

Q 5.8

$$(a) Y^k = \{e_1, e_2, \dots, e_n, 0\}$$

hence,  $L = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$

Now to compute determinant of L,

we subtract  $e_n$  to each of the columns to get,

$$\det(L) = \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

Now by swapping the columns so as to make it identity matrix we get

$$= (-1)^{n-1} \times (-1) |I|$$

$$= (-1)^n$$

$$\text{hence, } \det(L) = (-1)^n$$

$$\text{hence, } \text{Vol}(\mathcal{X}^k) = \frac{|\det(L)|}{n!} = \frac{|(-1)^n|}{n!} = \frac{1}{n!}$$

$$\text{and, } \text{diam}(\mathcal{X}^k) = \sqrt{r^2 + l^2} = \sqrt{2}$$

$$\text{and, so, } \text{Vol}(\mathcal{X}^k) = \frac{\text{Vol}(\mathcal{X}^k)}{(\text{diam}(\mathcal{X}))^n} = \frac{1}{n! (\sqrt{2})^n}$$

Now, for reflection step,

$$x^c = \frac{1}{n} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

And so,

$$\begin{aligned} x^n &= x^c + (x^c - y^n) \\ &= \frac{2}{n} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}, \quad x^n - y^0 = \begin{bmatrix} \frac{2}{n} - 1 \\ \frac{2}{n} \\ \frac{2}{n} \\ \vdots \\ \frac{2}{n} \end{bmatrix} = \frac{2}{n} \begin{bmatrix} \frac{(2-n)}{2} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}$$

NOW,

$$L^+ = \frac{2}{n} \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 & \frac{(2-n)}{2} \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$C_n \rightarrow C_n - \sum_{i=1}^{n-1} C_i$$

$$\det(L^+) = \frac{2}{n} \begin{vmatrix} -1 & -1 & -1 & \dots & -1 & \frac{n}{2} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 & \dots & -1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

and, so,

$$\det(L^+) = (-1)^{n-1}$$

$$\text{So, Volume } (\mathcal{X}^{k+1}) = \frac{|(-1)^{n-1}|}{n!} = \frac{1}{n!}$$

$$\text{diam } (\mathcal{X}^{k+1}) = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{Vol } (\mathcal{X}^{k+1}) = \frac{\text{Vol } (\mathcal{X}^{k+1})}{(\text{diam } (\mathcal{X}^{k+1}))^n} = \frac{1}{n!} \cdot \frac{1}{(\sqrt{2})^n} = \frac{1}{n! (\sqrt{2})^n}$$

(b)  $y^0 = 0$  and for every  $i$  from 1 to  $n$ , set  $y^i = y^{i-1} + e_i$ , where  $e_i$  is the  $i$ th coordinate vector.

$$\text{S.t. } \mathbf{y}^k = \sum_{i=0}^k e_i = e_0 + e_1 + \dots + e_k$$

$$\text{Sol. } \mathbb{X}^k = \{0, e_1, e_1 + e_2, \dots, \sum_{i=1}^n e_i\}$$

hence,

$$L = \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & 0 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

NOW, by  $C_n \rightarrow C_n - C_{n-1}$ ,  $C_{n-1} \rightarrow C_{n-1} - C_{n-2}$  ...,  $C_2 \rightarrow C_2 - C_1$ , we get

$$\det(L) = \begin{vmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{vmatrix}$$

$$= |I|$$

hence,  $\det(L) = 1$ .

$$\text{So, } \text{Vol.}(\mathbb{X}^k) = \frac{1}{n!}$$

$$\text{and, } \text{diam}(\mathbb{X}^k) = \sqrt{1^2 + 1^2 + \dots + 1^2}$$

$$= \sqrt{n-1}$$

$$\text{hence, } \text{Vol}(\mathbb{X}^k) = \frac{\text{Vol}(\mathbb{X}^k)}{(\text{diam}(\mathbb{X}^k))^n} = \frac{1}{n!} \times \frac{1}{(\sqrt{n-1})^n} = \frac{1}{n! (n-1)^{n/2}}$$

$$\text{num} , \text{num II} = \frac{\text{num - 1}}{(\text{diam}(x^*))^n} = n! (\sqrt{n-1}) \cdot n! (n-1)^{n/2}$$

Now, for reflection step,

$$x^c = \frac{1}{n} \begin{bmatrix} n-1 \\ n-2 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad x^n = x^c + (x^c - y^n)$$

$$= 2x^c - y^n$$

$$= \frac{2}{n} \begin{bmatrix} n-1 \\ n-2 \\ \vdots \\ n-(n-1) \\ n-n \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \frac{2}{n} \\ 2 - \frac{4}{n} \\ \vdots \\ 2 - \frac{2(n-1)}{n} \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{2}{n} \\ 1 - \frac{4}{n} \\ \vdots \\ 1 - \frac{2(n-1)}{n} \\ -1 \end{bmatrix}$$

$$L^+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 - \frac{2}{n} \\ 0 & 1 & 1 & \cdots & 1 - \frac{4}{n} \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 - \frac{2(n-1)}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & -\frac{1-2(n-1)}{n} \\ 0 & 1 & 0 & \cdots & -1 \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

