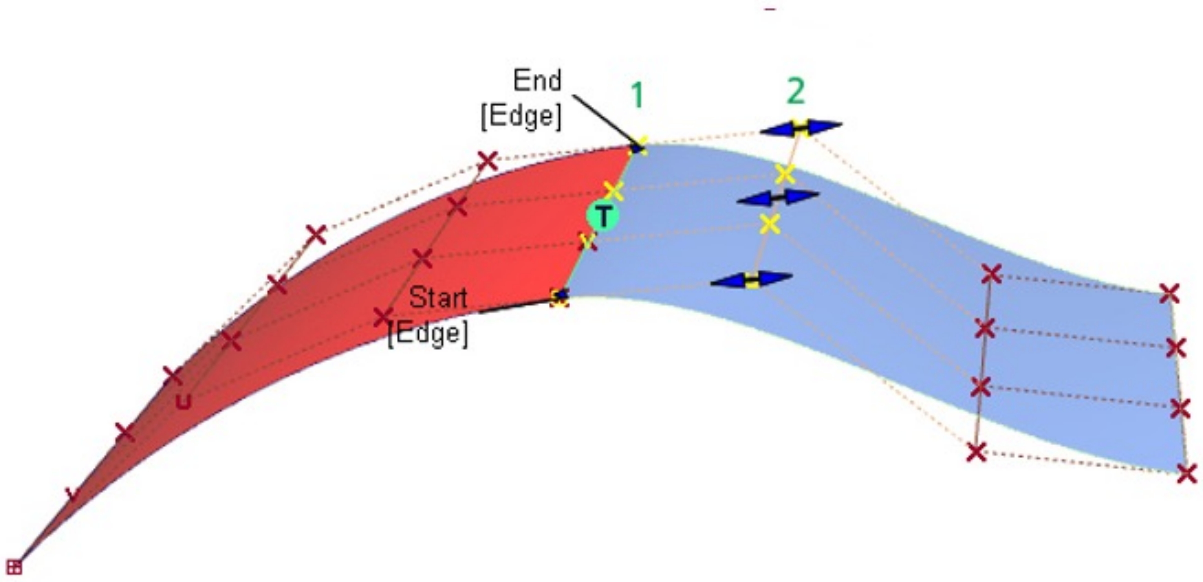


Figure 2.3 - Cone Z(xy – (x + y – z)^2) meeting Z(x) and Z(y) with G^1 - continuity

G^1 Geometric continuity

The shown cubic curves cannot be the boundary curves of two C^1 cubic Bezier triangles since no suitable pair of domain triangles can be found.

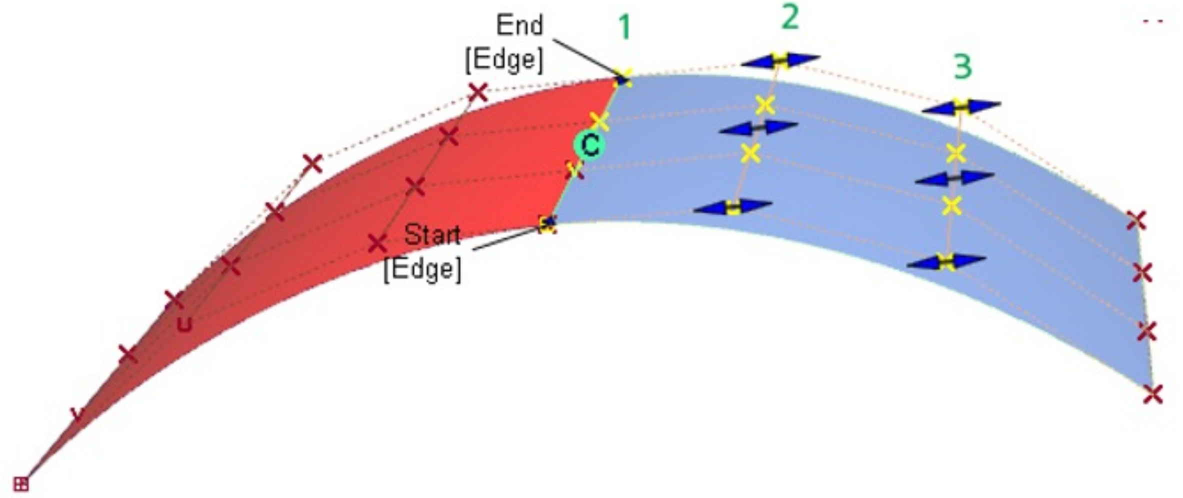




G^2 Geometric continuity

A tangent plane continuous, G^1 , surface is said to be curvature continuous, denoted by G^2 , if every point on the surface has a unique Dupin indicatrix, which varies continuously on the surface.

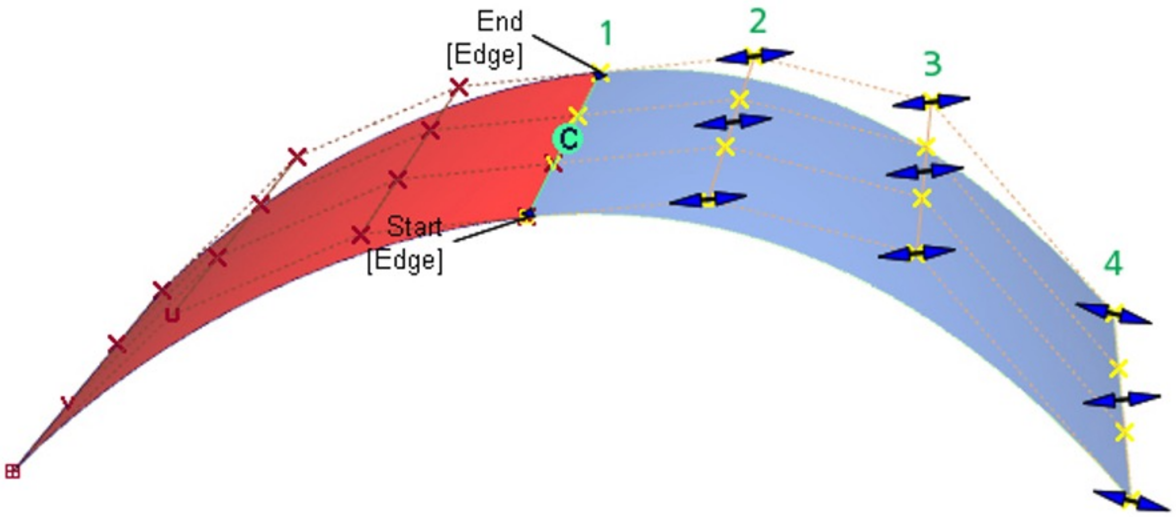
Such a surface is also said to be geometrically continuous of order two.



G^3 Geometric continuity

G^3 *continuity* follows the same process as its predecessors but controls the rate of the curvature along the curve as it transitions from one curve or surface to the other.

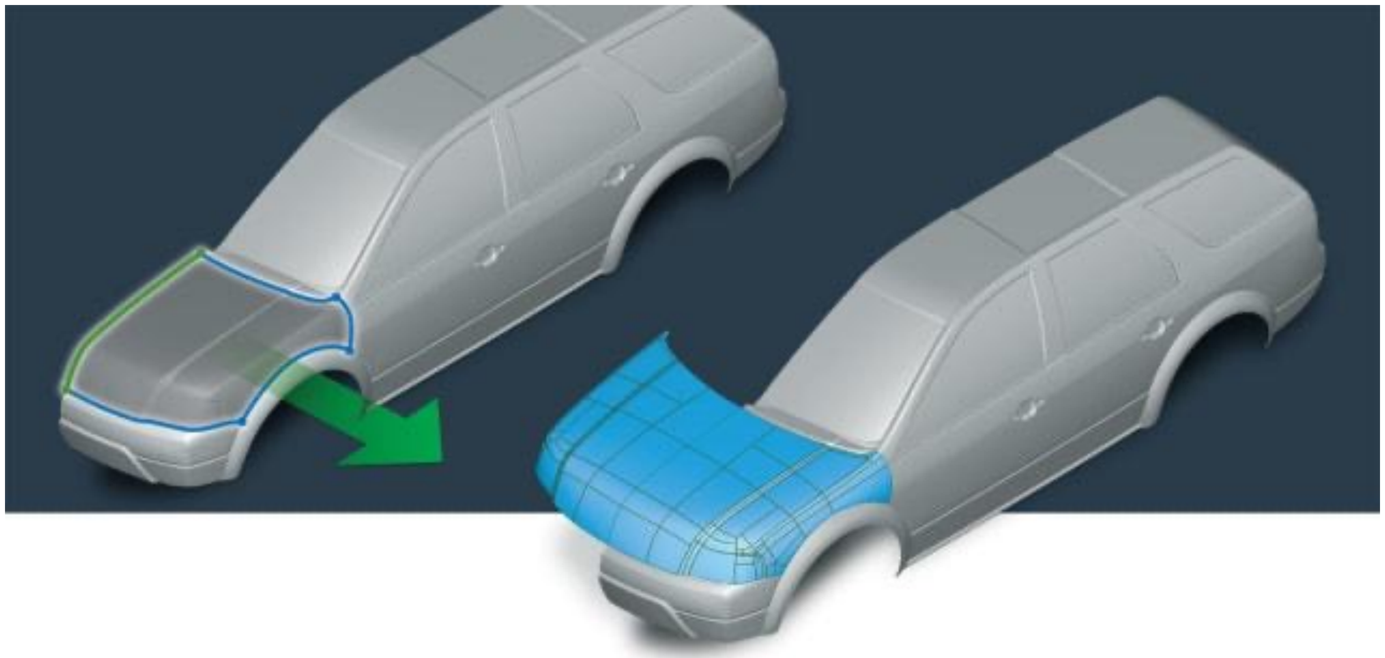
G^3 is looking for balance on the rate of curvature – in other words, that the max value of the curvature hits its peak about the middle of the transition area.



Surface model

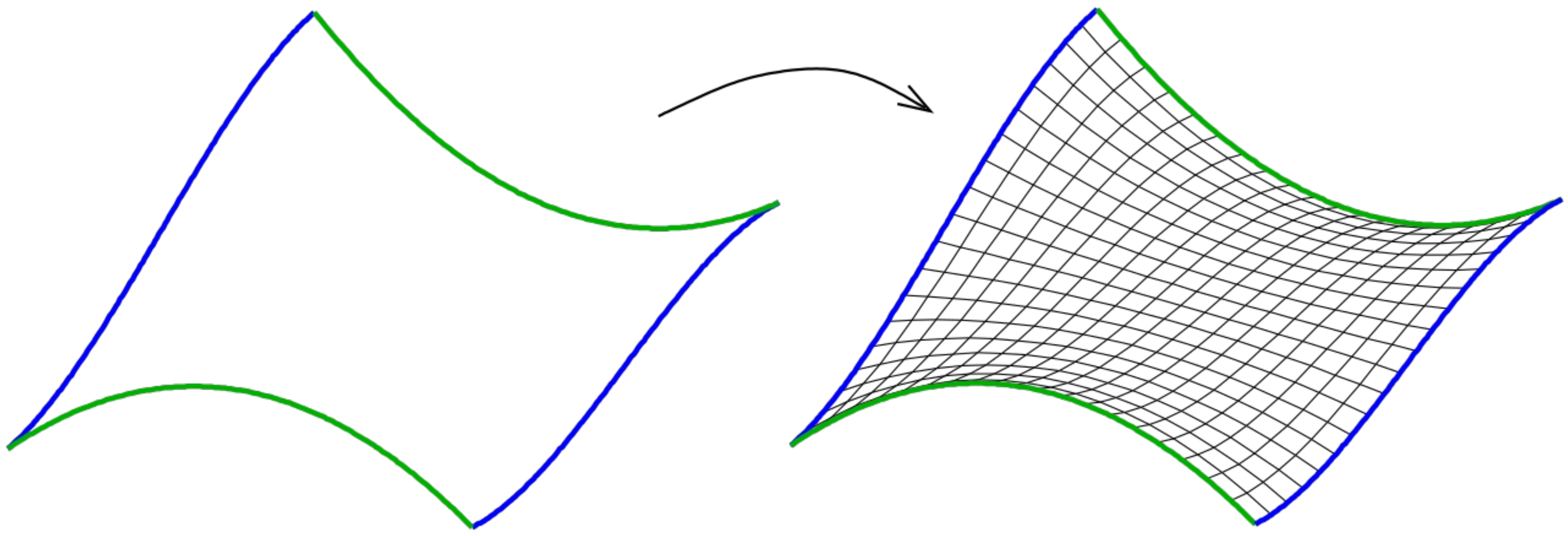
A surface model represents the skin of an object, these skins have no thickness or material type.

Surface models define the surface features, as well as the edges, of objects.



Patch definition

Complicated surfaces are usually broken down into smaller units, called patches. For example, a bicubic spline surface consists of a collection of bicubic patches.



Surface definition

In Computer Graphics, surface is an area within which every position is defined by mathematical methods.

Polynomial patches	Tensor product patches	Composite surfaces

- Planar surface
- Ruled (lofted) surface
    - Cylindrical surface
    - Conical surface
    - Developed surface
- Coons patch
- Bicubic Hermite patch
- Gordon surface
	- Bezier patches
	- B-spline surfaces
	- NURB surfaces
		- Rational Bezier surface
		- Rational B-spline surface
		- Revolved surface
		- CONS and Trimmed surface

Surface modeling

Surface modeling is more sophisticated than wireframe modeling in that it defines not only the edges of a 3D object, but also its surfaces. Surface modeling gives designers a great amount of control and flexibility.

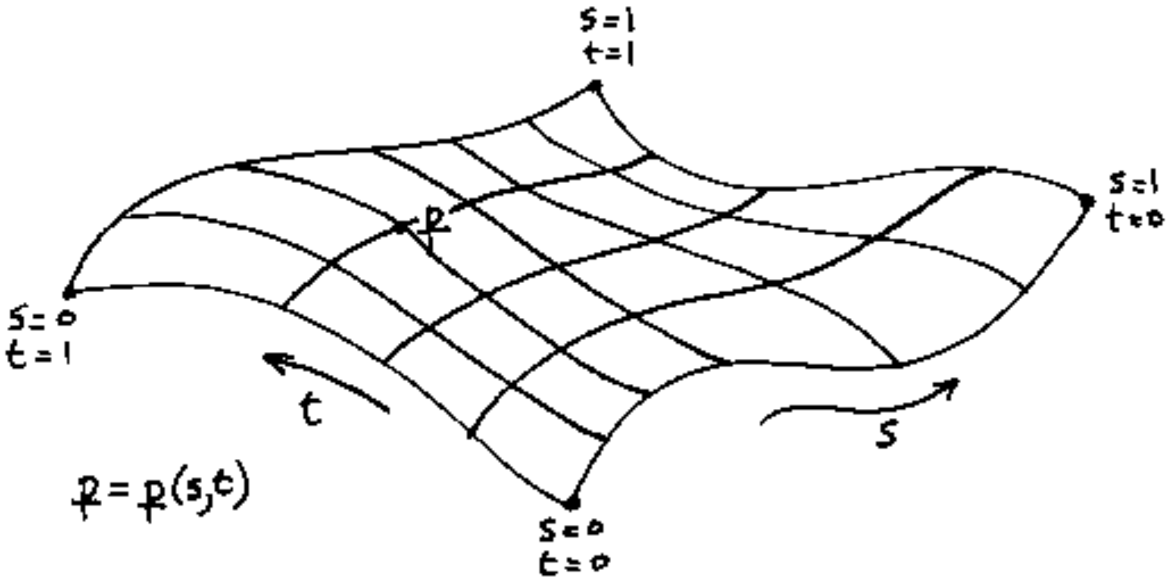
Analytic surface	Synthetic surface
- plane surfaces,	- Bezier patches
- ruled surfaces,	- B-spline surfaces
- surface of revolution,	- NURB surfaces
- tabulated surfaces	

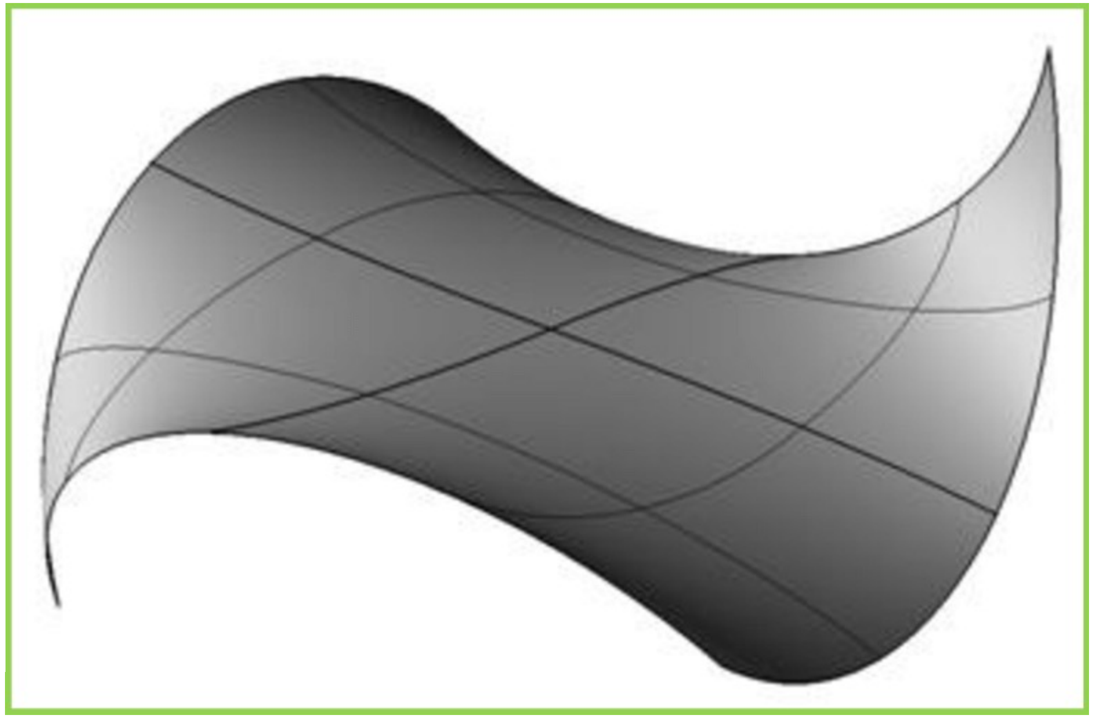
Analytic Surfaces

- **plane surfaces** : This is the simplest surface, requires 3 non-coincidental points to define an infinite plane
- **ruled surfaces** : This is a linear surface. It interpolates linearly between two boundary curves that define the surface.
- **surface of revolution** : This is an axi-symmetric surface that can model axi-symmetric objects. It is generated by rotating a planar wire frame entity in space about the axis of symmetry of a given angle.
- **tabulated surfaces** : This is a surface generated by translating a planar curve a given distance along a specified direction.

Sculptured Surfaces

These are also called Free Form Surfaces. These are created by spline curves in one or both directions in a 3-D space. These surfaces are used in the manufacture of car body panels, aircraft structures..

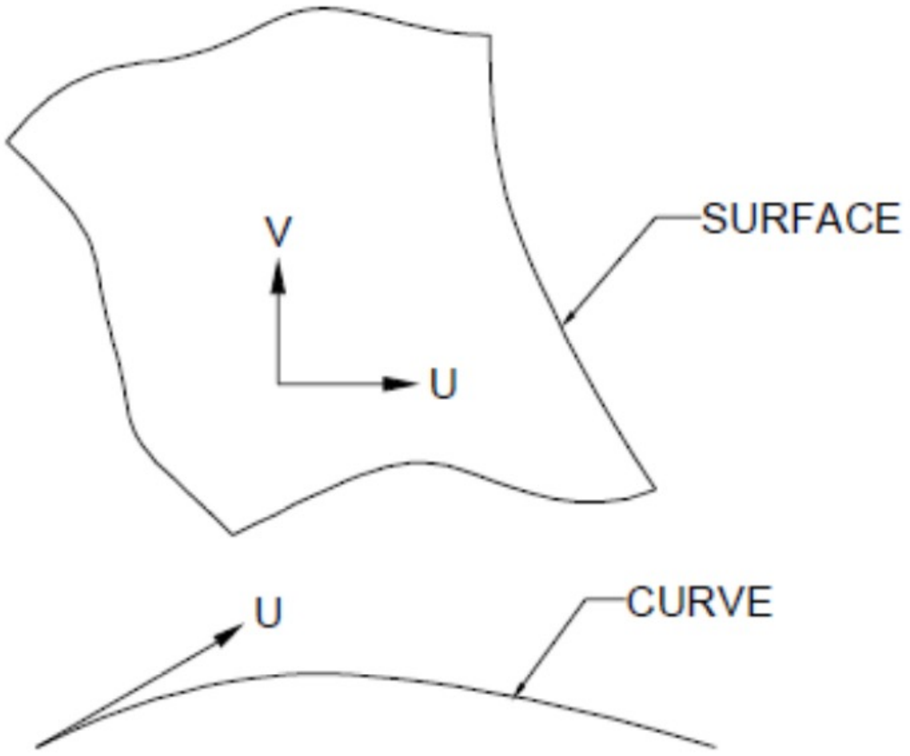




Degree of freedom

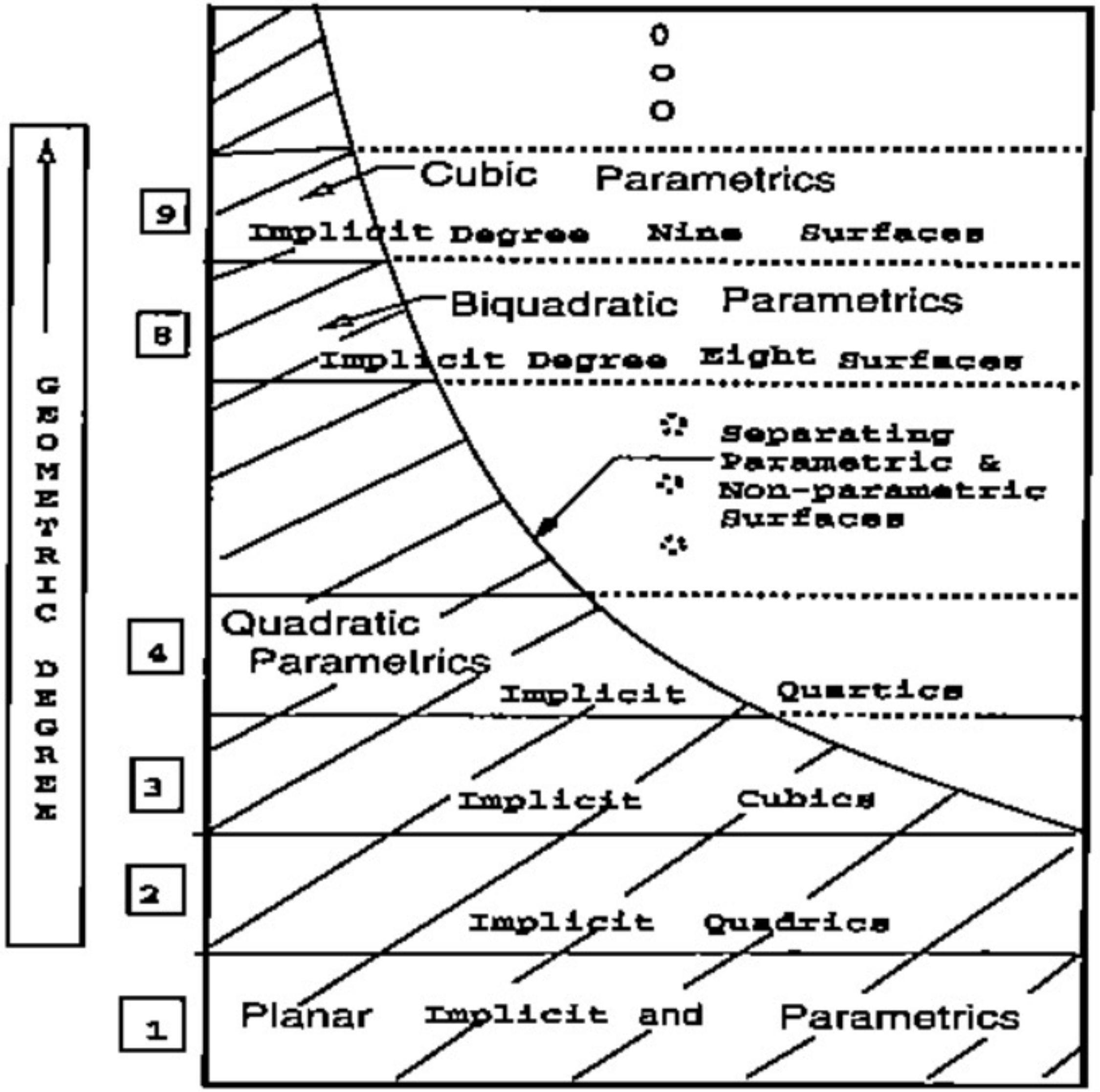
A curve has one degree of freedom while a surface has two degrees of freedom.

