MODULE 2

Public-Key Cryptography and RSA

Other Public-Key Cryptosystems

9.1 PRINCIPLES OF PUBLIC-KEY CRYPTOSYSTEMS

The concept of public-key cryptography evolved from an attempt to attack two of the most difficult problems associated with symmetric encryption. The first problem is that of key distribution, which is examined in some detail in Chapter 14.

As Chapter 14 discusses, key distribution under symmetric encryption requires either (1) that two communicants already share a key, which somehow has been distributed to them; or (2) the use of a key distribution center. Whitfield Diffie, one of the discoverers of public-key encryption (along with Martin Hellman, both at Stanford University at the time), reasoned that this second requirement negated the very essence of cryptography: the ability to maintain total secrecy over your own communication. As Diffie put it [DIFF88], "what good would it do after all to develop impenetrable cryptosystems, if their users were forced to share their keys with a KDC that could be compromised by either burglary or subpoena?"

The second problem that Diffie pondered, and one that was apparently unrelated to the first, was that of *digital signatures*. If the use of cryptography was to become widespread, not just in military situations but for commercial and private purposes, then electronic messages and documents would need the equivalent of signatures used in paper documents. That is, could a method be devised that would stipulate, to the satisfaction of all parties, that a digital message had been sent by a particular person? This is a somewhat broader requirement than that of authentication, and its characteristics and ramifications are explored in Chapter 13.

Diffie and Hellman achieved an astounding breakthrough in 1976 [DIFF76 a, b] by coming up with a method that addressed both problems and was radically different from all previous approaches to cryptography, going back over four millennia.[1]

In the next subsection, we look at the overall framework for public-key cryptography. Then we examine the requirements for the encryption/decryption algorithm that is at the heart of the scheme.

Public-Key Cryptosystems

Asymmetric algorithms rely on one key for encryption and a different but related key for decryption. These algorithms have the following important characteristic.

- It is computationally infeasible to determine the decryption key given only knowledge of the cryptographic algorithm and the encryption key.

In addition, some algorithms, such as RSA, also exhibit the following characteristic.

- Either of the two related keys can be used for encryption, with the other used for decryption.

A **public-key encryption** scheme has six ingredients (Figure 9.1a; compare with Figure 2.1).

- **Plaintext:** This is the readable message or data that is fed into the algorithm as input.

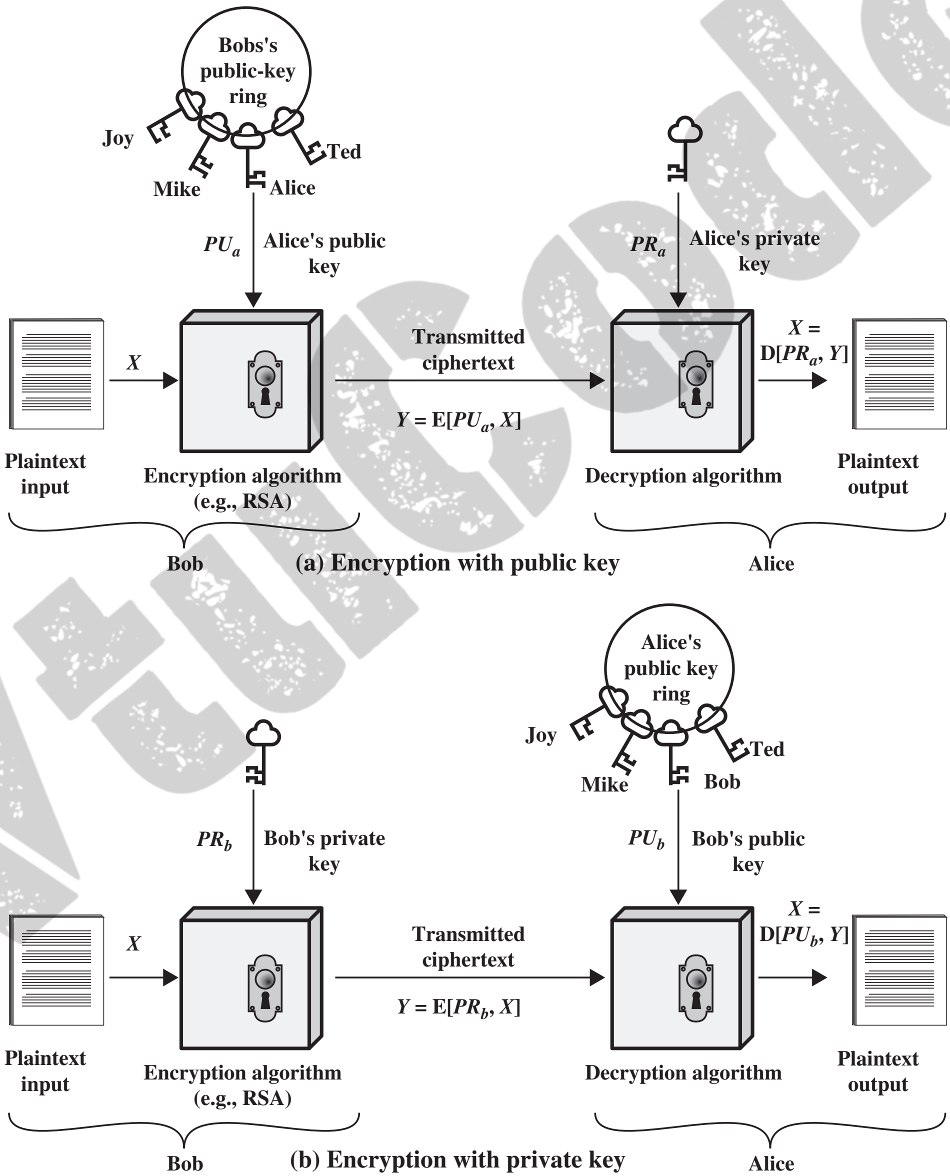


Figure 9.1 Public-Key Cryptography

- **Encryption algorithm:** The encryption algorithm performs various transformations on the plaintext.
- **Public and private keys:** This is a pair of keys that have been selected so that if one is used for encryption, the other is used for decryption. The exact transformations performed by the algorithm depend on the public or private key that is provided as input.
- **Ciphertext:** This is the scrambled message produced as output. It depends on the plaintext and the key. For a given message, two different keys will produce two different ciphertexts.
- **Decryption algorithm:** This algorithm accepts the ciphertext and the matching key and produces the original plaintext.

The essential steps are the following.

1. Each user generates a pair of keys to be used for the encryption and decryption of messages.
2. Each user places one of the two keys in a public register or other accessible file. This is the public key. The companion key is kept private. As Figure 9.1a suggests, each user maintains a collection of public keys obtained from others.
3. If Bob wishes to send a confidential message to Alice, Bob encrypts the message using Alice's public key.
4. When Alice receives the message, she decrypts it using her private key. No other recipient can decrypt the message because only Alice knows Alice's private key.

With this approach, all participants have access to public keys, and private keys are generated locally by each participant and therefore need never be distributed. As long as a user's private key remains protected and secret, incoming communication is secure. At any time, a system can change its private key and publish the companion public key to replace its old public key.

Table 9.2 summarizes some of the important aspects of symmetric and public-key encryption. To discriminate between the two, we refer to the key used in symmetric encryption as a **secret key**. The two keys used for asymmetric encryption are referred to as the **public key** and the **private key**.[2] Invariably, the private key is kept secret, but it is referred to as a private key rather than a secret key to avoid confusion with symmetric encryption.

Let us take a closer look at the essential elements of a public-key encryption scheme, using Figure 9.2 (compare with Figure 2.2). There is some source A that produces a message in plaintext, $X = [X_1, X_2, \ldots, X_M]$. The M elements of X are letters in some finite alphabet. The message is intended for destination B. B generates

Table 9.2 Conventional and Public-Key Encryption

Conventional Encryption	Public-Key Encryption
Needed to Work: 1. The same algorithm with the same key is used for encryption and decryption. 2. The sender and receiver must share the algorithm and the key. *Needed for Security:* 1. The key must be kept secret. 2. It must be impossible or at least impractical to decipher a message if the key is kept secret. 3. Knowledge of the algorithm plus samples of ciphertext must be insufficient to determine the key.	*Needed to Work:* 1. One algorithm is used for encryption and a related algorithm for decryption with a pair of keys, one for encryption and one for decryption. 2. The sender and receiver must each have one of the matched pair of keys (not the same one). *Needed for Security:* 1. One of the two keys must be kept secret. 2. It must be impossible or at least impractical to decipher a message if one of the keys is kept secret. 3. Knowledge of the algorithm plus one of the keys plus samples of ciphertext must be insufficient to determine the other key.

a related pair of keys: a public key, PU_b, and a private key, PR_b. PR_b is known only to B, whereas PU_b is publicly available and therefore accessible by A.

With the message X and the encryption key PU_b as input, A forms the ciphertext $Y = [Y_1, Y_2, \ldots, Y_N]$:

$$Y = \mathrm{E}(PU_b, X)$$

The intended receiver, in possession of the matching private key, is able to invert the transformation:

$$X = \mathrm{D}(PR_b, Y)$$

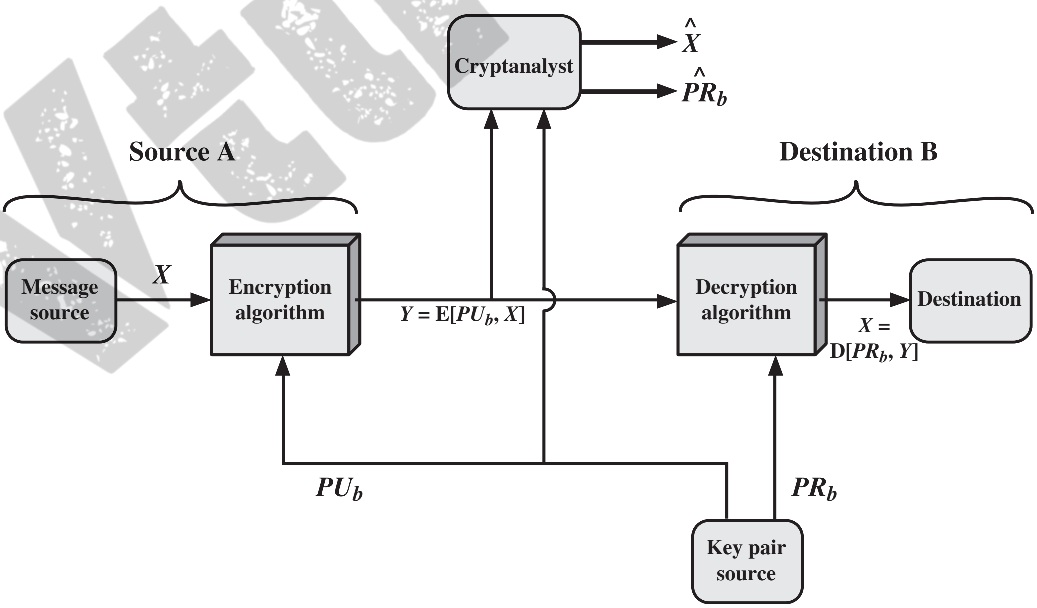


Figure 9.2 Public-Key Cryptosystem: Secrecy

An adversary, observing Y and having access to PU_b, but not having access to PR_b or X, must attempt to recover X and/or PR_b. It is assumed that the adversary does have knowledge of the encryption (E) and decryption (D) algorithms. If the adversary is interested only in this particular message, then the focus of effort is to recover X by generating a plaintext estimate $\hat{X}$. Often, however, the adversary is interested in being able to read future messages as well, in which case an attempt is made to recover PR_b by generating an estimate $\hat{PR}_b$.

We mentioned earlier that either of the two related keys can be used for encryption, with the other being used for decryption. This enables a rather different cryptographic scheme to be implemented. Whereas the scheme illustrated in Figure 9.2 provides confidentiality, Figures 9.1b and 9.3 show the use of public-key encryption to provide authentication:

$$Y = \mathrm{E}(PR_a, X)$$
$$X = \mathrm{D}(PU_a, Y)$$

In this case, A prepares a message to B and encrypts it using A's private key before transmitting it. B can decrypt the message using A's public key. Because the message was encrypted using A's private key, only A could have prepared the message. Therefore, the entire encrypted message serves as a **digital signature**. In addition, it is impossible to alter the message without access to A's private key, so the message is authenticated both in terms of source and in terms of data integrity.

