Lectures 2-5
Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

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Intended Learning Outcomes

At the end of the lectures, the students will be able to

- Describe basic discrete mathematical structures such as sets, functions, sequences and matrices
- Explain the mathematical operations on basic discrete mathematical structures
- Apply basic discrete mathematical structures to model simple computing applications
- Employ discrete mathematical techniques to solve simple problems

Topics

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Matrices
  - Matrix Arithmetic

Section 1:
Sets

Section Outline

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

  $$S = \{a,b,c,d\} = \{b,c,a,d\}$$
- Each distinct object is either a member or not; listing more than once does not change the set.

  $$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $$S = \{a,b,c,d,\ \ldots\ldots,z\}$$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots\ldots\ldots,99\}$

- Set of all integers less than 0:

  $S = \{\ldots., -3,-2,-1\}$

Some Important Sets

$\mathbf{N}$ = *natural numbers* = $\{0,1,2,3\ldots.\}$

$\mathbf{Z}$ = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z}^{+}$ = *positive integers* = $\{1,2,3,\ldots..\}$

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R}^{+}$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers*.

$\mathbf{Q}$ = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid \mathrm{P}(x)\}$$

- Example: $S = \{x \mid \mathrm{Prime}(x)\}$
- Positive rational numbers:

  $\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

- Intervals

$$[a,b] = \{x \mid a \le x \le b\}$$

$$[a,b) = \{x \mid a \le x < b\}$$

$$(a,b] = \{x \mid a < x \le b\}$$

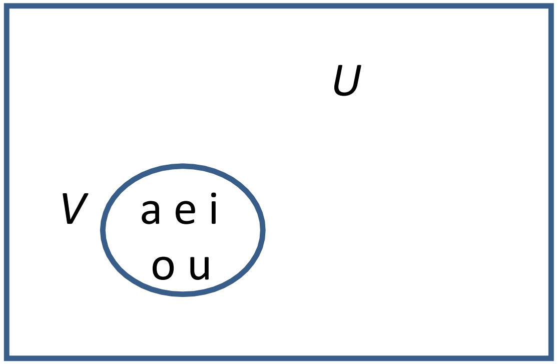
$$(a,b) = \{x \mid a < x < b\}$$

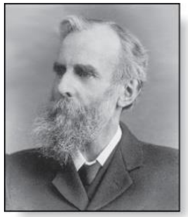
- *closed interval* [a,b]
- *open interval* (a,b)

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - – Sometimes implicit
  - – Sometimes explicitly stated.
  - – Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram





John Venn
(1834-1923)

Russell's Paradox

- Let *S* be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is *S* a member of itself?"
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Bertrand Russell (1872-1970)

Some things to remember

- Sets can be elements of sets.

  $$\{\{1,2,3\},a,\{b,c\}\}$$

  $$\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$$

- The empty set is different from a set containing the empty set.

  $$\emptyset \neq \{\ \emptyset\ \}$$

Set Equality

- **Definition**: Two sets are *equal* if and only if they have the same elements.
  - Therefore if A and B are sets, then A and B are equal if and only if .

$$\forall x(x \in A \leftrightarrow x \in B)$$

  - We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

- **Definition**: The set *A* is a *subset* of *B*, if and only if every element of *A* is also an element of *B*.
  - The notation $A \subseteq B$ is used to indicate that A is a subset of the set B. $\forall x(x \in A \rightarrow x \in B)$
  - $A \subseteq B$ holds if and only if is true.

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set *S*.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set *S*.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets *A* and *B* are *equal*, denoted by *A* = *B*, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that *A* = *B* iff

$$\forall x[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \qquad \text{and} \qquad B \subseteq A$$

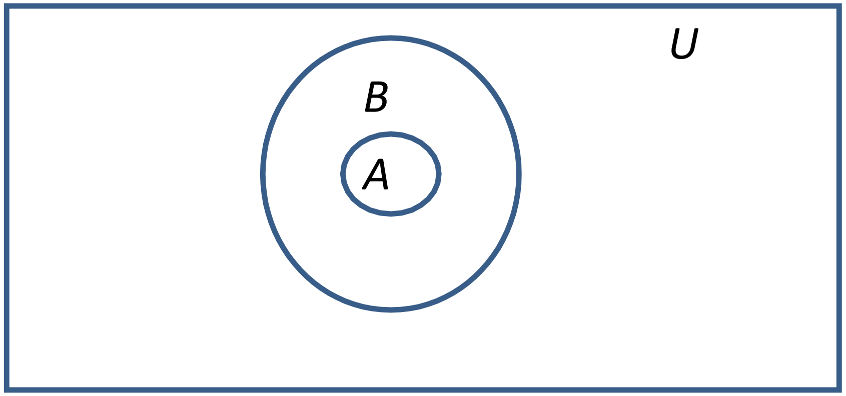
Proper Subsets

- **Definition**: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

- **Definition**: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.
- **Definition**: The *cardinality* of a finite set *A*, denoted by $|A|$, is the number of (distinct) elements of *A*.
- **Examples**:
  1. $|\emptyset| = 0$
  2. Let S be the letters of the English alphabet. Then $|S| = 26$
  3. $|\{1,2,3\}| = 3$
  4. $|\{\emptyset\}| = 1$
  5. The set of integers is infinite.

Power Sets

- **Definition**: The set of all subsets of a set *A*, denoted P**(***A***)**, is called the *power set* of *A*.
- **Example**: If *A* = {a,b} then

  $\mathcal{P}$(A) = {ø, {a},{b},{a,b}}

- If a set has *n* elements, then the cardinality of the power set is 2^n. (Later on, we will discuss different ways to show this.)

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

- **Definition**: The *Cartesian Product* of two sets *A* and *B*, denoted by *A* × *B* is the set of ordered pairs (a,b) where *a* ∈ *A* and *b* ∈ *B* .

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

- **Example**:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **Definition**: A subset *R* of the Cartesian product *A* × *B* is called a *relation* from the set A to the set B. (Relations will be covered in depth later on.)

René Descartes
(1596-1650)

Cartesian Product

- **Definition**: The cartesian products of the sets A_1,A_2, ..., A_n, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered *n*-tuples $(a_1,a_2,\ldots,a_n)$ where a_i belongs to A_i for *i* = 1, ..., n.

$$A_1 \times A_2 \times \cdots \times A_n = \\ \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

- **Example**: What is *A* × *B* × C where *A* = {0,1}, *B* = {1,2} and *C* = {0,1,2}
- **Solution:** *A* × *B* × C = {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)}

Truth Sets of Quantifiers

- Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

- **Example**: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "|*x*| = 1" is the set {-1,1}

Section 2:
Set Operations

Section Ouline

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Boolean Algebra

- Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra* (Covered in depth later).
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set *U*. All sets are assumed to be subsets of *U*.

Union

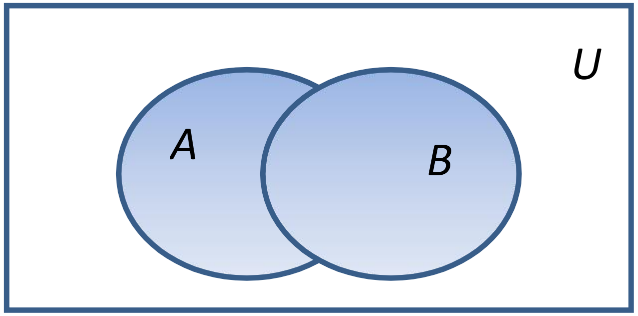
- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A* ∪ *B*, is the set:

$$\{x|x \in A \lor x \in B\}$$

- **Example**: What is $\{1,2,3\}$ ∪ $\{3, 4, 5\}$?

- **Solution**: $\{1,2,3,4,5\}$

Venn Diagram for *A* ∪ B



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Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by *A* ∩ *B*, is

$$\{x | x \in A \wedge x \in B\}$$

- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

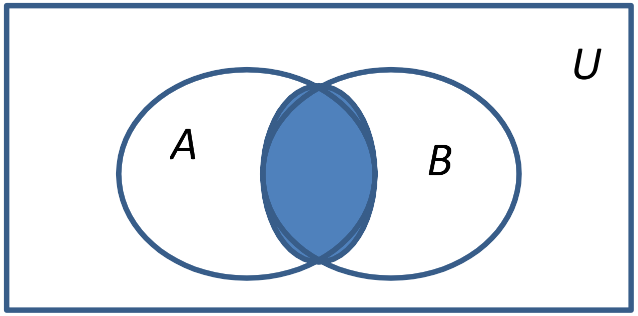
  Solution: $\{3\}$

- **Example:**What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: ∅

Venn Diagram for *A* ∩*B*



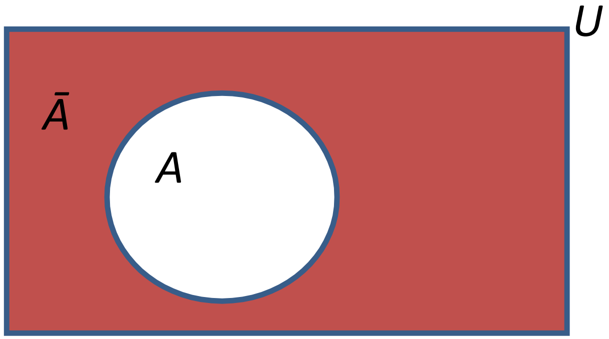
Complement

- **Definition**: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

$$\bar{A} = \{x \in U \mid x \notin A\}$$

- The complement of A is sometimes denoted by A^c.
- **Example**: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

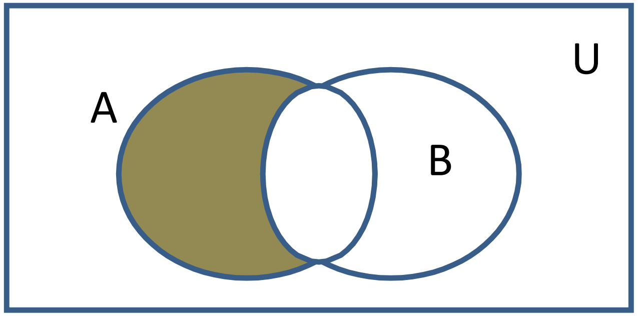


Venn Diagram for Complement

Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in \mathrm{A} \wedge x \notin B\} = A \cap \overline{B}$$

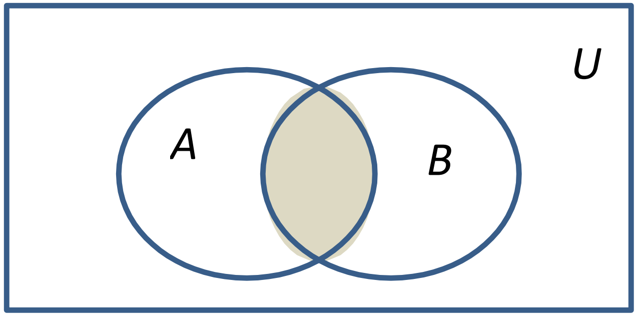


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- **Inclusion-Exclusion Formula:**

  $|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

- **Example**: Let *A* be the math majors in your class and *B* be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

- We will return to this principle later where we will derive a formula for the cardinality of the union of *n* sets, where *n* is a positive integer.