This means that a point on a curve can be moved in only one independent direction while on surfaces it has two independent directions to move.



Polynomial patches

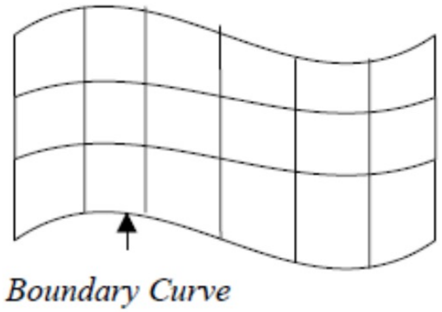
A Classification of Low Degree Algebraic Surfaces



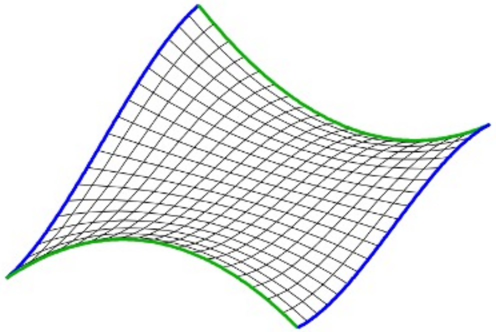
Polynomial patches

Planar surface

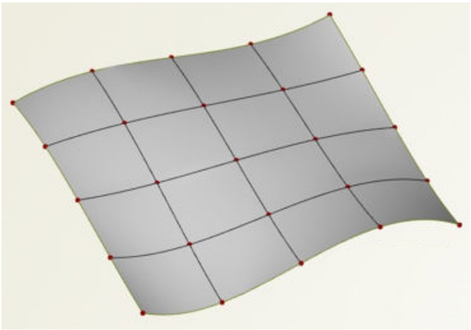
Ruled (lofted) surface



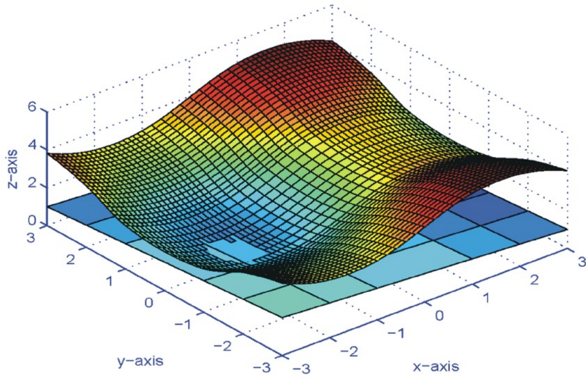
Coons patch



Gordon surface



Bicubic Hermite patch



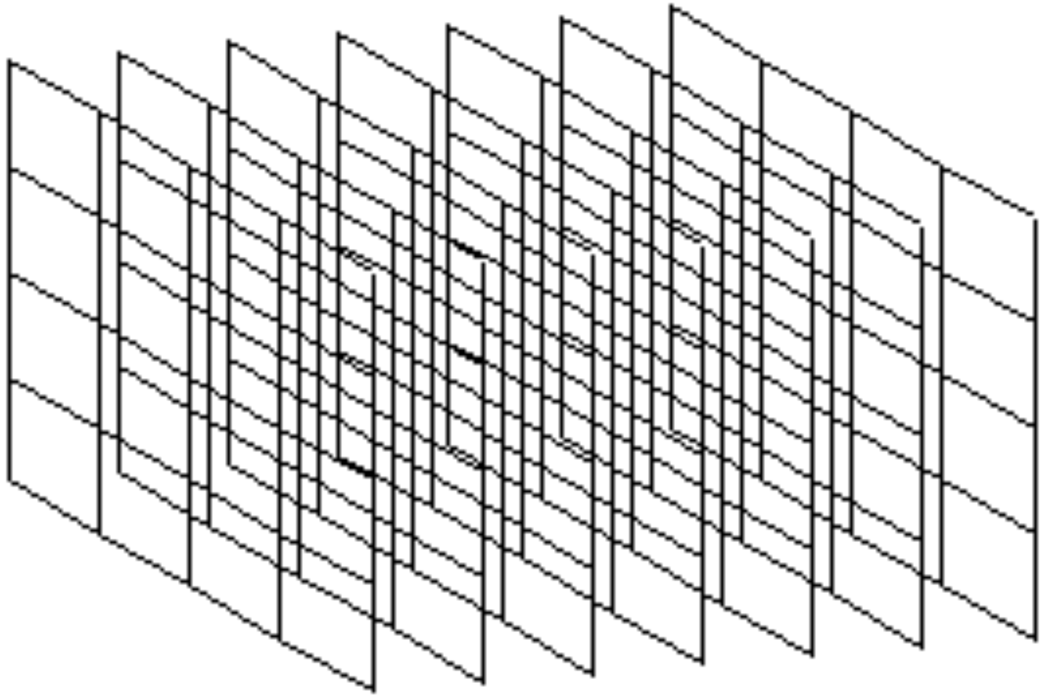
Planar surface

POLYNOMIAL PATCHES

Planar surface

A planar surface is a very simple mesh surface shading tool for basic geometry and surfaces.

The plane surface can be used to generate cross-sectional views by intersecting a surface model with it, generate cross sections for mass property calculations, or other similar applications where a plane is needed.



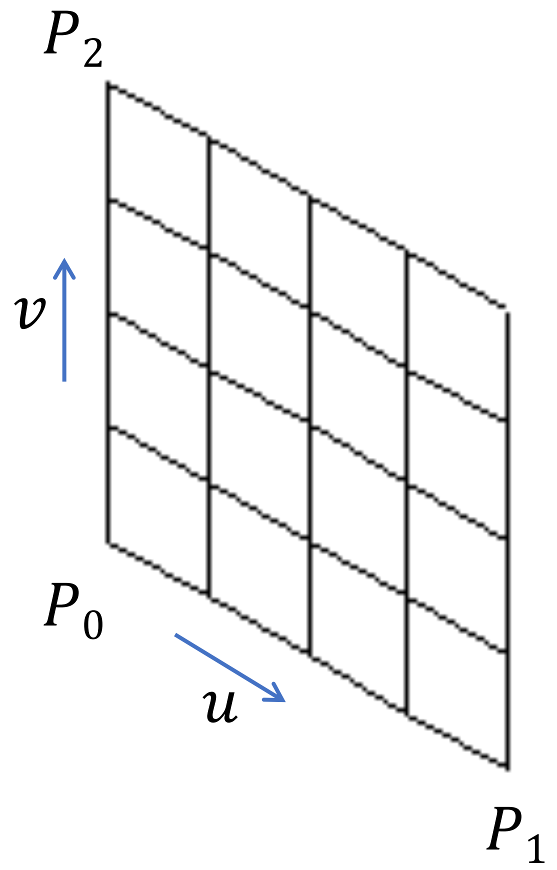
Planar surface

This is the simplest surface. It requires three non-coincident points to define an infinite plane.

A plane surface that passes through three points, P_0, P_1 and P_2 is given by

$$P(u, v) = P_0 + u(P_1 - P_0) + v(P_2 - P_0)$$

$$0 \leq u \leq 1$$
$$0 \leq v \leq 1$$



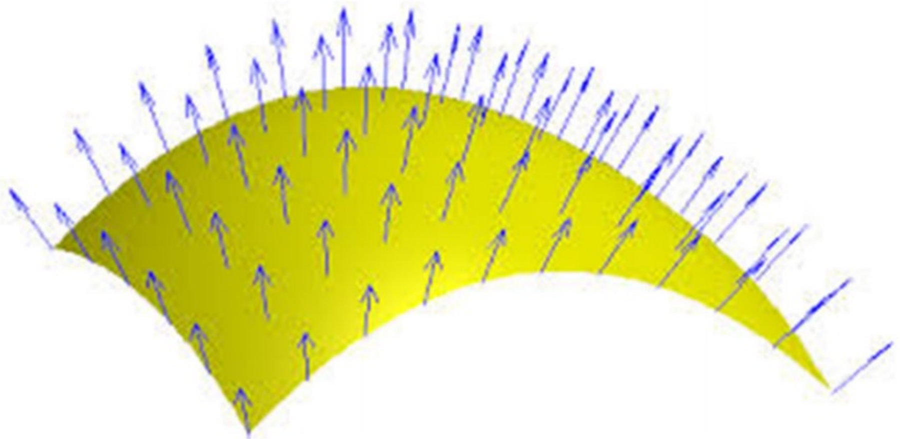
Planar surface

The surface normal vector then is

$$n(u, v) = \frac{(P_1 - P_0) \cdot (P_2 - P_0)}{|(P_1 - P_0) \cdot (P_2 - P_0)|}$$

Once the normal unit vector is known, the surface can be also expressed in non-parametric form as

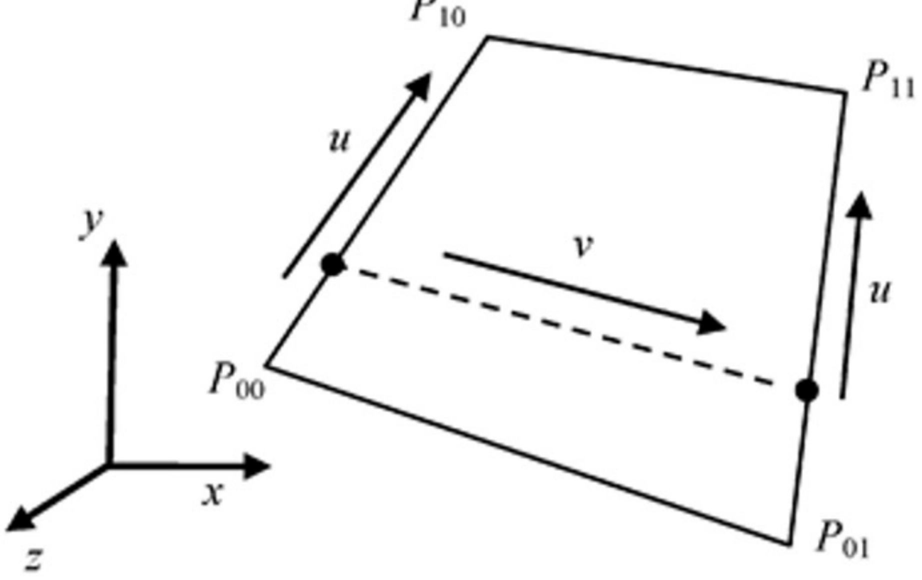
$$(\boldsymbol{P} - \boldsymbol{P}_0) \cdot \boldsymbol{n} = 0$$

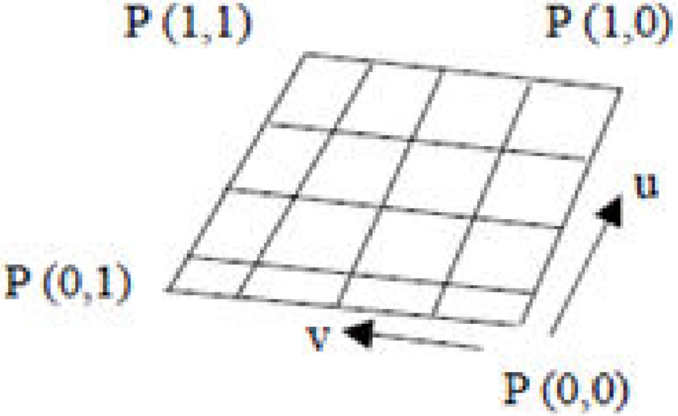


Bilinear surface

A bilinear surface is obtained by linear interpolation between four points, which may or may not lie in the same plane.

The four points appear as vertices or corner points and the parameter values u and v create lines at various intervals to provide the surface visibility.





Bilinear surface

The interpolated parametric equation of a bilinear surface is given as:

$$P(u,v) = (1-u)(1-v)P(0,0) + u(1-v)P(1,0)$$
$$+(1-u)P(0,1) + uvP(1,1)$$

In matrix form :

$$P(u) = \begin{bmatrix}(1-u)(1-v) & u(1-v) & (1-u) & uv\end{bmatrix}\begin{bmatrix}P(0,0)\\P(1,0)\\P(0,1)\\P(1,1)\end{bmatrix}$$

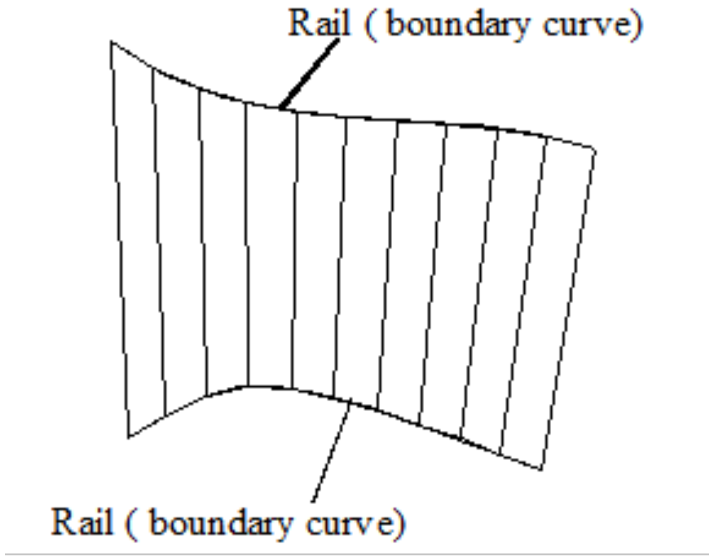
Ruled (lofted) surface

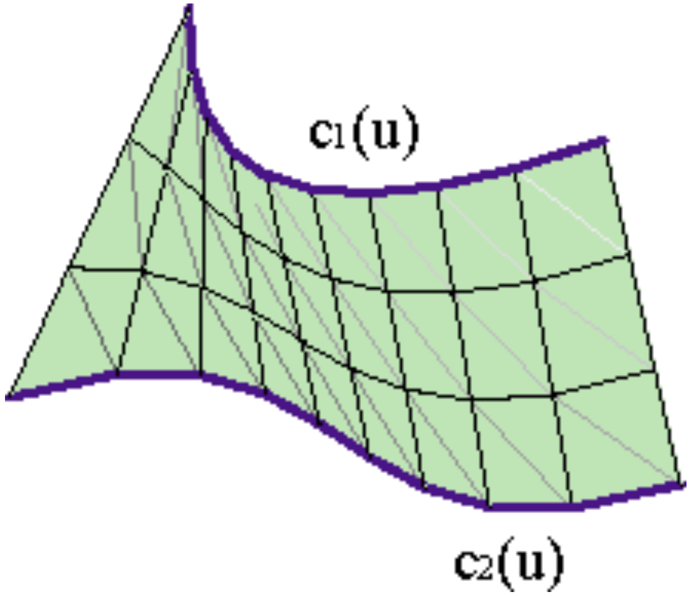
POLYNOMIAL PATCHES

Ruled (lofted) surface

Ruled Surfaces are surfaces that are generated using two curves with a straight line connecting each curve. The two driving curves can be 3D Curves or existing edges of parts or other surfaces.

A ruled surface is generated by joining two space curves (rails) with a straight line (ruling or generator).

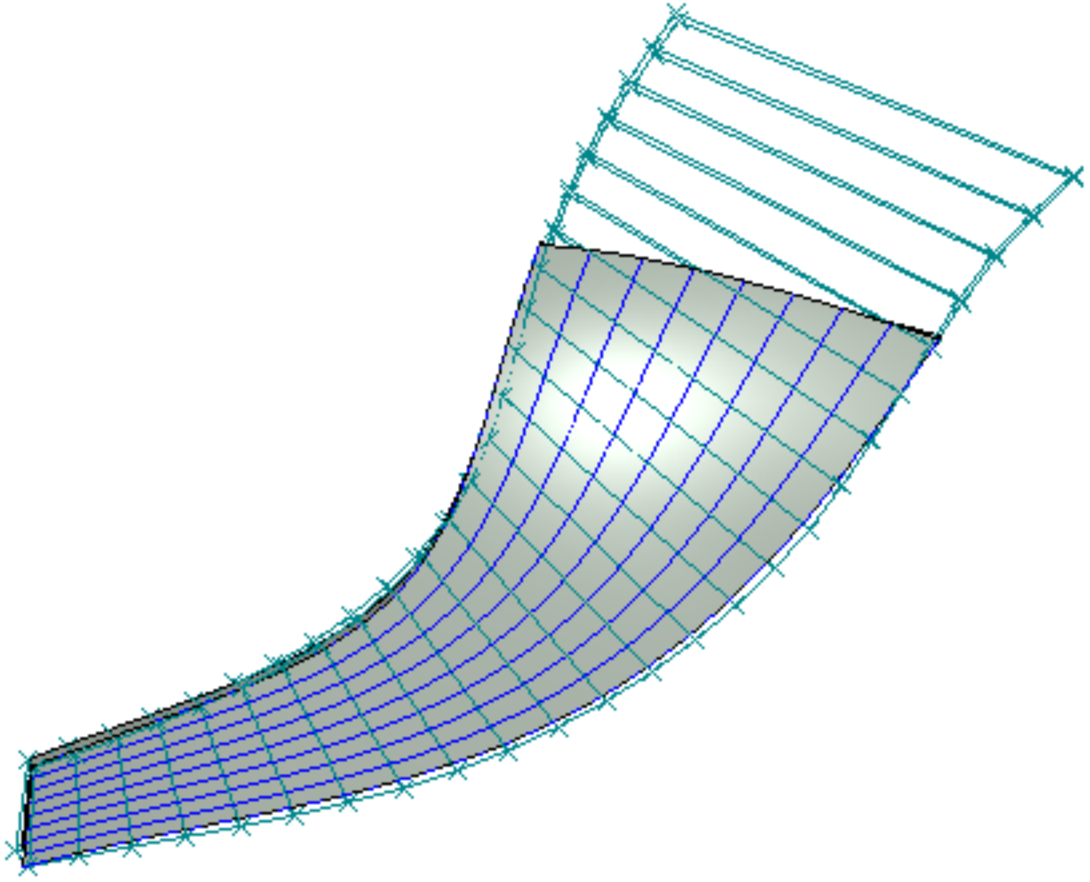




Ruled (lofted) surface

A ruled surface can be generated by the motion of a line in space, similar to the way a curve can be generated by the motion of a point.

A 3D surface is called ruled if through each of its points passes at least one line that lies entirely on that surface.

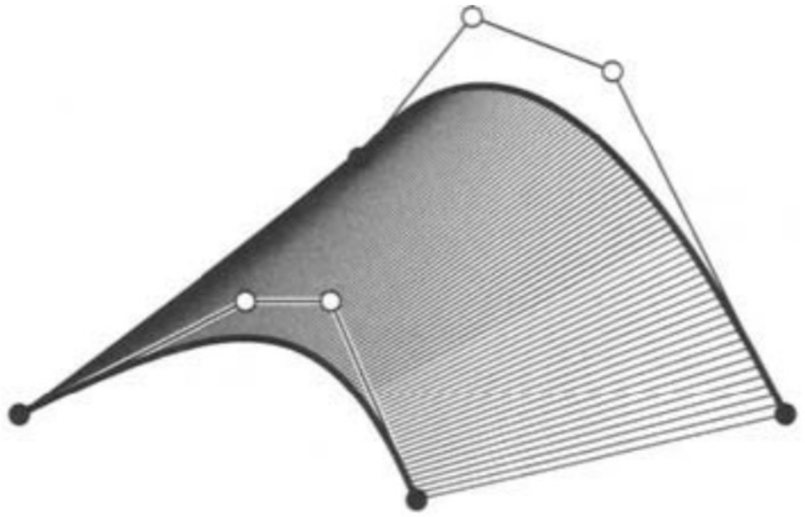


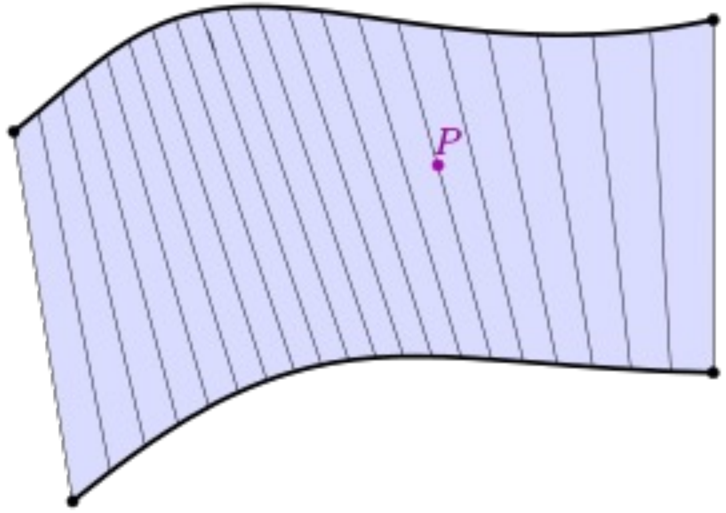
Ruled (lofted) surface

The word *lofted* has an interesting history.

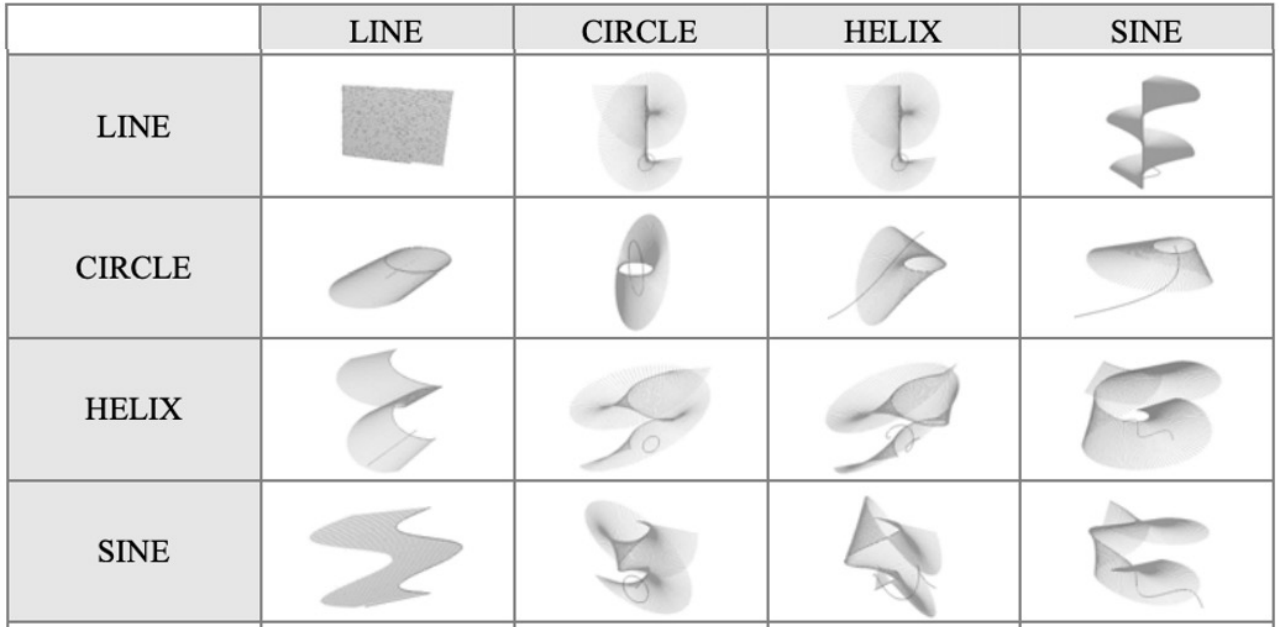
In the days of completely manual ship design, full-scale drawings were difficult to handle in the design office.