In the preceding scheme, the entire message is encrypted, which, although validating both author and contents, requires a great deal of storage. Each document

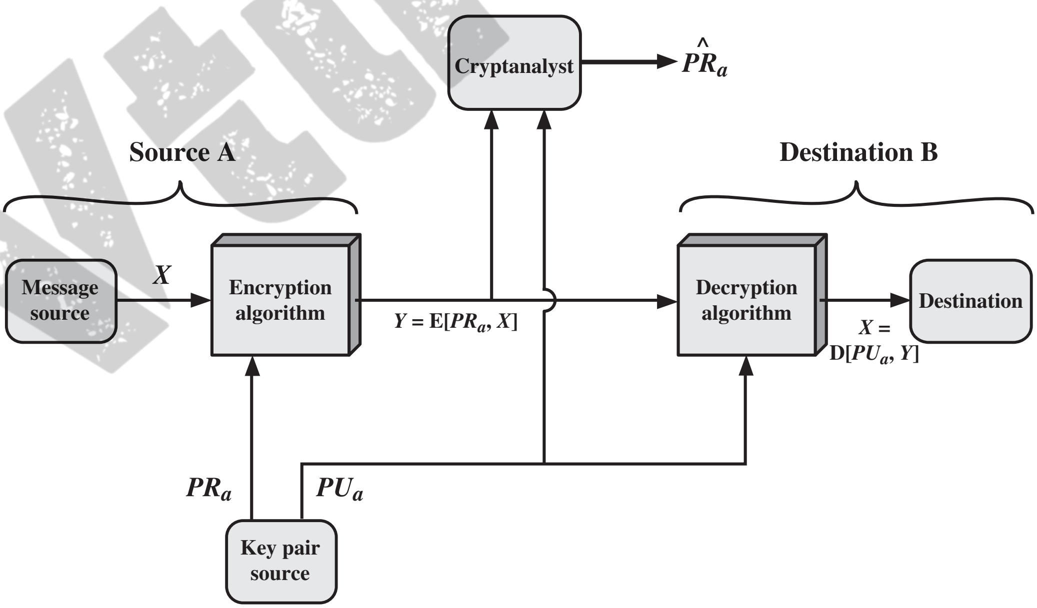


Figure 9.3 Public-Key Cryptosystem: Authentication

must be kept in plaintext to be used for practical purposes. A copy also must be stored in ciphertext so that the origin and contents can be verified in case of a dispute. A more efficient way of achieving the same results is to encrypt a small block of bits that is a function of the document. Such a block, called an authenticator, must have the property that it is infeasible to change the document without changing the authenticator. If the authenticator is encrypted with the sender's private key, it serves as a signature that verifies origin, content, and sequencing. Chapter 13 examines this technique in detail.

It is important to emphasize that the encryption process depicted in Figures 9.1b and 9.3 does not provide confidentiality. That is, the message being sent is safe from alteration but not from eavesdropping. This is obvious in the case of a signature based on a portion of the message, because the rest of the message is transmitted in the clear. Even in the case of complete encryption, as shown in Figure 9.3, there is no protection of confidentiality because any observer can decrypt the message by using the sender's public key.

It is, however, possible to provide both the authentication function and confidentiality by a double use of the public-key scheme (Figure 9.4):

$$Z = \mathrm{E}(PU_b, \mathrm{E}(PR_a, X))$$
$$X = \mathrm{D}(PU_a, \mathrm{D}(PR_b, Z))$$

In this case, we begin as before by encrypting a message, using the sender's private key. This provides the digital signature. Next, we encrypt again, using the receiver's public key. The final ciphertext can be decrypted only by the intended receiver, who alone has the matching private key. Thus, confidentiality is provided. The disadvantage of this approach is that the public-key algorithm, which is complex, must be exercised four times rather than two in each communication.

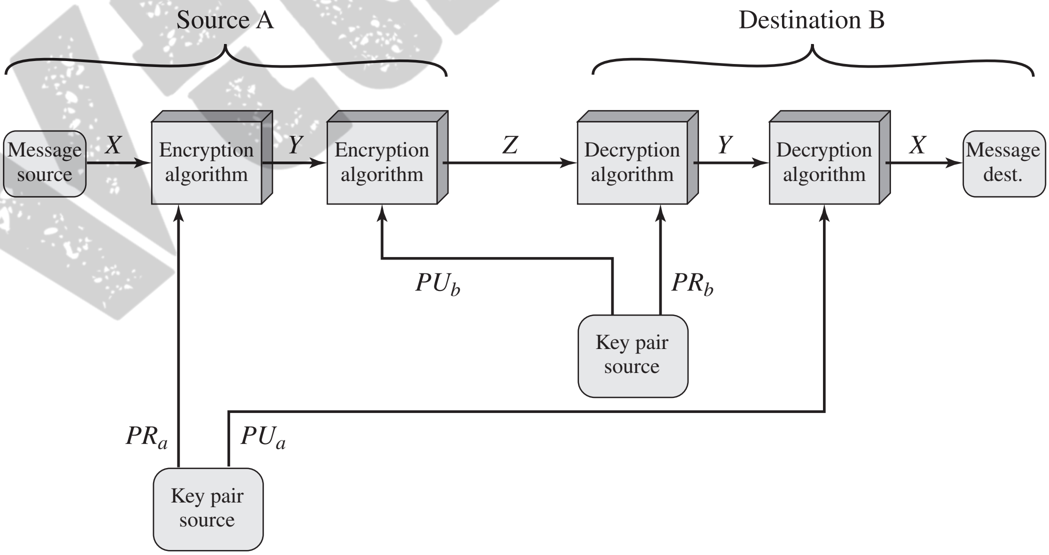


Figure 9.4 Public-Key Cryptosystem: Authentication and Secrecy

Applications for Public-Key Cryptosystems

Before proceeding, we need to clarify one aspect of public-key cryptosystems that is otherwise likely to lead to confusion. Public-key systems are characterized by the use of a cryptographic algorithm with two keys, one held private and one available publicly. Depending on the application, the sender uses either the sender's private key or the receiver's public key, or both, to perform some type of cryptographic function. In broad terms, we can classify the use of **public-key cryptosystems** into three categories

- **Encryption/decryption:** The sender encrypts a message with the recipient's public key.
- **Digital signature:** The sender "signs" a message with its private key. Signing is achieved by a cryptographic algorithm applied to the message or to a small block of data that is a function of the message.
- **Key exchange:** Two sides cooperate to exchange a session key. Several different approaches are possible, involving the private key(s) of one or both parties.

Some algorithms are suitable for all three applications, whereas others can be used only for one or two of these applications. Table 9.3 indicates the applications supported by the algorithms discussed in this book.

Requirements for Public-Key Cryptography

The cryptosystem illustrated in Figures 9.2 through 9.4 depends on a cryptographic algorithm based on two related keys. Diffie and Hellman postulated this system without demonstrating that such algorithms exist. However, they did lay out the conditions that such algorithms must fulfill [DIFF76b].

1. It is computationally easy for a party B to generate a pair (public key PU_b, private key PR_b).
2. It is computationally easy for a sender A, knowing the public key and the message to be encrypted, M, to generate the corresponding ciphertext:

$$C = \mathrm{E}(PU_b, M)$$

3. It is computationally easy for the receiver B to decrypt the resulting ciphertext using the private key to recover the original message:

$$M = \mathrm{D}(PR_b, C) = \mathrm{D}[PR_b, \mathrm{E}(PU_b, M)]$$

4. It is computationally infeasible for an adversary, knowing the public key, PU_b, to determine the private key, PR_b.

Table 9.3 Applications for Public-Key Cryptosystems

Algorithm	Encryption/Decryption	Digital Signature	Key Exchange
RSA	Yes	Yes	Yes
Elliptic Curve	Yes	Yes	Yes
Diffie-Hellman	No	No	Yes
DSS	No	Yes	No

5. It is computationally infeasible for an adversary, knowing the public key, PU_b, and a ciphertext, C, to recover the original message, M.

We can add a sixth requirement that, although useful, is not necessary for all public-key applications:

6. The two keys can be applied in either order:

$$M = \mathrm{D}[PU_b, \mathrm{E}(PR_b, M)] = \mathrm{D}[PR_b, \mathrm{E}(PU_b, M)]$$

These are formidable requirements, as evidenced by the fact that only a few algorithms (RSA, elliptic curve cryptography, Diffie-Hellman, DSS) have received widespread acceptance in the several decades since the concept of public-key cryptography was proposed.

Before elaborating on why the requirements are so formidable, let us first recast them. The requirements boil down to the need for a trap-door one-way function. A **one-way function**[3] is one that maps a domain into a range such that every function value has a unique inverse, with the condition that the calculation of the function is easy, whereas the calculation of the inverse is infeasible:

$$\begin{aligned} Y &= \mathrm{f}(X) && \text{easy} \\ X &= \mathrm{f}^{-1}(Y) && \text{infeasible} \end{aligned}$$

Generally, *easy* is defined to mean a problem that can be solved in polynomial time as a function of input length. Thus, if the length of the input is n bits, then the time to compute the function is proportional to n^a, where a is a fixed constant. Such algorithms are said to belong to the class **P**. The term *infeasible* is a much fuzzier concept. In general, we can say a problem is infeasible if the effort to solve it grows faster than polynomial time as a function of input size. For example, if the length of the input is n bits and the time to compute the function is proportional to 2^n, the problem is considered infeasible. Unfortunately, it is difficult to determine if a particular algorithm exhibits this complexity. Furthermore, traditional notions of computational complexity focus on the worst-case or average-case complexity of an algorithm. These measures are inadequate for cryptography, which requires that it be infeasible to invert a function for virtually all inputs, not for the worst case or even average case. A brief introduction to some of these concepts is provided in Appendix 9A.

We now turn to the definition of a **trap-door one-way function**, which is easy to calculate in one direction and infeasible to calculate in the other direction unless certain additional information is known. With the additional information the inverse can be calculated in polynomial time. We can summarize as follows: A trap-door one-way function is a family of invertible functions f_k, such that

$$\begin{aligned} Y &= \mathrm{f}_k(X) && \text{easy, if } k \text{ and } X \text{ are known} \\ X &= \mathrm{f}_k^{-1}(Y) && \text{easy, if } k \text{ and } Y \text{ are known} \\ X &= \mathrm{f}_k^{-1}(Y) && \text{infeasible, if } Y \text{ is known but } k \text{ is not known} \end{aligned}$$

Thus, the development of a practical public-key scheme depends on discovery of a suitable trap-door one-way function.

Public-Key Cryptanalysis

As with symmetric encryption, a public-key encryption scheme is vulnerable to a brute-force attack. The countermeasure is the same: Use large keys. However, there is a tradeoff to be considered. Public-key systems depend on the use of some sort of invertible mathematical function. The complexity of calculating these functions may not scale linearly with the number of bits in the key but grow more rapidly than that. Thus, the key size must be large enough to make brute-force attack impractical but small enough for practical encryption and decryption. In practice, the key sizes that have been proposed do make brute-force attack impractical but result in encryption/decryption speeds that are too slow for general-purpose use. Instead, as was mentioned earlier, public-key encryption is currently confined to key management and signature applications.

Another form of attack is to find some way to compute the private key given the public key. To date, it has not been mathematically proven that this form of attack is infeasible for a particular public-key algorithm. Thus, any given algorithm, including the widely used RSA algorithm, is suspect. The history of cryptanalysis shows that a problem that seems insoluble from one perspective can be found to have a solution if looked at in an entirely different way.

Finally, there is a form of attack that is peculiar to public-key systems. This is, in essence, a probable-message attack. Suppose, for example, that a message were to be sent that consisted solely of a 56-bit DES key. An adversary could encrypt all possible 56-bit DES keys using the public key and could discover the encrypted key by matching the transmitted ciphertext. Thus, no matter how large the key size of the public-key scheme, the attack is reduced to a brute-force attack on a 56-bit key. This attack can be thwarted by appending some random bits to such simple messages.