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

- **Example**: Prove that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- **Solution**: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and
2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

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Proof of Second De Morgan Law

1) These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

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Proof of Second De Morgan Law

2) These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Second De Morgan Law: Set-Builder Notation

$$
\begin{array}{lcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
 & = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
 & = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
 & = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
 & & & \text{for Prop Logic} \\
 & = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
 & = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
 & = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
 & = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

- **Example**: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- **Solution**:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- Let A_1, A_2 ,…, A_n be an indexed collection of sets. We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

- For i = 1,2,…, let A_i = {i, i + 1, i + 2, ….}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots..\} = A_n$$

Computer Representation of Sets

- Assume the universal set *U* is finite (not larger than the memory size of the computer being used)
- First specify an arbitrary ordering of the elements in *U*, for instance $a_1, a_2, \ldots, a_n$
- Represent a subset *A* of *U* with the bit string of length *n*, where the *i*th bit is *1* if a_i belongs to *A* and is *0* if a_i does not belong to *A*
- ***Example***: Let *U* ={1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of *U* has the elements in the increasing order.

  What bit strings represent the subset of all odd integers?

  What bit string represents the set of all even integers?

  What bit string represents the subset of integers not exceeding *5* in *U*?

  What is the bit string that represents the complement of a subset?

  What is the bit string that represents the union of two subsets?

  What is the bit string that represents the intersection of two subsets?

Section 3: Functions

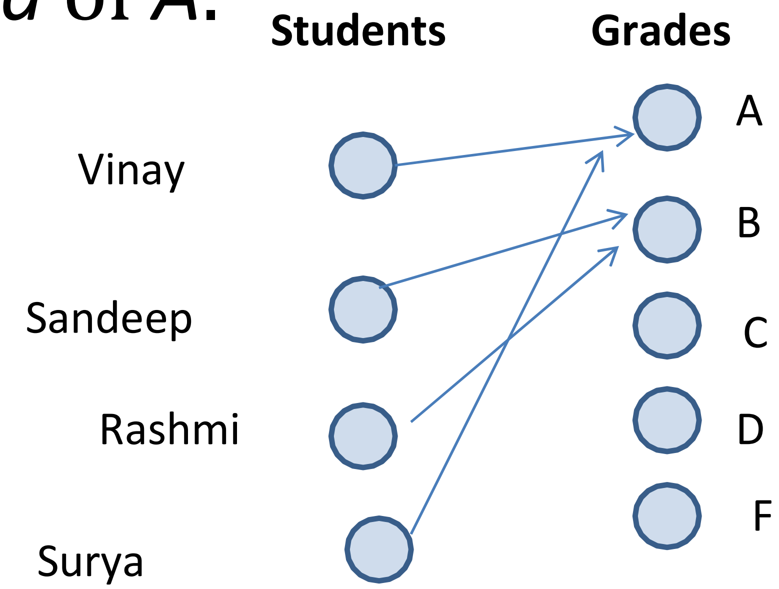
Section Outline

- Definition of a Function
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function *f*: *A* → *B* can also be defined as a subset of *A*×*B* (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element *a*∈ *A*.

$$\forall x[x \in A \to \exists y[y \in B \wedge (x, y) \in f]]$$

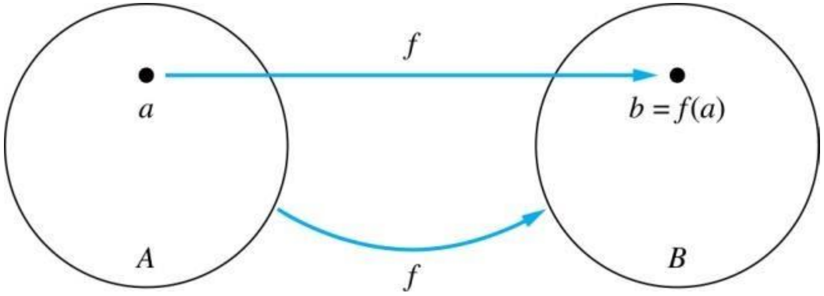
and $\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \to y_1 = y_2]$

Functions

Given a function f: $A \rightarrow B$:

- We say f *maps* A to B *or*

  f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Representing Functions

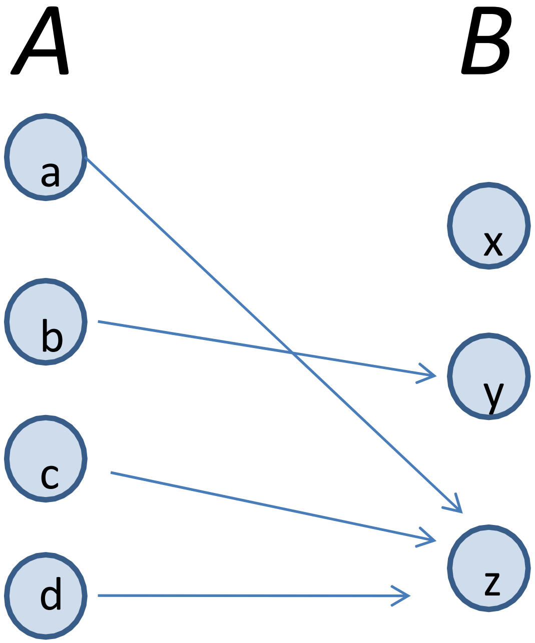
- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section).

Example

1. f(a) = z
2. The image of d is z
3. The domain of f is A
4. The codomain of f is B
5. The preimage of y is b
6. $f(A)$ = {z, y}
7. The preimage(s) of z is (are) {a, d, c}



Question on Functions and Sets

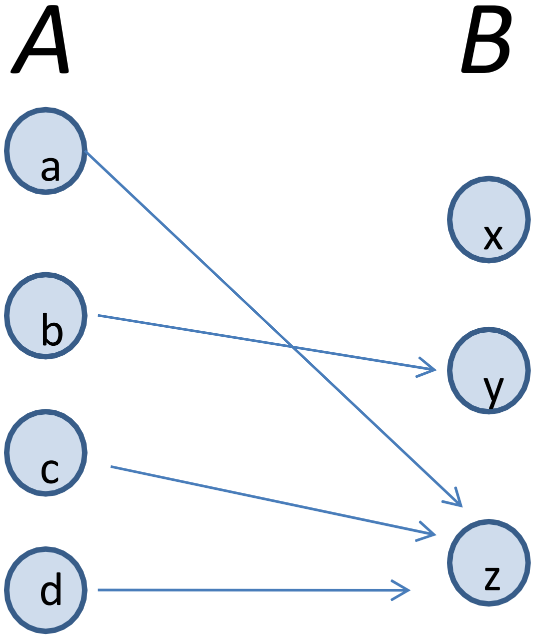
- If and S is a subset of A, then

$$f : A \to B$$

$$f(S) = \{f(s) | s \in S\}$$

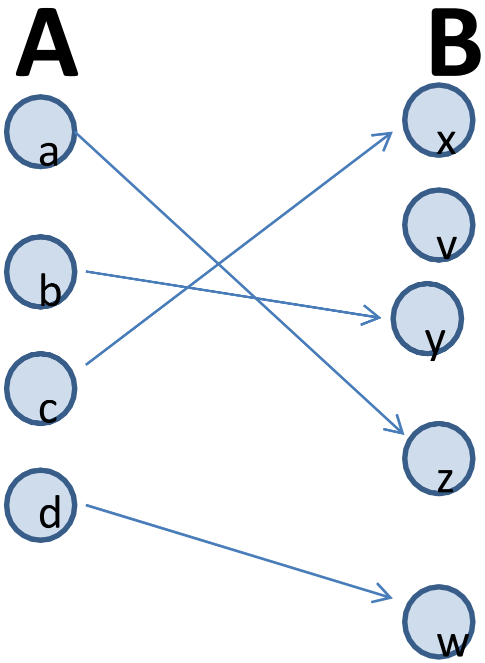
f ({a,b,c}) is {y, z}

f ({c,d}) is {z}



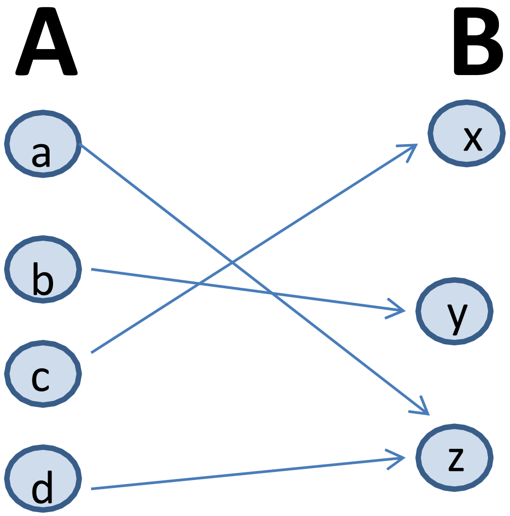
Injections

- **Definition**: A function f is said to be ***one-to-one***, or **injective**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an **injection** if it is one-to-one.



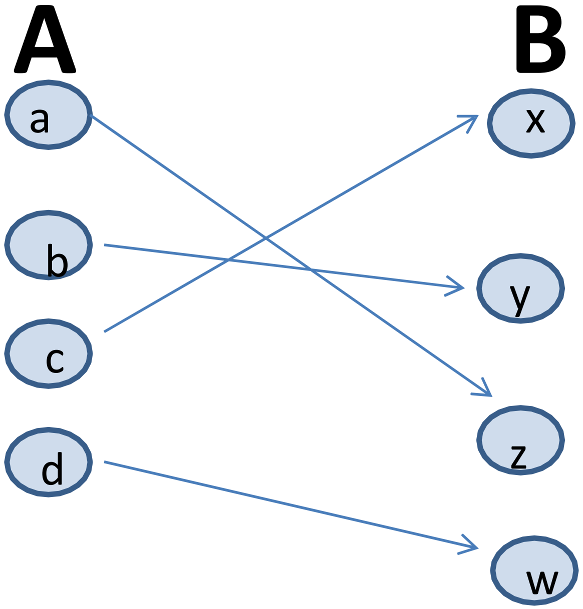
Surjections

- **Definition**: A function *f* from *A* to *B* is called **onto** or **surjective**, if and only if for every element $a \in A$there is an element $b \in B$ with $f(a) = b$
- A function *f* is called a **surjection** if it is onto.



