MATH604 - Assignment 5 - Deepak Kumar,73217151

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Q 5.8)

(a) $\gamma^K = \{ e_1, e_2, \ldots, e_n, 0 \}$

hence,
$$L = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Now to compute determinant of L,
we subtract C_n to each of the column to get,

$$\det(L) = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

Now by swapping the columns so as to make it identity matrix we get

$$= (-1)^{n-1} \times (-1) |I|$$

$$= (-1)^n$$

hence, $\det(L) = (-1)^n$

hence, $vol(\gamma^k) = \dfrac{|\det(L)|}{n!} = \dfrac{|(-1)^n|}{n!} = \dfrac{1}{n!}$

and, $diam(\gamma^k) = \sqrt{1^2 + 1^2} = \sqrt{2}$

and, so, $Von(\gamma^k) = \dfrac{vol(\gamma^k)}{(diam(\gamma))^n} = \dfrac{1}{n!(\sqrt{2})^n}$

Now, for reflection step,

$$x^c = \frac{1}{n}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and so,

$$x^r = x^c + (x^c - y^n)$$

$$= \frac{2}{n}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$, \quad x^r - y^0 = \begin{bmatrix} \frac{2}{n} - 1 \\ \frac{2}{n} \\ 2/n \\ \vdots \\ 2/n \end{bmatrix} = \frac{2}{n}\begin{bmatrix} \frac{(2-n)}{2} \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Now,

$$L^+ = \frac{2}{n}\begin{bmatrix} -1 & -1 & -1 & \cdots & -1 & \left(\frac{2-n}{2}\right) \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

$$C_n \longrightarrow C_n - \sum_{i=1}^{n-1} C_i$$

$$\det(L^{+}) = \frac{2}{n}\begin{vmatrix} -1 & -1 & -1 & \cdots & -1 & \frac{n}{2} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 & \cdots & -1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

and, so,

$$\det(L^{+}) = (-1)^{n-1}$$

So, $\text{Volume}(\gamma^{k+1}) = \dfrac{|(-1)^{n-1}|}{n!} = \dfrac{1}{n!}$

$$\text{diam}(\gamma^{k+1}) = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\text{Von}(\gamma^{k+1}) = \frac{\text{Vol}(\gamma^{k+1})}{(\text{diam}(\gamma^{k+1}))^n} = \frac{1}{n!}\cdot\frac{1}{(\sqrt{2})^n} = \frac{1}{n!\,(\sqrt{2})^n}$$

(b) $y^0 = 0$ and for every i from 1 to n, set $y^i = y^{i-1} + e_i$, where e_i is the ith co-ordinate vector.

(c) $\gamma^k = \{0,\ e_1,\ e_1+e_2,\ \ldots,\ \sum^{n} e_i\}$

Sol. $\gamma^k = \{0, e_1, e_1+e_2, \ldots, \sum_{i=1}^{n} e_i\}$

hence,

$$L = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Now, by $C_n \to C_n - C_{n-1}$, $C_{n-1} \to C_{n-1} - C_{n-2} \ldots$, $C_2 \to C_2 - C_1$, we get

$$\det(L) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 \cdots & 1 \end{vmatrix}$$

$$= |I|$$

hence, $\det(L) = 1$.

So, $\text{vol.}(\gamma^k) = \frac{1}{n!}$

and, $\text{diam}(\gamma^k) = \sqrt{1^2 + 1^2 \ldots 1^2}$

$$= \sqrt{n}$$

hence, $\text{Von}(\gamma^k) = \frac{\text{Vol.}(\gamma^k)}{(\text{diam}(\gamma^k))^n} = \frac{1}{n!} \times \frac{1}{(\sqrt{n})^n} = \frac{1}{n!} \times \frac{1}{n^{n/2}}$

hence, vol(T) = $\dfrac{\text{vol}(\cdot)}{(\text{diam}(\mathcal{T}^{k}))^{n}}$ = $n!\,(\sqrt{n})^{n}$ $n!$ $n^{n/2}$

Now, for reflection step,

$$x^{c} = \frac{1}{n}\begin{bmatrix} n-1 \\ n-2 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad x^{r} = x^{c} + (x^{c} - y^{n})$$

$$= 2x^{c} - y^{n}$$

$$= \frac{2}{n}\begin{bmatrix} n-1 \\ n-2 \\ \vdots \\ n-(n-1) \\ n-n \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\frac{2}{n} \\ 2-\frac{4}{n} \\ \vdots \\ 2-\frac{2(n-1)}{n} \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{2}{n} \\ 1-\frac{4}{n} \\ \vdots \\ 1-\frac{2(n-1)}{n} \\ -1 \end{bmatrix}$$

$$L^{+} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1-2/n \\ 0 & 1 & 1 & \cdots & 1-4/n \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & 1-\frac{2(n-1)}{n} \end{bmatrix}$$

$$\begin{bmatrix} \vdots & \vdots & \vdots & & \vdots & 1-\frac{2(n-1)}{n} \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