9.2 THE RSA ALGORITHM

The pioneering paper by Diffie and Hellman [DIFF76b] introduced a new approach to cryptography and, in effect, challenged cryptologists to come up with a cryptographic algorithm that met the requirements for public-key systems. A number of algorithms have been proposed for public-key cryptography. Some of these, though initially promising, turned out to be breakable.[4]

One of the first successful responses to the challenge was developed in 1977 by Ron Rivest, Adi Shamir, and Len Adleman at MIT and first published in 1978 [RIVE78].[5] The Rivest-Shamir-Adleman (RSA) scheme has since that time reigned supreme as the most widely accepted and implemented general-purpose approach to public-key encryption.

The **RSA** scheme is a cipher in which the plaintext and ciphertext are integers between 0 and $n - 1$ for some n. A typical size for n is 1024 bits, or 309 decimal digits. That is, n is less than 2^{1024}. We examine RSA in this section in some detail, beginning with an explanation of the algorithm. Then we examine some of the computational and cryptanalytical implications of RSA.

Description of the Algorithm

RSA makes use of an expression with exponentials. Plaintext is encrypted in blocks, with each block having a binary value less than some number n. That is, the block size must be less than or equal to $\log_2(n) + 1$; in practice, the block size is i bits, where $2^i < n \leq 2^{i+1}$. Encryption and decryption are of the following form, for some plaintext block M and ciphertext block C.

$$C = M^e \bmod n$$
$$M = C^d \bmod n = (M^e)^d \bmod n = M^{ed} \bmod n$$

Both sender and receiver must know the value of n. The sender knows the value of e, and only the receiver knows the value of d. Thus, this is a public-key encryption algorithm with a public key of $PU = \{e, n\}$ and a private key of $PR = \{d, n\}$. For this algorithm to be satisfactory for public-key encryption, the following requirements must be met.

1. It is possible to find values of e, d, and n such that $M^{ed} \bmod n = M$ for all $M < n$.
2. It is relatively easy to calculate $M^e \bmod n$ and $C^d \bmod n$ for all values of $M < n$.
3. It is infeasible to determine d given e and n.

For now, we focus on the first requirement and consider the other questions later. We need to find a relationship of the form

$$M^{ed} \bmod n = M$$

The preceding relationship holds if e and d are multiplicative inverses modulo $\phi(n)$, where $\phi(n)$ is the Euler totient function. It is shown in Chapter 8 that for p, q prime, $\phi(pq) = (p - 1)(q - 1)$. The relationship between e and d can be expressed as

$$ed \bmod \phi(n) = 1 \tag{9.1}$$

This is equivalent to saying

$$ed \equiv 1 \bmod \phi(n)$$
$$d \equiv e^{-1} \bmod \phi(n)$$

That is, e and d are multiplicative inverses mod $\phi(n)$. Note that, according to the rules of modular arithmetic, this is true only if d (and therefore e) is relatively prime to $\phi(n)$. Equivalently, $\gcd(\phi(n), d) = 1$. See Appendix R for a proof that Equation (9.1) satisfies the requirement for RSA.

We are now ready to state the RSA scheme. The ingredients are the following:

p, q, two prime numbers	(private, chosen)
$n = pq$	(public, calculated)
e, with $\gcd(\phi(n), e) = 1; 1 < e < \phi(n)$	(public, chosen)
$d \equiv e^{-1} (\bmod\ \phi(n))$	(private, calculated)

The private key consists of $\{d, n\}$ and the public key consists of $\{e, n\}$. Suppose that user A has published its public key and that user B wishes to send the message M to A. Then B calculates $C = M^e \bmod n$ and transmits C. On receipt of this ciphertext, user A decrypts by calculating $M = C^d \bmod n$.

Figure 9.5 summarizes the RSA algorithm. It corresponds to Figure 9.1a: Alice generates a public/private key pair; Bob encrypts using Alice's public key; and Alice decrypts using her private key. An example from [SING99] is shown in Figure 9.6. For this example, the keys were generated as follows.

1. Select two prime numbers, $p = 17$ and $q = 11$.
2. Calculate $n = pq = 17 \times 11 = 187$.
3. Calculate $\phi(n) = (p - 1)(q - 1) = 16 \times 10 = 160$.
4. Select e such that e is relatively prime to $\phi(n) = 160$ and less than $\phi(n)$; we choose $e = 7$.
5. Determine d such that $de \equiv 1 \pmod{160}$ and $d < 160$. The correct value is $d = 23$, because $23 \times 7 = 161 = (1 \times 160) + 1$; d can be calculated using the extended Euclid's algorithm (Chapter 4).

The resulting keys are public key $PU = \{7, 187\}$ and private key $PR = \{23, 187\}$. The example shows the use of these keys for a plaintext input of $M = 88$. For encryption, we need to calculate $C = 88^7 \bmod 187$. Exploiting the properties of modular arithmetic, we can do this as follows.

$$
\begin{aligned}
88^7 \bmod 187 &= [(88^4 \bmod 187) \times (88^2 \bmod 187) \\
&\quad \times (88^1 \bmod 187)] \bmod 187 \\
88^1 \bmod 187 &= 88 \\
88^2 \bmod 187 &= 7744 \bmod 187 = 77 \\
88^4 \bmod 187 &= 59{,}969{,}536 \bmod 187 = 132 \\
88^7 \bmod 187 &= (88 \times 77 \times 132) \bmod 187 = 894{,}432 \bmod 187 = 11
\end{aligned}
$$

For decryption, we calculate $M = 11^{23} \bmod 187$:

$$
\begin{aligned}
11^{23} \bmod 187 &= [(11^1 \bmod 187) \times (11^2 \bmod 187) \times (11^4 \bmod 187) \\
&\quad \times (11^8 \bmod 187) \times (11^8 \bmod 187)] \bmod 187 \\
11^1 \bmod 187 &= 11 \\
11^2 \bmod 187 &= 121 \\
11^4 \bmod 187 &= 14{,}641 \bmod 187 = 55 \\
11^8 \bmod 187 &= 214{,}358{,}881 \bmod 187 = 33 \\
11^{23} \bmod 187 &= (11 \times 121 \times 55 \times 33 \times 33) \bmod 187 \\
&= 79{,}720{,}245 \bmod 187 = 88
\end{aligned}
$$

We now look at an example from [HELL79], which shows the use of RSA to process multiple blocks of data. In this simple example, the plaintext is an alphanumeric string. Each plaintext symbol is assigned a unique code of two decimal

Key Generation by Alice	
Select p, q	p and q both prime, $p \neq q$
Calculate $n = p \times q$	
Calcuate $\phi(n) = (p - 1)(q - 1)$	
Select integer e	$\gcd(\phi(n), e) = 1; 1 < e < \phi(n)$
Calculate d	$d \equiv e^{-1} (\bmod \phi(n))$
Public key	$PU = \{e, n\}$
Private key	$PR = \{d, n\}$

Encryption by Bob with Alice's Public Key	
Plaintext:	$M < n$
Ciphertext:	$C = M^e \bmod n$

Decryption by Alice with Alice's Public Key	
Ciphertext:	C
Plaintext:	$M = C^d \bmod n$

Figure 9.5 The RSA Algorithm

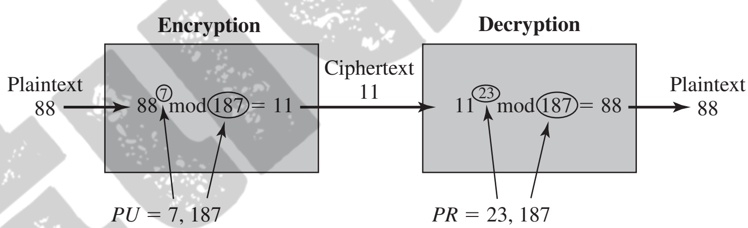


Figure 9.6 Example of RSA Algorithm

digits (e.g., a = 00, A = 26).[6] A plaintext block consists of four decimal digits, or two alphanumeric characters. Figure 9.7a illustrates the sequence of events for the encryption of multiple blocks, and Figure 9.7b gives a specific example. The circled numbers indicate the order in which operations are performed.

Computational Aspects

We now turn to the issue of the complexity of the computation required to use RSA. There are actually two issues to consider: encryption/decryption and key generation. Let us look first at the process of encryption and decryption and then consider key generation.

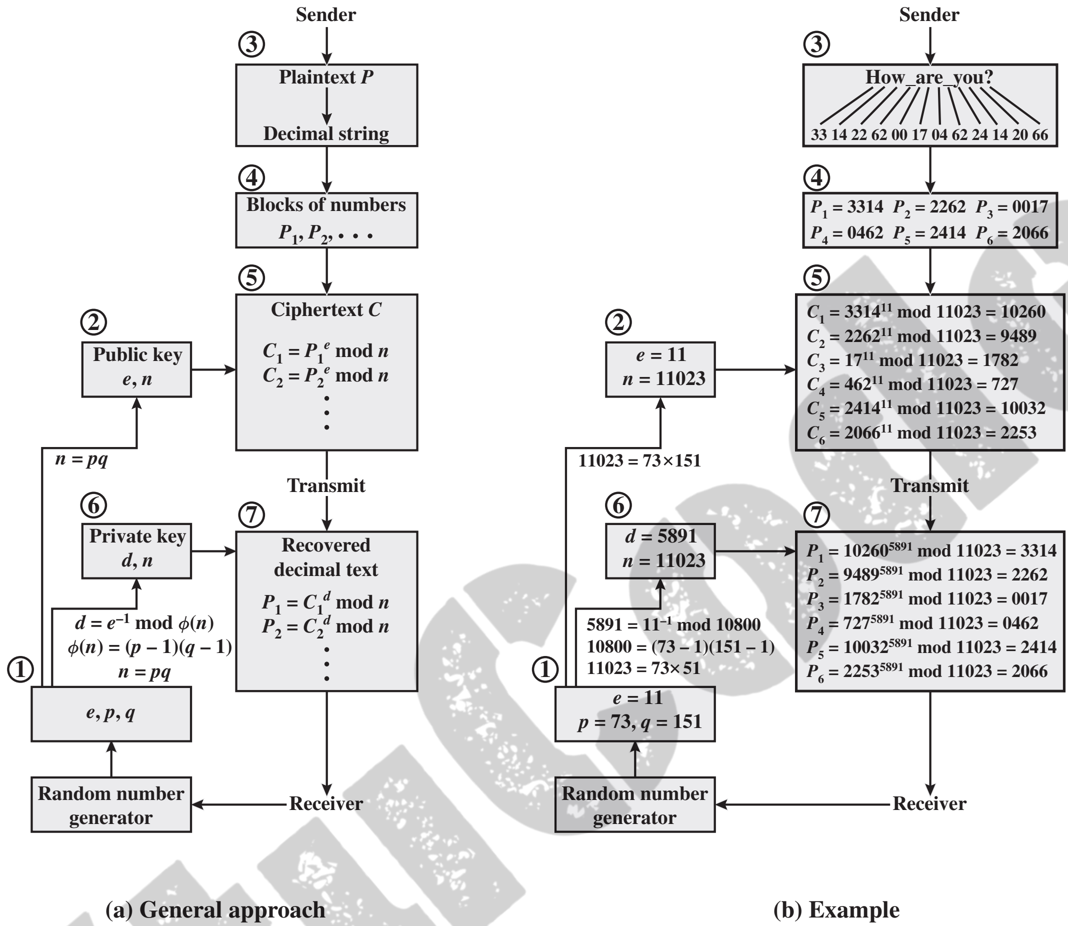


Figure 9.7 RSA Processing of Multiple Blocks

Exponentiation in Modular Arithmetic Both encryption and decryption in RSA involve raising an integer to an integer power, mod n. If the exponentiation is done over the integers and then reduced modulo n, the intermediate values would be gargantuan. Fortunately, as the preceding example shows, we can make use of a property of modular arithmetic:

$$[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$$

Thus, we can reduce intermediate results modulo n. This makes the calculation practical.

Another consideration is the efficiency of exponentiation, because with RSA, we are dealing with potentially large exponents. To see how efficiency might be increased, consider that we wish to compute x^{16}. A straightforward approach requires 15 multiplications:

$$x^{16} = x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x$$

However, we can achieve the same final result with only four multiplications if we repeatedly take the square of each partial result, successively forming (x^2, x^4, x^8, x^{16}). As another example, suppose we wish to calculate x^{11} mod n for some integers x and n. Observe that $x^{11} = x^{1+2+8} = (x)(x^2)(x^8)$. In this case, we compute x mod n, x^2 mod n, x^4 mod n, and x^8 mod n and then calculate $[(x \bmod n) \times (x^2 \bmod n) \times (x^8 \bmod n)] \bmod n$.