Bijections

- **Definition**: A function f is a **one-to-one correspondence**, or a **bijection**, if it is **both** one-to-one and onto (surjective and injective).



Showing that *f* is one-to-one or onto

- Suppose that f : A → B.
- To show that f is injective: Show that if f (x) = f (y) for arbitrary x, y ∈ A with x != y, then x = y.
- To show that f is not injective: Find particular elements x, y ∈ A such that x != y and f (x) = f (y).
- To show that f is surjective: Consider an arbitrary element y ∈ B and find an element x ∈ A such that f (x) = y.
- To show that f is not surjective: Find a particular y ∈ B such that f (x) != y for all x ∈ A.

Showing that *f* is one-to-one or onto

- **Example 1**: Let *f* be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is *f* an onto function?

- **Example 2**: Is the function $f(x) = x^2$ from the set of integers onto?

Showing that *f* is one-to-one or onto

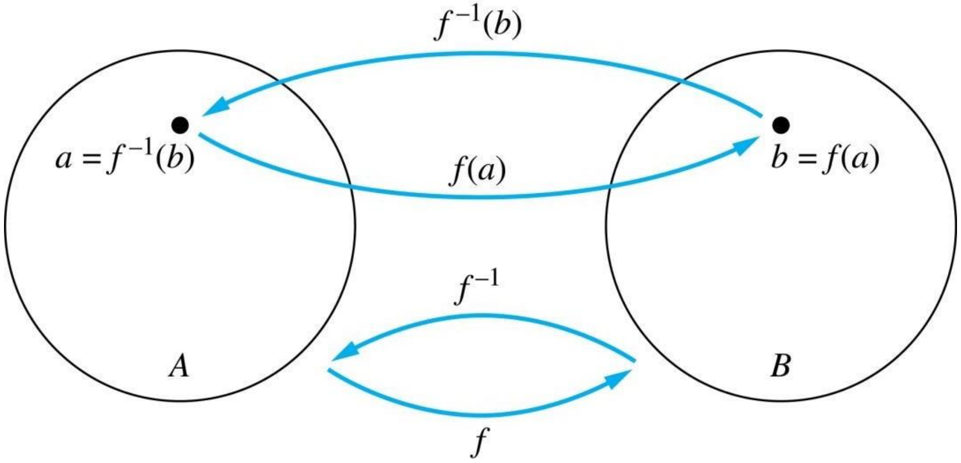
- **Example 1**: Let *f* be the function from {*a*,*b*,*c*,*d*} to $\{1,2,3\}$ defined by *f*(*a*) = 3, *f*(*b*) = 2, *f*(*c*) = 1, and *f*(*d*) = 3. Is *f* an onto function?
- **Solution**: Yes, *f* is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, *f* would not be onto.
- **Example 2**: Is the function *f*(*x*) = x^2 from the set of integers onto?
- **Solution**: No, *f* is not onto because there is no integer *x* with $x^2 = -1$, for example.

Inverse Functions

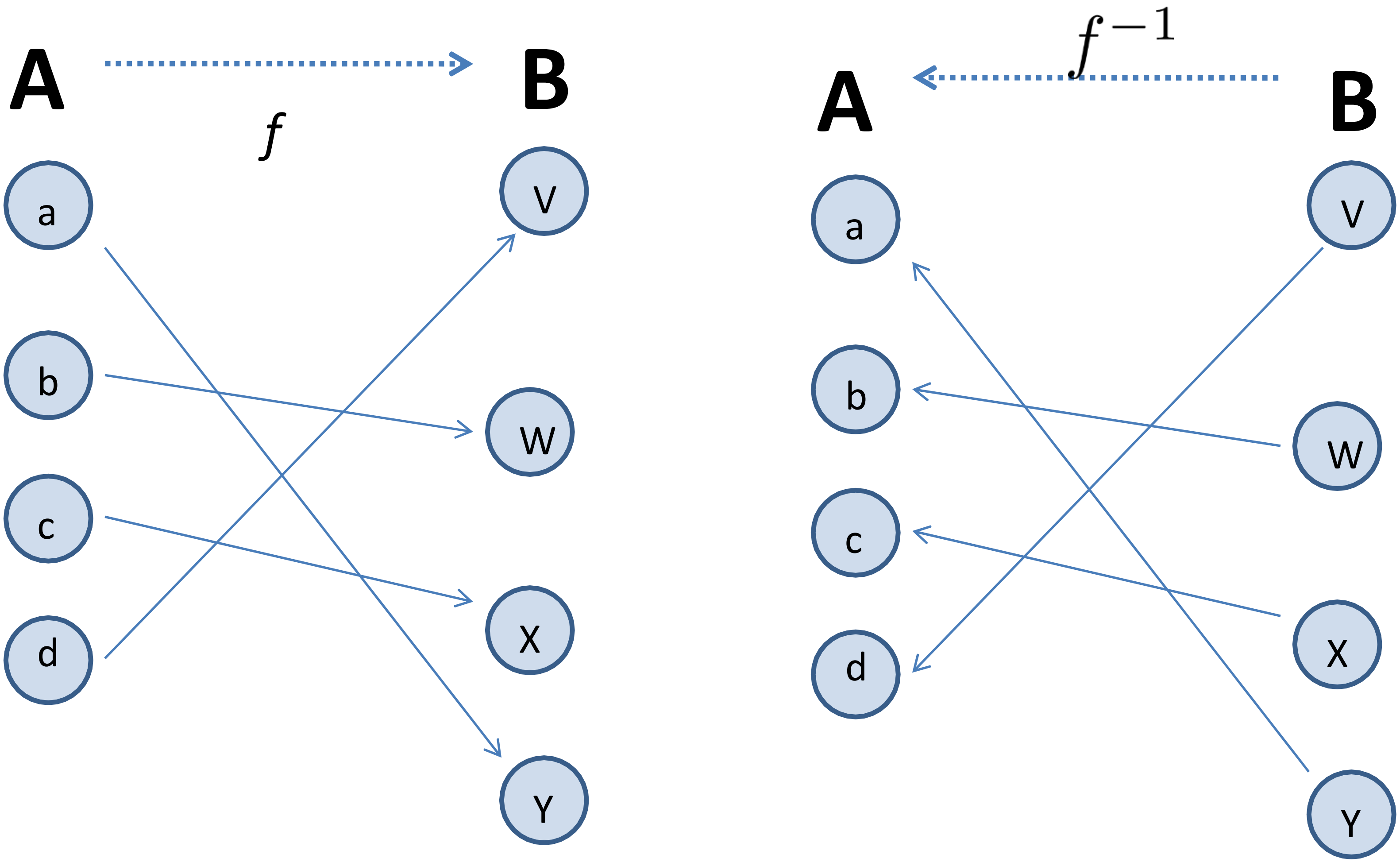
- **Definition**: Let *f* be a bijection from *A* to *B*. Then the **inverse** of *f*, denoted f^{-1} , is the function from *B* to *A* defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

- No inverse exists unless *f* is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Questions

- **Example 1**: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?
- **Solution**: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

- **Example 2**: Let f: **Z** → **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Questions

- **Example 2**: Let *f*: **Z** → **Z** be such that $f(x) = x + 1$. Is *f* invertible, and if so, what is its inverse?
- **Solution**: The function *f* is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{--1}(y) = y - 1$.

Questions

- **Example 3**: Let *f*: **R** → **R** be such that

$$f(x) = x^2$$

- Is *f* invertible, and if so, what is its inverse?

Questions

- **Example 3**: Let *f*: **R** → **R** be such that

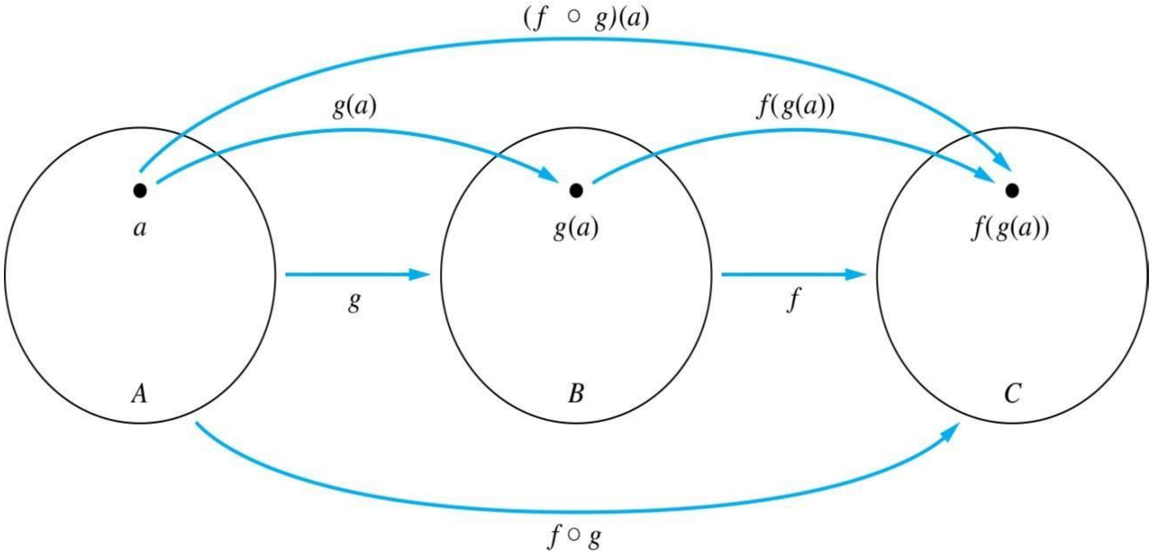
$$f(x) = x^2$$

- Is *f* invertible, and if so, what is its inverse?
- **Solution**: The function *f* is not invertible because it is not one-to-one.

