MATH604 - Assignment 6 - Deepak Kumar,73217151

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Q 6.1 →

(a) Prove f is a convex function.

Sol. $f(x) = \max\{|x_1|, |x_2|\}$.

let $x, y \in \mathbb{R}^2$, then for some $z = \alpha x + (1-\alpha)y$, $\alpha \in (0,1)$

then,

$$f(z) = f(\alpha x + (1-\alpha)y)$$

$$= \max\{|\alpha x_1 + (1-\alpha)y_1|, |\alpha x_2 + (1-\alpha)y_2|\}$$

by using triangular inequality, we get

$$\leq \max\{\alpha|x_1| + (1-\alpha)|y_1|, \alpha|x_2| + (1-\alpha)|y_2|\}$$

$$\leq \max\{\alpha|x_1|, \alpha|x_2|\} + \max\{(1-\alpha)|y_1|, (1-\alpha)|y_2|\}$$

$$\leq \alpha \max\{|x_1|, |x_2|\} + (1-\alpha)\max\{|y_1|, |y_2|\}$$

$$\leq \alpha f(x) + (1-\alpha) f(y)$$

hence, f is convex.

(b) Prove that R_θ is invertible.

Sol. (b) $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\det(R_\theta) = \begin{vmatrix} \cos\theta & -\sin\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$

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hence, R_θ is invertible.

Q 6.4 →

Sol. (6.4) let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(a) Exactly 6 vectors

Sol. (a) $D = \{e_1, e_2, e_3, -e_1, -e_2, -e_3\}$

(b) Exactly 5 vectors

Sol. (b) $D = \{e_1, e_2, e_3, -\frac{1}{2} \times (e_1 + e_2), -e_3\}$

(c) Exactly 4 vectors, and the angle between any two of them is identical.

Sol. (c) $D = \{\frac{1}{3}(e_1 - e_2 - e_3), \frac{1}{3}(e_2 - e_1 - e_3), \frac{1}{3}(e_3 - e_1 - e_2),$

Q 6.12 →

Sol. 6.12 → $f^\circ(x; d) = \lim\limits_{y \to x} \sup\limits_{t \downarrow 0} \dfrac{f(y + td) - f(y)}{t}$

(a) $f^\circ(x; \lambda d) = \lambda f^\circ(x; d)$ for every scalar $\lambda > 0$ and all $d \in \mathbb{R}^N$.

Sol. Since, $f^\circ(x; \lambda d) = \lim\limits_{y \to x} \sup\limits_{t \downarrow 0} \dfrac{f(y + \lambda t d) - f(y)}{t}$

let $u = \lambda t$, then $t \downarrow 0$, $u \downarrow 0$ as $\lambda > 0$.

$$= \lim_{y \to x} \sup_{u \downarrow 0} \frac{f(y+ud) - f(y)}{(u/\lambda)}$$

$$= \lambda \times \lim_{y \to x} \sup_{u \downarrow 0} \frac{f(y+ud) - f(y)}{u}.$$

$$= \lambda f^{\circ}(x;d).$$

hence Proved.

(b) $f^{\circ}(x;u+v) \leq f^{\circ}(x;u) + f^{\circ}(x;v)$

Sol.(b) Since, $f^{\circ}(x;u+v) = \lim_{y \to x} \sup_{t \downarrow 0} \frac{f(y+t(u+v)) - f(y)}{t}$

$$= \lim_{y \to x} \sup_{t \downarrow 0} \frac{f(y+tu+tv) - f(y+tu) + f(y+tu) - f(y)}{t}$$

$$\leq \lim_{y \to x} \sup_{t \downarrow 0} \frac{f(y+tu+tv) - f(y+tu)}{t} + \lim_{y \to x} \sup_{t \downarrow 0} \frac{f(y+tu) - f(y)}{t}$$

$$\leq f^{\circ}(x;v) + f^{\circ}(x;u).$$

(c) $d \longmapsto f^{\circ}(x;d)$ is a Convex function with respect to d.

Sol.(c) let $d_1, d_2 \in \mathbb{R}^N$,

then for some $d_3 = \alpha d_1 + (1-\alpha) d_2$, $\alpha \in (0,1)$

$$f^{\circ}(x;d_3) = f^{\circ}(x;\alpha d_1 + (1-\alpha) d_2)$$

then from part (b), we have

$$f^{\circ}(x;\alpha d_1 + (1-\alpha) d_2) \leq f^{\circ}(x;\alpha d_1) + f^{\circ}(x;(1-\alpha) d_2)$$

From part (a), we have

$$f^{\circ}(x; \alpha d_1 + (1-\alpha) d_2) \leq \alpha f^{\circ}(x; d_1) + (1-\alpha) f^{\circ}(x; d_2)$$

Hence, $f^{\circ}(x;d)$ is convex with respect to d.

(d) $f^{\circ}(x;-d) = (-f^{\circ})(x;d)$ for every $d \in \mathbb{R}^n$.