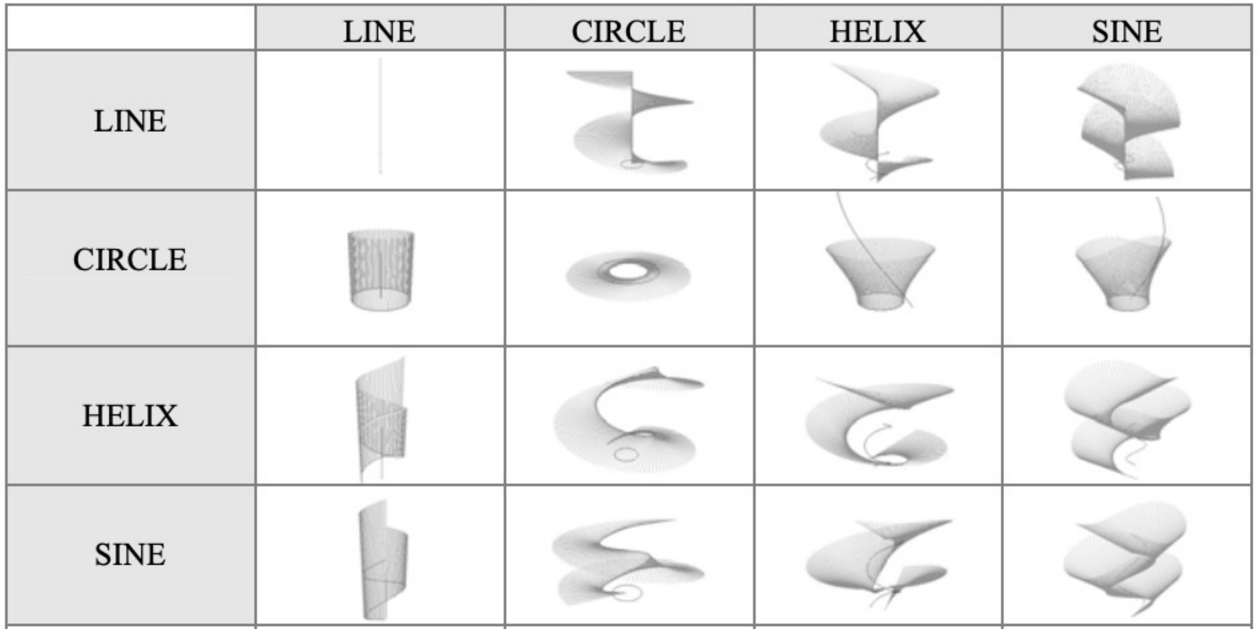
These drawings were stored and dealt with in large attics, called *lofts.*

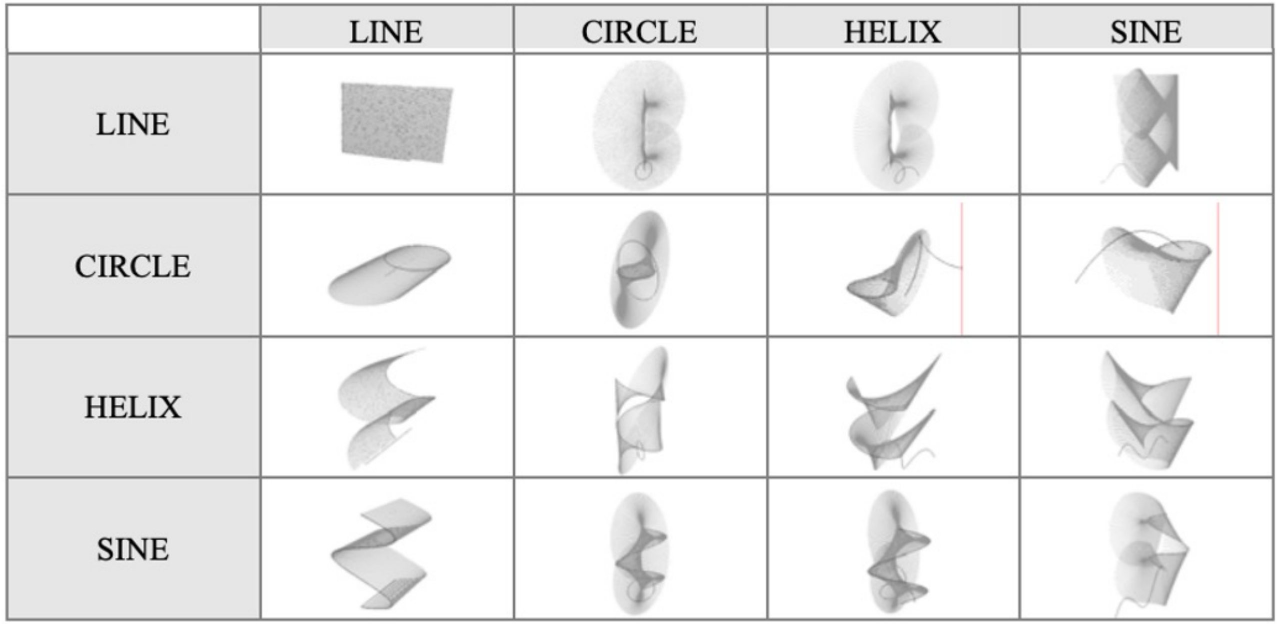




Ruled Surfaces





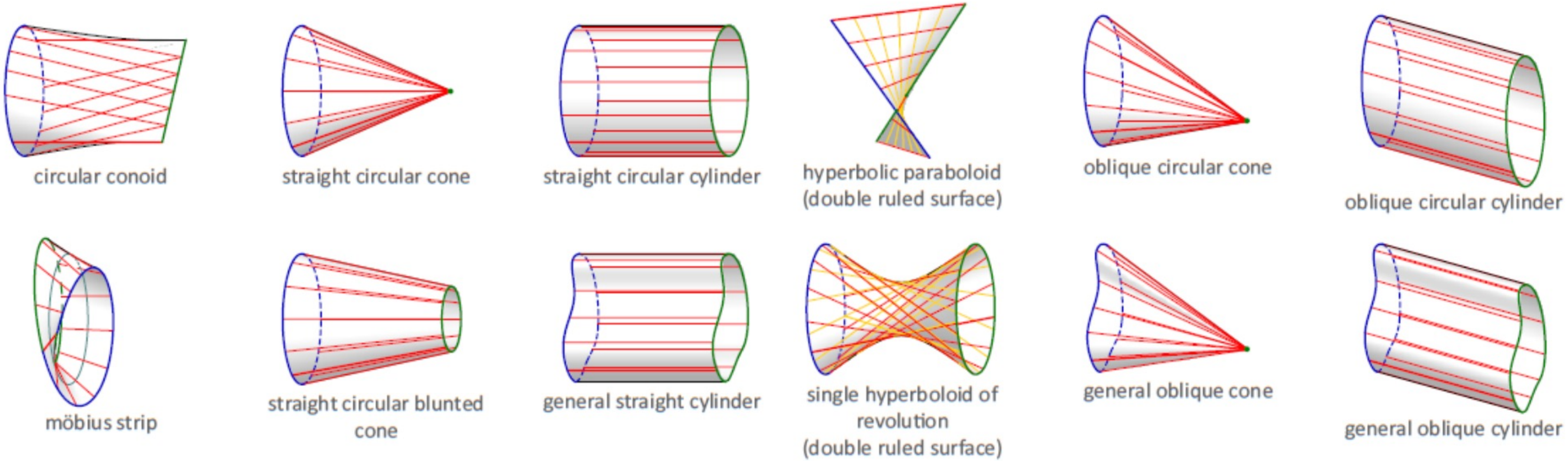


Ruled (lofted) surface

- Single ruled surfaces (lines in one direction)

  The straight lines on a ruled surface are called its *rulings.*
- Double ruled surfaces (lines in two directions)
  - Hyberbolic paraboloid
  - Single hyperboloid of revolution
  - A plane
- Developed surfaces
  - Cylindrical surface (If all rulings are parallel, our ruled surface is *cylindrical*)
  - conical surface (if all intersect in a point, we have a *conical* surface)
  - Swept surface (where the profile is a tangent to path)

Ruled (lofted) surface

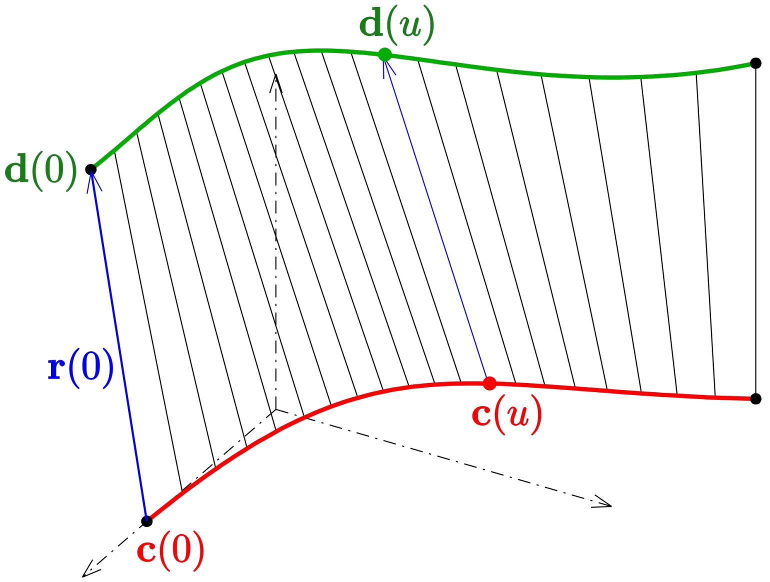


Ruled surface

A ruled surface can be described by a parametric representation of the form
$P(u, v) = c(u) + vr(u)$

Alternatively the ruled surface can be described by
$P(u, v) = (1 - v)c(u) + vd(u)$

where
$r(u) = d(c) - c(u)$

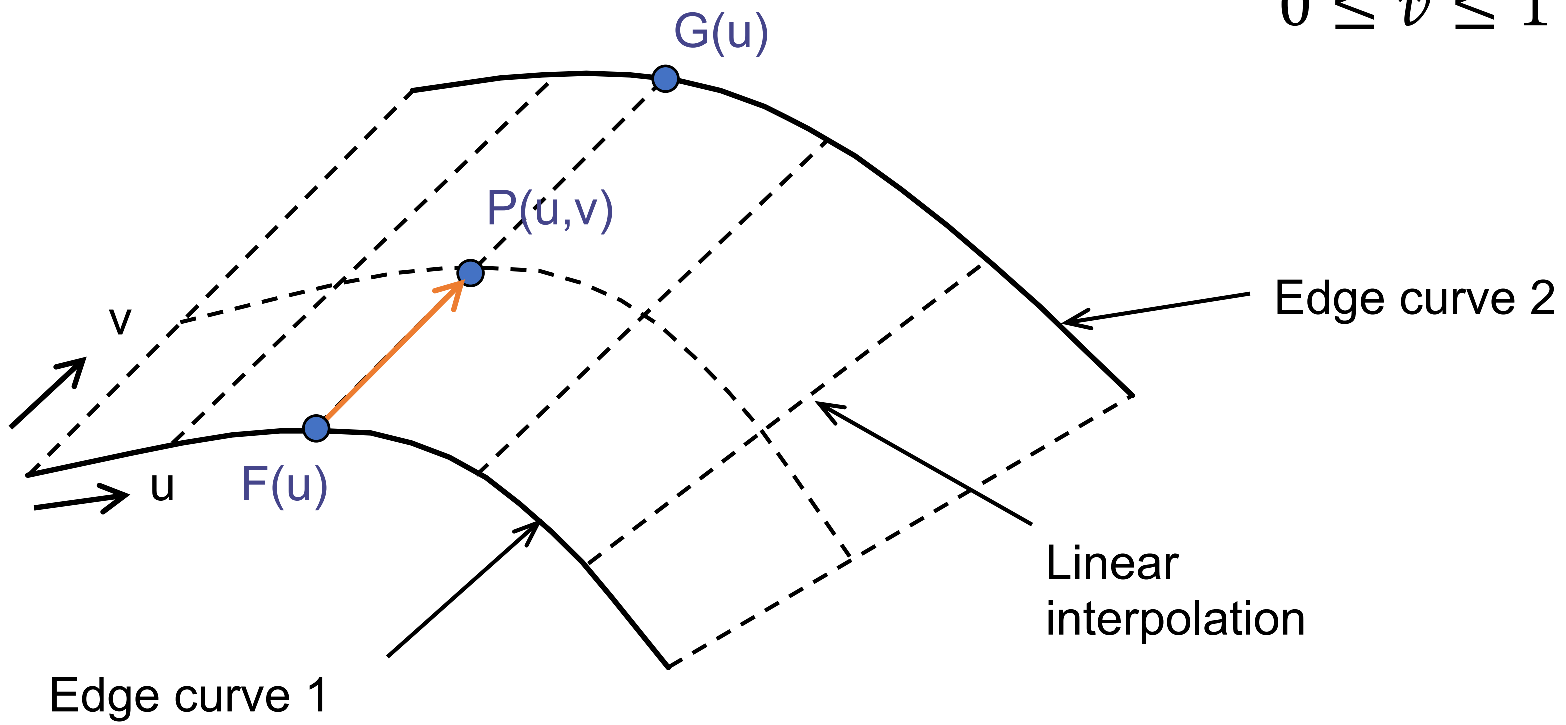


Ruled (lofted) surface

If two curves are denoted by $F(u)$ and $G(u)$ respectively, for a value of u, then the parametric equation is given by

$$P(u,v) = (1-v)G(u) + vF(u)$$

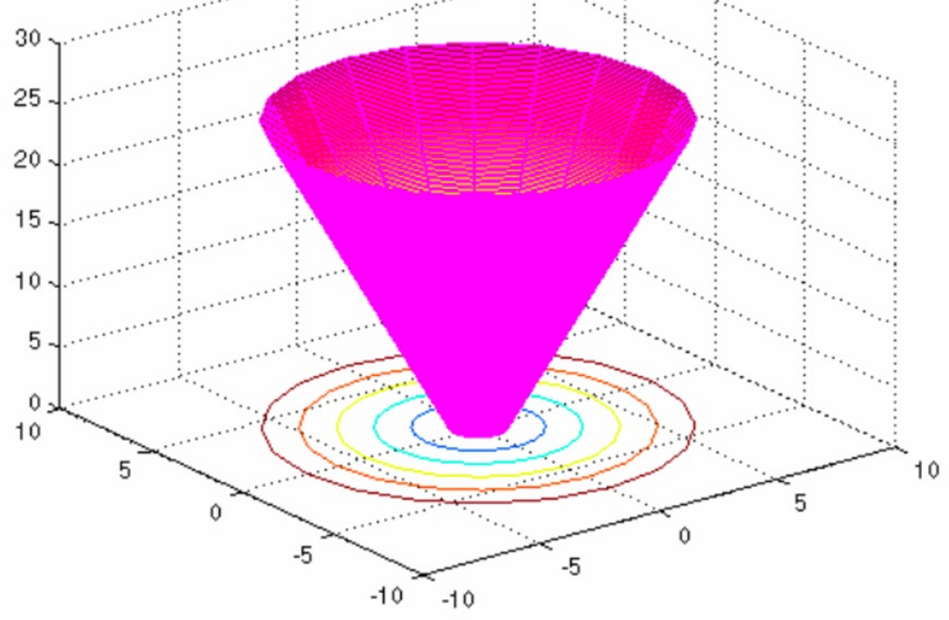
$$0 \le u \le 1$$
$$0 \le v \le 1$$



Ruled (lofted) surface

$$P_1(u) = [\cos 2\pi u \quad \sin 2\pi u \quad 0];$$

$$P_2(u) = [2\cos 2\pi u \quad 2\sin 2\pi u \quad 4] \qquad 0 \le u \le 1$$



$$Q(u,v) = (1-v)P_1(u) + vP_2(u) \qquad 0 \le v \le 1$$

Cylindrical surface

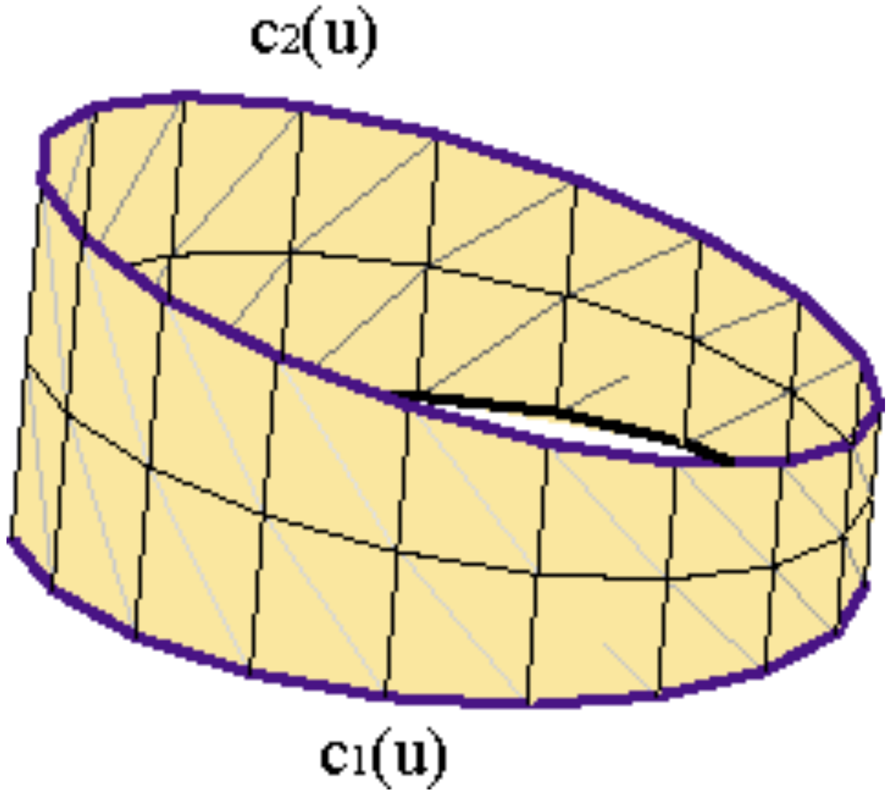
$$P(u,v) = (1-v)(a\cos u\,, a\sin u\,, 0)^T + v(a\cos u\,, a\sin u\,, 1)^T$$

where

$$c1(u) = (a\cos u\,, a\sin u\,, 0)^T$$

$$c2(u) = (a\cos u\,, a\sin u\,, 1)^T$$

$$r(u) = (0,0,1)^T$$



Conical surface

Another category of ruled surfaces is that of conic surfaces; their generators pass through a given point.

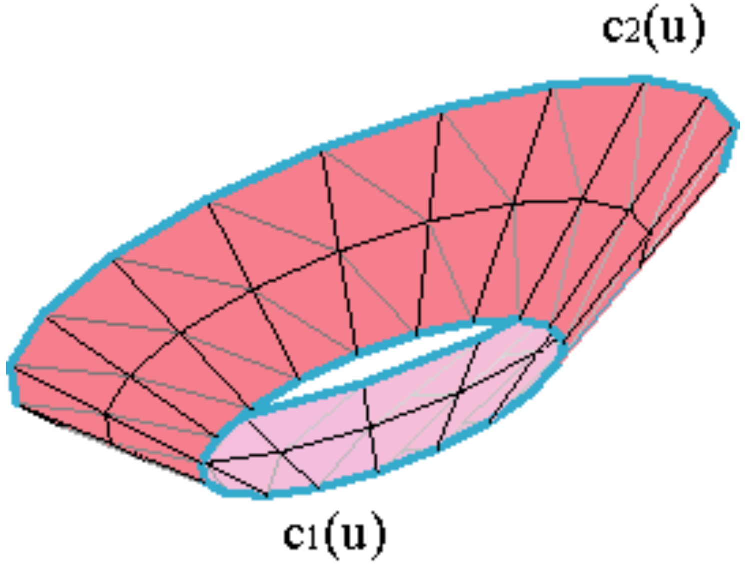
$$P(u,v) = (1-v)(\cos u\,, \sin u\,, 1)^T + v(2\cos u\,, 2\sin u\,, 2)^T$$

where

$$c1(u) = (\cos u\,, \sin u\,, 1)^T$$

$$c2(u) = (2\cos u\,, 2\sin u\,, 2)^T$$

$$r(u) = (\cos u\,, \sin u\,, 1)^T$$

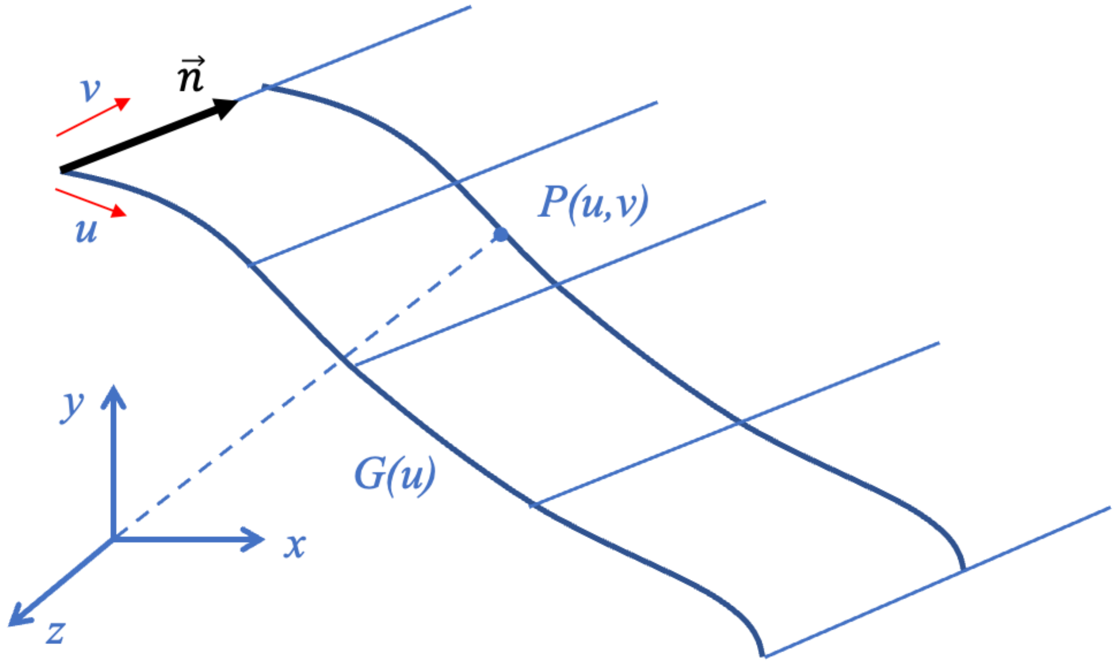


Tabulated cylinder

A tabulated cylinder has been defined as a surface that results from translating a space planar curve along a given direction.

The parametric equation of a tabulated cylinder is given as
$P(u, v) = G(u) + v\vec{n}$

$$0 \le u \le u_{max}$$
$$0 \le v \le v_{max}$$



$G(u)$: can be any wireframe entities to form the cylinder

v : the cylinder length

$\vec{n}$: the cylinder axis (defined by two points)

Coons patch

POLYNOMIAL PATCHES

Coons patch

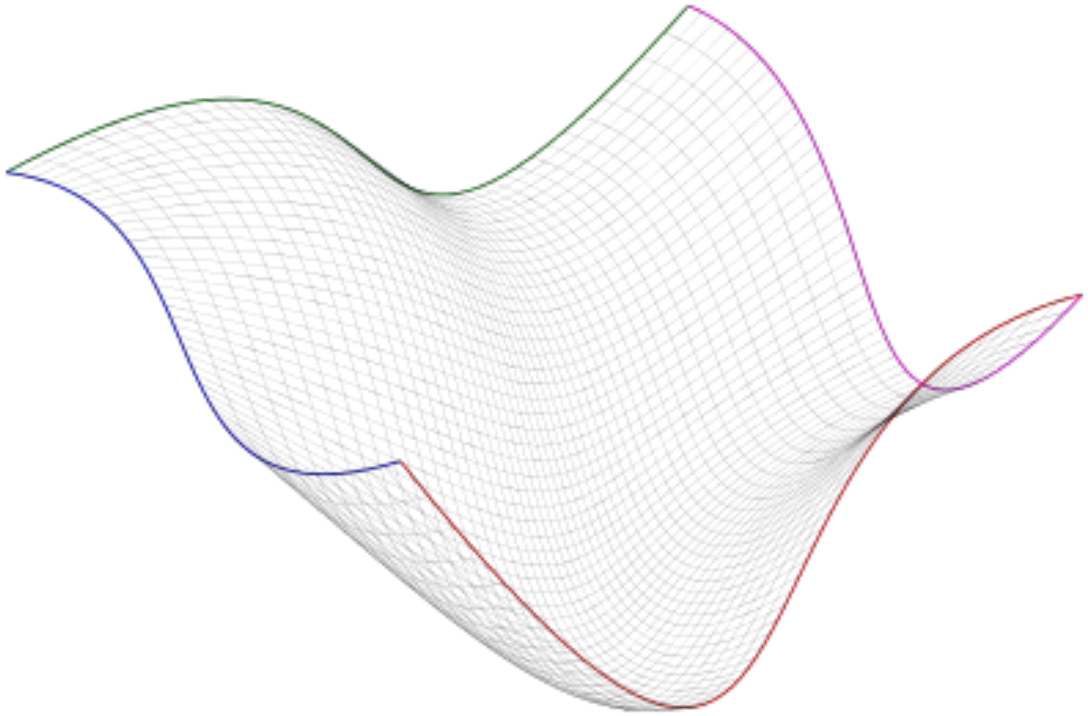
The other surfaces are used with either open boundaries or given data points.
The **Coons patch** is used to create a surface using curves that form closed boundaries.
Coons Surface is easy to create, and therefore, many 2-D CAD packages utilize it for generating models.

