Assignment 11

Thursday, March 30, 2017 12:54 AM

Q9.2 →

Sol. $Y = \{y^0, y^1, \dots, y^n\}$. Given $\gamma > 0$,

$Y_\gamma = \{y^0_\gamma, y^1_\gamma, \dots, y^n_\gamma\}$ where $y^i_\gamma = y^0 + \gamma(y^i - y^0)$

and,

$\Delta = \overline{\text{diam}}(Y)$, $L = [\, y^1 - y^0 \quad y^2 - y^0 \; \dots \; y^n - y^0 \,]$

and, $\Delta_\gamma = \overline{\text{diam}}(Y_\gamma)$, $L_\gamma = [\, y^1_\gamma - y^0_\gamma \quad y^2_\gamma - y^0_\gamma \; \dots \; y^n_\gamma - y^0_\gamma \,]$

Now, $\Delta_\gamma = \overline{\text{diam}}(Y_\gamma) = \max\{\|\gamma(y^i - y^0)\|, \|\gamma(y^i - y^0)\|, \dots\}$

$= \|\gamma\| \, \overline{\text{diam}}(Y)$

$= \gamma \cdot \overline{\text{diam}}(Y)$ as $\gamma > 0$

$= \gamma \cdot \Delta$

and, $L_\gamma = [\, y^0 + \gamma(y^i - y^0) - y_0 \,]$ $\forall\, i \in I,\ 0 \le i \le n$,

$= [\gamma(y^i - y^0)]$

$= \gamma\, [\, y^i - y^0 \,]$

$= \gamma \cdot L$

So, $\dfrac{L_\gamma}{\Delta_\gamma} = \dfrac{\gamma L}{\gamma \Delta} = \dfrac{L}{\Delta}$. Hence Proved.

Now, since,

$$\frac{L_r}{\Delta_r} = \frac{L}{\Delta}$$

$$\Rightarrow \frac{(L_r)^{-T}}{\Delta_r} = \frac{(L)^{-T}}{\Delta}$$

$$\Rightarrow \left\| \frac{(L_r)^{-T}}{\Delta_r} \right\| = \left\| \frac{(L)^{-T}}{\Delta} \right\|$$

Hence, Proved.

Q4.7 → $f(x) = x^3$ at $\bar{x} = 0.5$

Sol. true gradient at $\bar{x} = 0.5$, $\frac{\partial f(\bar{x})}{\partial x} = 2\bar{x}^2 = 2 \times 0.5 \times 0.5$

$$= \frac{1}{2}.$$

(b) Since, for linear regression problem,

$$\begin{bmatrix} \alpha_0 \\ \alpha \end{bmatrix} = \left(\begin{bmatrix} 1^T \\ \gamma \end{bmatrix} [1 \ \ \gamma^T] \right)^{-1} \begin{bmatrix} 1^T \\ \gamma \end{bmatrix} f(\gamma)$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.125 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1.5 \\ 1.5 & 1.25 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.125 \\ \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1.5 \\ 1.5 & 1.25 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.125 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8333 & 0.333 & -0.1667 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.125 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.125 \\ 1 \end{bmatrix}$$

Hence, Approximate gradient at $\bar{x} = 1$.

(c) Similarly, for $X = \{0.25, 0.5, 0.75\}$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 1 & 0.25 \\ 1 & 0.5 \\ 1 & 0.75 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 0.0156 \\ 0.1250 \\ 0.4219 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1.5 \\ 1.5 & 0.8750 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 0.0156 \\ 0.1250 \\ 0.4219 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2187 \\ 0.8125 \end{bmatrix}$$

and so, Approximate gradient at $\bar{x} = 0.8125$

(d) The true gradient at $\bar{x}$ is $1/2$, so,

(d) The true gradient at $\bar{x}$ is $1/2$, so,

error, $|\nabla f(\bar{x}) - \nabla_{approx.(b)} f| = |1/2 - 1| = 1/2$

while, for part c,

error, $|\nabla f(\bar{x}) - \nabla_{approx\ in} f| = |-0.8125 + 0.5| = 0.3125$

we can see the points in part(c) are much closer to $\bar{x}$ as compared in part (b) so we have slightly better approximation for part (c).