Composition

- **Definition**: Let *f*: *B* → *C*, *g*: *A* → *B*. The **composition of f with g**, denoted $f \circ g$ is the function from *A* to *C* defined by

$$f \circ g(x) = f(g(x))$$



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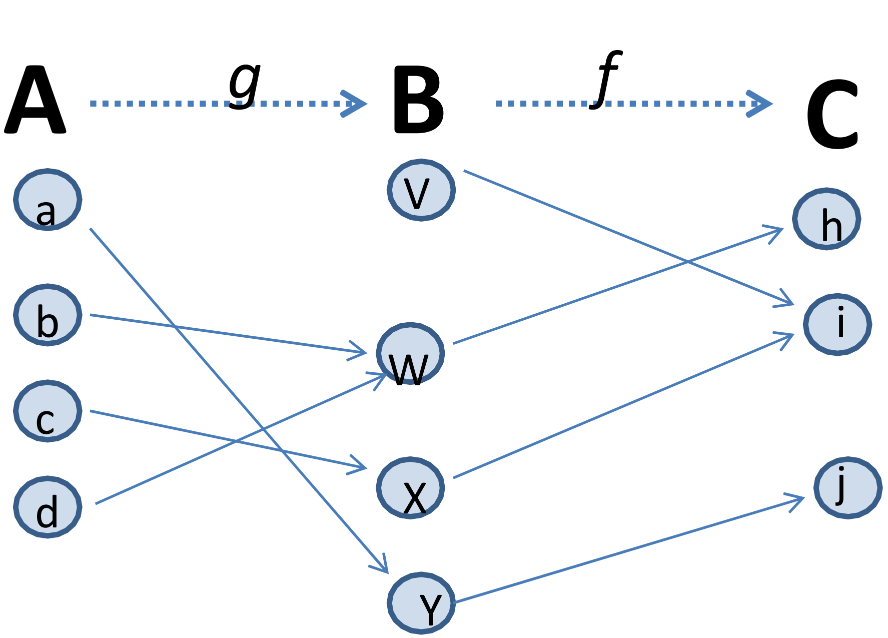
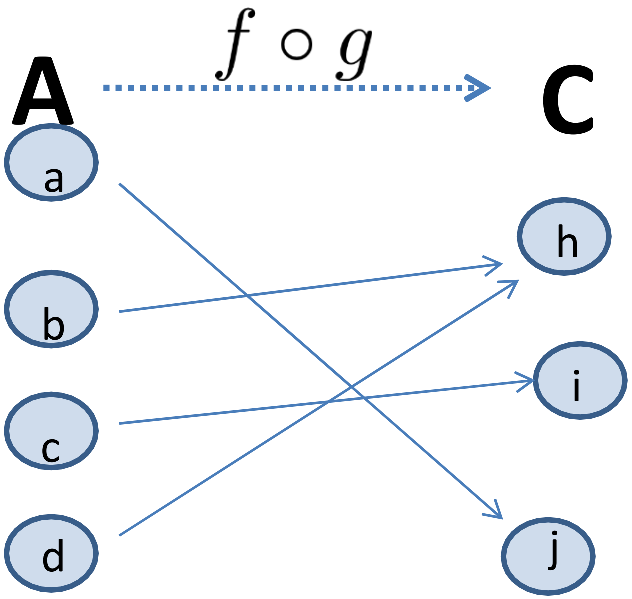
Composition





Composition

- **Example 1**: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

- **Example 2**: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.
- What is the composition of f and g, and what is the composition of g and f?

Composition Questions

- **Example 2**: Let *g* be the function from the set {*a,b,c*} to itself such that *g*(*a*) = *b*, *g*(*b*) = *c*, and *g*(*c*) = *a*. Let *f* be the function from the set {*a,b,c*} to the set {1,2,3} such that *f*(*a*) = 3, *f*(*b*) = 2, and *f*(*c*) = 1.

  What is the composition of *f* and *g*, and what is the composition of *g* and *f*.

- **Solution:** The composition $f \circ g$ is defined by

  $f \circ g\ (a) = f(g(a)) = f(b) = 2.$

  $f \circ g\ (b) = f(g(b)) = f(c) = 1.$

  $f \circ g\ (c) = f(g(c)) = f(a) = 3.$

  Note that $g \circ f$ is not defined, because the range of *f* is not a subset of the domain of *g*.

Composition Questions

- **Example 2**: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.
- What is the composition of f and g, and also the composition of g and f ?

Composition Questions

- **Example 2**: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

  What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

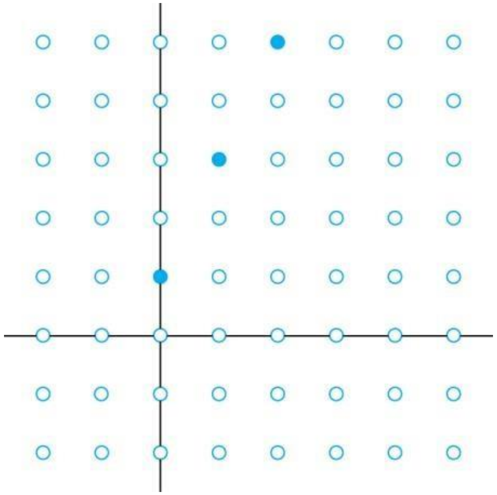
- **Solution:**

  $$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

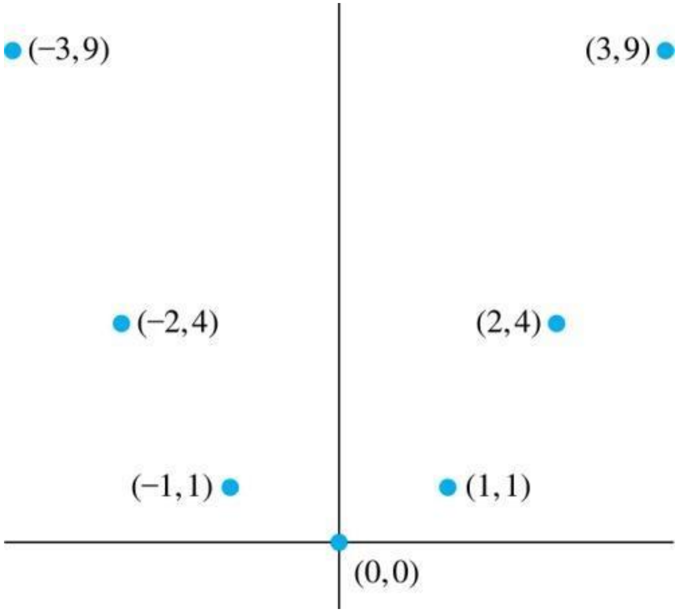
  $$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Graphs of Functions

- Let *f* be a function from the set *A* to the set *B*. The *graph* of the function *f* is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of *f*(*n*) = $2n + 1$ from Z to Z



Graph of *f*(*x*) = x^2 from Z to Z

Some Important Functions

- The **floor** function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

- The **ceiling** function, denoted

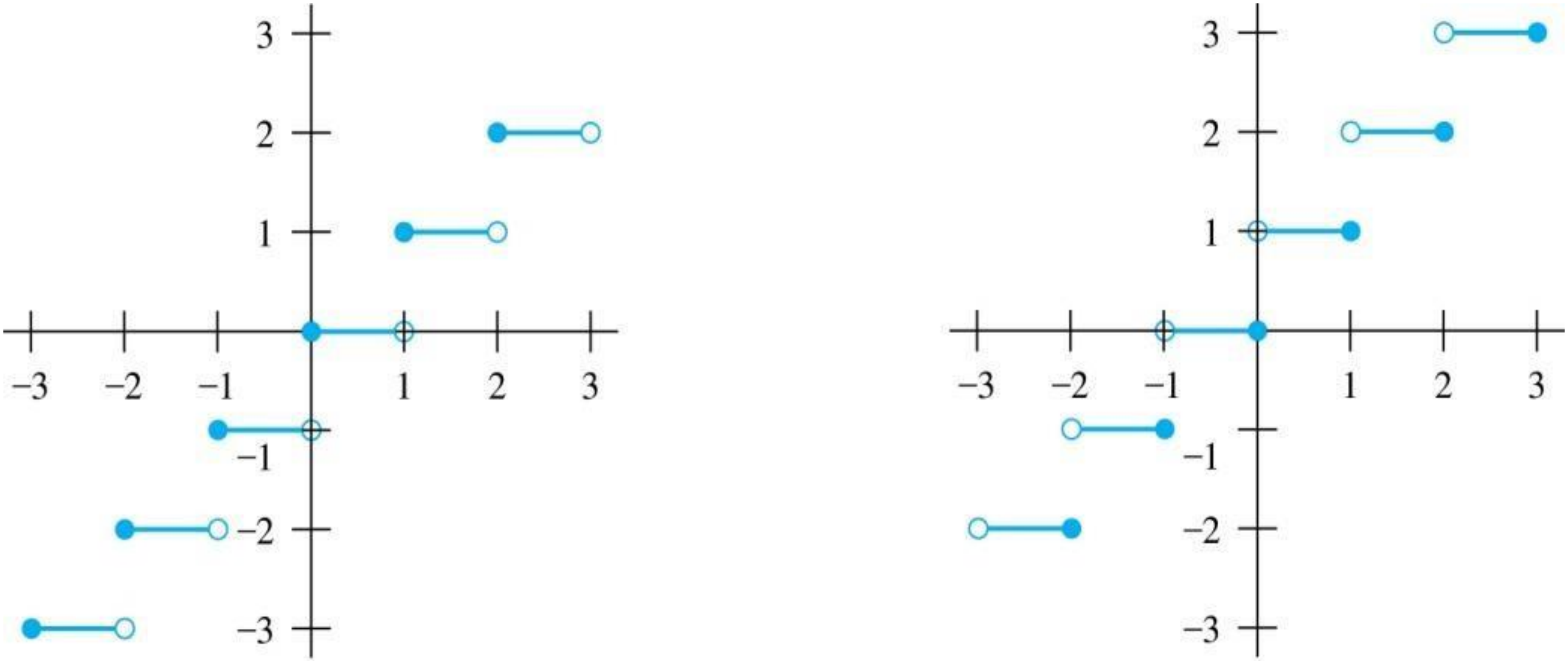
$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to *x*

Examples: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



(a) $y = [x]$

(b) $y = [x]$

Graph of (a) Floor and (b) Ceiling Functions

$$\lceil -1.5 \rceil = -1 \quad \lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions

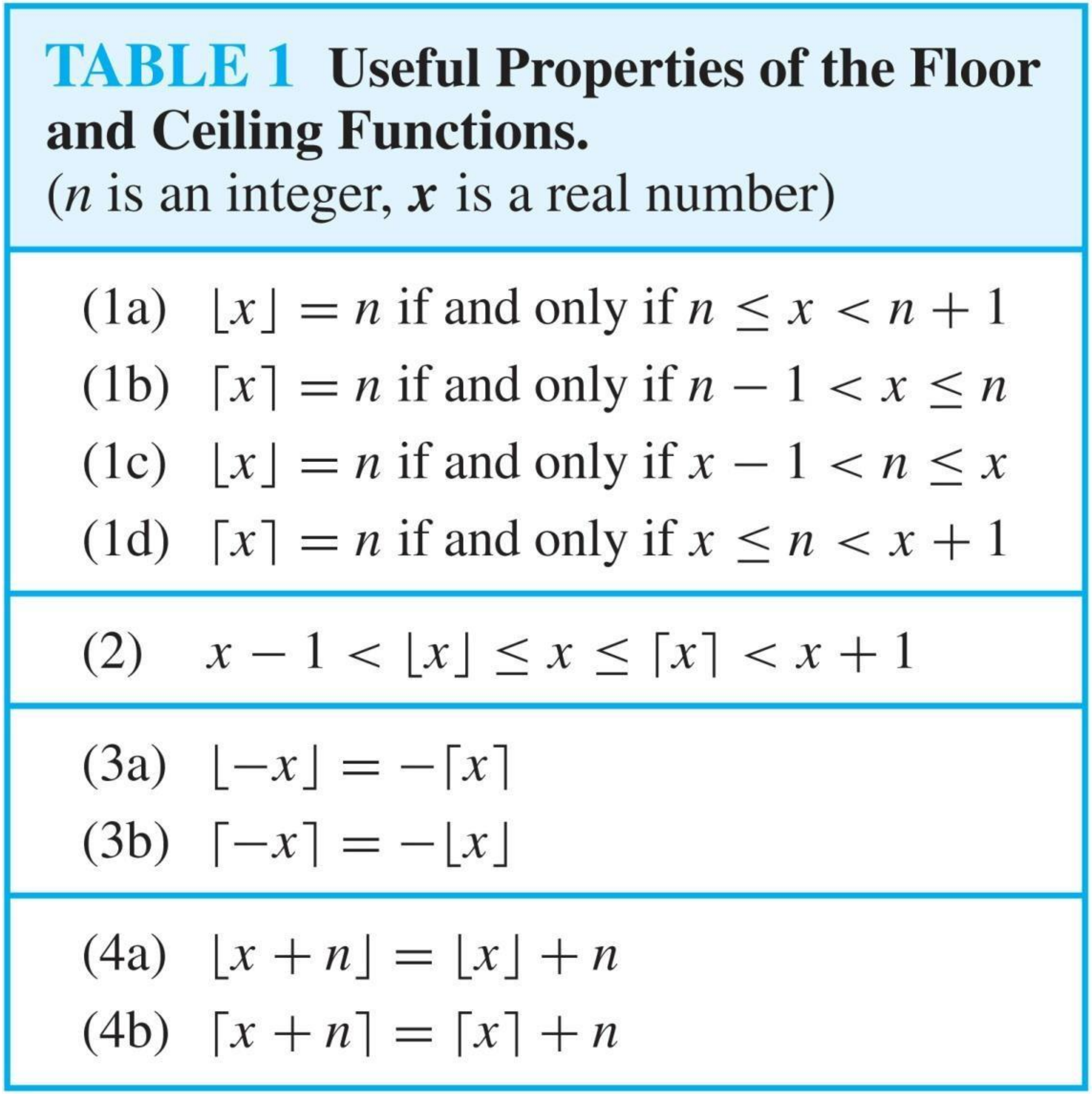


TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Computer Applications of Floor and Ceiling Functions

- Example: Data stored on a computer or transmitted over a network are represented as a string of *bytes*. Each *byte* consists of 8 *bits*. How many bytes are required to encode 100 bits of data?
- Example: In asynchronous transfer mode (ATM) (a communication protocol used in backbone networks), data are organized into *cells* of 53 bytes. How many *ATM cells* can be transmitted in 1 minute over a connection that transmits data at a rate of 500 kilobits per second?

Proving Properties of Functions

- **Example**: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + ½ \rfloor$$

Proving Properties of Functions

- **Example**: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

- **Solution**: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Factorial Function

- **Definition:** f: **N** → **Z**$^+$, denoted by *f*(*n*) = *n*! is the product of the first *n* positive integers when *n* is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

- **Examples:**

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$

Stirling’s Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Section 4:
Sequences and Summations

Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations

Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

- **Definition**: A *sequence* is a function from a subset of the integers (usually either the set $\{0, 1, 2, 3, 4, ...\}$ or $\{1, 2, 3, 4, ...\}$) to a set *S*.
- The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of *f(n)* where *f* is a function from $\{0,1,2,.....\}$ to *S*. We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

- **Definition**: A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

- **Examples**:

1. Let *a* = 1 and *r* = −1. Then:

$$\{b_n\} \;=\; \{b_0, b_1, b_2, b_3, b_4, \ldots\} \;=\; \{1, -1, 1, -1, 1, \ldots\}$$

2. Let *a* = 2 and *r* = 5. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let *a* = 6 and *r* = 1/3. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

- **Definition**: A *arithmetic progression* is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

- **Examples**:

1. Let *a* = −1 and *d* = 4:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let *a* = 7 and *d* = −3:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let *a* = 1 and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Sequences Used in Computer Science: Strings

- **Definition**: A *string* is a finite sequence of characters from a finite set (an alphabet).
- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by *λ*.
- The string *abcde* has *length* 5.
- The (bit) string 0011 1100 has length 8.

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Recurrence Relations

- **Definition:** A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation.
- The ***initial conditions*** for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Questions on Recurrence Relations

- **Example** 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,…. and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3?

  [Here $a_0 = 2$ is the initial condition.]

- **Solution**: We see from the recurrence relation that

$$\begin{aligned} a_1 &= a_0 + 3 = 2 + 3 = 5 \\ a_2 &= 5 + 3 = 8 \\ a_3 &= 8 + 3 = 11 \end{aligned}$$

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Questions on Recurrence Relations

- **Example** 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} – a_{n-2}$ for n = 2,3,4,…. and suppose that a_0 = 3 and a_1 = 5. What are a_2 and a_3?

  [Here the initial conditions are a_0 = 3 and a_1 = 5.]

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Questions about Recurrence Relations

- **Example** 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2,3,4,…. and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?
  [Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]
- **Solution**: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Fibonacci Sequence

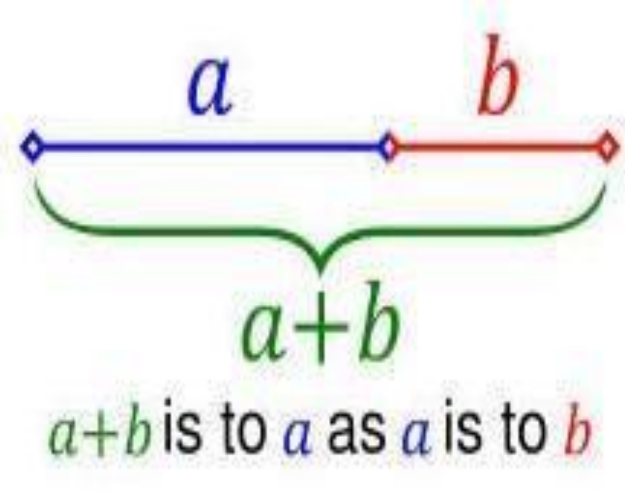
- **Definition**: Define the *Fibonacci sequence*, f_0, f_1, f_2, …, by:
  - Initial Conditions: $f_0 = 0$, $f_1 = 1$
  - Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$
- **Example**: Find f_2, f_3, f_4, f_5 and f_6 .
- **Answer:**

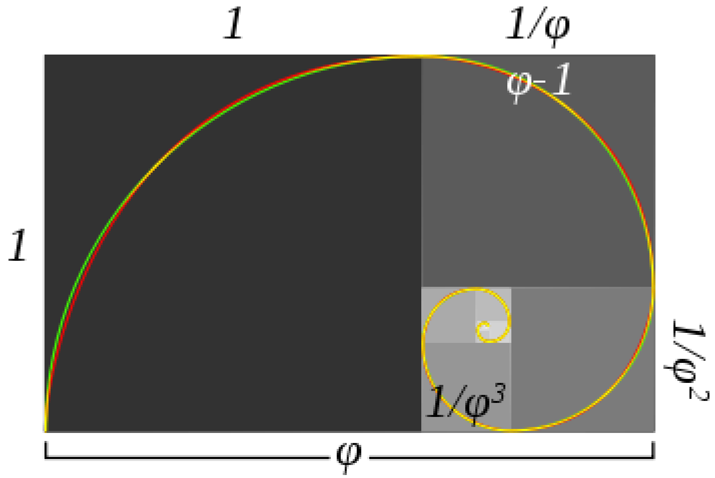
$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Fibonacci Sequence and the Golden Ratio

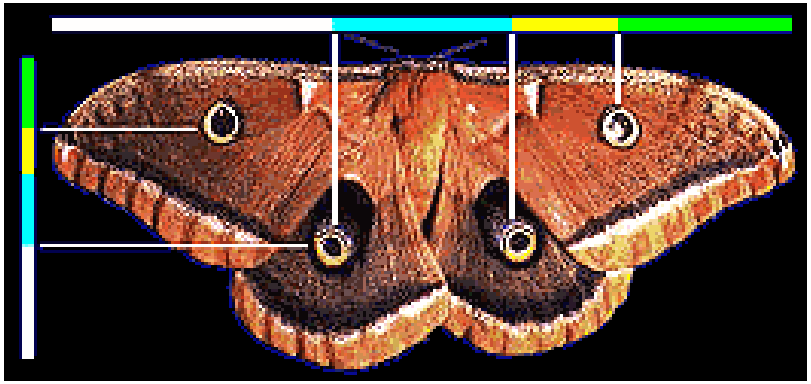
- Golden Ratio : (a+b)/a = a/b = ϕ
- Solution ϕ = (1+ √5)/2 = 1.61...
- $\lim_{n \to \infty} (f_{n+1} / f_n) = \phi$