Steven Anson Coons
(1912 –1979)

S. Coons. Graphical and analytical methods as applied to aircraft design. J. of Eng. Education, 37(10), 1947.
S. Coons. Surfaces for computer aided design. Technical report, MIT, 1964.
S. Coons. Rational bicubic surface patches. Technical report, MIT, 1968. Project MAC.

Coons patch

Coons decided to search for an expression $P(u, w)$ of the surface that satisfies



(1) it is symmetric in u and w and
(2) it is an interpolation of $P(u, 0)$ and $P(u, 1)$ in one direction and of $P(0, w)$ and P$(1, w)$ in the other direction.

He found a surprisingly simple, two-step solution.

Coons patch

The first step :

to construct two lofted surfaces from the two sets of opposite boundary curves.

$$P_a(u,w) = P(0,w)(1-u) + P(1,w)u$$

$$P_b(u,w) = P(u,0)(1-w) + P(u,1)w$$

Coons patch

The second step :

to tentatively attempt to create the final surface $P(u, w)$ as the sum

$$P(u, w) = P_a(u, w) + P_b(u, w)$$

This is not the expression we are looking for because it does not converge to the right curves at the boundaries.

Coons patch

The sum converges to

For $u \ = \ 0 \ \rightarrow P(u,w) = P(0,w) + P(0,0)(1-w) + P(0,1)w$

For $u \ = \ 1 \rightarrow P(u,w) = P(1,w) + P(1,0)(1-w) + P(1,1)w$

For w= $0 \ \rightarrow P(u,w) = P(u,0) + P(0,0)(1-u) + P(1,0)u$

For w= $1 \ \rightarrow P(u,w) = P(u,1) + P(0,1)(1-u) + P(1,1)u$

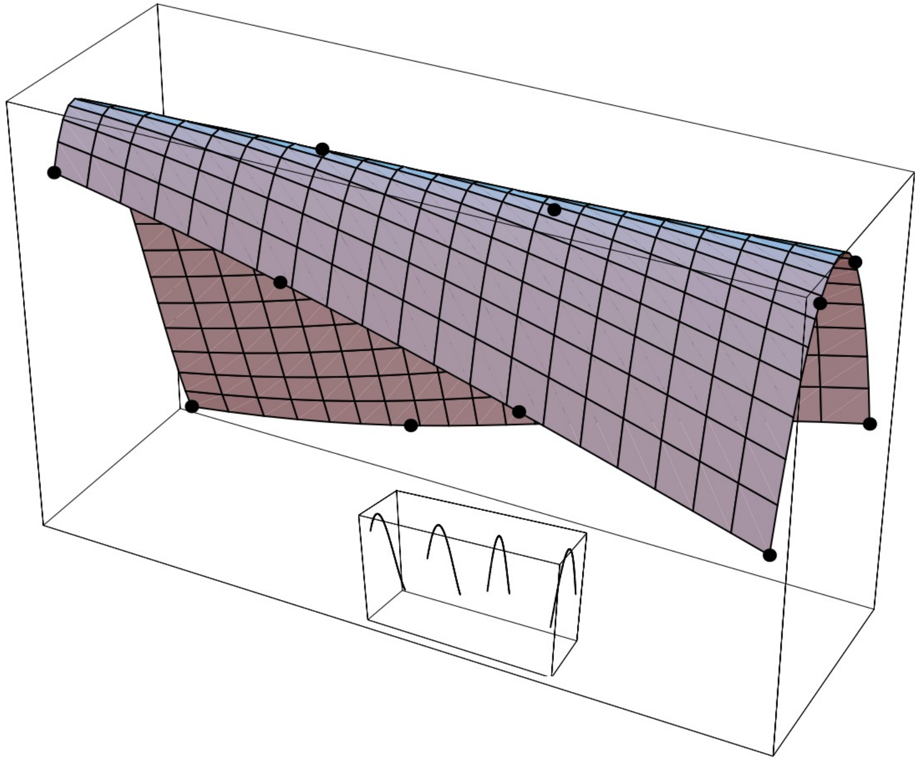
Coons patch

This type of surface is known as the linear Coons surface. Its expression is

$$P(u,w) = P_a(u,w) + P_b(u,w) - P_{ab}(u,w)$$

Where

$$\begin{aligned} P_{ab}(u,w) = P_{00}&(1-u)(1-w) \\ &+P_{01}(1-u)w + P_{10}u(1-w) \\ &+P_{11}uw \end{aligned}$$



P_a and P_b are lofted surfaces,

whereas P_{ab} is a bilinear surface.

Coons patch

The final expression is

$$P(u,w) = P_a(u,w) + P_b(u,w) - P_{ab}(u,w)$$

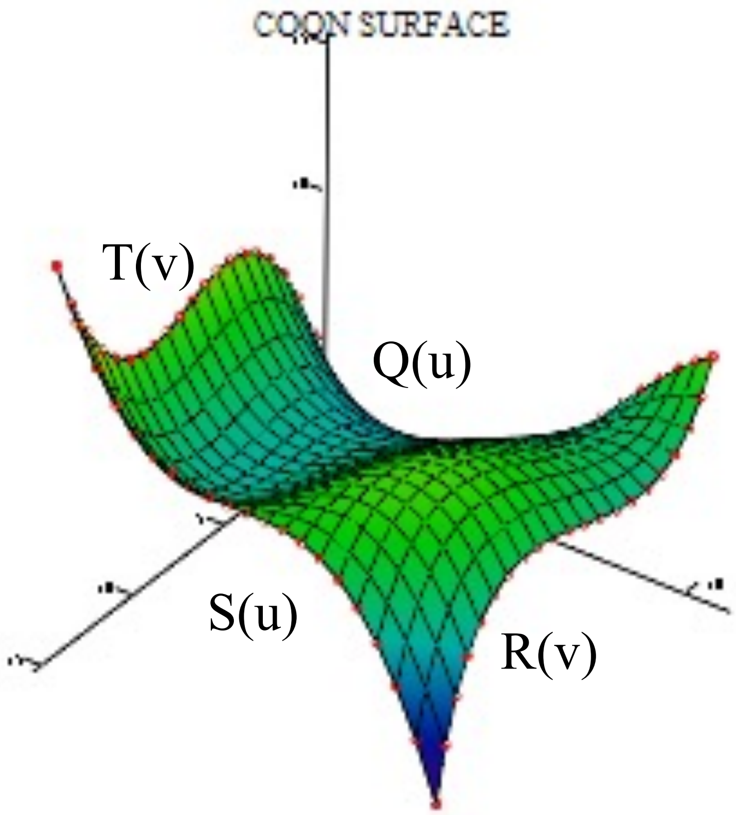
$$P(u,w) = (1-u,u)\begin{bmatrix} P(0,w) \\ P(1,w) \end{bmatrix} + (1-w,w)\begin{bmatrix} P(u,0) \\ P(u,1) \end{bmatrix}$$

$$-(1-u,u)\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}\begin{bmatrix} 1-w \\ w \end{bmatrix}$$

$$+(1-u,u,1)\begin{bmatrix} -P_{00} & -P_{01} & P(0,w) \\ -P_{10} & -P_{11} & P(1,w) \\ P(u,0) & P(u,1) & (0,0,0) \end{bmatrix}\begin{bmatrix} 1-w \\ w \\ 1 \end{bmatrix}$$

Linearly Blended Coons patch

- Surface is defined by linearly interpolating between the boundary curves.
- Simple, but doesn't allow adjacent patches to be joined smoothly.

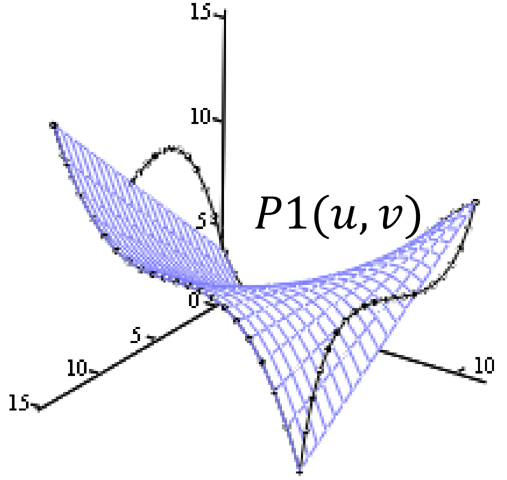


Linearly Blended Coons patch

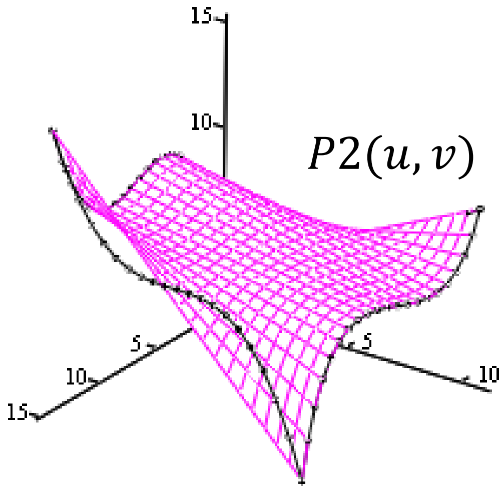
$$P1(u,v) = (1-u)T(v) + uR(v)$$

$$P2(u,v) = (1-v)Q(u) + vR(u)$$

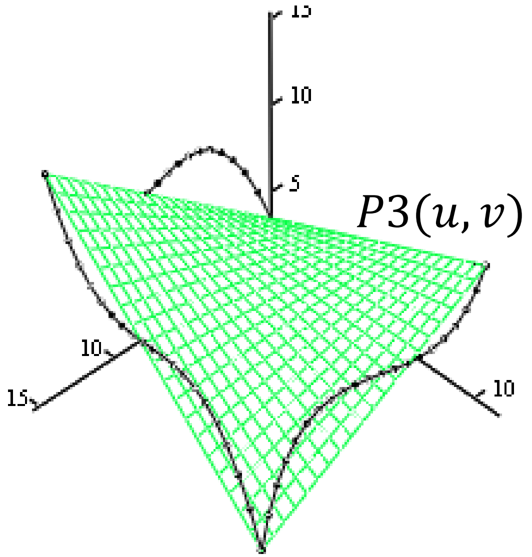
$$P3(u,v) = (1-v)[(1-u)T(0) + uR(0)] + v[(1-u)T(1) + uR(1)]$$



$\left(E^{\langle 0\rangle}, E^{\langle 1\rangle}, E^{\langle 2\rangle}\right), (X1, Y1, Z1)$



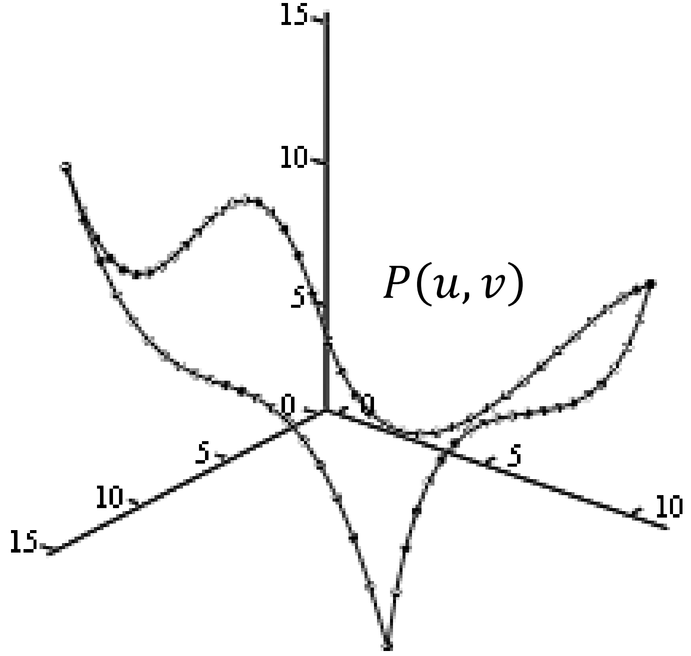
$\left(E^{\langle 0\rangle}, E^{\langle 1\rangle}, E^{\langle 2\rangle}\right), (X2, Y2, Z2)$



$\left(E^{\langle 0\rangle}, E^{\langle 1\rangle}, E^{\langle 2\rangle}\right), (X3, Y3, Z3)$

Linearly Blended Coons patch

$$P(u,v) = P1(u,v) + P2(u,v) - P3(u,v)$$



$$\left(E^{\langle 0 \rangle}, E^{\langle 1 \rangle}, E^{\langle 2 \rangle}\right)$$

Coons patch

In matrix form :

$$Q(u,v) = [(1-u) \quad u \quad 1] \begin{bmatrix} -P(0,0) & -P(0,1) & P(0,v) \\ -P(1,0) & -P(1,1) & P(1,v) \\ P(u,0 & P(u,1) & 0 \end{bmatrix} \begin{bmatrix} (1-v) \\ v \\ 1 \end{bmatrix}$$

Example: Given the four corner points $P_{01} = (-1, -1, 0)$, $P_{10} = (1, -1, 0)$ and $P_{11} = (1, 1, 0)$ (notice that they lie on the xy plane), we calculate the four boundary curves of a linear Coons surface patch

We select boundary curve $P(0, w)$ as the straight line from P_{00} to P_{01} :

$$P(0, w) = P_{00}(1 - w) + P_{01}w = (-1, 2w - 1, 0)$$

Between P_{10} and $P_{11}(1, -0.5, 0.5)$ and $(1, 0.5, -0.5)$ calculate boundary curve $P(1, w)$ as the cubic Lagrange polynomial determined by these four points

$$P(1, w) = \frac{1}{2}(w^3, w^2, w, 1)\begin{bmatrix} -9 & -27 & 27 & 9 \\ 18 & -45 & 36 & -9 \\ 11 & 18 & 9 & 2 \\ 2 & 0 & 9 & 0 \end{bmatrix}\begin{bmatrix} (1, -1, 0) \\ (1, -0.5, 0.5) \\ (1, 0.5, -0.5) \\ (1, 1, 0) \end{bmatrix}$$

$$P(1, w) = (1, (-4 - w + 27w^2 - 18w^3)/4, 27(w - 3w^2 + 2w^3)/4)$$

Solution

$(0,-1,-0.5)$ between P_{00} and P_{10} boundary curve $P(u,0)$ is calculated as the quadratic Lagrange polynomial determined by these three points:

$$P(u,0) = (u^2, u, 1)\begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} (-1,-1,0) \\ (0,-1,-0.5) \\ (1,-1,0) \end{bmatrix}$$

$$P(u,0) = (2u - 1, -1, 2u^2 - 2u)$$

Solution

$(0,1,0.5)$ between P_{01} and P_{11} boundary curve $P(u,1)$ is calculated as the quadratic Lagrange polynomial determined by these three points:

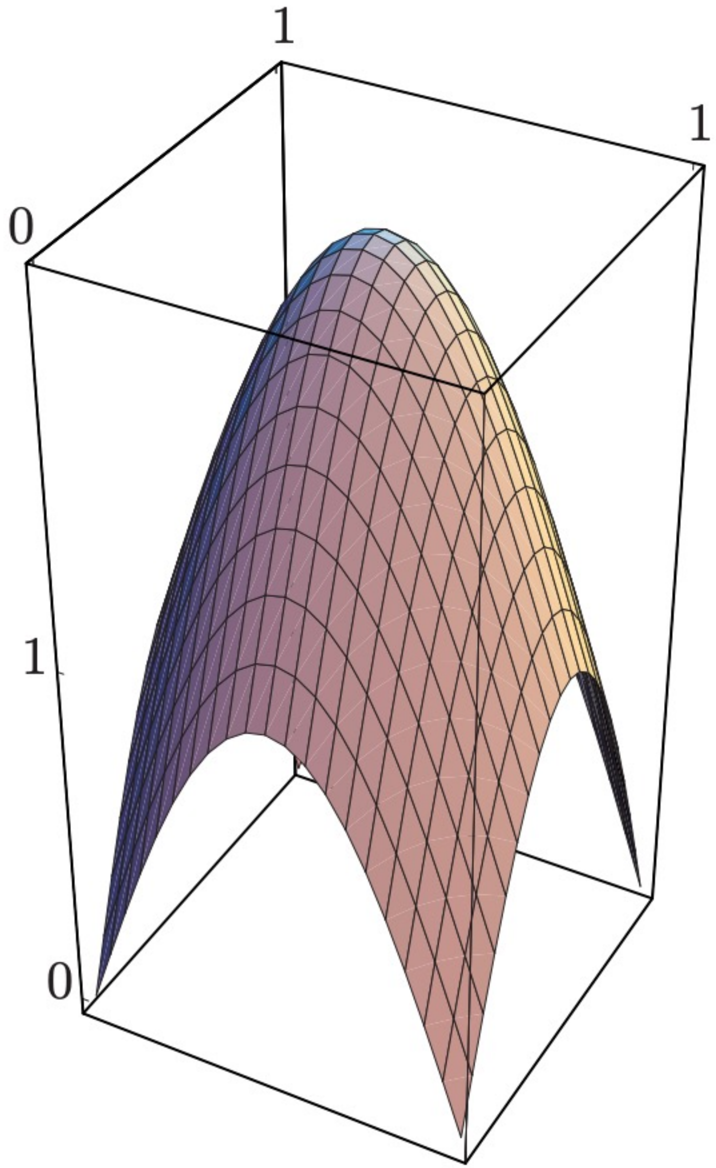
$$P(u,1) = (u^2,u,1)\begin{bmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} (-1,-1,0) \\ (0,1,0.5) \\ (1,-1,0) \end{bmatrix}$$

$$P(u,1) = (2u-1,1,-2u^2+2u)$$

Solution

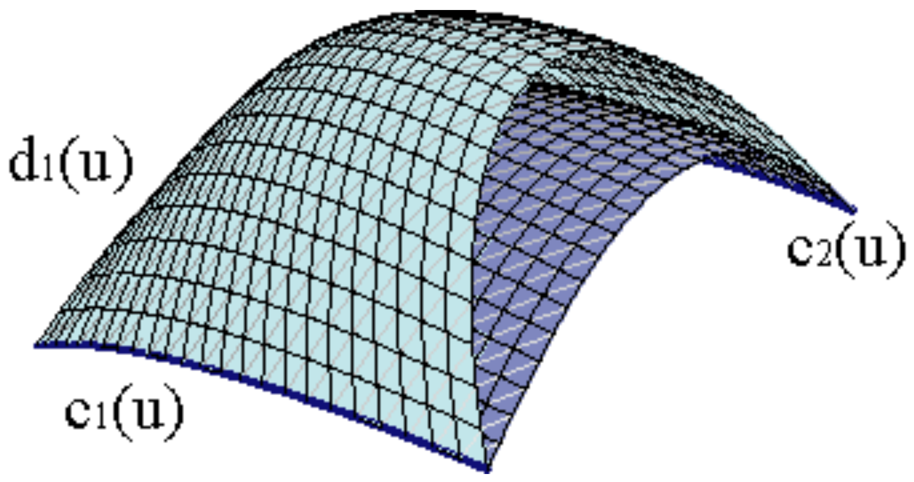
The four boundary curves and the four corner points now become the linear Coons surface patch

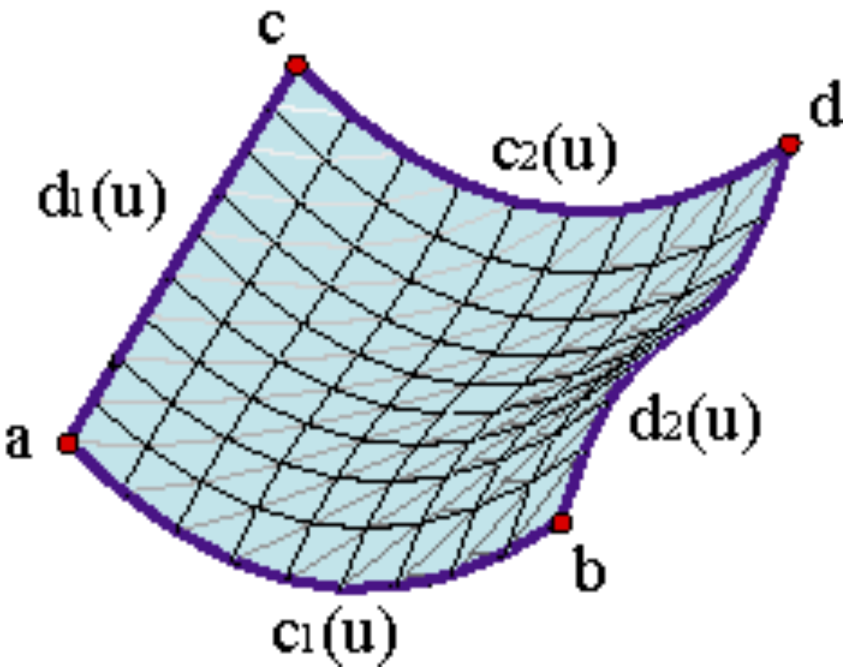
$$
\begin{aligned}
&P(u,w)\\
&= (-1 + 2u + (1-u)(1-w)\\
&+ (-1+2u)(1-w) + (1-u)w - uw + (-1\\
&+ 2u)w,\\
&\quad -1 + (1-u)(1-w) + u(1-w) + 2w\\
&\quad - (1-u)w - uw + (1-u)(-1+2w)\\
&\quad + u(-4 - w + 27w^2 - 18w^3)/4\,,\\
&\ (-2u + 2u^2)w + 27u(w - 3w^2 + 2w^3)/4)
\end{aligned}
$$



Coons patch

Coons patches: an example for $n = 11$ and $m = 9$.





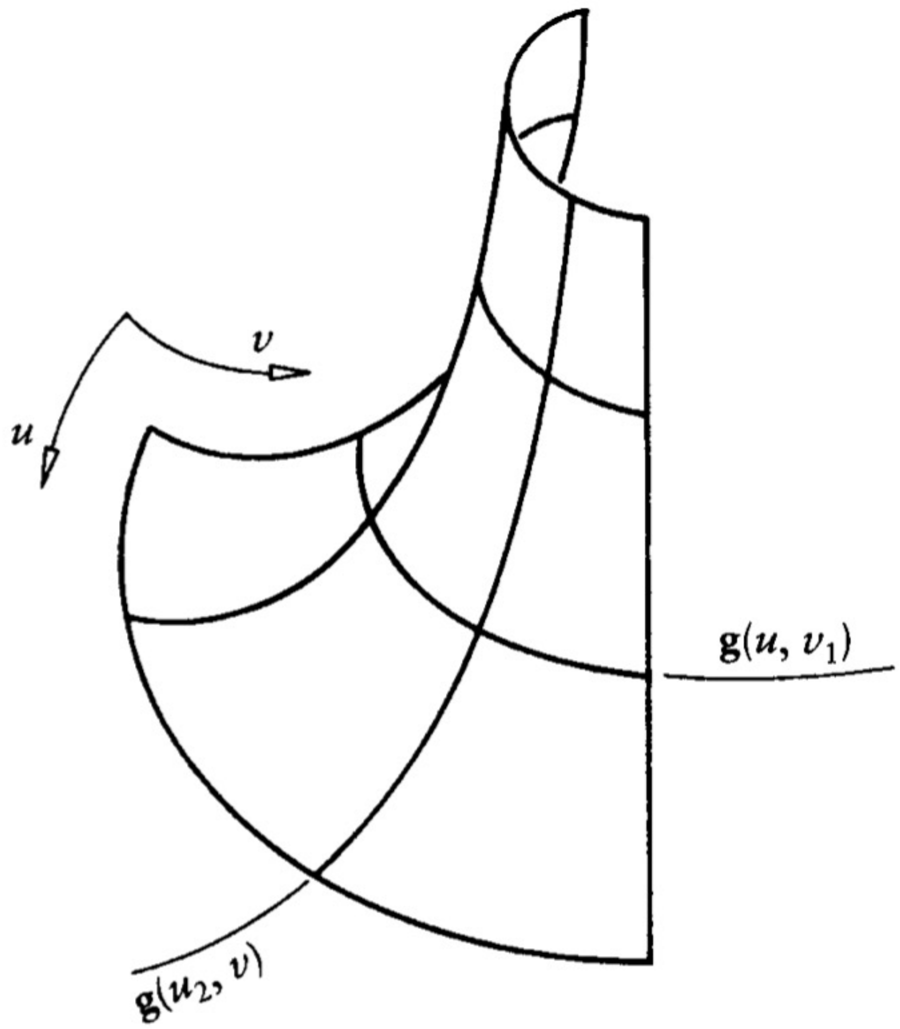
Gordon surface

POLYNOMIAL PATCHES

Gordon surface

Gordon surfaces are a generalization of Coons patches. They were developed in 1969 by W. Gordon.