Now, by  $C_{n-1} \rightarrow C_{n-1} - C_{n-2}, C_{n-2} \rightarrow C_{n-2} - C_{n-3}, \dots, C_2 \rightarrow C_2 - C_1$ , we get

$$\det(L^+) = \begin{vmatrix} 1 & 0 & 0 & \cdots & -\frac{1-2(n-1)}{n} \\ 0 & 1 & 0 & \cdots & -1 \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

Now, By  $C_n \rightarrow C_n - \sum_{i=1}^{n-1} C_i$ , we get

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & -2n \\ 0 & 1 & 0 & \cdots & -4n \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2\frac{(n-1)}{n} \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

$$= -\frac{2}{n} \begin{vmatrix} 1 & 0 & 0 & \cdots & \frac{4}{2} \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & n^2 \end{vmatrix}$$

Now, By  $C_n \rightarrow C_n - \sum i \cdot C_i$

$$= -\frac{2}{n} \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & n! \\ 0 & 0 & 0 & \dots & \dots \end{vmatrix}$$

$$= \left(-\frac{2}{n}\right) \left(\frac{n}{2}\right) |I|$$

$$= (-1) |I|$$

$$\det(I^+) = -1$$

$$\text{Vol.}(Y^{k+1}) = \frac{|-1|}{n!} = 1/n!$$

$$\text{diam}(Y^{k+1}) = \sqrt{1^2 + 1^2 + \dots + 1^2} = \sqrt{n-2}$$

$$\text{Vol}(Y^{k+1}) = \frac{\text{Vol}(Y^{k+1})}{(\text{diam}(Y^{k+1}))^n} = \frac{1}{n!} \times \frac{1}{(n-2)^{n!}}$$

Q5.10 → Consider the McKinnon Function

$$f(x_1, x_2) = \begin{cases} 360x_1^2 + x_2 + (x_2)^3 & \text{if } x_1 \leq 0 \\ 6x_1^2 + x_2 + (x_2)^3 & \text{if } x_1 > 0. \end{cases}$$

(a) Prove that  $f \in C^1$

Sol. When  $x_1 \leq 0$

when  $x_1 > 0$

$$\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix},$$

$$\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\left[ \begin{array}{c} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{array} \right] = \left[ \begin{array}{c} 72x_1 \\ 2x_2 + 1 \end{array} \right], \quad \left[ \begin{array}{c} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{array} \right] = \left[ \begin{array}{c} 12x_1 \\ 2x_2 + 1 \end{array} \right]$$

hence,  $\nabla f(x_1, x_2) = \lim_{x_1 \rightarrow 0^-} \nabla f(x_1, x_2) = \lim_{x_1 \rightarrow 0^+} \nabla f(x_1, x_2) = \left[ \begin{array}{c} 0 \\ 2x_2 + 1 \end{array} \right]$

So,  $f \in C^1$ .

(b) Prove that  $f$  is convex.

Sol. When  $x_1 \leq 0$  and when  $x_1 > 0$

$$\nabla^2 f(x_1, x_2) = \left[ \begin{array}{cc} 72 & 0 \\ 0 & 2 \end{array} \right], \quad \nabla^2 f(x_1, x_2) = \left[ \begin{array}{cc} 12 & 0 \\ 0 & 2 \end{array} \right]$$

Since,  $\nabla^2 f(x_1, x_2) \in S_2^+$

hence,  $f$  is convex.

(c) Find minimizer of  $f$ .

Sol. Since,  $f$  is convex, hence,

$\nabla f(x_1, x_2) = 0, \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right]$  will be the minimizer.

So, when  $x_1 \leq 0$  and when  $x_1 \geq 0$

$$\Rightarrow \nabla f(x_1, x_2) = \left[ \begin{array}{c} 72x_1 \\ 2x_2 + 1 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \quad \nabla f(x_1, x_2) = \left[ \begin{array}{c} 12x_1 \\ 2x_2 + 1 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right]$$

$$\Rightarrow \nabla f(x_0, x_1) - \begin{bmatrix} 1 \\ 2x_0 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \nabla^2 f(x_0, x_1) \begin{bmatrix} 1 & 2x_0 + 1 \\ 2x_0 + 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_0 = 0, x_1 = -1/2, \quad x_0 = 0, x_1 = -1/2$$

hence, minimizer of  $f$  would be  $\begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$ .