More generally, suppose we wish to find the value a^b mod n with a, b, and m positive integers. If we express b as a binary number $b_k b_{k-1} \ldots b_0$, then we have

$$b = \sum_{b_i \neq 0} 2^i$$

Therefore,

$$a^b = a^{\left(\sum_{b_i \neq 0} 2^i\right)} = \prod_{b_i \neq 0} a^{(2^i)}$$

$$a^b \bmod n = \left[\prod_{b_i \neq 0} a^{(2^i)}\right] \bmod n = \left(\prod_{b_i \neq 0}\left[a^{(2^i)} \bmod n\right]\right) \bmod n$$

We can therefore develop the algorithm[7] for computing a^b mod n, shown in Figure 9.8. Table 9.4 shows an example of the execution of this algorithm. Note that the variable c is not needed; it is included for explanatory purposes. The final value of c is the value of the exponent.

Efficient Operation Using the Public Key To speed up the operation of the RSA algorithm using the public key, a specific choice of e is usually made. The most common choice is 65537 ($2^{16} + 1$); two other popular choices are 3 and 17. Each of these choices has only two 1 bits, so the number of multiplications required to perform exponentiation is minimized.

```
c ← 0; f ← 1
for i ← k downto 0
    do  c ← 2 × c
        f ← (f × f) mod n
    if  bi = 1
        then c ← c + 1
             f ← (f × a) mod n
return f
```

Note: The integer b is expressed as a binary number $b_k b_{k-1} \ldots b_0$.

Figure 9.8 Algorithm for Computing a^b mod n

Table 9.4 Result of the Fast Modular Exponentiation Algorithm for $a^b \bmod n$, where $a = 7$, $b = 560 = 1000110000$, and $n = 561$

i	9	8	7	6	5	4	3	2	1	0
b_i	1	0	0	0	1	1	0	0	0	0
c	1	2	4	8	17	35	70	140	280	560
f	7	49	157	526	160	241	298	166	67	1

However, with a very small public key, such as $e = 3$, RSA becomes vulnerable to a simple attack. Suppose we have three different RSA users who all use the value $e = 3$ but have unique values of n, namely (n_1, n_2, n_3). If user A sends the same encrypted message M to all three users, then the three ciphertexts are $C_1 = M^3 \bmod n_1$, $C_2 = M^3 \bmod n_2$, and $C_3 = M^3 \bmod n_3$. It is likely that n_1, n_2, and n_3 are pairwise relatively prime. Therefore, one can use the Chinese remainder theorem (CRT) to compute $M^3 \bmod (n_1 n_2 n_3)$. By the rules of the RSA algorithm, M is less than each of the n_i; therefore $M^3 < n_1 n_2 n_3$. Accordingly, the attacker need only compute the cube root of M^3. This attack can be countered by adding a unique pseudorandom bit string as padding to each instance of M to be encrypted. This approach is discussed subsequently.

The reader may have noted that the definition of the RSA algorithm (Figure 9.5) requires that during key generation the user selects a value of e that is relatively prime to $\phi(n)$. Thus, if a value of e is selected first and the primes p and q are generated, it may turn out that $\gcd(\phi(n), e) \neq 1$. In that case, the user must reject the p, q values and generate a new p, q pair.

Efficient Operation Using the Private Key We cannot similarly choose a small constant value of d for efficient operation. A small value of d is vulnerable to a brute-force attack and to other forms of cryptanalysis [WIEN90]. However, there is a way to speed up computation using the CRT. We wish to compute the value $M = C^d \bmod n$. Let us define the following intermediate results:

$$V_p = C^d \bmod p \quad V_q = C^d \bmod q$$

Following the CRT using Equation (8.8), define the quantities

$$X_p = q \times (q^{-1} \bmod p) \quad X_q = p \times (p^{-1} \bmod q)$$

The CRT then shows, using Equation (8.9), that

$$M = (V_p X_p + V_q X_q) \bmod n$$

Furthermore, we can simplify the calculation of V_p and V_q using Fermat's theorem, which states that $a^{p-1} \equiv 1 \pmod p$ if p and a are relatively prime. Some thought should convince you that the following are valid.

$$V_p = C^d \bmod p = C^{d \bmod (p-1)} \bmod p \quad V_q = C^d \bmod q = C^{d \bmod (q-1)} \bmod q$$

The quantities $d \bmod (p - 1)$ and $d \bmod (q - 1)$ can be precalculated. The end result is that the calculation is approximately four times as fast as evaluating $M = C^d \bmod n$ directly [BONE02].

Key Generation Before the application of the public-key cryptosystem, each participant must generate a pair of keys. This involves the following tasks.

- Determining two prime numbers, p and q.
- Selecting either e or d and calculating the other.

First, consider the selection of p and q. Because the value of $n = pq$ will be known to any potential adversary, in order to prevent the discovery of p and q by exhaustive methods, these primes must be chosen from a sufficiently large set (i.e., p and q must be large numbers). On the other hand, the method used for finding large primes must be reasonably efficient.

At present, there are no useful techniques that yield arbitrarily large primes, so some other means of tackling the problem is needed. The procedure that is generally used is to pick at random an odd number of the desired order of magnitude and test whether that number is prime. If not, pick successive random numbers until one is found that tests prime.

A variety of tests for primality have been developed (e.g., see [KNUT98] for a description of a number of such tests). Almost invariably, the tests are probabilistic. That is, the test will merely determine that a given integer is *probably* prime. Despite this lack of certainty, these tests can be run in such a way as to make the probability as close to 1.0 as desired. As an example, one of the more efficient and popular algorithms, the Miller-Rabin algorithm, is described in Chapter 8. With this algorithm and most such algorithms, the procedure for testing whether a given integer n is prime is to perform some calculation that involves n and a randomly chosen integer a. If n "fails" the test, then n is not prime. If n "passes" the test, then n may be prime or nonprime. If n passes many such tests with many different randomly chosen values for a, then we can have high confidence that n is, in fact, prime.

In summary, the procedure for picking a prime number is as follows.

1. Pick an odd integer n at random (e.g., using a pseudorandom number generator).
2. Pick an integer $a < n$ at random.
3. Perform the probabilistic primality test, such as Miller-Rabin, with a as a parameter. If n fails the test, reject the value n and go to step 1.
4. If n has passed a sufficient number of tests, accept n; otherwise, go to step 2.

This is a somewhat tedious procedure. However, remember that this process is performed relatively infrequently: only when a new pair (PU, PR) is needed.

It is worth noting how many numbers are likely to be rejected before a prime number is found. A result from number theory, known as the prime number theorem, states that the primes near N are spaced on the average one every ln (N) integers. Thus, on average, one would have to test on the order of ln(N) integers before a prime is found. Actually, because all even integers can be immediately rejected, the correct figure is ln(N)/2. For example, if a prime on the order of magnitude of 2^{200} were sought, then about $\ln(2^{200})/2 = 70$ trials would be needed to find a prime.

Having determined prime numbers p and q, the process of key generation is completed by selecting a value of e and calculating d or, alternatively, selecting a value of d and calculating e. Assuming the former, then we need to select an e such that $\gcd(\phi(n), e) = 1$ and then calculate $d \equiv e^{-1} \pmod{\phi(n)}$. Fortunately, there is a single algorithm that will, at the same time, calculate the greatest common divisor of two integers and, if the gcd is 1, determine the inverse of one of the integers modulo the other. The algorithm, referred to as the extended Euclid's algorithm, is explained in Chapter 4. Thus, the procedure is to generate a series of random numbers, testing each against $\phi(n)$ until a number relatively prime to $\phi(n)$ is found. Again, we can ask the question: How many random numbers must we test to find a usable number, that is, a number relatively prime to $\phi(n)$? It can be shown easily that the probability that two random numbers are relatively prime is about 0.6; thus, very few tests would be needed to find a suitable integer (see Problem 8.2).

The Security of RSA

Five possible approaches to attacking the RSA algorithm are

- **Brute force:** This involves trying all possible private keys.
- **Mathematical attacks:** There are several approaches, all equivalent in effort to factoring the product of two primes.
- **Timing attacks:** These depend on the running time of the decryption algorithm.
- **Hardware fault-based attack:** This involves inducing hardware faults in the processor that is generating digital signatures.
- **Chosen ciphertext attacks:** This type of attack exploits properties of the RSA algorithm.

The defense against the brute-force approach is the same for RSA as for other cryptosystems, namely, to use a large key space. Thus, the larger the number of bits in d, the better. However, because the calculations involved, both in key generation and in encryption/decryption, are complex, the larger the size of the key, the slower the system will run.

In this subsection, we provide an overview of mathematical and timing attacks.

THE FACTORING PROBLEM We can identify three approaches to attacking RSA mathematically.

1. Factor n into its two prime factors. This enables calculation of $\phi(n) = (p - 1) \times (q - 1)$, which in turn enables determination of $d \equiv e^{-1} \pmod{\phi(n)}$.
2. Determine $\phi(n)$ directly, without first determining p and q. Again, this enables determination of $d \equiv e^{-1} \pmod{\phi(n)}$.
3. Determine d directly, without first determining $\phi(n)$.

Most discussions of the cryptanalysis of RSA have focused on the task of factoring n into its two prime factors. Determining $\phi(n)$ given n is equivalent to factoring n [RIBE96]. With presently known algorithms, determining d given e and n appears to be at least as time-consuming as the factoring problem [KALI95]. Hence, we can use factoring performance as a benchmark against which to evaluate the security of RSA.

For a large n with large prime factors, factoring is a hard problem, but it is not as hard as it used to be. A striking illustration of this is the following. In 1977, the three inventors of RSA dared *Scientific American* readers to decode a cipher they printed in Martin Gardner's "Mathematical Games" column [GARD77]. They offered a \$100 reward for the return of a plaintext sentence, an event they predicted might not occur for some 40 quadrillion years. In April of 1994, a group working over the Internet claimed the prize after only eight months of work [LEUT94]. This challenge used a public key size (length of n) of 129 decimal digits, or around 428 bits. In the meantime, just as they had done for DES, RSA Laboratories had issued challenges for the RSA cipher with key sizes of 100, 110, 120, and so on, digits. The latest challenge to be met is the RSA-768 challenge with a key length of 232 decimal digits, or 768 bits. Table 9.5 shows the results to date. Million-instructions-per-second processor running for one year, which is about 3×10^{13} instructions executed. A 1 GHz Pentium is about a 250-MIPS machine.

A striking fact about the progress reflected in Table 9.5 concerns the method used. Until the mid-1990s, factoring attacks were made using an approach known as the quadratic sieve. The attack on RSA-130 used a newer algorithm, the generalized number field sieve (GNFS), and was able to factor a larger number than RSA-129 at only 20% of the computing effort.