He coined the term *transfinite interpolation* for this kind of surface.

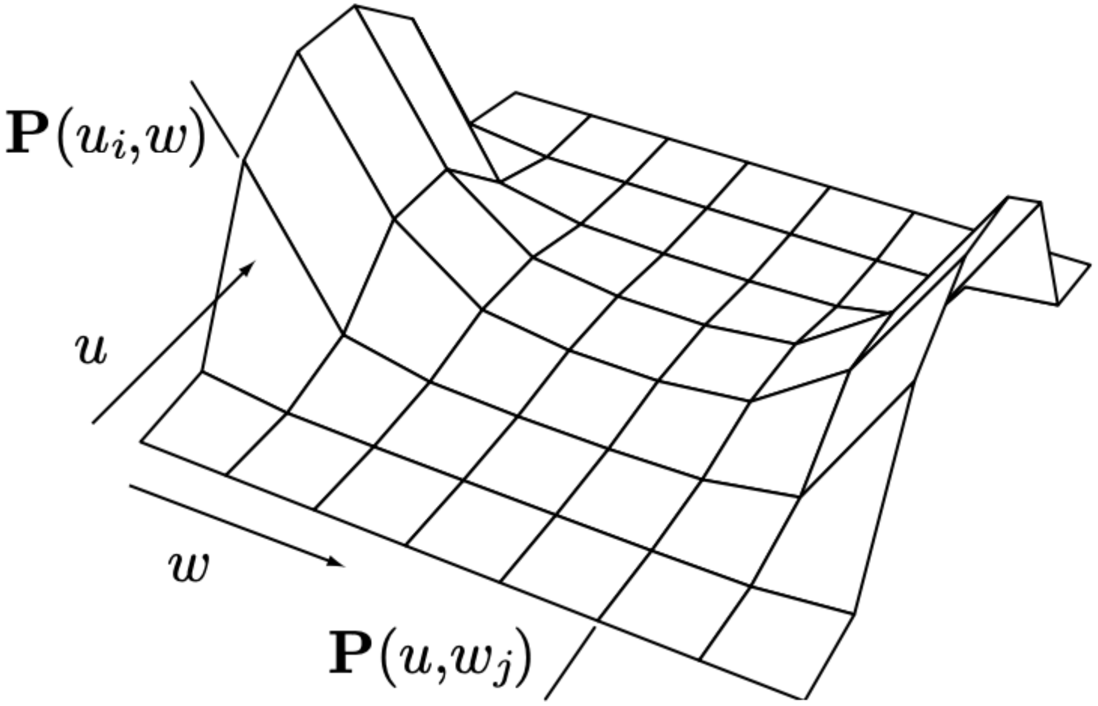


W. Gordon, "Spline-blended surface interpolation through curve networks", Journal of Math and Mechanics, vol.18, no.10, pp. 931- 952, 1969.

Gordon surface

A Gordon surface is defined by means of two families of curves, one in each of the u and w directions.

It can have very complex shapes and is a good candidate for use in applications where realism is important.



Gordon surface

A spline - blended surface interpolating the network of curves, known also as Gordon surface

$$G(u,v) = G_1(u,v) + G_2(u,v) - G_{12}(u,v)$$

where

$$G_1(u,v) = \sum_{i=1}^{m} G(u_i,v) L_i^m(u)$$

$$G_2(u,v) = \sum_{j=1}^{n} G(u,v_j) L_j^n(v)$$

$$G_{12}(u,v) = \sum_{i=1}^{m} \sum_{j=1}^{n} G(u_i,v_j) L_i^m(u) L_j^n(v)$$

Blending functions

$L_i^m(u)$ and $L_j^n(v)$ are blending functions satisfying :

$$L_i^m(u_i) = 1$$

$$L_i^m(u_k) = 0$$

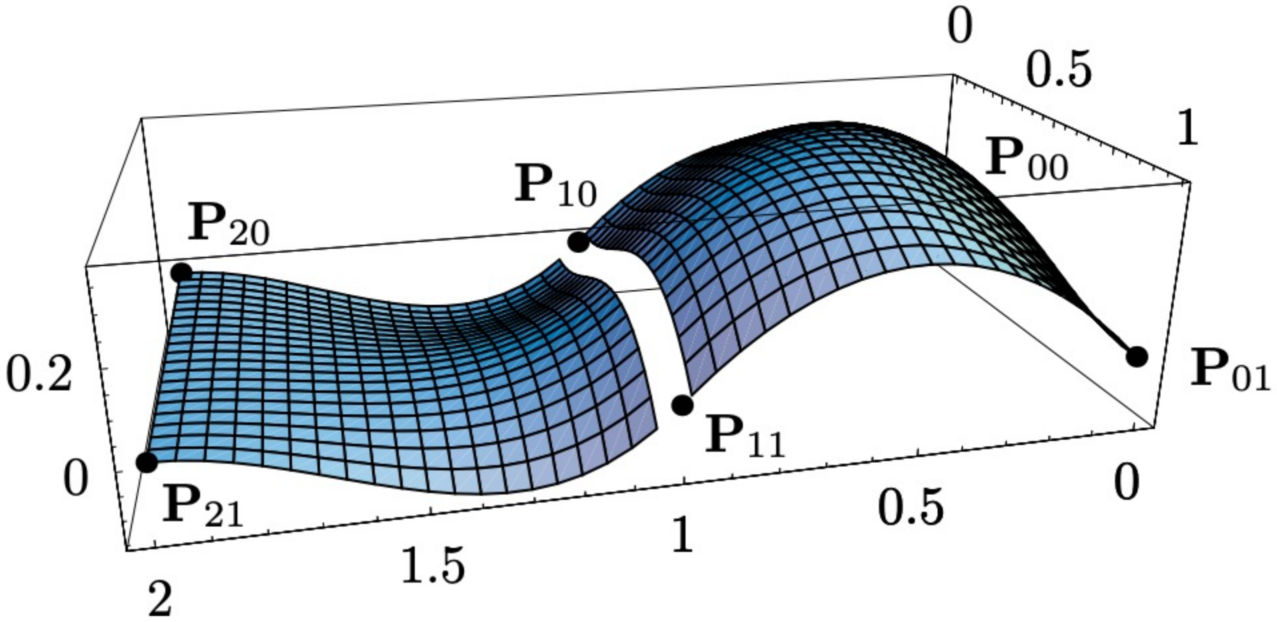
$$i \neq k$$

Bicubic Hermite Surface

POLYNOMIAL PATCHES

Bicubic Hermite Surface

The method is called Hermite interpolation after Charles Hermite who developed it and derived its blending functions in the 1870s, as part of his work on approximation and interpolation.



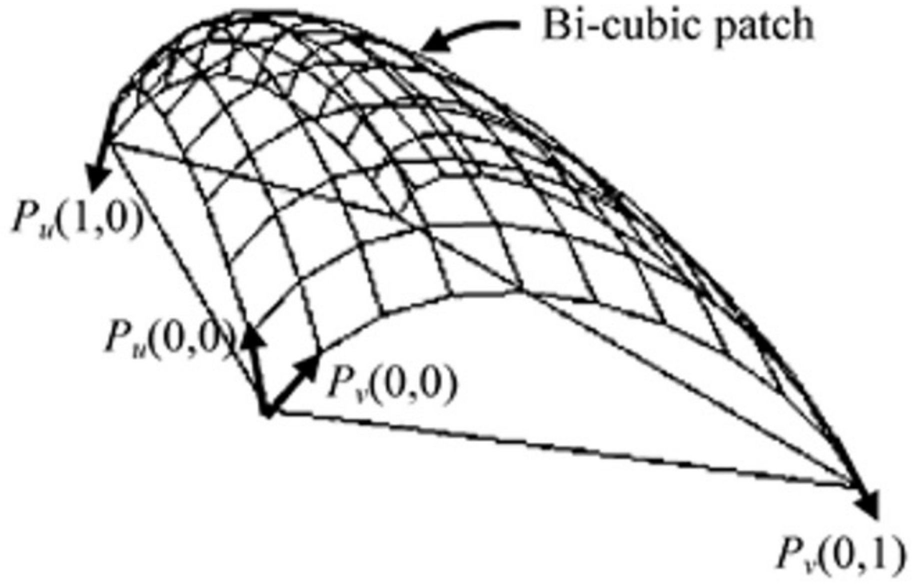
> [Hermite] had a kind of positive hatred of geometry and once curiously reproached me with having made a geometrical memoir.
>
> Jacques Hadamard

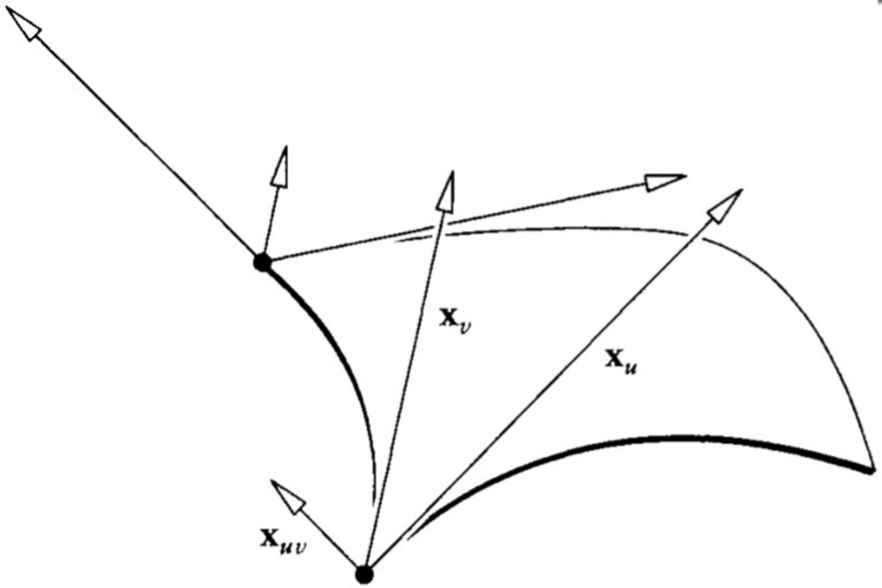
Bicubic Hermite Surface

This surface is formed by Hermite cubic splines running in two different directions.

It interpolates to a finite number of data points to form the surface.

The bicubic interpolation is an invaluable tool used in image processing.





Bicubic Hermite Surface

$$P(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} C_{ij} u^i v^j \qquad \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 1 \end{matrix}$$

In a matrix form it can be expressed

$$P(u,v) = U^T [C]\, V$$

$$P(u,v) = U^T [M_H][B][M_H]^T\, V$$

where

$U^T = \{u^3, u^2, u, 1\}$

$V = \{v^3, v^2, v, 1\}^T$

Bicubic Hermite Surface

Applying the boundary conditions (continuity and tangency) at data points determines all coefficients.

This matrix can be determined by imposing the smoothness conditions at data points joining two adjacent panels.

$$[\mathbf{B}] = \begin{bmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \left.\frac{\partial \mathbf{P}}{\partial v}\right|_{00} & \left.\frac{\partial \mathbf{P}}{\partial v}\right|_{01} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \left.\frac{\partial \mathbf{P}}{\partial v}\right|_{10} & \left.\frac{\partial \mathbf{P}}{\partial v}\right|_{11} \\ \left.\frac{\partial \mathbf{P}}{\partial u}\right|_{00} & \left.\frac{\partial \mathbf{P}}{\partial u}\right|_{01} & \left.\frac{\partial^2 \mathbf{P}}{\partial u \partial v}\right|_{00} & \left.\frac{\partial^2 \mathbf{P}}{\partial u \partial v}\right|_{01} \\ \left.\frac{\partial \mathbf{P}}{\partial u}\right|_{10} & \left.\frac{\partial \mathbf{P}}{\partial u}\right|_{11} & \left.\frac{\partial^2 \mathbf{P}}{\partial u \partial v}\right|_{10} & \left.\frac{\partial^2 \mathbf{P}}{\partial u \partial v}\right|_{11} \end{bmatrix}$$

Catmull–Rom Surface

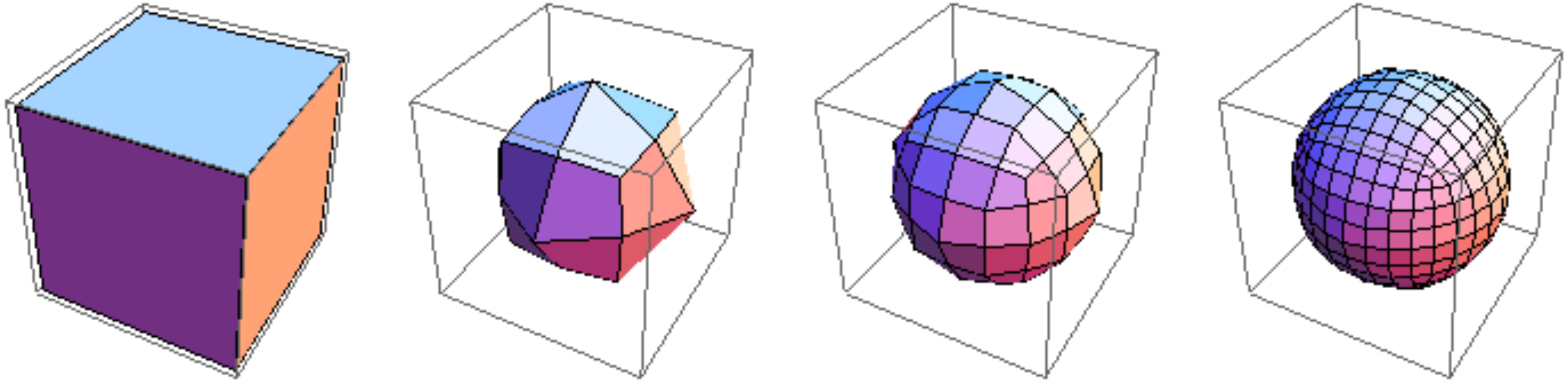
We start with a group of $m \times n$ data points roughly arranged in a rectangle. We look at all the overlapping groups that consist of 4×4 adjacent points, and we calculate a surface patch for each group.

$$
\begin{array}{llll}
\mathbf{P}_{40}\mathbf{P}_{41}\mathbf{P}_{42}\mathbf{P}_{43} & \mathbf{P}_{41}\mathbf{P}_{42}\mathbf{P}_{43}\mathbf{P}_{44} & \mathbf{P}_{42}\mathbf{P}_{43}\mathbf{P}_{44}\mathbf{P}_{45} \;\ldots & \mathbf{P}_{4,n-3}\mathbf{P}_{4,n-2}\mathbf{P}_{4,n-1}\mathbf{P}_{4n} \\
\mathbf{P}_{30}\mathbf{P}_{31}\mathbf{P}_{32}\mathbf{P}_{33} & \mathbf{P}_{31}\mathbf{P}_{32}\mathbf{P}_{33}\mathbf{P}_{34} & \mathbf{P}_{32}\mathbf{P}_{33}\mathbf{P}_{34}\mathbf{P}_{35} \;\ldots & \mathbf{P}_{3,n-3}\mathbf{P}_{3,n-2}\mathbf{P}_{3,n-1}\mathbf{P}_{3n} \\
\mathbf{P}_{20}\mathbf{P}_{21}\mathbf{P}_{22}\mathbf{P}_{23} & \mathbf{P}_{21}\mathbf{P}_{22}\mathbf{P}_{23}\mathbf{P}_{24} & \mathbf{P}_{22}\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25} \;\ldots & \mathbf{P}_{2,n-3}\mathbf{P}_{2,n-2}\mathbf{P}_{2,n-1}\mathbf{P}_{2n} \\
\mathbf{P}_{10}\mathbf{P}_{11}\mathbf{P}_{12}\mathbf{P}_{13} & \mathbf{P}_{11}\mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14} & \mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}\mathbf{P}_{15} \;\ldots & \mathbf{P}_{1,n-3}\mathbf{P}_{1,n-2}\mathbf{P}_{1,n-1}\mathbf{P}_{1n} \\
 & & & \\
\mathbf{P}_{30}\mathbf{P}_{31}\mathbf{P}_{32}\mathbf{P}_{33} & \mathbf{P}_{31}\mathbf{P}_{32}\mathbf{P}_{33}\mathbf{P}_{34} & \mathbf{P}_{32}\mathbf{P}_{33}\mathbf{P}_{34}\mathbf{P}_{35} \;\ldots & \mathbf{P}_{3,n-3}\mathbf{P}_{3,n-2}\mathbf{P}_{3,n-1}\mathbf{P}_{3n} \\
\mathbf{P}_{20}\mathbf{P}_{21}\mathbf{P}_{22}\mathbf{P}_{23} & \mathbf{P}_{21}\mathbf{P}_{22}\mathbf{P}_{23}\mathbf{P}_{24} & \mathbf{P}_{22}\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25} \;\ldots & \mathbf{P}_{2,n-3}\mathbf{P}_{2,n-2}\mathbf{P}_{2,n-1}\mathbf{P}_{2n} \\
\mathbf{P}_{10}\mathbf{P}_{11}\mathbf{P}_{12}\mathbf{P}_{13} & \mathbf{P}_{11}\mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14} & \mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}\mathbf{P}_{15} \;\ldots & \mathbf{P}_{1,n-3}\mathbf{P}_{1,n-2}\mathbf{P}_{1,n-1}\mathbf{P}_{1n} \\
\mathbf{P}_{00}\mathbf{P}_{01}\mathbf{P}_{02}\mathbf{P}_{03} & \mathbf{P}_{01}\mathbf{P}_{02}\mathbf{P}_{03}\mathbf{P}_{04} & \mathbf{P}_{02}\mathbf{P}_{03}\mathbf{P}_{04}\mathbf{P}_{05} \;\ldots & \mathbf{P}_{0,n-3}\mathbf{P}_{0,n-2}\mathbf{P}_{0,n-1}\mathbf{P}_{0n}
\end{array}
$$

Points for a Catmull–Rom Surface Patch

Catmull–Clark Surfaces

The method described here is due to Edwin Catmull and Jim Clark and is an extension of the method to arbitrary polygonal surfaces. A Catmull–Clark surface patch starts with an arbitrary polygonal surface and subdivides it by generating new face, edge, and vertex points and connecting them in a simple way.



Catmull E. and Clark J., Recursively generated B-spline surfaces on arbitrary topological meshes. Computer Aided Design, 9(6), pp. 350-355, 1978.

Catmull–Clark Surfaces

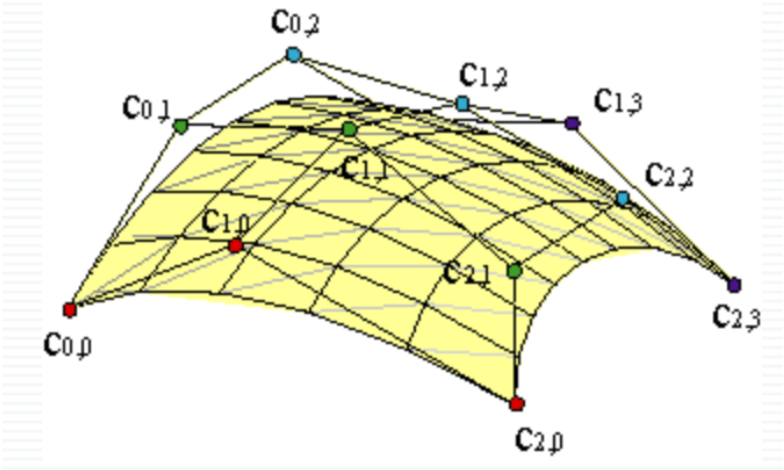
The rules for generating the points are the following:

1. A face point is calculated for each face of the original mesh. The point is simply the average of all the points that bound the face.
2. An edge point is created for each interior edge of the polygonal surface. The point is the average of the midpoint of the edge and of the two face points on both sides of the edge.
3. A vertex point is generated for each interior vertex P of the original mesh.

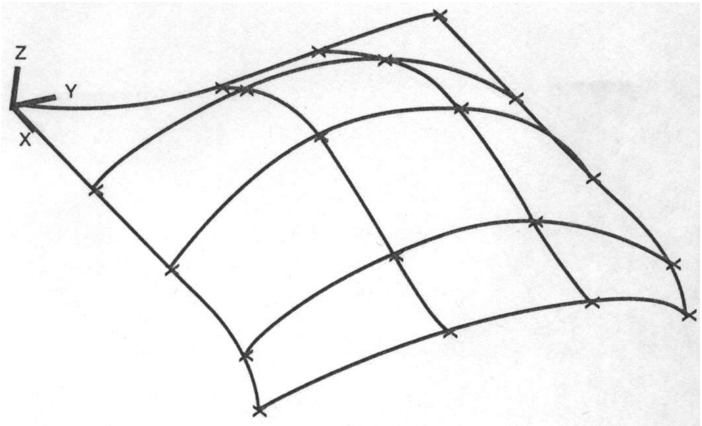
Tensor product patches

Tensor product patches

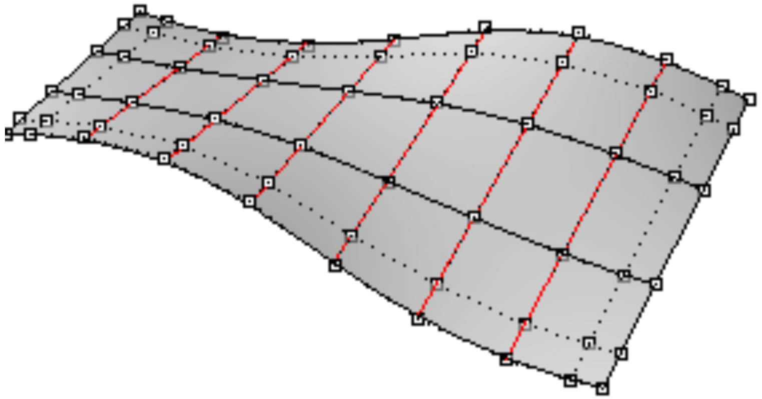
Bezier patch



B-spline surface



NURBS surface

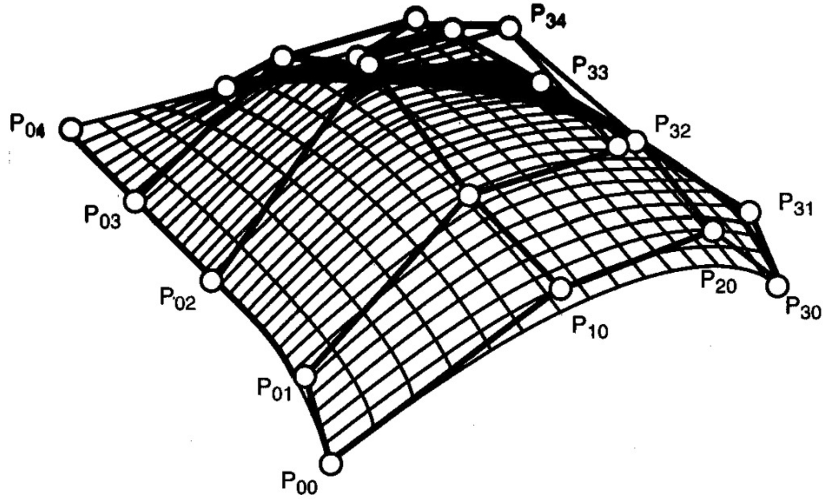


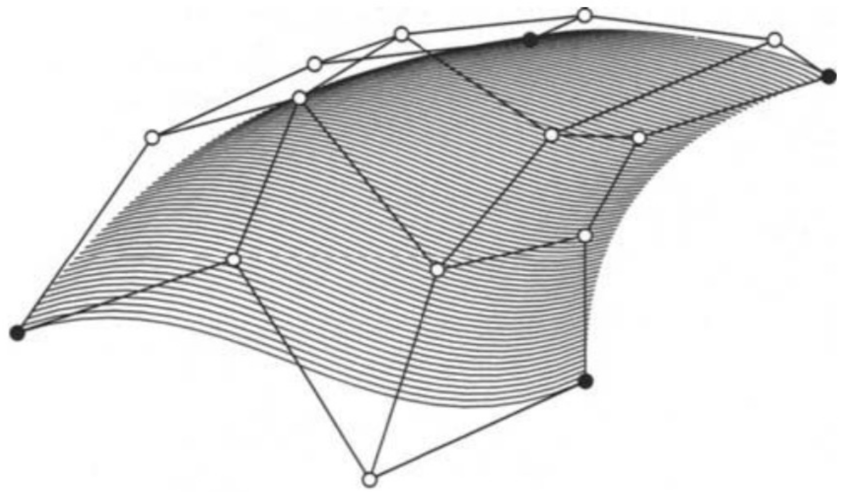
Bezier patch

TENSOR PRODUCT PATCHES

Bezier surface

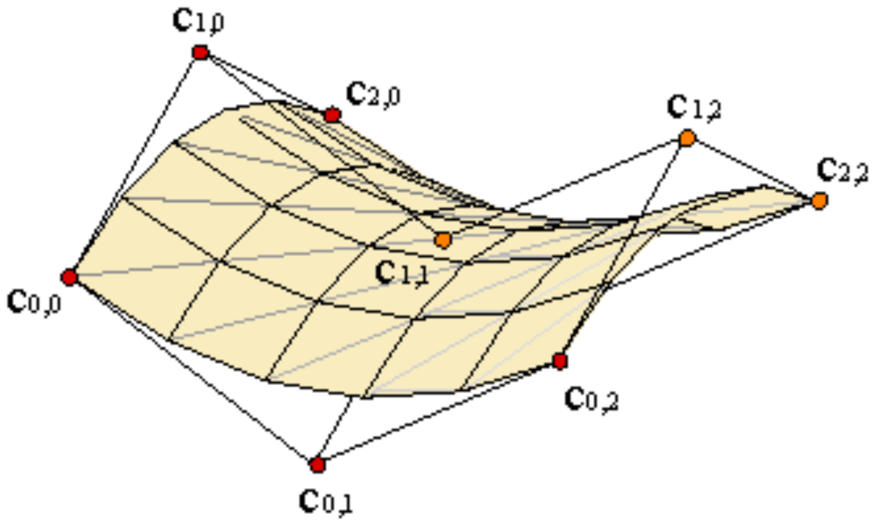
The Bezier surface patch, like its relative the Bezier curve, is popular and is commonly used in practice.