Sol (d) Since,

$$f^{\circ}(x;-d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y-td) - f(y)}{t}$$

let $u = y - td$

$$= \limsup_{\substack{u \to x \\ t \downarrow 0}} \frac{f(u) - f(u+td)}{t}$$

$$= \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{-f(u+td) - (-f(u))}{t}$$

$$= - \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(u+td) - f(u)}{t}$$

$$= - f^{\circ}(x;d)$$

Q 6.7 →

$$f(x) = \begin{cases} x \cos(\log(|x|)) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

(a)

Sol. (a) Since, $\lim\limits_{k \to \infty} \frac{s^k \cos(\log|s^k|)}{s^k} = -1$

Sol. (a) Since, $\lim_{k\to\infty} \dfrac{s \cos(\log|s^k|)}{s^k} = -1$

$$\Rightarrow \cos(\log|s^\infty|) = -1$$

$$\Rightarrow |s^\infty| = e^{-\pi}$$

hence, $\{s^k\} = e^{-k\pi}$

also, $\lim_{k\to\infty} \dfrac{t^k \cos(\log|t^k|)}{t^k} = 1$

$$\Rightarrow \cos(\log|t^\infty|) = 1$$

$$\Rightarrow |t^\infty| = e^{-2\pi}$$

hence, $\{t^k\} = e^{-2k\pi}$

(b)

Sol (b) Since for $x = 0$, we are able to find two sequences $\{s^k\}$ and $\{t^k\}$ such that

$$\lim_{k\to\infty} \frac{t^k \cos(\ln|t^k|) - f(t^k)}{t^k} \neq \lim_{k\to\infty} \frac{s^k \cos(\ln|s^k|)}{s^k}$$

hence, $f'(0;1)$ doesnot exists.

(c)

Sol. (c) Since, $\nabla f(u^k) = \sqrt{2}$ as $u^k \searrow 0$

$$\frac{\partial}{\partial x} f(x) = \cos(\ln|x|) + x(-\sin(\ln|x|)) \cdot \frac{1}{x} \qquad \text{for } x > 0$$

$$= \cos(\ln x) - \sin(\ln|x|)$$

let $\ln(x) = \theta$.

$\Rightarrow \cos\theta - \sin\theta = \sqrt{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = 1$

$\Rightarrow \sin(\pi/4).\cos\theta - \cos(\pi/4).\sin\theta = 1$

$\Rightarrow \sin(\pi/4 - \theta) = 1$

$\Rightarrow \frac{\pi}{4} - \theta = \frac{\pi}{2}$

$\Rightarrow \theta = -\pi/4$

hence, $a_k = e^{-k\pi/4}$

Q 6.3 →

(b) $D_{max} = \{e_1, e_2, e_3, \dots e_n, -e_1, -e_2, \dots, -e_n\}$

Sol. To prove that D_{max} is a positive basis of $\mathbb{R}^N$, we need to show that pspan$(D_{max}) = \mathbb{R}^N$ and, D_{max} is p.l.i.

Part I (pspan): let $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^N$

To show x is in pspan(D_{max}) we need to create x using D_{max}.

let $m_1 = \min\{x_1, x_2, \dots, x_n\}$.

if $m_1 \geq 0$, then $x = \sum_{i=1}^{n} x_i e_i$ with $x_i \geq 0$ for all i.

if $m_2 = \max\{x_1, x_2, \dots, x_n\} < 0$

then,

$x = \sum_{i=1}^{n} x_i e_i \Rightarrow x = \sum_{i=1}^{n} |x_i| \cdot -e_i$ $\quad \forall -e_i \in D_{max}$ and, $\lambda_i = |x_i| \geq 0$ for all i.

hence, $x \in$ pspan(D_{max}).

If x contains both positive and negative x_i's,

then,

$$x = \sum_{i \in \{x_i \geq 0\}} x_i e_i + \sum_{i \in \{x_i < 0\}} |x_i| -e_i$$

$\forall\ e_i, -e_i \in D_{max}$ and $\lambda_i = |x_i| \geq 0$ for all i.

hence, $x \in \text{pspan}(D_{max})$.

<u>Part II</u> (positive linear Independent)

Suppose $d^i \in \text{pspan}(D_{max} \setminus \{d^i\})$

if $d^i = e_i \quad (1 \leq i \leq n)$, then

$$d^i = \sum_{j \neq i}^{n} \lambda_j . e_j + \sum_{k=n+1}^{2n} \lambda_k . (-e_k)$$

$$= \sum_{k=n+1}^{2n} \lambda_k . (-e_k)$$

$$= (-\lambda_a) . e_a \qquad [\because e_a = e_i]$$

hence a contradiction.

if $d_i = -e_i \quad (1 \leq i \leq n)$ then,

$$d_i = \sum_{j=1}^{n} \lambda_j . e_j + \sum_{k=n+1}^{2n} \lambda_k (-e_k)$$

$$= \sum_{j=1}^{n} \lambda_j . e_i$$

$$= (-\lambda_a) . (-e_i)$$

hence a contradiction.

So D_{max} is a positive basis of $\mathbb{R}^n$.

Extra question: Prove $||x| - |y|| \leq |x - y|$ for any x and y in $\mathbb{R}^1$

Sol. Since, we can write $x = x + y - y$

$$\Rightarrow |x| = |x + y - y|$$

Now, By triangular inequality, we have

$$|x| = |x + y - y| \leq |x - y| + |y|$$

$$\Rightarrow |x| \leq |x - y| + |y|$$

$$\Rightarrow |x| - |y| \leq |x - y|$$

$$\Rightarrow ||x| - |y|| \leq |x - y|$$

Hence proved.