The threat to larger key sizes is twofold: the continuing increase in computing power and the continuing refinement of factoring algorithms. We have seen that the move to a different algorithm resulted in a tremendous speedup. We can expect further refinements in the GNFS, and the use of an even better algorithm is also a possibility. In fact, a related algorithm, the special number field

Table 9.5 Progress in RSA Factorization

Number of Decimal Digits	Number of Bits	Date Achieved
100	332	April 1991
110	365	April 1992
120	398	June 1993
129	428	April 1994
130	431	April 1996
140	465	February 1999
155	512	August 1999
160	530	April 2003
174	576	December 2003
200	663	May 2005
193	640	November 2005
232	768	December 2009

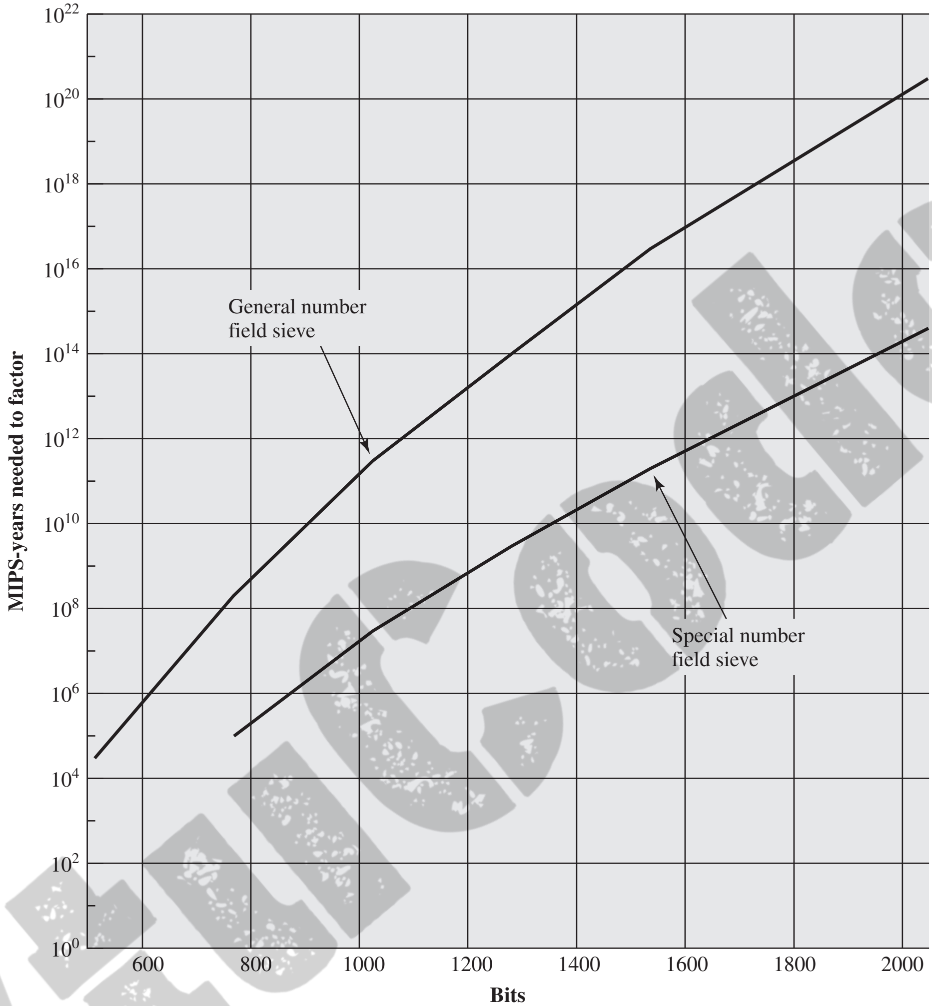


Figure 9.9 MIPS-years Needed to Factor

sieve (SNFS), can factor numbers with a specialized form considerably faster than the generalized number field sieve. Figure 9.9 compares the performance of the two algorithms. It is reasonable to expect a breakthrough that would enable a general factoring performance in about the same time as SNFS, or even better [ODLY95]. Thus, we need to be careful in choosing a key size for RSA. The team that produced the 768-bit factorization made the following observation [KLEI10]:

> Factoring a 1024-bit RSA modulus would be about a thousand times harder than factoring a 768-bit modulus, and a 768-bit RSA modulus is several thousands times harder to factor than a 512-bit one. Because the first factorization of a 512-bit RSA modulus

> was reported only a decade, it is not unreasonable to expect that 1024-bit RSA moduli can be factored well within the next decade by an academic effort such as ours. Thus, it would be prudent to phase out usage of 1024-bit RSA within the next three to four years.

In addition to specifying the size of n, a number of other constraints have been suggested by researchers. To avoid values of n that may be factored more easily, the algorithm's inventors suggest the following constraints on p and q.

1. p and q should differ in length by only a few digits. Thus, for a 1024-bit key (309 decimal digits), both p and q should be on the order of magnitude of 10^{75} to 10^{100}.
2. Both $(p - 1)$ and $(q - 1)$ should contain a large prime factor.
3. $\gcd(p - 1, q - 1)$ should be small.

In addition, it has been demonstrated that if $e < n$ and $d < n^{1/4}$, then d can be easily determined [WIEN90].

Timing Attacks If one needed yet another lesson about how difficult it is to assess the security of a cryptographic algorithm, the appearance of timing attacks provides a stunning one. Paul Kocher, a cryptographic consultant, demonstrated that a snooper can determine a private key by keeping track of how long a computer takes to decipher messages [KOCH96, KALI96b]. Timing attacks are applicable not just to RSA, but to other public-key cryptography systems. This attack is alarming for two reasons: It comes from a completely unexpected direction, and it is a ciphertext-only attack.

A **timing attack** is somewhat analogous to a burglar guessing the combination of a safe by observing how long it takes for someone to turn the dial from number to number. We can explain the attack using the modular exponentiation algorithm of Figure 9.8, but the attack can be adapted to work with any implementation that does not run in fixed time. In this algorithm, modular exponentiation is accomplished bit by bit, with one modular multiplication performed at each iteration and an additional modular multiplication performed for each 1 bit.

As Kocher points out in his paper, the attack is simplest to understand in an extreme case. Suppose the target system uses a modular multiplication function that is very fast in almost all cases but in a few cases takes much more time than an entire average modular exponentiation. The attack proceeds bit-by-bit starting with the leftmost bit, b_k. Suppose that the first j bits are known (to obtain the entire exponent, start with $j = 0$ and repeat the attack until the entire exponent is known). For a given ciphertext, the attacker can complete the first j iterations of the **for** loop. The operation of the subsequent step depends on the unknown exponent bit. If the bit is set, $d \leftarrow (d \times a) \bmod n$ will be executed. For a few values of a and d, the modular multiplication will be extremely slow, and the attacker knows which these are. Therefore, if the observed time to execute the decryption algorithm is always slow when this particular iteration is slow with a 1 bit, then this bit is assumed to be 1. If a number of observed execution times for the entire algorithm are fast, then this bit is assumed to be 0.

In practice, modular exponentiation implementations do not have such extreme timing variations, in which the execution time of a single iteration can exceed the mean execution time of the entire algorithm. Nevertheless, there is enough variation to make this attack practical. For details, see [KOCH96].

Although the timing attack is a serious threat, there are simple countermeasures that can be used, including the following.

- **Constant exponentiation time:** Ensure that all exponentiations take the same amount of time before returning a result. This is a simple fix but does degrade performance.
- **Random delay:** Better performance could be achieved by adding a random delay to the exponentiation algorithm to confuse the timing attack. Kocher points out that if defenders don't add enough noise, attackers could still succeed by collecting additional measurements to compensate for the random delays.
- **Blinding:** Multiply the ciphertext by a random number before performing exponentiation. This process prevents the attacker from knowing what ciphertext bits are being processed inside the computer and therefore prevents the bit-by-bit analysis essential to the timing attack.

RSA Data Security incorporates a blinding feature into some of its products. The private-key operation $M = C_d \bmod n$ is implemented as follows.

1. Generate a secret random number r between 0 and $n - 1$.
2. Compute $C' = C(r^e) \bmod n$, where e is the public exponent.
3. Compute $M' = (C')^d \bmod n$ with the ordinary RSA implementation.
4. Compute $M = M' r^{-1} \bmod n$. In this equation, r^{-1} is the multiplicative inverse of $r \bmod n$; see Chapter 4 for a discussion of this concept. It can be demonstrated that this is the correct result by observing that $r^{ed} \bmod n = r \bmod n$.

RSA Data Security reports a 2 to 10% performance penalty for blinding.

Fault-Based Attack Still another unorthodox approach to attacking RSA is reported in [PELL10]. The approach is an attack on a processor that is generating RSA digital signatures. The attack induces faults in the signature computation by reducing the power to the processor. The faults cause the software to produce invalid signatures, which can then be analyzed by the attacker to recover the private key. The authors show how such an analysis can be done and then demonstrate it by extracting a 1024-bit private RSA key in approximately 100 hours, using a commercially available microprocessor.

The attack algorithm involves inducing single-bit errors and observing the results. The details are provided in [PELL10], which also references other proposed hardware fault-based attacks against RSA.

This attack, while worthy of consideration, does not appear to be a serious threat to RSA. It requires that the attacker have physical access to the target machine and that the attacker is able to directly control the input power to the processor. Controlling the input power would for most hardware require more than simply controlling the AC power, but would also involve the power supply control hardware on the chip.

CHOSEN CIPHERTEXT ATTACK AND OPTIMAL ASYMMETRIC ENCRYPTION PADDING The basic RSA algorithm is vulnerable to a **chosen ciphertext attack** (CCA). CCA is defined as an attack in which the adversary chooses a number of ciphertexts and is then given the corresponding plaintexts, decrypted with the target's private key. Thus, the adversary could select a plaintext, encrypt it with the target's public key, and then be able to get the plaintext back by having it decrypted with the private key. Clearly, this provides the adversary with no new information. Instead, the adversary exploits properties of RSA and selects blocks of data that, when processed using the target's private key, yield information needed for cryptanalysis.

A simple example of a CCA against RSA takes advantage of the following property of RSA:

$$E(PU, M_1) \times E(PU, M_2) = E(PU, [M_1 \times M_2]) \tag{9.2}$$

We can decrypt $C = M^e \bmod n$ using a CCA as follows.

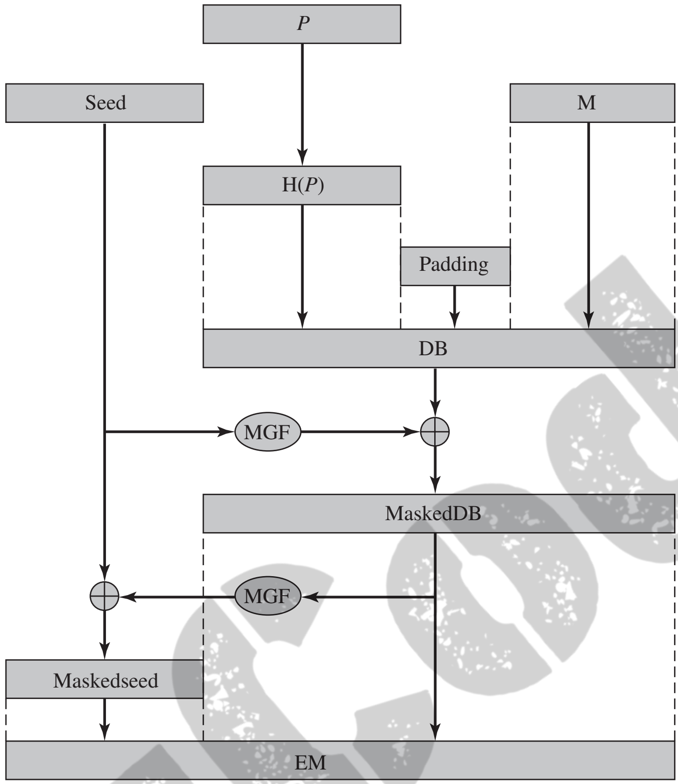
1. Compute $X = (C \times 2^e) \bmod n$.
2. Submit X as a chosen ciphertext and receive back $Y = X^d \bmod n$.

But now note that

$$\begin{aligned} X &= (C \bmod n) \times (2^e \bmod n) \\ &= (M^e \bmod n) \times (2^e \bmod n) \\ &= (2M)^e \bmod n \end{aligned}$$

Therefore, $Y = (2M) \bmod n$. From this, we can deduce M. To overcome this simple attack, practical RSA-based cryptosystems randomly pad the plaintext prior to encryption. This randomizes the ciphertext so that Equation (9.2) no longer holds. However, more sophisticated CCAs are possible, and a simple padding with a random value has been shown to be insufficient to provide the desired security. To counter such attacks, RSA Security Inc., a leading RSA vendor and former holder of the RSA patent, recommends modifying the plaintext using a procedure known as **optimal asymmetric encryption padding** (OAEP). A full discussion of the threats and OAEP are beyond our scope; see [POIN02] for an introduction and [BELL94] for a thorough analysis. Here, we simply summarize the OAEP procedure.

Figure 9.10 depicts OAEP encryption. As a first step, the message M to be encrypted is padded. A set of optional parameters, P, is passed through a hash function, H.[8] The output is then padded with zeros to get the desired length in the overall data block (DB). Next, a random seed is generated and passed through another hash function, called the mask generating function (MGF). The resulting hash value is bit-by-bit XORed with DB to produce a maskedDB. The maskedDB is in turn passed through the MGF to form a hash that is XORed with the seed to produce the maskedseed. The concatenation of the maskedseed and the maskedDB forms the encoded message EM. Note that the EM includes the padded message, masked by the seed, and the seed, masked by the maskedDB. The EM is then encrypted using RSA.