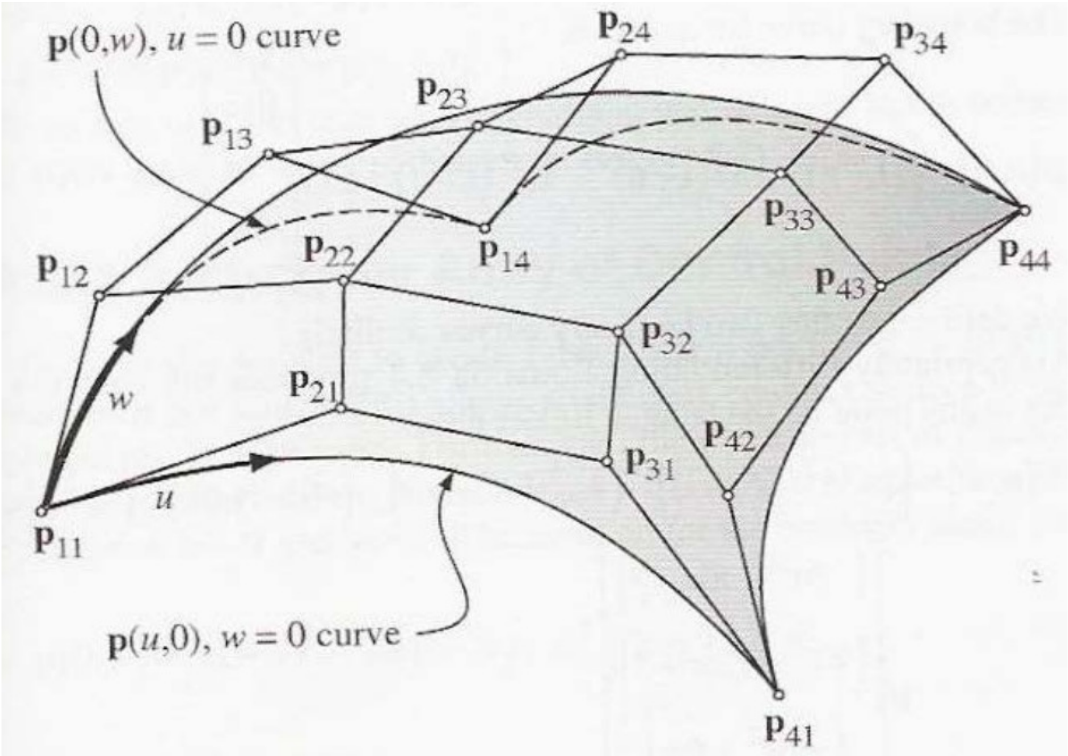




Bezier surface

This is a surface that approximates given input data. It is different from the previous surfaces in that it is a synthetic surface, it does not pass through all given data points.





Rectangular Bezier Surface

We start with an $(m\ +\ 1) \times (n\ +\ 1)$ grid of control points arranged in a roughly rectangular grid :

$$\begin{matrix} \mathbf{P}_{m,0} & \mathbf{P}_{m,1} & \cdots & \mathbf{P}_{m,n} \\ \vdots & \vdots & & \vdots \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \cdots & \mathbf{P}_{1,n} \\ \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \cdots & \mathbf{P}_{0,n} \end{matrix}$$

construct the rectangular Bezier surface patch for the points by applying the technique of Cartesian product to the Bezier curve produces

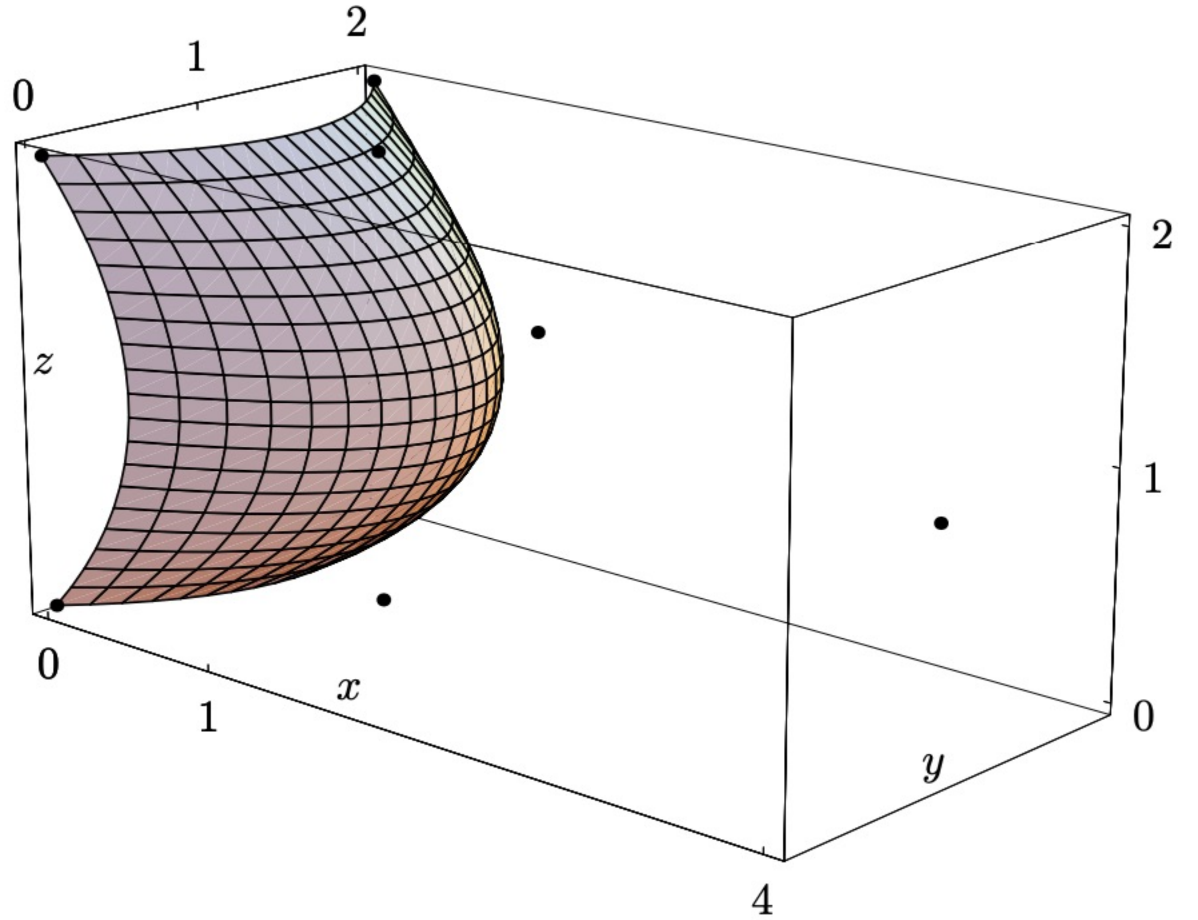
$$P(u,w) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u)\ P_{i,j} B_{n,j}(w)$$

Rectangular Bezier Surface

$$P(u, w) = B_m(u)\ P\ B_n(w)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \cdots & \mathbf{P}_{0,n} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \cdots & \mathbf{P}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{m,0} & \mathbf{P}_{m,1} & \cdots & \mathbf{P}_{m,n} \end{pmatrix}$$



It is anchored at the four corner points and employs the other grid points to determine its shape.

Nonparametric Rectangular Patches

The explicit representation of a surface is $z = f(x, y)$. The rectangular Bezier surface is, of course, parametric, but it can be represented in a nonparametric form, similar to explicit surfaces. Given $(n + 1)\times(m + 1)$ real values (not points) P_{ij}, we start with the double polynomial

$$s(u, w) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{n,i}(u)\ P_{i,j}B_{m,j}(w)$$

to create the surface patch

$$P(u, w) = (u, w, s(u, w)) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{n,i}(u)\ \left(i/m\,, j/n\,, P_{i,j}\right)B_{m,j}(w)$$

Bicubic Bezier surface

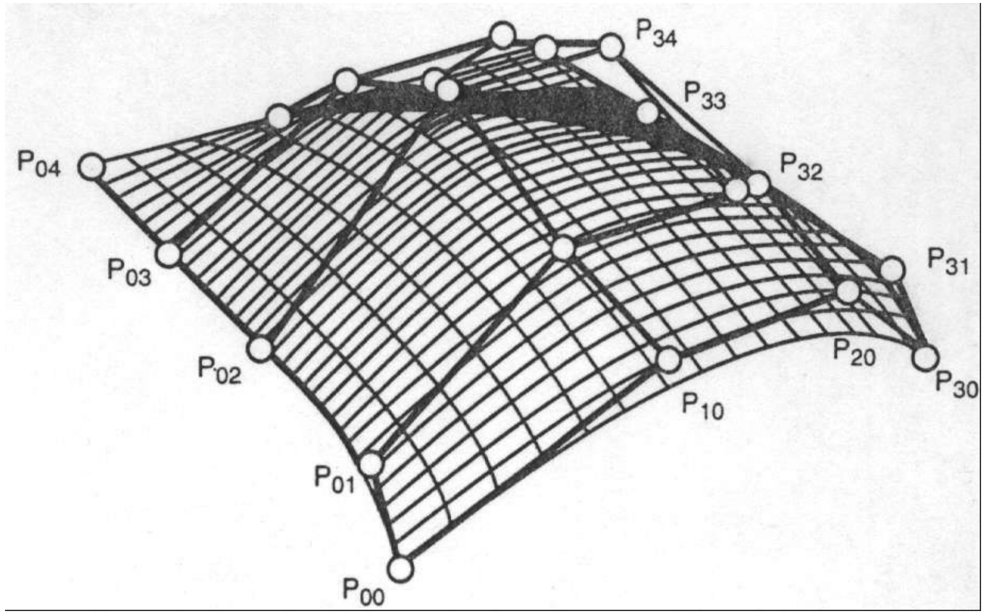
Bezier surface is an extension of Bezier curve and interpolates to a finite number of data point. It can be expressed as

$$P(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,m} B_{j,n}\, P_{i,j} \qquad \begin{array}{c} 0 \le u \le 1 \\ 0 \le v \le 1 \end{array}$$

where

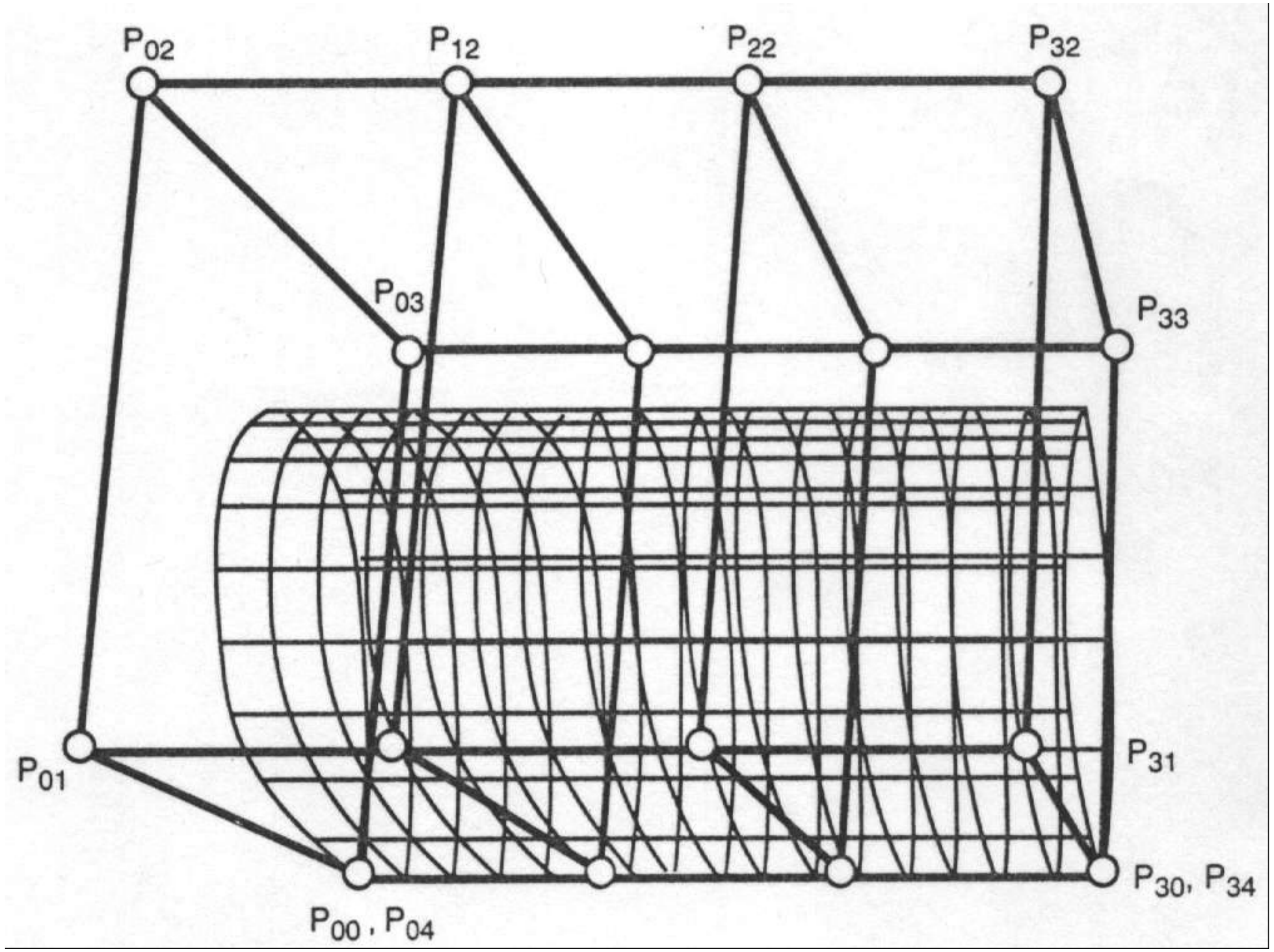
$$B(u,i) = C_{n,i}\, u^{i}\, (1-u)^{n-i}$$

$$C_{n,i} = \frac{n!}{i!\,(n-i)!}$$



Bezier patch with 5 x 4 array of points

Closed Bezier patch

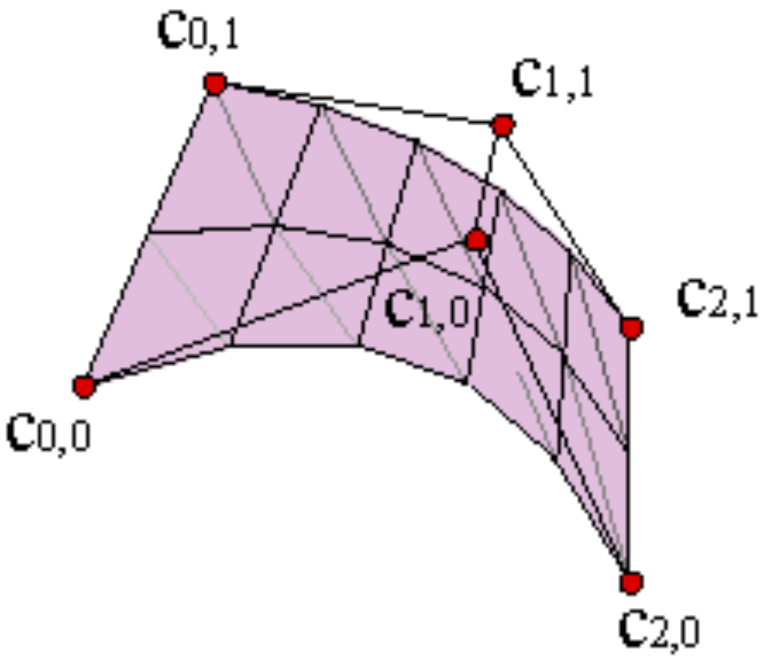


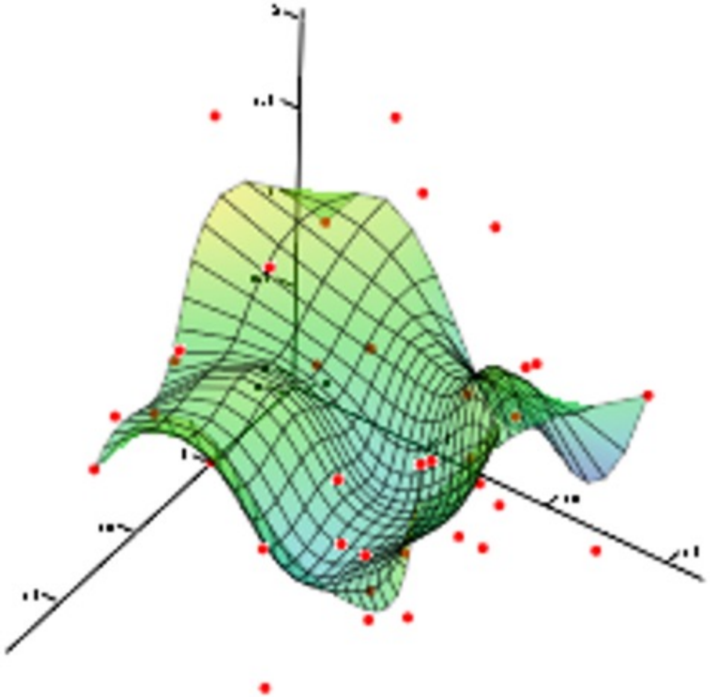
B-spline surface

TENSOR PRODUCT PATCHES

B-spline surface

This is a surface that can approximate or interpolate given input data.
It is a synthetic surface.
It is a general surface like the Bezier surface but with the advantage of permitting local control of the surface.





B-spline surface

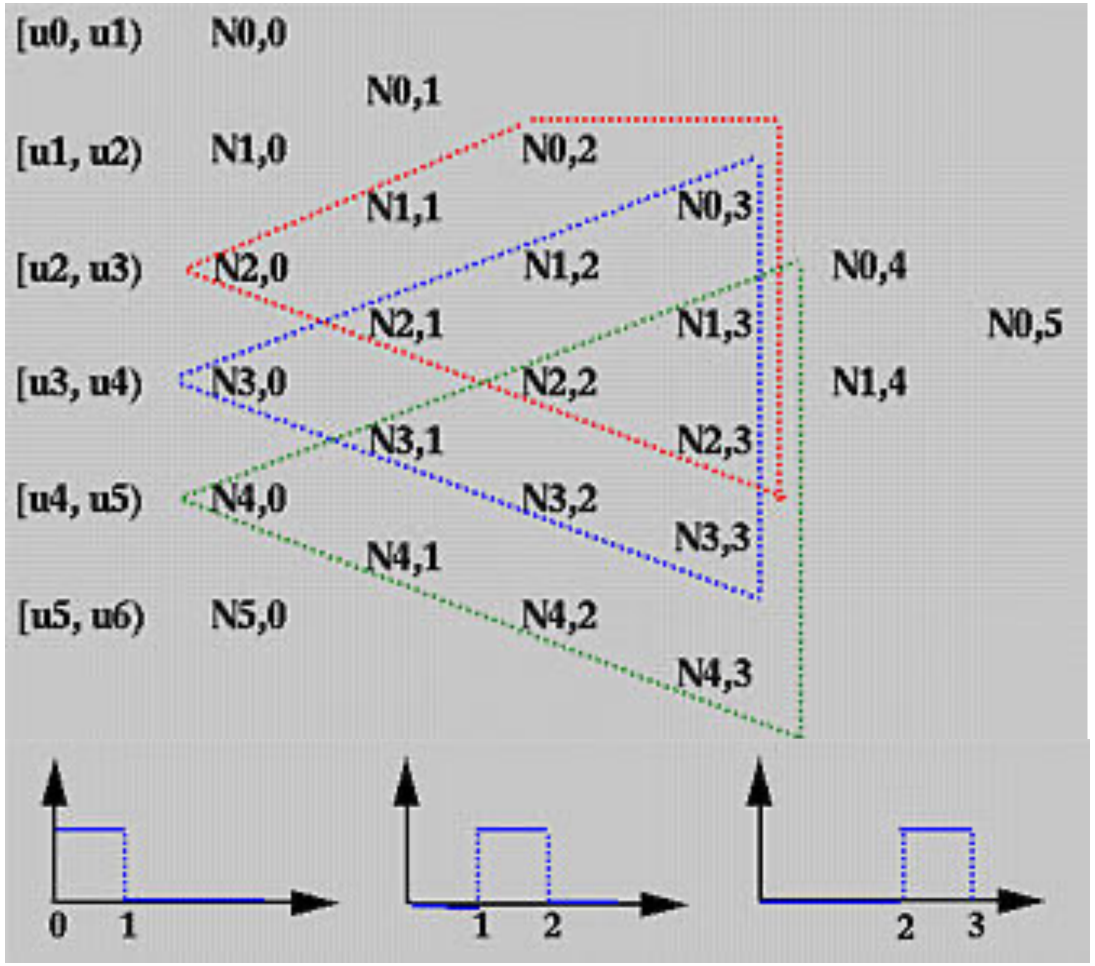
- As with curves, B-spline surfaces are a generalization of Bezier surfaces
- The surface approximates a control polygon
- Open and closed surfaces can be represented