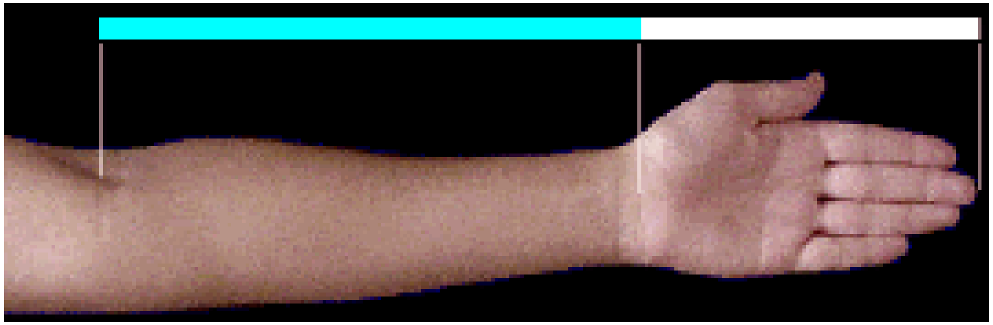


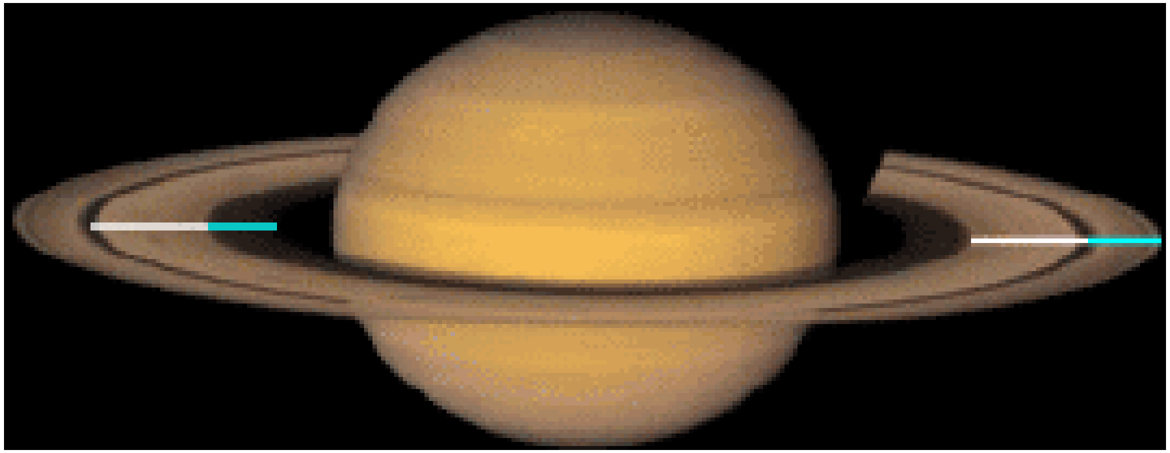


Golden Ratio in Nature

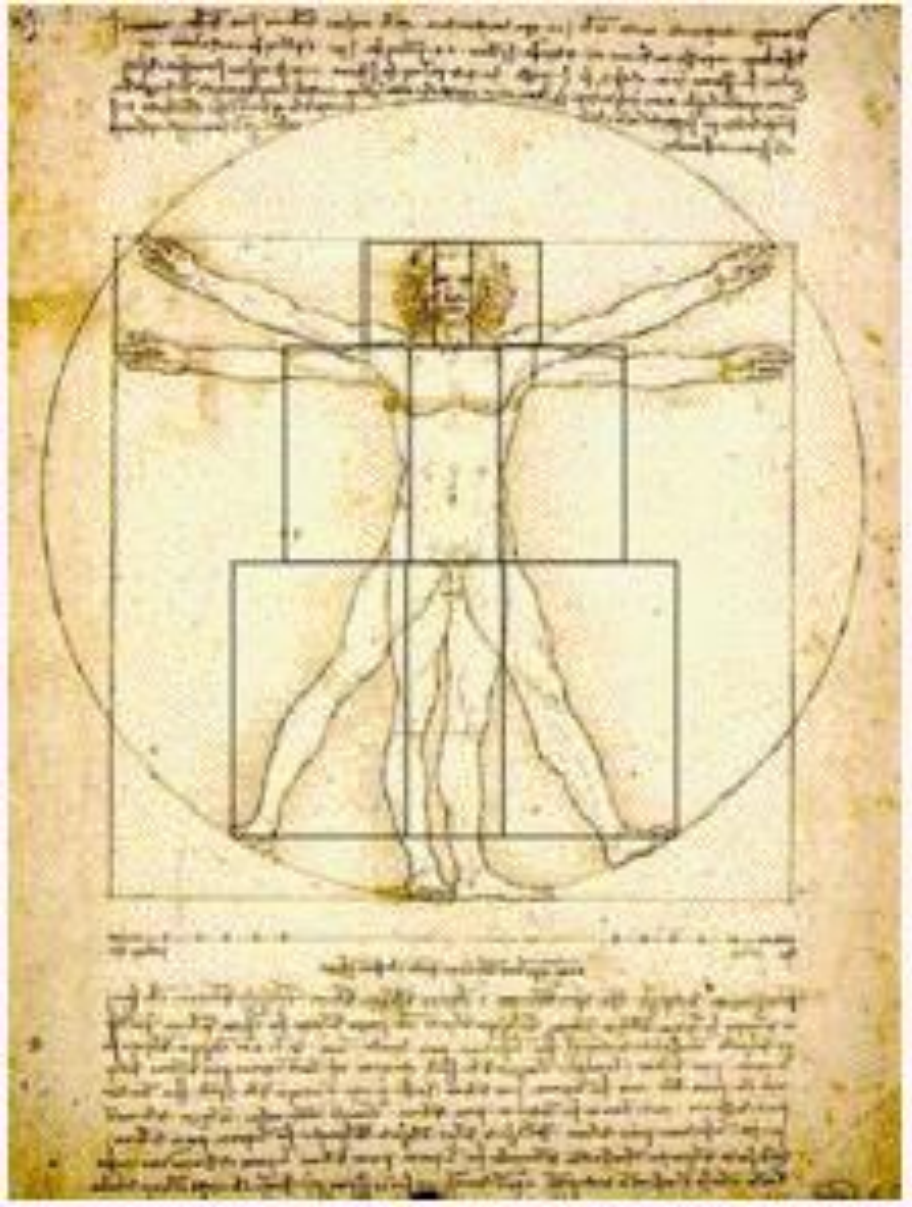
Golden Ratio in Nature



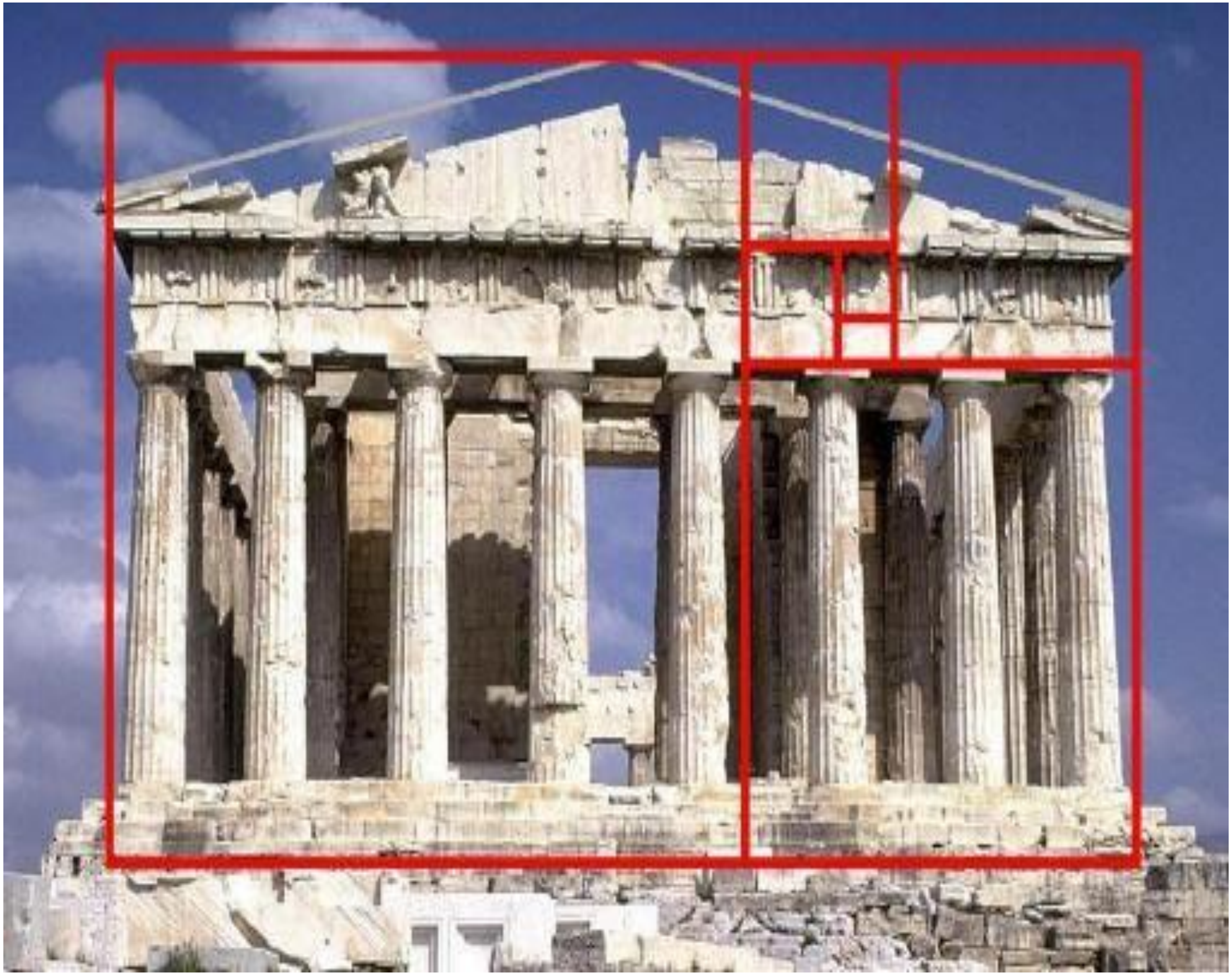




Golden Ratio in Art



Golden Ratio in Architecture



Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations are covered later on when recurrence relations are studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of *mathematical induction*.

Iterative Solution Example

- **Method 1**: Working upward, forward substitution
  Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,....$ and suppose that $a_1 = 2.$

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

.

.

.

$$a_n = a_{n\text{-}1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

- **Method 2**: Working downward, backward substitution
  Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3\,) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Financial Application

- **Example**: Suppose that a person deposits Rs10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?
- Let P_n denote the amount in the account after n years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\ P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Continued on next slide →

Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\ P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

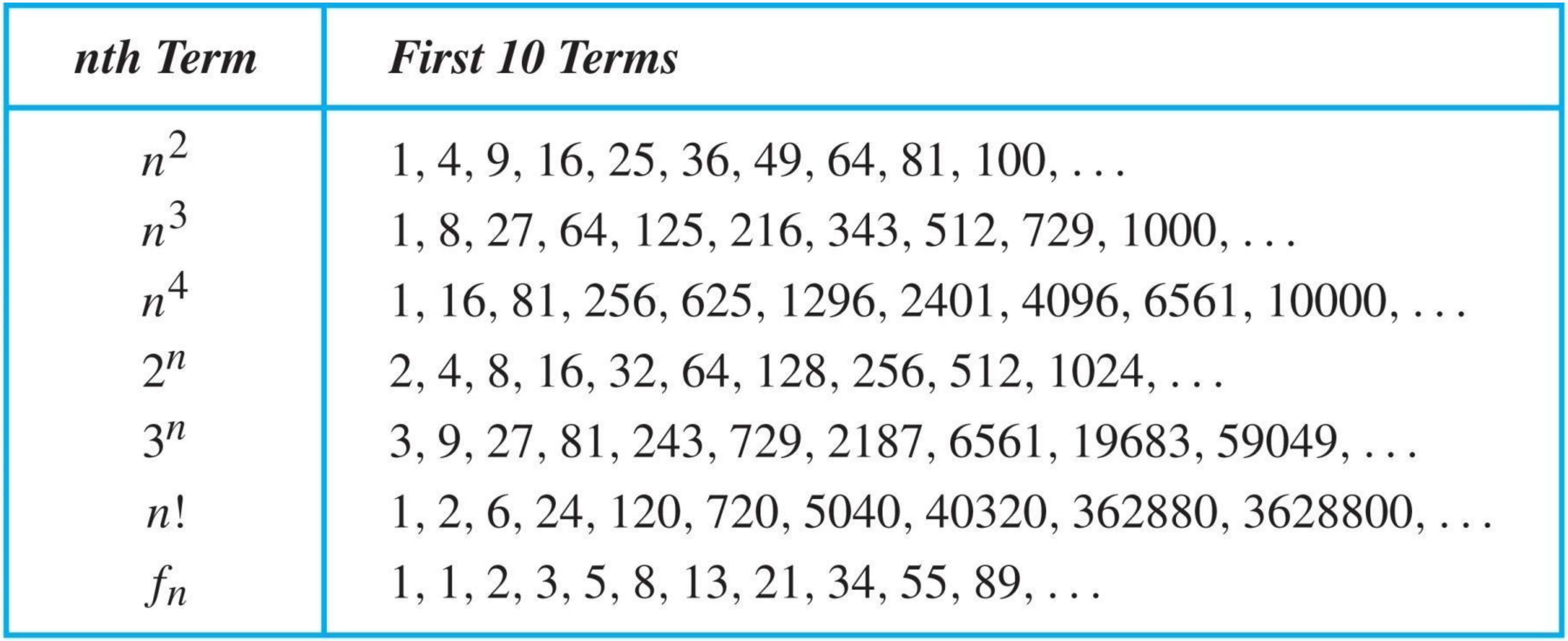
:

$P_n = (1.11)P_{n-1} = (1.11)^n\mathrm{P}_0 = (1.11)^n\ 10{,}000$

$P_n = (1.11)^n\ 10{,}000$ (Can be proved by induction)

$P_{30} = (1.11)^{30}\ 10{,}000 = 228{,}992.97$

Useful Sequences



nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$
  from the sequence $\{a_n\}$
- The notation:

$$\sum_{j=m}^{n} a_j \quad \textstyle\sum_{j=m}^{n} a_j \quad \sum_{m \leq j \leq n} a_j$$

  represents

$$a_m + a_{m+1} + \cdots + a_n$$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations

- More generally for a set *S*:

$$\sum_{j \in S} a_j$$

- **Examples:**

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \le j \le n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Continued on next slide →

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

Continued on next slide →

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Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$

From previous slide.

$$= \sum_{k=1}^{n+1} ar^{k}$$

Shifting the index of summation with *k* = *j* + 1.

$$= \left(\sum_{k=0}^{n} ar^{k} \right) + (ar^{n+1} - a)$$

Removing *k* = *n* + 1 term and adding *k* = 0 term.

$$= S_n + (ar^{n+1} - a)$$

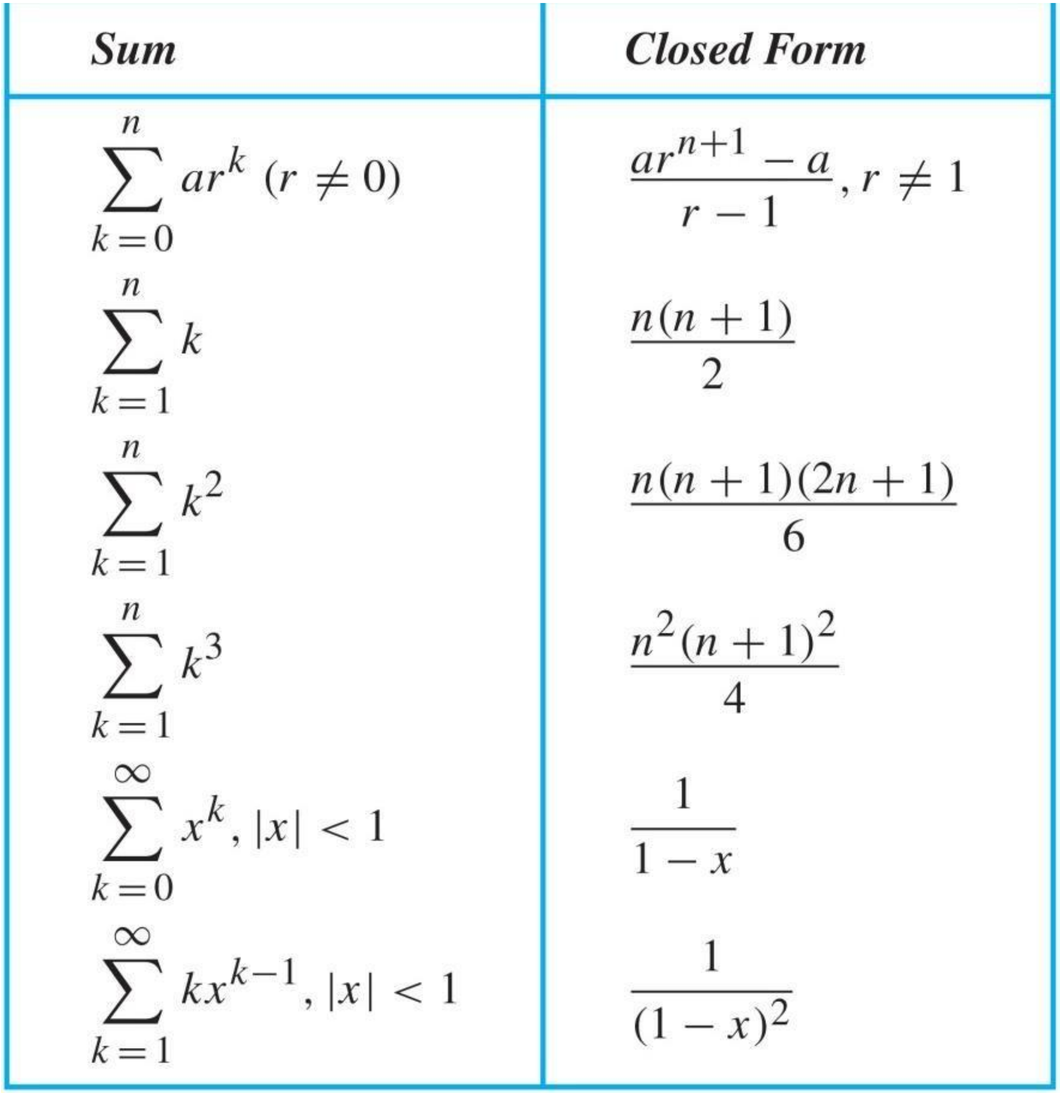
Substituting *S* for summation formula

$$\therefore \qquad rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{if r} \neq 1$$

$$S_n = \sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} a = (n+1)a \qquad \text{if r} = 1$$

Some Useful Summation Formulae



Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \; (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)