P = encoding parameters
M = message to be encoded
H = hash function
DB = data block
MGF = mask generating function
EM = encoded message

Figure 9.10 Encryption Using Optimal Asymmetric Encryption Padding (OAEP)

10.1 DIFFIE-HELLMAN KEY EXCHANGE

The first published public-key algorithm appeared in the seminal paper by Diffie and Hellman that defined public-key cryptography [DIFF76b] and is generally referred to as Diffie-Hellman key exchange.[1] A number of commercial products employ this key exchange technique.

The purpose of the algorithm is to enable two users to securely exchange a key that can then be used for subsequent symmetric encryption of messages. The algorithm itself is limited to the exchange of secret values.

The Diffie-Hellman algorithm depends for its effectiveness on the difficulty of computing discrete logarithms. Briefly, we can define the discrete logarithm in the following way. Recall from Chapter 8 that a primitive root of a prime number p is one whose powers modulo p generate all the integers from 1 to $p - 1$. That is, if a is a primitive root of the prime number p, then the numbers

$$a \bmod p,\ a^2 \bmod p, \ldots,\ a^{p-1} \bmod p$$

are distinct and consist of the integers from 1 through $p - 1$ in some permutation.

For any integer b and a primitive root a of prime number p, we can find a unique exponent i such that

$$b \equiv a^i \pmod{p} \qquad \text{where } 0 \leq i \leq (p - 1)$$

The exponent i is referred to as the **discrete logarithm** of b for the base a, mod p. We express this value as $\text{dlog}_{a,p}(b)$. See Chapter 8 for an extended discussion of discrete logarithms.

The Algorithm

Figure 10.1 summarizes the Diffie-Hellman key exchange algorithm. For this scheme, there are two publicly known numbers: a prime number q and an integer α that is a primitive root of q. Suppose the users A and B wish to create a shared key.

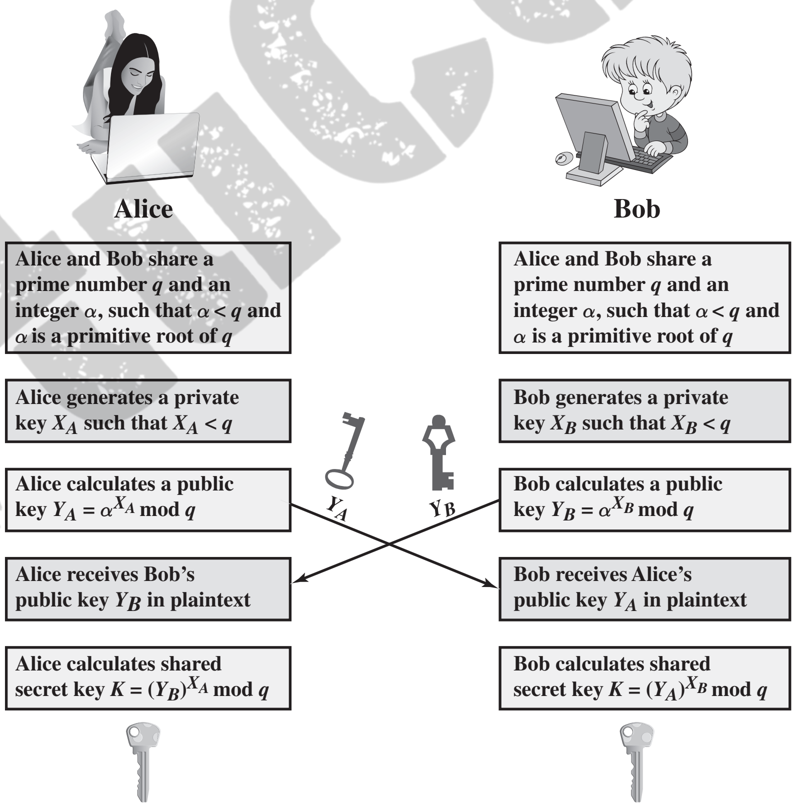


Figure 10.1 The Diffie-Hellman Key Exchange

User A selects a random integer $X_A < q$ and computes $Y_A = \alpha^{X_A} \bmod q$. Similarly, user B independently selects a random integer $X_B < q$ and computes $Y_B = \alpha^{X_B} \bmod q$. Each side keeps the X value private and makes the Y value available publicly to the other side. Thus, X_A is A's private key and Y_A is A's corresponding public key, and similarly for B. User A computes the key as $K = (Y_B)^{X_A} \bmod q$ and user B computes the key as $K = (Y_A)^{X_B} \bmod q$. These two calculations produce identical results:

$$\begin{aligned}
K &= (Y_B)^{X_A} \bmod q \\
&= (\alpha^{X_B} \bmod q)^{X_A} \bmod q \\
&= (\alpha^{X_B})^{X_A} \bmod q && \text{by the rules of modular arithmetic} \\
&= \alpha^{X_B X_A} \bmod q \\
&= (\alpha^{X_A})^{X_B} \bmod q \\
&= (\alpha^{X_A} \bmod q)^{X_B} \bmod q \\
&= (Y_A)^{X_B} \bmod q
\end{aligned}$$

The result is that the two sides have exchanged a secret value. Typically, this secret value is used as shared symmetric secret key. Now consider an adversary who can observe the key exchange and wishes to determine the secret key K. Because X_A and X_B are private, an adversary only has the following ingredients to work with: q, α, Y_A, and Y_B. Thus, the adversary is forced to take a discrete logarithm to determine the key. For example, to determine the private key of user B, an adversary must compute

$$X_B = \mathrm{dlog}_{\alpha,q}(Y_B)$$

The adversary can then calculate the key K in the same manner as user B calculates it. That is, the adversary can calculate K as

$$K = (Y_A)^{X_B} \bmod q$$

The security of the Diffie-Hellman key exchange lies in the fact that, while it is relatively easy to calculate exponentials modulo a prime, it is very difficult to calculate discrete logarithms. For large primes, the latter task is considered infeasible.

Here is an example. Key exchange is based on the use of the prime number $q = 353$ and a primitive root of 353, in this case $\alpha = 3$. A and B select private keys $X_A = 97$ and $X_B = 233$, respectively. Each computes its public key:

A computes $Y_A = 3^{97} \bmod 353 = 40$.
B computes $Y_B = 3^{233} \bmod 353 = 248$.

After they exchange public keys, each can compute the common secret key:

A computes $K = (Y_B)^{X_A} \bmod 353 = 248^{97} \bmod 353 = 160$.
B computes $K = (Y_A)^{X_B} \bmod 353 = 40^{233} \bmod 353 = 160$.

We assume an attacker would have available the following information:

$$q = 353;\ \alpha = 3;\ Y_A = 40;\ Y_B = 248$$

In this simple example, it would be possible by brute force to determine the secret key 160. In particular, an attacker E can determine the common key by discovering a solution to the equation $3^a \bmod 353 = 40$ or the equation $3^b \bmod 353 = 248$. The brute-force approach is to calculate powers of 3 modulo 353, stopping when the result equals either 40 or 248. The desired answer is reached with the exponent value of 97, which provides $3^{97} \bmod 353 = 40$.

With larger numbers, the problem becomes impractical.

Key Exchange Protocols

Figure 10.1 shows a simple protocol that makes use of the Diffie-Hellman calculation. Suppose that user A wishes to set up a connection with user B and use a secret key to encrypt messages on that connection. User A can generate a one-time private key X_A, calculate Y_A, and send that to user B. User B responds by generating a private value X_B, calculating Y_B, and sending Y_B to user A. Both users can now calculate the key. The necessary public values q and α would need to be known ahead of time. Alternatively, user A could pick values for q and α and include those in the first message.

As an example of another use of the Diffie-Hellman algorithm, suppose that a group of users (e.g., all users on a LAN) each generate a long-lasting private value X_i (for user i) and calculate a public value Y_i. These public values, together with global public values for q and α, are stored in some central directory. At any time, user j can access user i's public value, calculate a secret key, and use that to send an encrypted message to user A. If the central directory is trusted, then this form of communication provides both confidentiality and a degree of authentication. Because only i and j can determine the key, no other user can read the message (confidentiality). Recipient i knows that only user j could have created a message using this key (authentication). However, the technique does not protect against replay attacks.

Man-in-the-Middle Attack

The protocol depicted in Figure 10.1 is insecure against a man-in-the-middle attack. Suppose Alice and Bob wish to exchange keys, and Darth is the adversary. The attack proceeds as follows (Figure 10.2).

1. Darth prepares for the attack by generating two random private keys X_{D1} and X_{D2} and then computing the corresponding public keys Y_{D1} and Y_{D2}.
2. Alice transmits Y_A to Bob.
3. Darth intercepts Y_A and transmits Y_{D1} to Bob. Darth also calculates $K2 = (Y_A)^{X_{D2}} \bmod q$.
4. Bob receives Y_{D1} and calculates $K1 = (Y_{D1})^{X_B} \bmod q$.
5. Bob transmits Y_B to Alice.
6. Darth intercepts Y_B and transmits Y_{D2} to Alice. Darth calculates $K1 = (Y_B)^{X_{D1}} \bmod q$.
7. Alice receives Y_{D2} and calculates $K2 = (Y_{D2})^{X_A} \bmod q$.

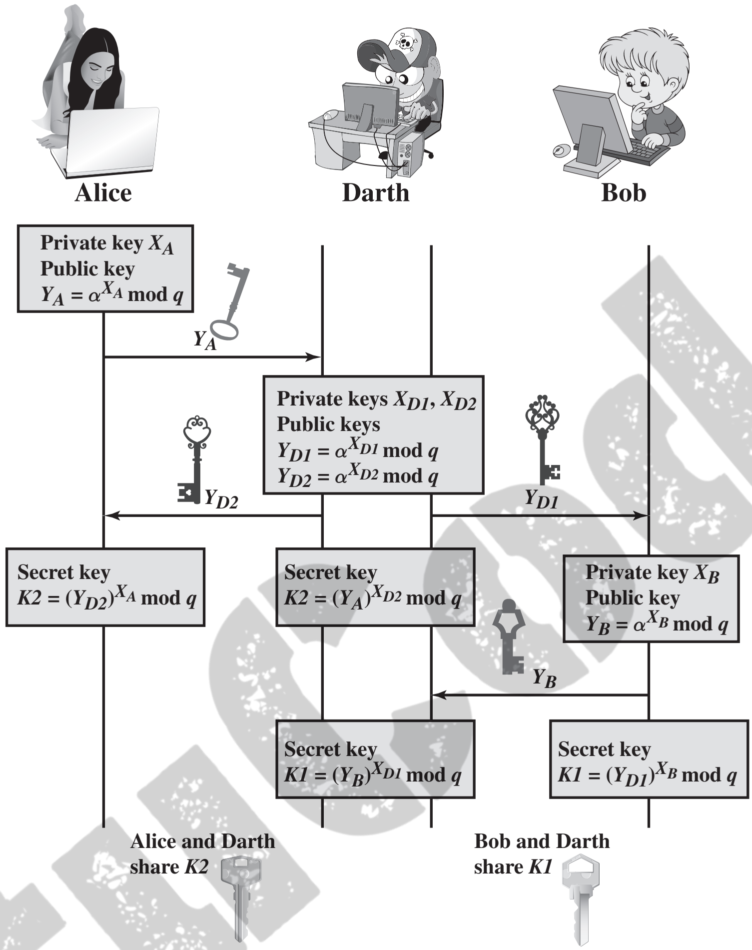


Figure 10.2 Man-in-the-Middle Attack

At this point, Bob and Alice think that they share a secret key, but instead Bob and Darth share secret key $K1$ and Alice and Darth share secret key $K2$. All future communication between Bob and Alice is compromised in the following way.

1. Alice sends an encrypted message M: $E(K2, M)$.
2. Darth intercepts the encrypted message and decrypts it to recover M.
3. Darth sends Bob $E(K1, M)$ or $E(K1, M')$, where M' is any message. In the first case, Darth simply wants to eavesdrop on the communication without altering it. In the second case, Darth wants to modify the message going to Bob.