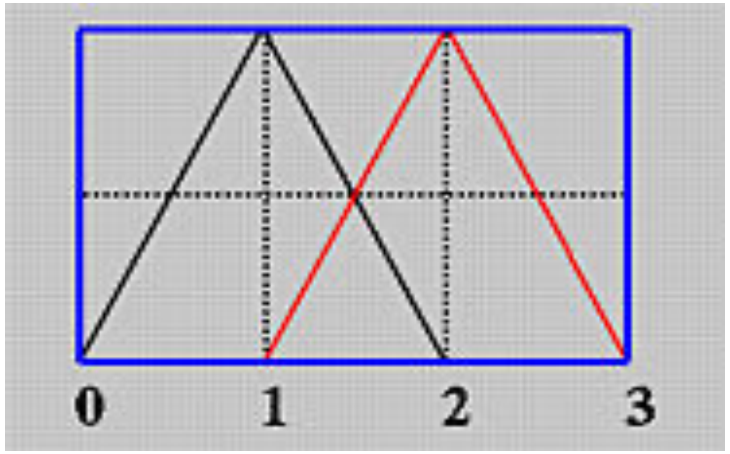
$$P(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k}(u) N_{j,l}(v) P_{i,j} \qquad \begin{array}{c} 0 \le u \le u_{max} \\ 0 \le v \le v_{max} \end{array}$$

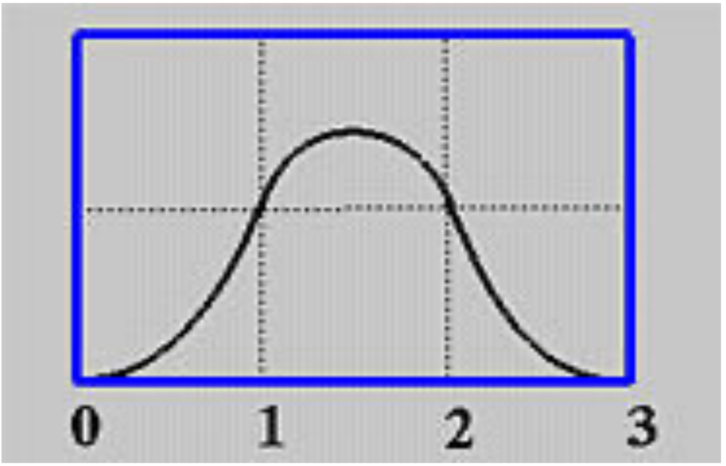
where

$$N_{i,k}(u) = (u - u_i) \frac{N_{i,k-1}}{u_{i+k-1} - u_i} + (u_{i+k} - u) \frac{N_{i+1,k-1}}{u_{i+k} - u_{i+1}}$$

Basis functions







Uniform B-spline surface

The uniform B-Spline surface patch is constructed as a Cartesian product of two uniform B-spline curves. The biquadratic B-spline surface patch, for example, is fully defined by nine control points and is constructed as the Cartesian product with itself

$$\mathbf{P}(u,w) = \left(\frac{1}{2}\right)^2 (u^2, u, 1) \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^T \begin{pmatrix} w^2 \\ w \\ 1 \end{pmatrix}.$$

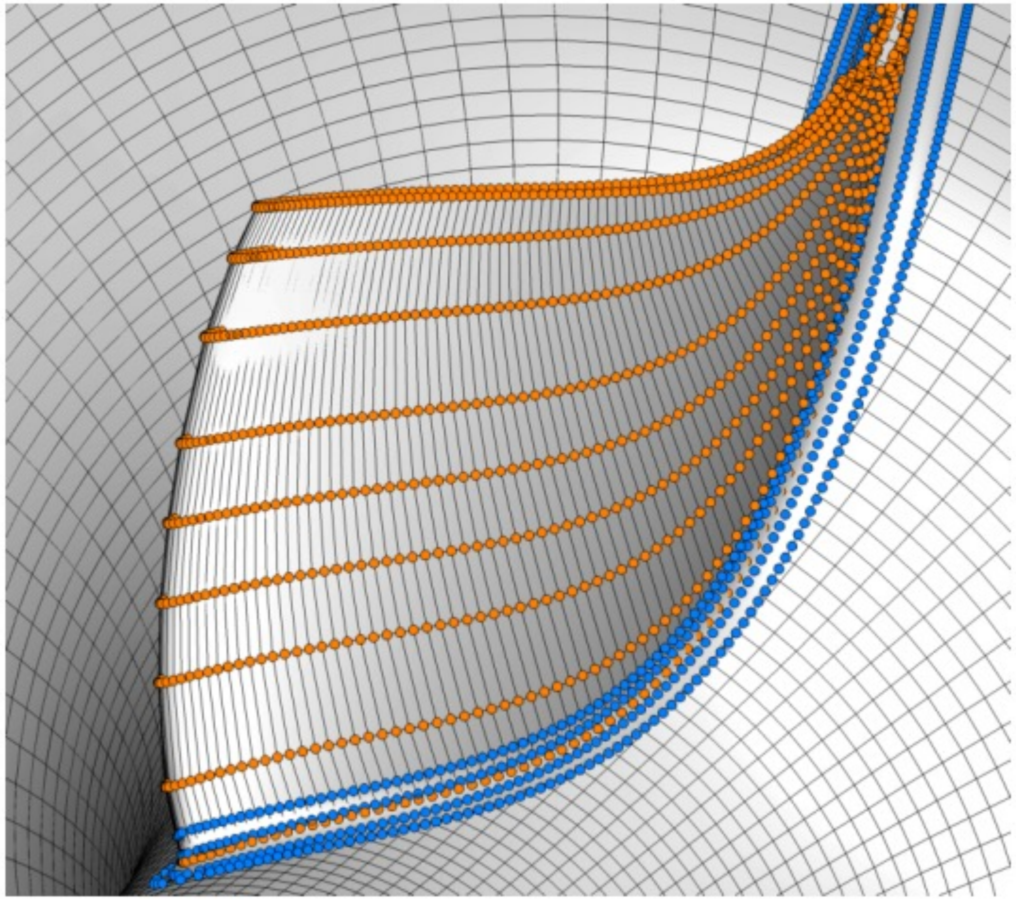
NURBS surface

TENSOR PRODUCT PATCHES

NURBS surfaces

NURBS surfaces are included as a particular case. However, some authors identify rational B-spline surfaces to NURBS surfaces.

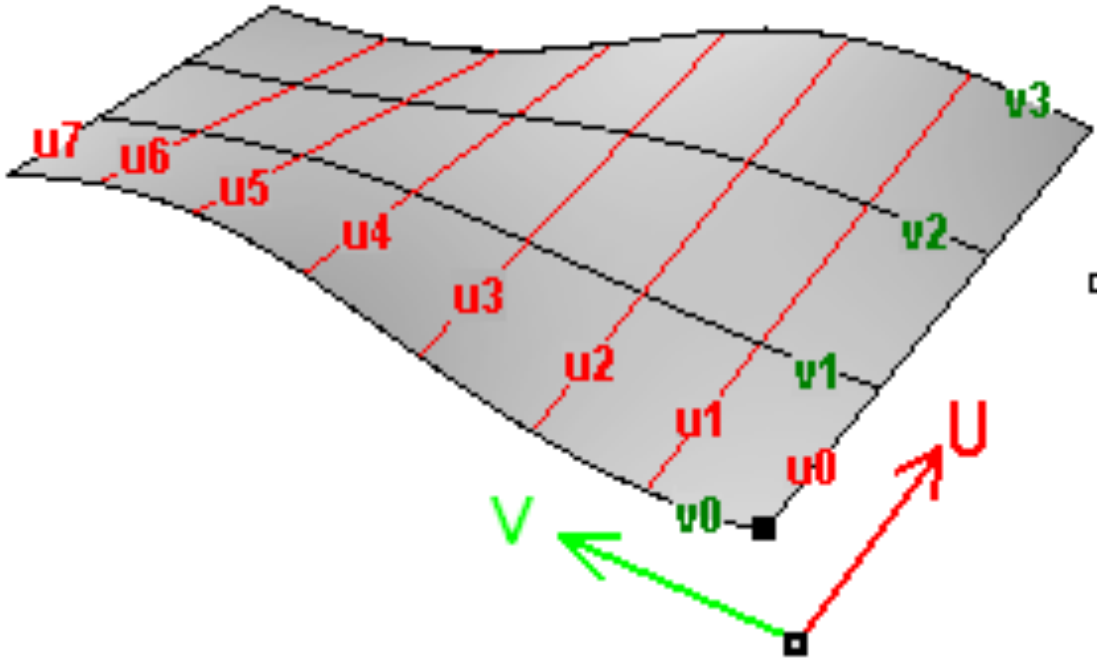
From this point of view, NURBS surfaces are a key topic in computer graphics, being incorporated in most of the computer design systems.



NURBS surfaces of the blade

NURBS surfaces

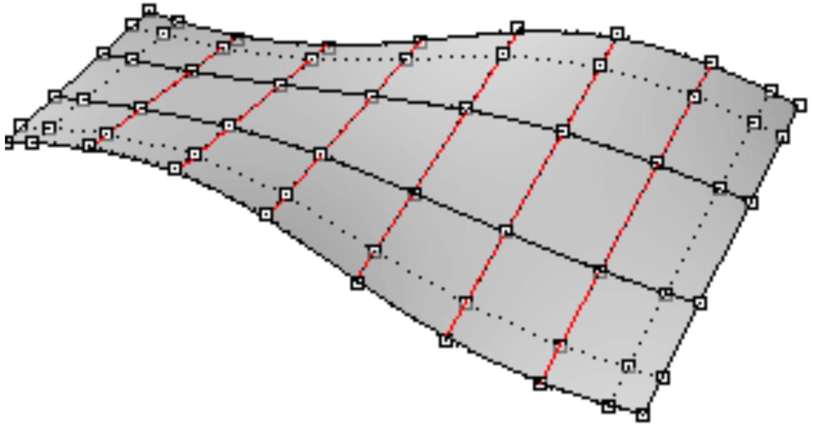
The shape of the surface is defined by a number of control points and the degree of that surface in each one of the two directions (u- and v-directions).

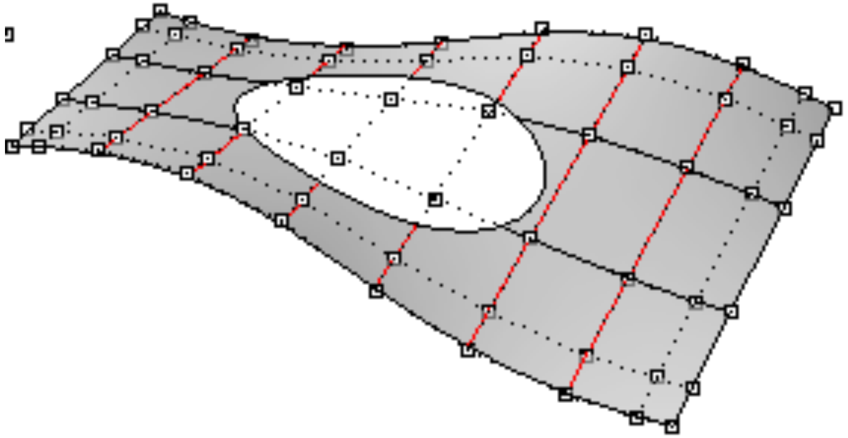


NURBS surfaces

NURBS surfaces can be trimmed or untrimmed. Each surface has one closed curve that define the outer border (*OuterLoop*) and as many non-intersecting closed inner curves that define holes (*InnerLoops*).

We refer to a surface with outer loop that is the same as that of its underlying NURBS surface and that has no holes as **untrimmed surface**.





NURBS surfaces

Non Uniform Rational B-Splines, are parametric tensor product curves or surfaces defined by the following expression

$$S(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}P_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}}$$

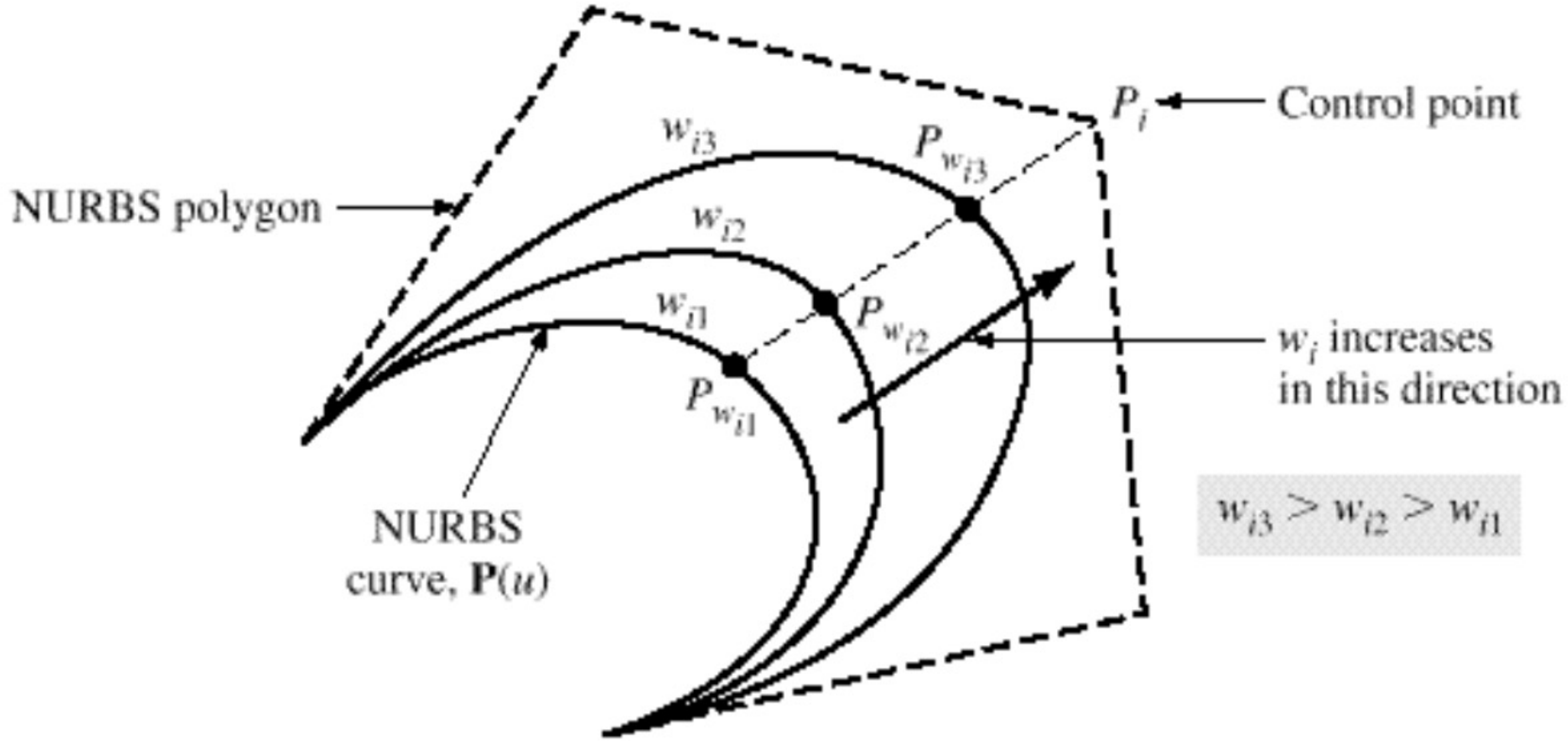
Where

w_{ij} : the weights

P_{ij} : control points

$N_{ip}(u)$, $N_{jq}(v)$: B-spline basis functions of order p and q.

Effects of changing the weight



Example: Right arc (quarter of a circle)

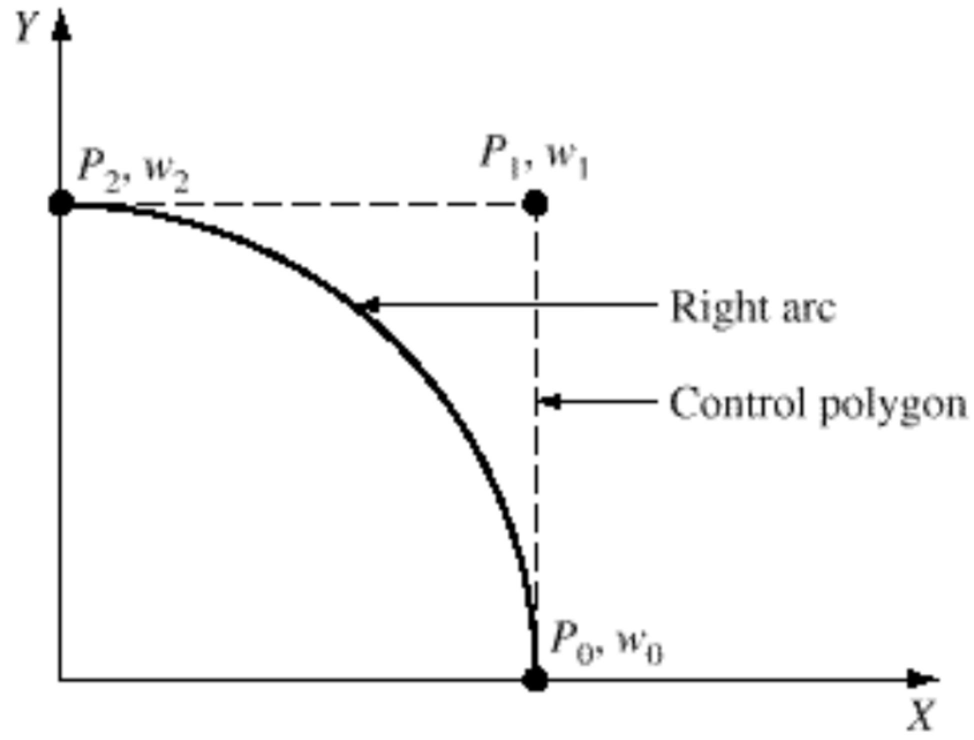
A circular arc is considered part of a circle. A NURBS circular arc has its own equation.

The arc is defined by three points P_0, P_1, P_2 that have weights w_1, w_2, w_3 respectively.

$k = 3$

$n = 2$

$m = 6$ (the length of the knot vector)



Example: Right arc (quarter of a circle)

NURBS curve uses knots and has a knot vector (KV) as

$$KV = [u_0 u_1 \quad u]^T = [u_0 \quad u_0 u_k \quad u_{m-k} \quad u]^T$$
$$= \begin{bmatrix} 0 & 0 u_k & u_{m-k} & 1 & 1 \\ k & & & & k \end{bmatrix}^T$$

Utilizing this equation gives the knot vector :

$$KV = [u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5]^T = [0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1]^T$$

Example: Right arc (quarter of a circle)

The equation of the NURBS arc is

$$P(u) = \sum_{i=0}^{2} P_i \; R_{i,3}(u) = P_0 R_{0,3}(u) + P_1 R_{1,3}(u) + P_2 R_{2,3}(u)$$

The pyramid of the basis functions $N_{i,k}(u)$:

$$\begin{array}{ccccccccc}
 & & N_{0,3} & & N_{1,3} & & N_{2,3} & & \\
 & N_{0,2} & & N_{1,2} & & N_{2,2} & & N_{3,2} & \\
N_{0,1} & & N_{1,1} & & N_{2,1} & & N_{3,1} & & N_{4,1}
\end{array}$$

Example: Right arc (quarter of a circle)

Evaluate the $N_{i,k}(u)$ basis functions as follows:

$$N_{0\;1} = \begin{matrix} 1 & u = 0 \\ 0 & \text{elsewhere} \end{matrix} \qquad N_{1\;1} = \begin{matrix} 1 & u = 0 \\ 0 & \text{elsewhere} \end{matrix}$$

$$N_{2\;1} = \begin{matrix} 1 & 0 \quad u \quad 1 \\ 0 & \text{elsewhere} \end{matrix} \qquad N_{3\;1} = \begin{matrix} 1 & u = 1 \\ 0 & \text{elsewhere} \end{matrix} \qquad N_{4\;1} = \begin{matrix} 1 & u = 1 \\ 0 & \text{elsewhere} \end{matrix}$$

$$N_{0\;2} = \frac{u - u_0 \;\; N_{0\;1}}{u_1 - u_0} + \frac{u_2 - u \;\; N_{1\;1}}{u_2 - u_1} = 0$$

$$N_{1\;2} = \frac{u - u_1 \;\; N_{1\;1}}{u_2 - u_1} + \frac{u_3 - u \;\; N_{2\;1}}{u_3 - u_2} = \;\; 1 - u \;\; N_{2\;1}$$

$$N_{2\;2} = \frac{u - u_2 \;\; N_{2\;1}}{u_3 - u_2} + \frac{u_4 - u \;\; N_{3\;1}}{u_4 - u_3} = uN_{2\;1}$$

$$N_{3\;2} = \frac{u - u_3 \;\; N_{3\;1}}{u_4 - u_3} + \frac{u_5 - u \;\; N_{4\;1}}{u_5 - u_4} = 0$$

$$N_{0\;3} = \frac{u - u_0 \;\; N_{0\;2}}{u_2 - u_0} + \frac{u_3 - u \;\; N_{1\;2}}{u_3 - u_1} = \;\; 1 - u \;\; {}^2 N_{2\;1}$$

$$N_{1\;3} = \frac{u - u_1 \;\; N_{1\;2}}{u_3 - u_1} + \frac{u_4 - u \;\; N_{2\;2}}{u_4 - u_2} = u \;\; 1 - u \;\; N_{2\;1} + \;\; 1 - u \;\; uN_{2\;1} = 2u \;\; 1 - u \;\; N_{2\;1}$$

$$N_{2\;3} = \frac{u - u_2 \;\; N_{2\;2}}{u_4 - u_2} + \frac{u_5 - u \;\; N_{3\;2}}{u_5 - u_3} = u^2 N_{2\;1}$$

Example: Right arc (quarter of a circle)

Substituting the $N_{i,3}$ into the equation $R_{i,k}(u) = \frac{w_i\ N_{i,k}(u)}{\sum_{i=0}^{n} w_i\ N_{i,k}(u)}$

$$P(u) = \frac{P_0\ w_0\ N_{0,3} + P_1\ w_1\ N_{1,3} + P_2\ w_2\ N_{2,3}}{w_0\ N_{0,3} + w_1\ N_{1,3} + w_2\ N_{2,3}}$$

$$P(u) = \frac{P_0\ w_0\ (1-u^2) + P_1\ w_1\ 2u(1-u) + P_2\ w_2\ u^2}{w_0(1-u^2) + w_1 2u(1-u) + +w_2 u^2}$$

$0 \leq u \leq 1$

Example: Right arc (quarter of a circle)

If we use $w_0 = w_1 = w_2 = 1$, it reduces to

$$P(u) = P_0\,(1 - u^2) + P_1\,2u(1-u) + P_2\,u^2$$

If we use $w_0 = w_1 = 1$ and $w_2 = 2$, it gives

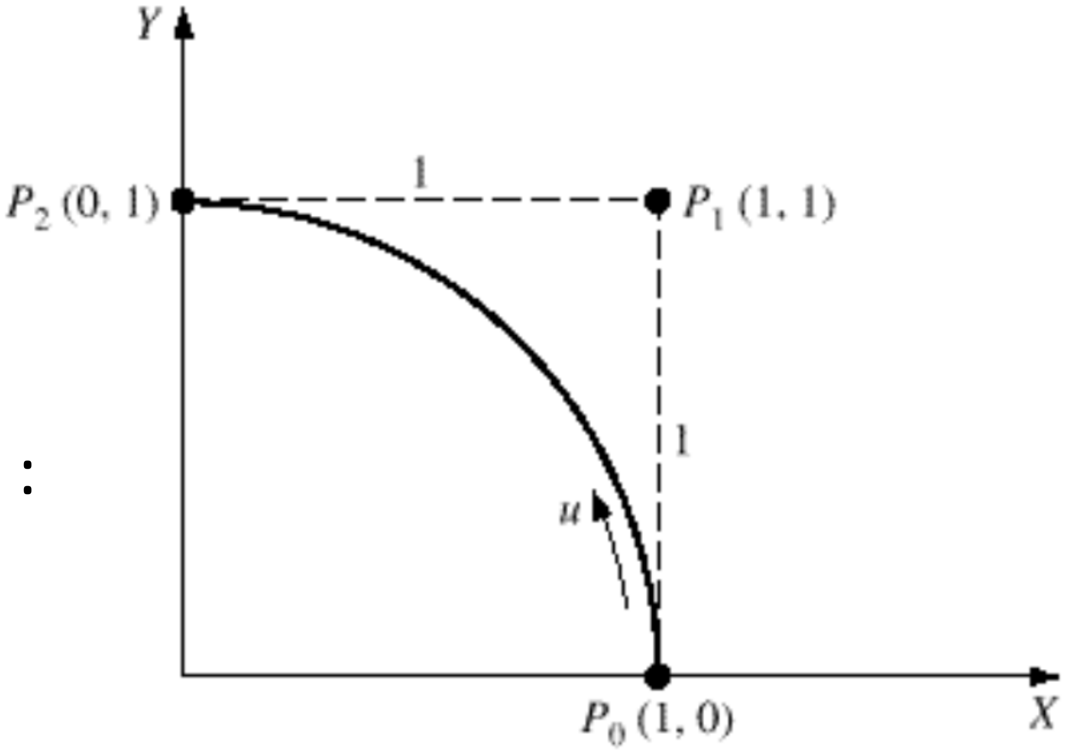
$$P(u) = \frac{P_0\,(1 - u^2) + P_1\,2u(1-u) + P_2\,2u^2}{(1-u^2) + 2u(1-u) + +2u^2}$$

$$P(u) = \frac{P_0\,(1 - u^2) + P_1\,2u(1-u) + P_2\,2u^2}{(1+u^2)}$$

Example: Right arc (quarter of a circle)