Double Summation

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

- Like a nested for loop
- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
        for ( int j = 1; j <= 3; j++ )
             sum += i*j;
```

Section 5:
Cardinality of Sets

Cardinality

- **Definition**: The cardinality of a set A is equal to the cardinality of a set B, denoted

  $$|A| = |B|,$$

  if and only if there is a bijection from A to B.
- If there is an injection from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

Cardinality

- **Definition:** A set that is either finite or has the same cardinality as the set of positive integers (Z+) is called **countable**. A set that is not count able is **uncountable.**
- The set of all finite strings over the alphabet of lowercase letters is countable.
- The set of real numbers **R** is an uncountable set.

Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a bijection f from the set of positive integers to a set S can be expressed in terms of a sequence a1, a2, …, an ,…

  where a1 = f(1), a2 = f(2),…, an = f(n),…

Showing that a Set is Countable

- Example 1: Show that the set of positive even integers is countable set.
- Proof: Let E be the set of even integers and f(x) = 2x be a function from N to E:

1	2	3	4	5	6	.....
2	4	6	8	10	12	......

- Then f is a bijection from N to E since f is both one-to-one and onto. To show that it is injective, suppose that f(n) = f(m). Then 2n = 2m, and so n = m. To see that it is surjective, suppose that t is some even positive integer. Then t = 2k for some positive integer k and f(k) = t.

Showing that a Set is Countable

- Example 2: Show that the set of all integers Z is countable.
- Proof: We can list the integers in a sequence:

  0, 1, − 1, 2, − 2, 3, − 3 ,…

  Let f be a function from N to Z defined as:

  When n is even: $f(n) = n/2$

  When n is odd: $f(n) = -(n-1)/2$

  that generates this list.

  We now show that this function is a bijection. First we show that it is injective by case analysis on the parity of N.

Continued ...

Showing that a Set is Countable

- Let m and n be two even natural number, then f(m) = m/2 and f(n) = n/2, it follows that f(m)=f(n) implies m =n.
- Let m and n be two odd natural numbers, then f(m) = -(m-1)/2 and f(n) = -(n-1)/2, it follows that f(m)=f(n) implies m=n

Therefore, f is injective. We now show that f is surjective by case analysis on the sign of some integer t in Z.

- Let t be positive, then t will appear in an even position in the sequence, thus f(2k)=2k/2=t with t=k. This implies that for every positive value t in Z there is a natural number 2k.
- Let t be negative, then t will appear in an odd position in the sequence, thus f(2k-1)=-(2k-1-1)/2=t with t=-k. This implies that for every negative value t in Z there is a natural number 2k-1.

Therefore, f is surjective. (QED)

Strings

- **Example 4:** Show that the set of finite strings S over the lowercase letters is countably infinite.
- **Proof:** Show that the strings can be listed in a seq uence. First list
  1. All the strings of length 0 in alphabetical order.
  2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
  3. Then all the strings of length 2 in lexicographic order.
  4. And so on.
- This implies a bijection from N to S and hence it is a countably infinite set.

Set of Java Programs is Countable

- **Example 5:** Show that the set of all Java programs is countable.
- **Solution**: Let S be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:
  1. Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
  2. If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
  3. We move on to the next string.
- In this way we construct an implied bijection from N to the set of Java programs. Hence, the set of Java programs is countable.

Real Numbers are Uncountable

- **Example:** Show that the real numbers are not countable.
- **Proof:** It is sufficient to show that the real numbers between 0 and 1 are not countable. Proof by contradiction: Assume that they are countable. Then we can list them:
  - r1 = $0.d_{11}d_{12}d_{13}\ldots d_{1n}\ldots$
  - r2 = $0.d_{21}d_{22}d_{23}\ldots d_{2n}\ldots$
  - r2 = $0.d_{31}d_{32}d_{33}\ldots d_{3n}\ldots$
  - ....
  - r2 = $0.d_{n1}d_{n2}d_{n3}\ldots d_{nn}\ldots$

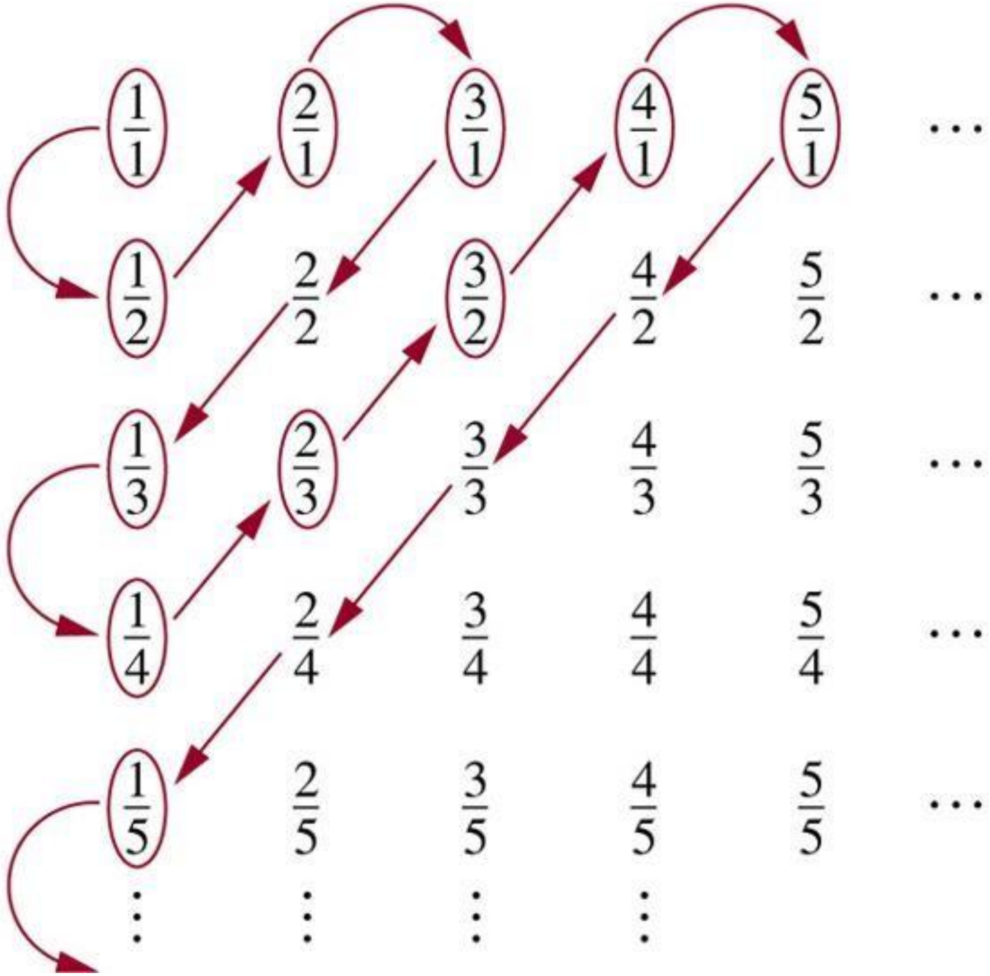
  Cantor's diagonal argument: From this list, we can construct a new real number

  $$r_q = 0.d_{q1}d_{q2}d_{q3}\ldots d_{qn}\ldots$$

  with $d_{q1} != d_{11,}$ $d_{q2} != d_{22,}$ $d_{q3} != d_{33}$, ...
- The real number r_q differs from *any* real number in the list in at least one position. This is a contradiction. (QED)

Positive Rational Numbers are Countable

Terms not circled are not listed because they repeat previously listed terms



$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	⋯
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	⋯
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	⋯
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	⋯
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	⋯
⋮	⋮	⋮	⋮		

Section 6: Matrices

Section Outline

- Definition of a Matrix
- Matrix Arithmetic
- Transposes and Powers
- Zero-One Matrices

Matrices

- Matrices are useful discrete structures that can be used in many ways. For example, they are used to:
  - Describe certain types of functions known as linear transformations.
  - Express which vertices of a graph are connected by edges.
- In later chapters, we will see matrices used to build models of:
  - Transportation systems.
  - Communication networks.
- Algorithms based on matrix models will be presented in later chapters.
- Here we cover the aspect of matrix arithmetic that will be needed later.

Matrix

- **Definition**: A *matrix* is a rectangular array of numbers. A matrix with *m* rows and *n* columns is called an m x n matrix.
  - A matrix with the same number of rows as columns is called *square*.
  - Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

$$3 \times 2 \text{ matrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

Notation

- Let *m* and *n* be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The *i*th row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$. The *j*th column of **A** is the $m \times 1$ *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}$$

- The (*i,j*)th *element* or *entry* of **A** is the element a_{ij}. We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (*i,j*)th element equal to a_{ij}.

Matrix Arithmetic: Addition

- **Defintion**: Let **A** = [a_{ij}] and **B** = [b_{ij}] be $m \times n$ matrices. The sum of **A** and **B**, denoted by **A** + **B**, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words, **A** + **B** = [$a_{ij} + b_{ij}$].
- **Example**:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

- Note that matrices of different sizes can not be added.

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Matrix Multiplication

- **Definition**: Let **A** be an n k matrix and **B** be a k n matrix. The *product* of **A** and **B**, denoted by **AB**, is the $\times$ m n matrix that has its (i,j)th element equal to the sum of the products of the corresponding elements from the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$. In other words, if **AB** = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1\mathrm{j}} + a_{i2}b_{2j} + \ldots + a_{i_k}b_{\mathrm{k}j}$.
- **Example**:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

- The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.

Illustration of Matrix Multiplication

- The Product of **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & a_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & \cdot & & \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & c_{ij} & \cdot \\ \cdot & \cdot & & \cdot \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

Matrix Multiplication Algorithm

- The definition for matrix multiplication can be expressed as an algorithm; **C** = **A** **B** where **C** is an $m \times n$ matrix that is the product of the m ×k matrix **A** and the k n ×matrix **B**.
- This algorithm carries out matrix multiplication based on its definition.

```
procedure matrix multiplication(A,B: matrices)
  for $i$ := 1 to $m$
    for $j$ := 1 to $n$
      $c_{ij}$ := 0
      for $q$ := 1 to $k$
        $c_{ij}$ := $c_{ij} + a_{iq} b_{qj}$
return C{C = [$c_{ij}$] is the product of A and B}
```

$\mathbf{A} = [a_{ij}]$ is a $m \times k$ matrix
$\mathbf{B} = [b_{ij}]$ is a $k \times n$ matrix

Matrix Multiplication is not Commutative

- **Example**: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does **AB = BA**?

- **Solution:**

$$\mathbf{AB} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

AB $\neq$ **BA**

Identity Matrix and Powers of Matrices

- **Definition**: The *identity matrix of order n* is the *m* *n* matrix $\mathbf{I}_n$ = $[\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- $\mathbf{A}\,\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$
  when **A** is an $m \times n$ matrix

- Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \qquad \mathbf{A}^r = \mathbf{AAA}\cdots\mathbf{A}$$

r terms

Transposes of Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an $m \times n$ matrix. The *transpose* of **A**, denoted by $\mathbf{A}^t$,is the $n \times m$ matrix obtained by interchanging the rows and columns of **A**.
  - If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1,2,\ldots,n$ and $j = 1,2, \ldots,m$.

$$\text{The transpose of the matrix } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is the matrix } \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Transposes of Matrices

- **Definition**: A square matrix **A** is called symmetric if $\mathbf{A} = \mathbf{A}^t$. Thus **A** = $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

$$\text{The matrix } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is square.}$$

- Square matrices do not change when their rows and columns are interchanged.

Zero-One Matrices

- **Definition**: A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*.
- Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

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Zero-One Matrices

- **Definition**: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be an $m \times n$ zero-one matrices.
  - The ***join*** of **A** and **B** is the zero-one matrix with (*i,j*)th entry $a_{ij} \vee b_{ij}$. The *join* of **A** and **B** is denoted by **A** ∨ **B**.
  - The ***meet*** of of **A** and **B** is the zero-one matrix with (*i,j*)th entry $a_{ij} \wedge b_{ij}$. The *meet* of **A** and **B** is denoted by **A** ∧ **B**.

Joins and Meets of Zero-One Matrices

- **Example**: Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Joins and Meets of Zero-One Matrices

- **Example**: Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

- **Solution**: The join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an $m \times k$ zero-one matrix and **B** = $[b_{ij}]$ be a $k \times n$ zero-one matrix. The *Boolean product* of **A** and **B**, denoted by **A** ⊙ **B**, is the $m \times n$ zero-one matrix with (i,j)th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

- **Example**: Find the Boolean product of **A** and **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

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Boolean Product of Zero-One Matrices

- **Solution**: The Boolean product **A** ⊙ **B** is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Boolean Powers of Zero-One Matrices

- **Definition**: Let **A** be a square zero-one matrix and let *r* be a positive integer. The *r*th Boolean power of **A** is the Boolean product of *r* factors of **A**, denoted by $\mathbf{A}^{[r]}$. Hence,

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{r \text{ times}}.$$

  We define $\mathbf{A}^{[r]}$ to be $\mathbf{I}_n$.

- The Boolean product is well defined because the Boolean product of matrices is associative.

Boolean Powers of Zero-One Matrices

- **Example**: Let $$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

  Find $\mathbf{A}^n$ for all positive integers *n*.

- **Solution**:

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[\mathbf{n}]} = \mathbf{A}^{\mathbf{5}} \quad \text{for all positive integers } n \text{ with } n \geq 5.$$

Summary

- A *set* is an unordered collection of objects
- Roster and set builder are two ways to describing sets
- Sets can be represented using bit strings: for subset *A* of *U* can be represented by a bit string of length *n*, where the *i*th bit is *1* if a_i belongs to *A* and is *0* if a_i does not belong to *A*
- Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*

Summary

- A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B
- Functions can composed or inverted as long as it is meaningful to do so
- The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$
- Sequences can be defined by an explicit formula (e.g., progressions) or recursively (e.g., Fibinocci sequence)

Summary

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation* and the solution as *closed formula* (or, *closed form solution*)
- The iteration method of solving recurrence relations involves guessing the formula for the *nth* term of the sequence and proving it using mathematical induction
- Cardinality of a set is the number of elements in it

Summary

- The cardinality of a set A is equal to the cardinality of a set B, denoted|A| = |B|, if and only if there is a bijection from A to B
- The above definition is used to establish cardinalities of infinite sets such as Q
- The set of rational numbers Q is countable (i.e., has a cardinality same as N, the set of natural numbers)
- The set of Real Numbers R is *uncountable* as there is no bijection possible between R and N

Summary

- Powers of matrices are defined recursively using multiplication of two matrices
- A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*
- One can apply Binary Operators of AND and OR to develop the arithmetic of zero-one matrices: join, meet, Boolean product and Boolean Products of zero-one matrices