Q 9.12 →

$$Y = \begin{bmatrix} 1 & 0 & -1 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1 & 0 & -1 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Sol. (a) Y is poised for quadratic interpolation if the system

$$\alpha_0 + \alpha^T y^i + \frac{1}{2}(y^i)^T H y^i = f(y^i) \quad , i = 0,1,2,\dots m$$

has unique solution for α_0, α and $H = H^T$.

let $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and $H = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. then,

$$\frac{1}{2}\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} ay_1 + by_2 & by_1 + dy_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2}(ay_1^2 + 2y_1y_2 b + dy_2^2)$$

So, the system can be written as,

$$
\begin{bmatrix}
1 & 1 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1/2 \\
1 & -1 & 0 & 1/2 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 1/2 \\
1 & 1/\sqrt{2} & 1/\sqrt{2} & 1/4 & 1/2 & 1/4 \\
1 & -1/\sqrt{2} & -1/\sqrt{2} & 1/4 & 1/2 & 1/4
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\ a_1 \\ \alpha_2 \\ a \\ b \\ d
\end{bmatrix}
= f(y^i)
$$

$\Rightarrow \quad Ax = b.$

Now, $\det(A) = \dfrac{1}{2^3}
\begin{vmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & -1 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 1 \\
1 & 1/\sqrt{2} & 1/\sqrt{2} & 1/2 & 1 & 1/2 \\
1 & -1/\sqrt{2} & -1/\sqrt{2} & 1/2 & 1 & 1/2
\end{vmatrix}$

$R_1 \longrightarrow R_1 - R_3, \quad R_2 \to R_2 - R_4$

$$
= \frac{1}{2^3}
\begin{vmatrix}
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 1 \\
1 & 1/\sqrt{2} & 1/\sqrt{2} & 1/2 & 1 & 1/2 \\
1 & -1/\sqrt{2} & -1/\sqrt{2} & 1/2 & 1 & 1/2
\end{vmatrix}
$$

$R_5 \to R_5 - R_6$

$$= \frac{1}{8} \begin{vmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 1 & -1/\sqrt{2} & -1/\sqrt{2} & 1/2 & 1 & 1/2 \end{vmatrix}$$

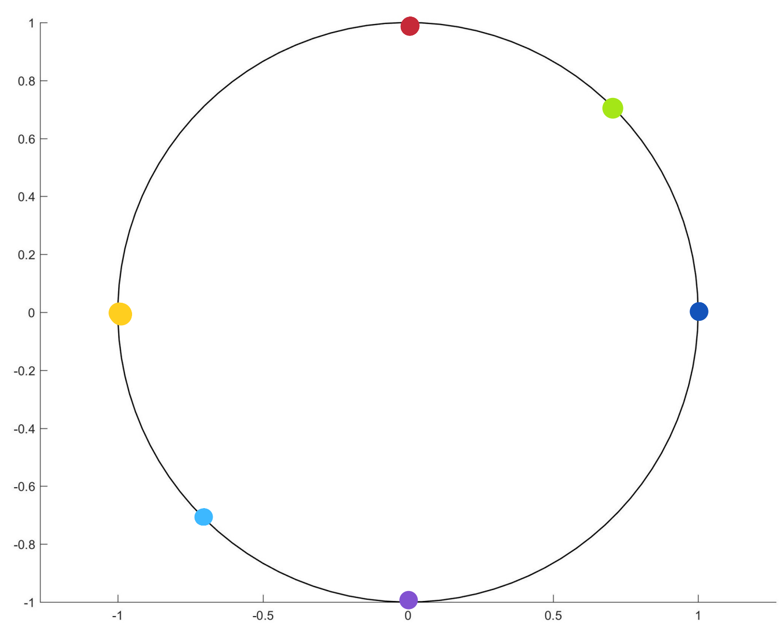
Now, By $R_5 \to \sqrt{2} R_5 - R_2 - R_1$, R_5 becomes a zero row.

So,

$\det(M) = 0$.

Hence, Y is not poised for quadratic Interpolation.

(b) .



Y is not poised for quadratic interpolation because all the points in Y lie on the unit circle and so we can find infinite number of conic sections [illegible] [illegible] this circle and so, we don't have a unique solution.

lie on the unit [illegible] a unique solution.

which contains this circle and so, we don't have