The key exchange protocol is vulnerable to such an attack because it does not authenticate the participants. This vulnerability can be overcome with the use of digital signatures and public-key certificates; these topics are explored in Chapters 13 and 14.

10.2 ELGAMAL CRYPTOGRAPHIC SYSTEM

In 1984, T. Elgamal announced a public-key scheme based on discrete logarithms, closely related to the Diffie-Hellman technique [ELGA84, ELGA85]. The Elgamal[2] cryptosystem is used in some form in a number of standards including the digital signature standard (DSS), which is covered in Chapter 13, and the S/MIME e-mail standard (Chapter 19).

As with Diffie-Hellman, the global elements of Elgamal are a prime number q and α, which is a primitive root of q. User A generates a private/public key pair as follows:

1. Generate a random integer X_A, such that $1 < X_A < q - 1$.
2. Compute $Y^A = \alpha^{X_A} \bmod q$.
3. A's private key is X_A and A's public key is $\{q, \alpha, Y_A\}$.

Any user B that has access to A's public key can encrypt a message as follows:

1. Represent the message as an integer M in the range $0 \le M \le q - 1$. Longer messages are sent as a sequence of blocks, with each block being an integer less than q.
2. Choose a random integer k such that $1 \le k \le q - 1$.
3. Compute a one-time key $K = (Y_A)^k \bmod q$.
4. Encrypt M as the pair of integers (C_1, C_2) where

$$C_1 = \alpha^k \bmod q; C_2 = KM \bmod q$$

User A recovers the plaintext as follows:

1. Recover the key by computing $K = (C_1)^{X_A} \bmod q$.
2. Compute $M = (C_2 K^{-1}) \bmod q$.

These steps are summarized in Figure 10.3. It corresponds to Figure 9.1a: Alice generates a public/private key pair; Bob encrypts using Alice's public key; and Alice decrypts using her private key.

Let us demonstrate why the Elgamal scheme works. First, we show how K is recovered by the decryption process:

$$\begin{aligned}
K &= (Y_A)^k \bmod q && K \text{ is defined during the encryption process} \\
K &= (\alpha^{X_A} \bmod q)^k \bmod q && \text{substitute using } Y_A = \alpha^{X_A} \bmod q \\
K &= \alpha^{kX_A} \bmod q && \text{by the rules of modular arithmetic} \\
K &= (C_1)^{X_A} \bmod q && \text{substitute using } C_1 = \alpha^k \bmod q
\end{aligned}$$

Next, using K, we recover the plaintext as

$$\begin{aligned}
&C_2 = KM \bmod q \\
&(C_2 K^{-1}) \bmod q = KMK^{-1} \bmod q = M \bmod q = M
\end{aligned}$$

Global Public Elements	
q	prime number
α	$\alpha < q$ and α a primitive root of q

Key Generation by Alice	
Select private X_A	$X_A < q - 1$
Calculate Y_A	$Y_A = \alpha^{X_A} \bmod q$
Public key	$\{q, \alpha, Y_A\}$
Private key	X_A

Encryption by Bob with Alice's Public Key	
Plaintext:	$M < q$
Select random integer k	$k < q$
Calculate K	$K = (Y_A)^k \bmod q$
Calculate C_1	$C_1 = \alpha^k \bmod q$
Calculate C_2	$C_2 = KM \bmod q$
Ciphertext:	(C_1, C_2)

Decryption by Alice with Alice's Private Key	
Ciphertext:	(C_1, C_2)
Calculate K	$K = (C_1)^{X_A} \bmod q$
Plaintext:	$M = (C_2 K^{-1}) \bmod q$

Figure 10.3 The Elgamal Cryptosystem

We can restate the Elgamal process as follows, using Figure 10.3.

1. Bob generates a random integer k.
2. Bob generates a one-time key K using Alice's public-key components Y_A, q, and k.
3. Bob encrypts k using the public-key component α, yielding C_1. C_1 provides sufficient information for Alice to recover K.
4. Bob encrypts the plaintext message M using K.
5. Alice recovers K from C_1 using her private key.
6. Alice uses K^{-1} to recover the plaintext message from C_2.

Thus, K functions as a one-time key, used to encrypt and decrypt the message.

For example, let us start with the prime field GF(19); that is, $q = 19$. It has primitive roots {2, 3, 10, 13, 14, 15}, as shown in Table 8.3. We choose $\alpha = 10$.

Alice generates a key pair as follows:

1. Alice chooses $X_A = 5$.
2. Then $Y_A = \alpha^{X_A} \bmod q = \alpha^5 \bmod 19 = 3$ (see Table 8.3).
3. Alice's private key is 5 and Alice's public key is $\{q, \alpha, Y_A\} = \{19, 10, 3\}$.

Suppose Bob wants to send the message with the value $M = 17$. Then:

1. Bob chooses $k = 6$.
2. Then $K = (Y_A)^k \bmod q = 3^6 \bmod 19 = 729 \bmod 19 = 7$.
3. So
   $C_1 = \alpha^k \bmod q = \alpha^6 \bmod 19 = 11$
   $C_2 = \text{KM} \bmod q = 7 \times 17 \bmod 19 = 119 \bmod 19 = 5$
4. Bob sends the ciphertext (11, 5).

For decryption:

1. Alice calculates $K = (C_1)^{X_A} \bmod q = 11^5 \bmod 19 = 161051 \bmod 19 = 7$.
2. Then K^{-1} in GF(19) is $7^{-1} \bmod 19 = 11$.
3. Finally, $M = (C_2 K^{-1}) \bmod q = 5 \times 11 \bmod 19 = 55 \bmod 19 = 17$.

If a message must be broken up into blocks and sent as a sequence of encrypted blocks, a unique value of k should be used for each block. If k is used for more than one block, knowledge of one block M_1 of the message enables the user to compute other blocks as follows. Let

$$C_{1,1} = \alpha^k \bmod q;\ C_{2,1} = KM_1 \bmod q$$
$$C_{1,2} = \alpha^k \bmod q;\ C_{2,2} = KM_2 \bmod q$$

Then,

$$\frac{C_{2,1}}{C_{2,2}} = \frac{KM_1 \bmod q}{KM_2 \bmod q} = \frac{M_1 \bmod q}{M_2 \bmod q}$$

If M_1 is known, then M_2 is easily computed as

$$M_2 = (C_{2,1})^{-1}\, C_{2,2}\, M_1 \bmod q$$

The security of Elgamal is based on the difficulty of computing discrete logarithms. To recover A's private key, an adversary would have to compute $X_A = \text{dlog}_{\alpha,q}(Y_A)$. Alternatively, to recover the one-time key K, an adversary would have to determine the random number k, and this would require computing the discrete logarithm $k = \text{dlog}_{\alpha,q}(C_1)$. [STIN06] points out that these calculations are regarded as infeasible if p is at least 300 decimal digits and $q - 1$ has at least one "large" prime factor.

10.3 ELLIPTIC CURVE ARITHMETIC

Most of the products and standards that use public-key cryptography for encryption and digital signatures use RSA. As we have seen, the key length for secure RSA use has increased over recent years, and this has put a heavier processing load on applications using RSA. This burden has ramifications, especially for electronic commerce sites that conduct large numbers of secure transactions. A competing system challenges RSA: elliptic curve cryptography (ECC). ECC is showing up in standardization efforts, including the IEEE P1363 Standard for Public-Key Cryptography.

The principal attraction of ECC, compared to RSA, is that it appears to offer equal security for a far smaller key size, thereby reducing processing overhead. On the other hand, although the theory of ECC has been around for some time, it is only recently that products have begun to appear and that there has been sustained cryptanalytic interest in probing for weaknesses. Accordingly, the confidence level in ECC is not yet as high as that in RSA.

ECC is fundamentally more difficult to explain than either RSA or Diffie-Hellman, and a full mathematical description is beyond the scope of this book. This section and the next give some background on elliptic curves and ECC. We begin with a brief review of the concept of abelian group. Next, we examine the concept of elliptic curves defined over the real numbers. This is followed by a look at elliptic curves defined over finite fields. Finally, we are able to examine elliptic curve ciphers.

The reader may wish to review the material on finite fields in Chapter 4 before proceeding.

Abelian Groups

Recall from Chapter 4 that an **abelian group** G, sometimes denoted by $\{G, \cdot\}$, is a set of elements with a binary operation, denoted by $\cdot$, that associates to each ordered pair (a, b) of elements in G an element $(a \cdot b)$ in G, such that the following axioms are obeyed:[3]

(A1) Closure:	If a and b belong to G, then $a \cdot b$ is also in G.
(A2) Associative:	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c in G.
(A3) Identity element:	There is an element e in G such that $a \cdot \mathrm{e} = \mathrm{e} \cdot a = a$ for all a in G.
(A4) Inverse element:	For each a in G there is an element a' in G such that $a \cdot a' = a' \cdot a = \mathrm{e}$.
(A5) Commutative:	$a \cdot b = b \cdot a$ for all a, b in G.

A number of public-key ciphers are based on the use of an abelian group. For example, Diffie-Hellman key exchange involves multiplying pairs of nonzero integers modulo a prime number q. Keys are generated by exponentiation over the group, with exponentiation defined as repeated multiplication. For example, $a^k \bmod q = \underbrace{(a \times a \times \ldots \times a)}_{k \text{ times}} \bmod q$. To attack Diffie-Hellman, the attacker must determine k given a and a^k; this is the discrete logarithm problem.

For elliptic curve cryptography, an operation over elliptic curves, called addition, is used. Multiplication is defined by repeated addition. For example,

$$a \times k = \underbrace{(a + a + \ldots + a)}_{k \text{ times}}$$

where the addition is performed over an elliptic curve. Cryptanalysis involves determining k given a and $(a \times k)$.

An **elliptic curve** is defined by an equation in two variables with coefficients. For cryptography, the variables and coefficients are restricted to elements in a finite field, which results in the definition of a finite abelian group. Before looking at this, we first look at elliptic curves in which the variables and coefficients are real numbers. This case is perhaps easier to visualize.

Elliptic Curves over Real Numbers

Elliptic curves are not ellipses. They are so named because they are described by cubic equations, similar to those used for calculating the circumference of an ellipse. In general, cubic equations for elliptic curves take the following form, known as a **Weierstrass equation:**

$$y^2 + axy + by = x^3 + cx^2 + dx + e$$

where a, b, c, d, e are real numbers and x and y take on values in the real numbers.[4] For our purpose, it is sufficient to limit ourselves to equations of the form

$$y^2 = x^3 + ax + b \tag{10.1}$$

Such equations are said to be cubic, or of degree 3, because the highest exponent they contain is a 3. Also included in the definition of an elliptic curve is a single element denoted O and called the *point at infinity* or the *zero point*, which we discuss subsequently. To plot such a curve, we need to compute

$$y = \sqrt{x^3 + ax + b}$$

For given values of a and b, the plot consists of positive and negative values of y for each value of x. Thus, each curve is symmetric about $y = 0$. Figure 10.4 shows two examples of elliptic curves. As you can see, the formula sometimes produces weird-looking curves.

Now, consider the set of points $\mathrm{E}(a, b)$ consisting of all of the points (x, y) that satisfy Equation (10.1) together with the element O. Using a different value of the pair (a, b) results in a different set $\mathrm{E}(a, b)$. Using this terminology, the two curves in Figure 10.4 depict the sets $\mathrm{E}(-1, 0)$ and $\mathrm{E}(1, 1)$, respectively.