Now, by $C_{n-1} \to C_{n-1} - C_{n-2},\ C_{n-2} \to C_{n-2} - C_{n-3}, \dots\dots, C_2 \to C_2 - C_1$, we get

$$\det(L^+) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 1-2/n \\ 0 & 1 & 1 & \cdots & 1-4/n \\ 0 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & 1-\frac{2(n-1)}{n} \\ 0 & 0 & \vdots & \cdots & -1 \end{vmatrix}$$

Now, By $C_n \to C_n - \sum_{i=1}^{n-1} C_i$, we get

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & -2/n \\ 0 & 1 & 0 & \cdots & -4/n \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & -\frac{2(n-1)}{n} \\ 0 & 0 & \cdots & & -1 \end{vmatrix}$$

$$= -\frac{2}{n} \begin{vmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & n-1 \\ 0 & 0 & 0 & \cdots & n/2 \end{vmatrix}$$

Now, By $C_n \to C_n - \sum i \cdot C_i$

$$= -\frac{2}{n}\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & n/2 \\ 0 & 0 & 0 & \cdots & \end{vmatrix}$$

$$= \left(-\frac{2}{n}\right)\left(\frac{n}{2}\right)|I|$$

$$= (-1)|I|$$

$$\det(C^{+}) = -1$$

$$\text{vol.}(\gamma^{k+1}) = \frac{|-1|}{n!} = 1/n!$$

$$\text{diam}(\gamma^{k+1}) = \sqrt{1^2+1^2+\ldots 1^2} = \sqrt{n-1}$$

$$\text{Von}(\gamma^{k+1}) = \frac{\text{Vol}(\gamma^{k+1})}{(\text{diam}(\gamma^{k+1}))^n} = \frac{1}{n!} \times \frac{1}{(n-1)^{n/2}}$$

Q 5.10 → Consider the McKinnon Function

$$f(x_1, x_2) = \begin{cases} 360x_1^2 + x_2 + (x_2)^2 & \text{if } x_1 \le 0 \\ 6x_1^2 + x_2 + (x_2)^2 & \text{if } x_1 > 0. \end{cases}$$

(a) Prove that $f \in C^1$

Sol. When $x_1 \le 0$, when $x_1 > 0$

$$\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} , \quad \nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x_2} \end{bmatrix} \qquad\qquad \begin{bmatrix} \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 720x_1 \\ 2x_2+1 \end{bmatrix} , \qquad = \begin{bmatrix} 12x_1 \\ 2x_2+1 \end{bmatrix}$$

hence, $\nabla f(x_1, x_2)_{\lim x_1 \to 0^-} = \nabla f(x_1, x_2)_{\lim x_1 \to 0^+} = \begin{bmatrix} 0 \\ 2x_2+1 \end{bmatrix}$

So, $f \in C^2$.

(b) Prove that f is convex.

Sol. when $x_1 \leq 0$ and when $x_1 > 0$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 720 & 0 \\ 0 & 2 \end{bmatrix} , \qquad \nabla^2 f(x_1, x_2) = \begin{bmatrix} 12 & 0 \\ 0 & 2 \end{bmatrix}$$

Since, $\nabla^2 f(x_1, x_2) \in S_2^+$

hence, f is convex.

(c) Find minimizer of f.

Sol. Since, f is convex, hence,

$\nabla f(x_1, x_2) = 0$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ will be the minimizer.

So, when $x_1 \leq 0$ and when $x_1 \geq 0$

$$\Rightarrow \nabla f(x_1, x_2) = \begin{bmatrix} 720x_1 \\ 2x_2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \qquad \nabla f(x_1, x_2) = \begin{bmatrix} 12x_1 \\ 2x_2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \nabla f(x_1, x_2) = \begin{bmatrix} \cdots \\ 2x_2+1 \end{bmatrix} = \begin{bmatrix} \cdots \\ 0 \end{bmatrix}, \quad \nabla f(x_1, x_2) = \begin{bmatrix} \cdots \\ 2x_2+1 \end{bmatrix} = \begin{bmatrix} \cdots \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 0,\ x_2 = -1/2, \quad x_1 = 0,\ x_2 = -1/2$$

hence, minimizer of f would be $\begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$.