Let us assume a unit arc in the XY coordinate system. Substituting the (x, y) coordinates of the control points into the last equation, the arc equation becomes

$$P(u) = \begin{bmatrix} x & u \\ y & u \end{bmatrix} = \begin{bmatrix} \dfrac{1-u^2}{1+u^2} \\ \dfrac{2u}{1+u^2} \end{bmatrix}$$



It satifsies the boundary conditions :

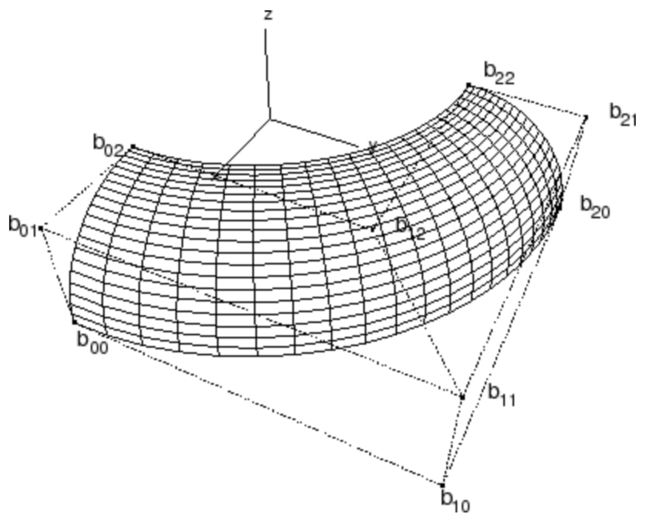
At $u \ = \ 0, P \ = \ P_0$

At $u \ = 1, P \ = P_2$

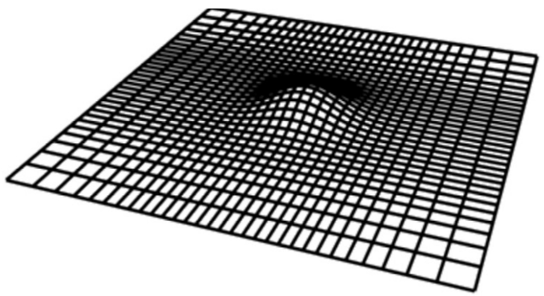
Composite surfaces

Composite surfaces

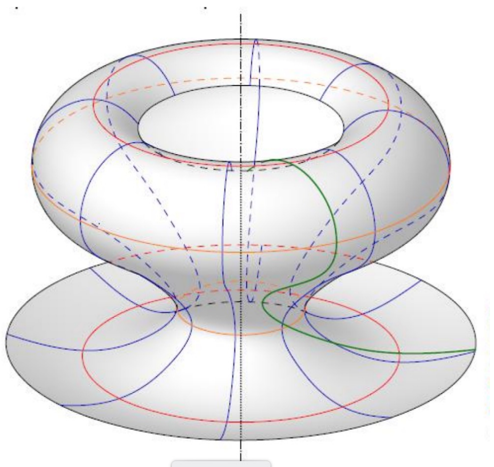
Rational Bezier surface



Rational B-spline surface

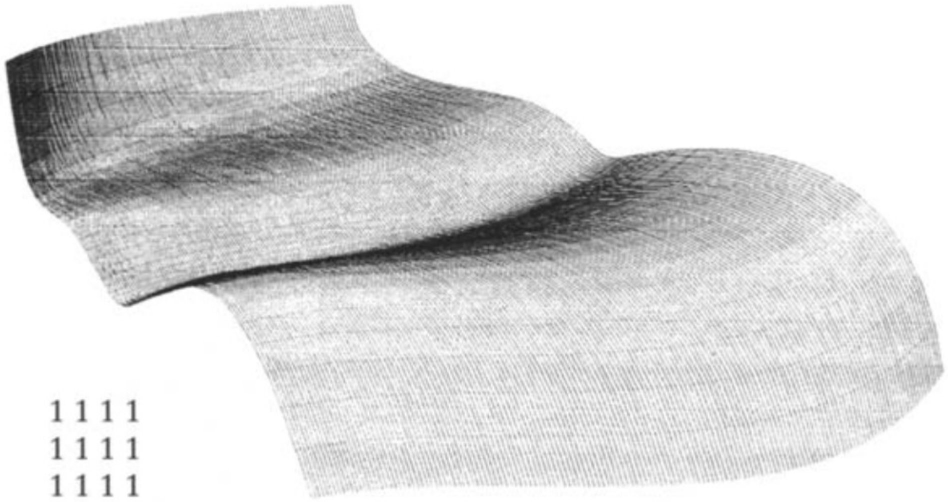


Revolved surface

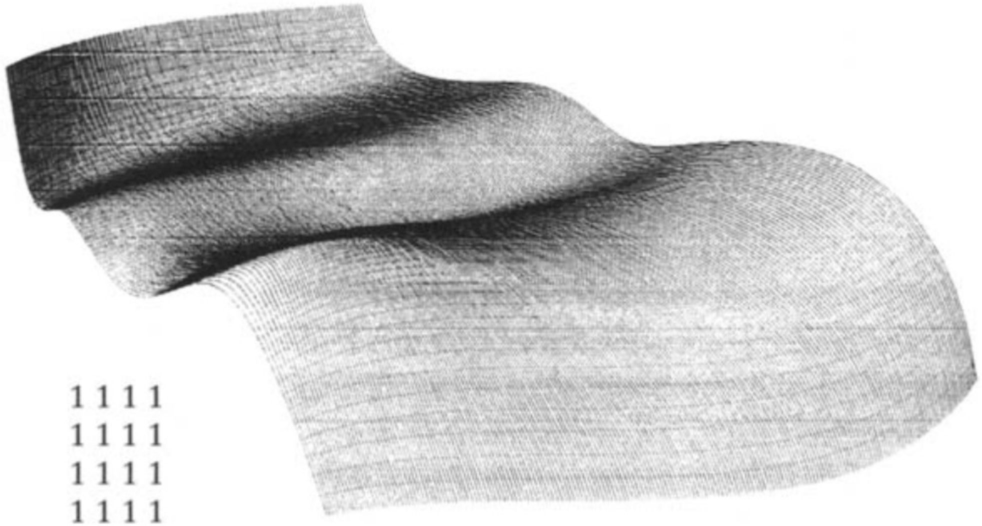


Rational Bezier and B-Spline Surfaces

We can generalize Bezier and B-spline surfaces to their rational counterparts in much the same way as we did for the curve cases. In other words, we define a rational Bezier or B-spline surface as the projection of a 4D tensor product Bezier or B-spline surface.



$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$



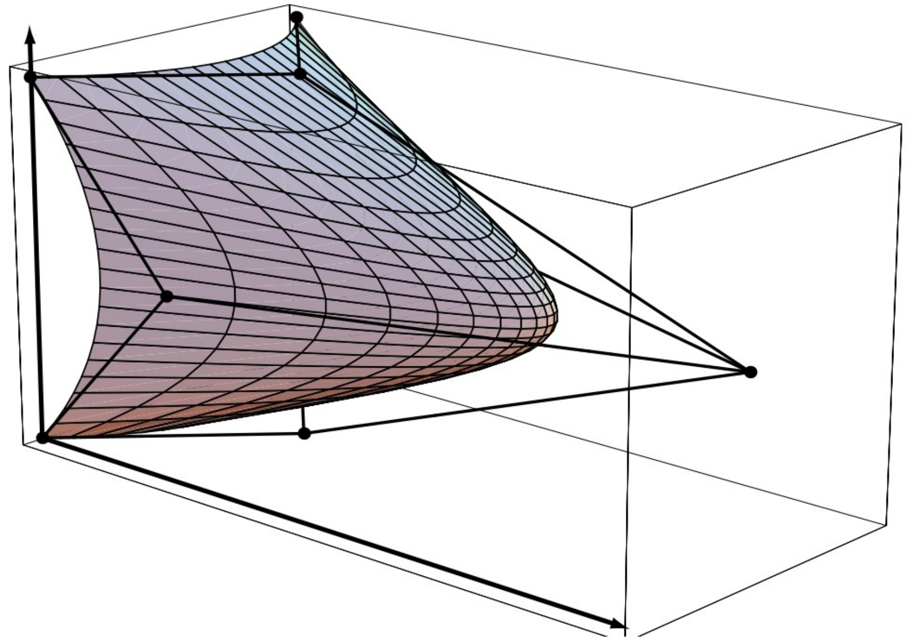
$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

Rational Bezier Surfaces

The principle of rational Bezier curve can be extended to surfaces. The rational Bezier patch takes the form :

$$P(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij} B_{n,i}(u) P_{ij} B_{m,j}(v)}{\sum_{k=0}^{n}\sum_{l=0}^{m} w_{kl} B_{n,k}(u) B_{m,l}(v)}$$

When all the weights w_{ij} are set to 1, equation reduces to the original rectangular Bezier surface patch.



Rational B-Spline Surfaces

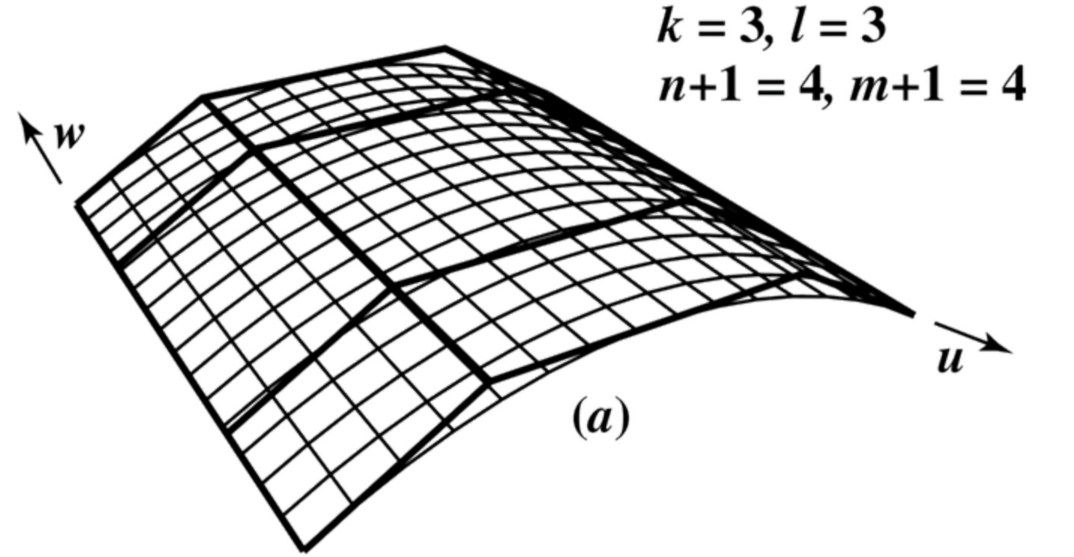
Rational B-spline surface is a single precise mathematical representation capable of representing common analytical surfaces – planes, conic surfaces including sphere, freeform surfaces, quadric and sculptured surfaces used in the CAD applications

Rational B-Spline Surfaces

A rational B-spline surface is the projection of a non rational (polynomial) B-spline defined in 4-dimensional homogenous coordinate space back into 3D physical space

Rational B-spline surface is written as

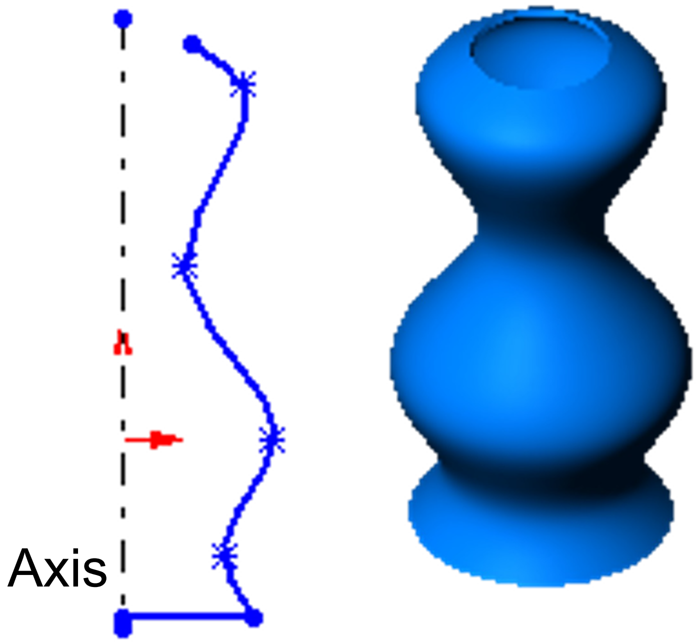
$$s(u,v) = \frac{\sum_i \sum_j w_{ij} d_{ij}\, N_i^m(u) N_j^n(v)}{\sum_i \sum_j w_{ij}\, N_i^m(u) N_j^n(v)}$$



(a)

Revolved surface

A revolved surface is generated a space curve about an axis of rotation.
This is an axisymmetric surface that can model axisymmetric objects.
It is generated by rotating a planar wireframe entity in space about the axis of symmetry a certain angle.



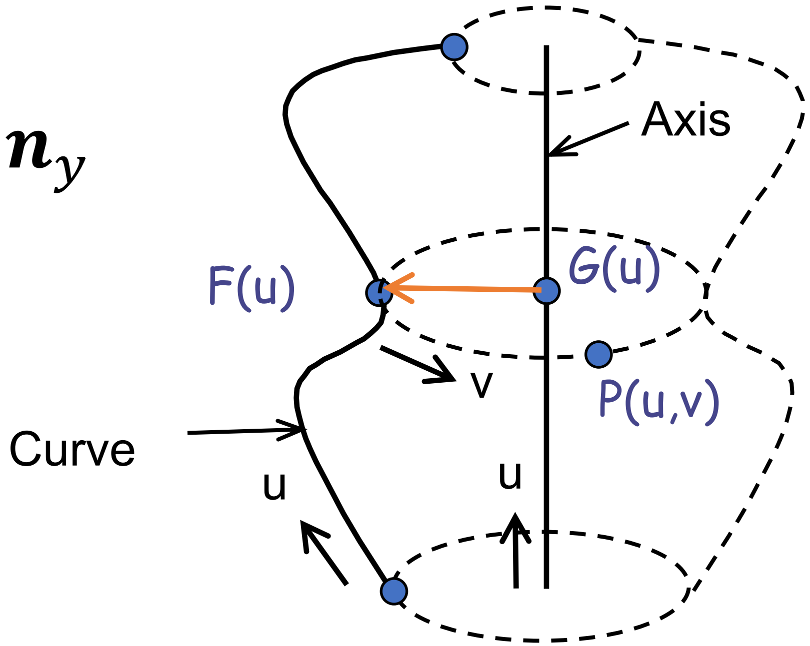
Revolved surface

The parametric representation of the curve in the working coordinate system that has x- and y-axes on the perpendicular plane to the axis of rotation that is z-axis can be expressed by

$$P(u,v) = F(u) + v(G(u) - F(u))$$

$$P(u,v) = r_z(u)\cos v\boldsymbol{n}_x + r_z(u)\sin v\boldsymbol{n}_y + z(u)\boldsymbol{n}_y$$

$$0 \le u \le 1$$
$$0 \le v \le 2\pi$$

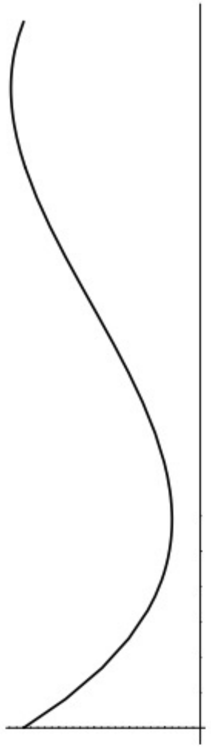


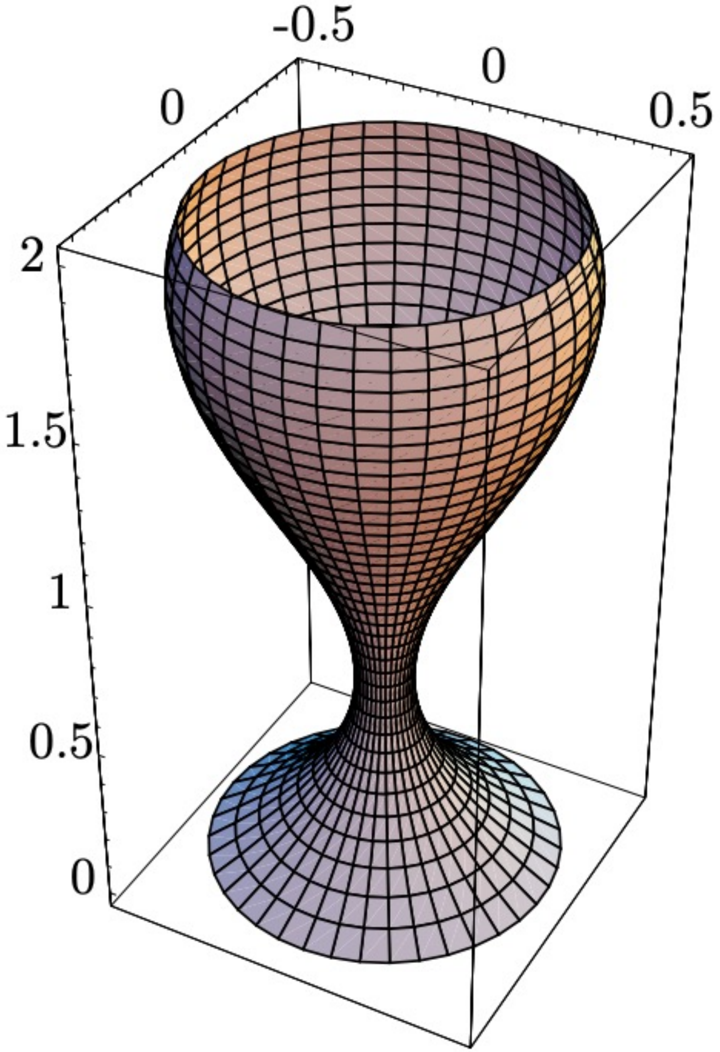
Positive rotation direction usually based upon direction of axis vector.

Surfaces of Revolution

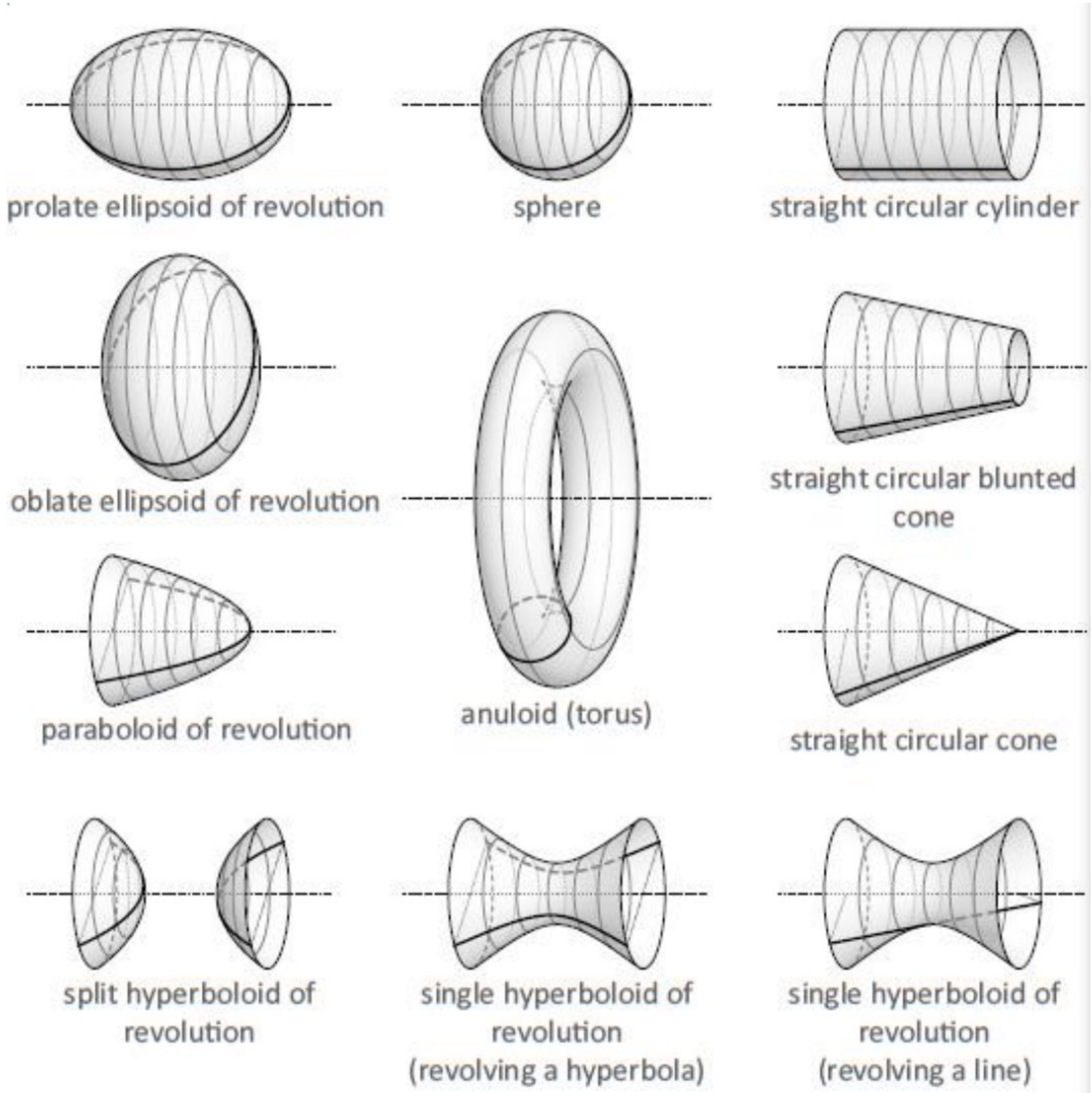
A surface of revolution is a special case of a swept surface. It is obtained when a space curve (termed the profile of the surface) is rotated about an axis $r = (r_x, r_y, r_z)$ in space.

The rotation angle can be 360◦ or less. A general rotation in three dimensions is fully specified by the axis of rotation (a vector) and the rotation angle (a number).





Surfaces of Revolution



Surfaces of Revolution

In general, the point matrix gives a point on the surface of revolution obtained by rotation around the z-axis,

$$P(t,\theta) = [x(t)\cos\theta \quad x(t)\sin\theta \quad z(t)]$$

In matrix form :

$$P(t,\theta) = [x(t) \quad 0 \quad z(t) \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0. & 0 & 0 & 1 \end{bmatrix}$$

CONS and Trimmed surface

Trimmed surfaces: certain parts of a tensor product surface are marked as "invalid" by a pair of CONS.

CONS are mainly used for a modification of tensor product surfaces by a technique known as trimming. A trimmed surface has certain areas of it marked as invalid or invisible by a set of closed CONS.

Literature on trimmed surfaces:

R. T. Farouki and J. K. Hinds. A hierarchy of geometric forms. IEEE Computer Graphics and Applications, pages 51–78, May 1985.

M.S. Casale and J.E. Bobrow, “A Set Operation Algorithm for Sculptured Solids Modeled with Trimmed Patches,” *Computer Aided Geometric Design,* Vol. 6, 1989, pp 235-247.

D. Lasser and G. R. Bonneau ‘’ Bézier Representation of Trim Curves’’, Geometric Modelling, Dagstuhl, Germany, 1993.

Abrrivations

CONS	Cuve on surface
CAGD	Computer Aided Geometric Design
NURBS	Nonuniform rational B-splines