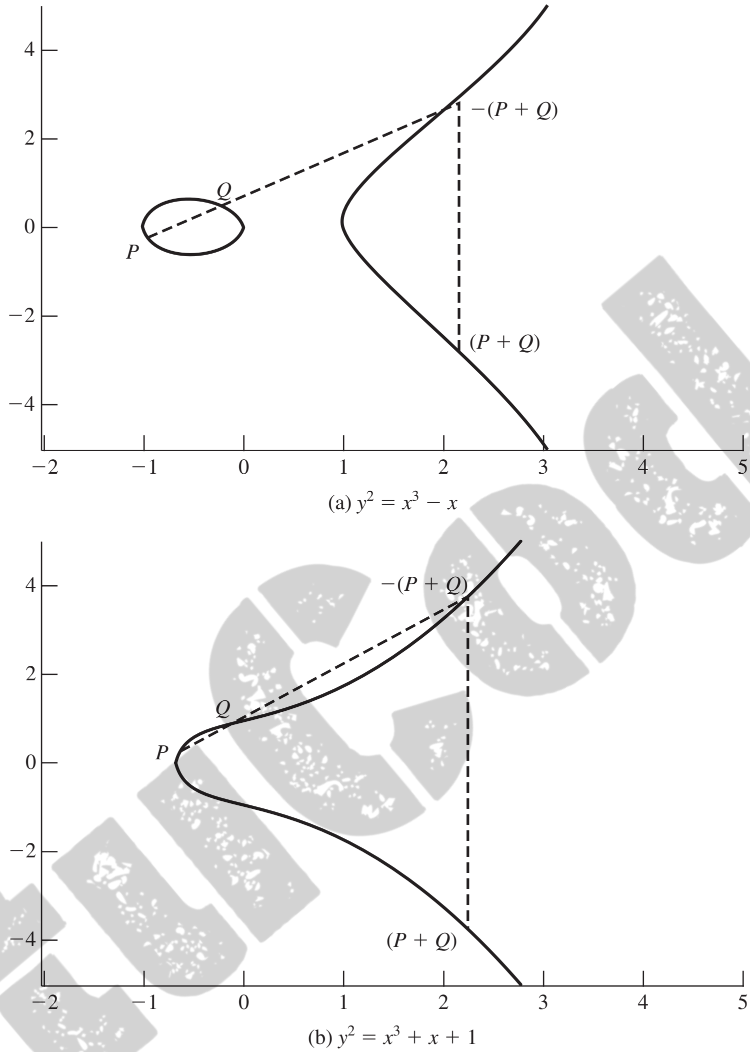


Figure 10.4 Example of Elliptic Curves

Geometric Description of Addition It can be shown that a group can be defined based on the set E(a, b) for specific values of a and b in Equation (10.1), provided the following condition is met:

$$4a^3 + 27b^2 \neq 0 \tag{10.2}$$

To define the group, we must define an operation, called addition and denoted by +, for the set E(a, b), where a and b satisfy Equation (10.2). In geometric terms, the rules for addition can be stated as follows: If three points on an elliptic curve lie on a straight line, their sum is O. From this definition, we can define the rules of addition over an elliptic curve.

1. O serves as the additive identity. Thus $O = -O$; for any point P on the elliptic curve, $P + O = P$. In what follows, we assume $P \neq O$ and $Q \neq O$.
2. The negative of a point P is the point with the same x coordinate but the negative of the y coordinate; that is, if $P = (x, y)$, then $-P = (x, -y)$. Note that these two points can be joined by a vertical line. Note that $P + (-P) = P - P = O$.
3. To add two points P and Q with different x coordinates, draw a straight line between them and find the third point of intersection R. It is easily seen that there is a unique point R that is the point of intersection (unless the line is tangent to the curve at either P or Q, in which case we take $R = P$ or $R = Q$, respectively). To form a group structure, we need to define addition on these three points: $P + Q = -R$. That is, we define $P + Q$ to be the mirror image (with respect to the x axis) of the third point of intersection. Figure 10.4 illustrates this construction.
4. The geometric interpretation of the preceding item also applies to two points, P and $-P$, with the same x coordinate. The points are joined by a vertical line, which can be viewed as also intersecting the curve at the infinity point. We therefore have $P + (-P) = O$, which is consistent with item (2).
5. To double a point Q, draw the tangent line and find the other point of intersection S. Then $Q + Q = 2Q = -S$.

With the preceding list of rules, it can be shown that the set $E(a, b)$ is an abelian group.

Algebraic Description of Addition In this subsection, we present some results that enable calculation of additions over elliptic curves.[5] For two distinct points, $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, that are not negatives of each other, the slope of the line l that joins them is $\Delta = (y_Q - y_P)/(x_Q - x_P)$. There is exactly one other point where l intersects the elliptic curve, and that is the negative of the sum of P and Q. After some algebraic manipulation, we can express the sum $R = P + Q$ as

$$\begin{aligned} x_R &= \Delta^2 - x_P - x_Q \\ y_R &= -y_P + \Delta(x_P - x_R) \end{aligned} \tag{10.3}$$

We also need to be able to add a point to itself: $P + P = 2P = R$. When $y_P \neq 0$, the expressions are

$$\begin{aligned} x_R &= \left(\frac{3x_P^2 + a}{2y_P}\right)^2 - 2x_P \\ y_R &= \left(\frac{3x_P^2 + a}{2y_P}\right)(x_P - x_R) - y_P \end{aligned} \tag{10.4}$$

Elliptic Curves over Z_p

Elliptic curve cryptography makes use of elliptic curves in which the variables and coefficients are all restricted to elements of a finite field. Two families of elliptic curves are used in cryptographic applications: prime curves over Z_p and binary curves over GF(2^m). For a **prime curve** over Z_p, we use a cubic equation in which the variables and coefficients all take on values in the set of integers from 0 through $p - 1$ and in which calculations are performed modulo p. For a **binary curve** defined over GF(2^m), the variables and coefficients all take on values in GF(2^m) and in calculations are performed over GF(2^m). [FERN99] points out that prime curves are best for software applications, because the extended bit-fiddling operations needed by binary curves are not required; and that binary curves are best for hardware applications, where it takes remarkably few logic gates to create a powerful, fast cryptosystem. We examine these two families in this section and the next.

There is no obvious geometric interpretation of elliptic curve arithmetic over finite fields. The algebraic interpretation used for elliptic curve arithmetic over real numbers does readily carry over, and this is the approach we take.

For elliptic curves over Z_p, as with real numbers, we limit ourselves to equations of the form of Equation (10.1), but in this case with coefficients and variables limited to Z_p:

$$y^2 \bmod p = (x^3 + ax + b) \bmod p \tag{10.5}$$

For example, Equation (10.5) is satisfied for $a = 1, b = 1, x = 9, y = 7, p = 23$:

$$\begin{aligned} 7^2 \bmod 23 &= (9^3 + 9 + 1) \bmod 23 \\ 49 \bmod 23 &= 739 \bmod 23 \\ 3 &= 3 \end{aligned}$$

Now consider the set $E_p(a, b)$ consisting of all pairs of integers (x, y) that satisfy Equation (10.5), together with a point at infinity O. The coefficients a and b and the variables x and y are all elements of Z_p.

For example, let $p = 23$ and consider the elliptic curve $y^2 = x^3 + x + 1$. In this case, $a = b = 1$. Note that this equation is the same as that of Figure 10.4b. The figure shows a continuous curve with all of the real points that satisfy the equation. For the set $E_{23}(1, 1)$, we are only interested in the nonnegative integers in the quadrant from (0, 0) through $(p - 1, p - 1)$ that satisfy the equation mod p. Table 10.1 lists the points (other than O) that are part of $E_{23}(1, 1)$. Figure 10.5 plots the points of $E_{23}(1, 1)$; note that the points, with one exception, are symmetric about $y = 11.5$.

Table 10.1 Points (other than O) on the Elliptic Curve E_{23} (1,1)

(0, 1)	(6, 4)	(12, 19)
(0, 22)	(6, 19)	(13, 7)
(1, 7)	(7, 11)	(13, 16)
(1, 16)	(7, 12)	(17, 3)
(3, 10)	(9, 7)	(17, 20)
(3, 13)	(9, 16)	(18, 3)
(4, 0)	(11, 3)	(18, 20)
(5, 4)	(11, 20)	(19, 5)
(5, 19)	(12, 4)	(19, 18)

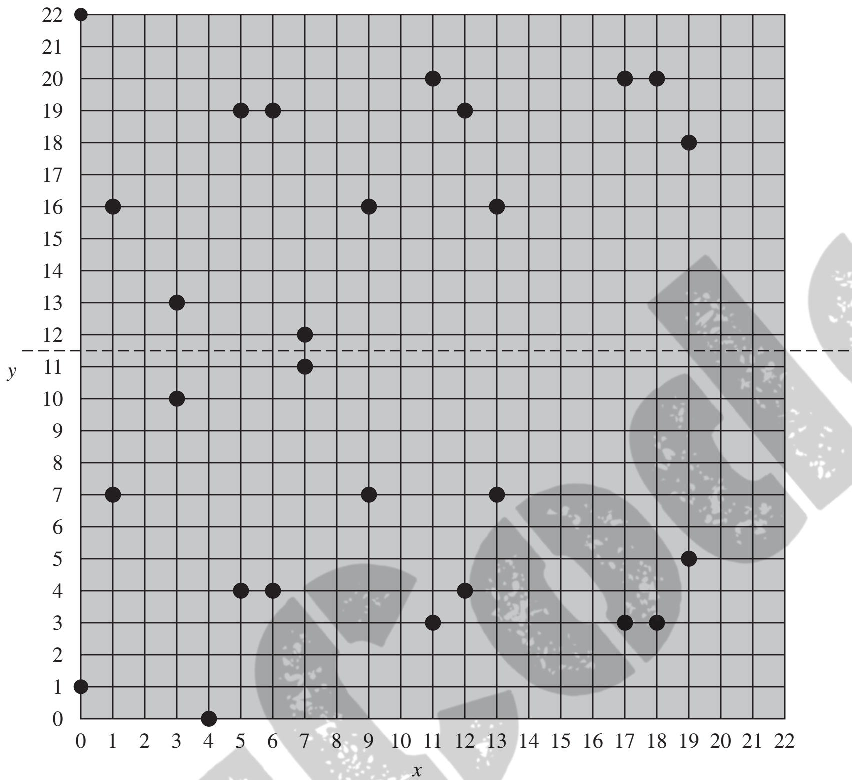


Figure 10.5 The Elliptic Curve $E_{23}(1, 1)$

It can be shown that a finite abelian group can be defined based on the set $E_p(a, b)$ provided that $(x^3 + ax + b) \bmod p$ has no repeated factors. This is equivalent to the condition

$$(4a^3 + 27b^2) \bmod p \neq 0 \bmod p \qquad \textbf{(10.6)}$$

Note that Equation (10.6) has the same form as Equation (10.2).

The rules for addition over $E_p(a, b)$, correspond to the algebraic technique described for elliptic curves defined over real numbers. For all points $P, Q \in E_p(a, b)$:

1. $P + O = P$.
2. If $P = (x_P, y_P)$, then $P + (x_P, -y_P) = O$. The point $(x_P, -y_P)$ is the negative of P, denoted as $-P$. For example, in $E_{23}(1, 1)$, for $P = (13, 7)$, we have $-P = (13, -7)$. But $-7 \bmod 23 = 16$. Therefore, $-P = (13, 16)$, which is also in $E_{23}(1, 1)$.
3. If $P = (x_p, y_p)$ and $Q = (x_Q, y_Q)$ with $P \neq -Q$, then $R = P + Q = (x_R, y_R)$ is determined by the following rules:

$$x_R = (\lambda^2 - x_P - x_Q) \bmod p$$
$$y_R = (\lambda(x_P - x_R) - y_P) \bmod p$$

where

$$\lambda = \begin{cases} \left(\dfrac{y_Q - y_P}{x_Q - x_P}\right) \bmod p & \text{if } P \neq Q \\ \left(\dfrac{3x_P^2 + a}{2y_P}\right) \bmod p & \text{if } P = Q \end{cases}$$

4. Multiplication is defined as repeated addition; for example, $4P = P + P + P + P$.

For example, let $P = (3, 10)$ and $Q = (9, 7)$ in $E_{23}(1, 1)$. Then

$$\lambda = \left(\frac{7 - 10}{9 - 3}\right) \bmod 23 = \left(\frac{-3}{6}\right) \bmod 23 = \left(\frac{-1}{2}\right) \bmod 23 = 11$$

$$x_R = (11^2 - 3 - 9) \bmod 23 = 109 \bmod 23 = 17$$

$$y_R = (11(3 - 17) - 10) \bmod 23 = -164 \bmod 23 = 20$$

So $P + Q = (17, 20)$. To find $2P$,

$$\lambda = \left(\frac{3(3^2) + 1}{2 \times 10}\right) \bmod 23 = \left(\frac{5}{20}\right) \bmod 23 = \left(\frac{1}{4}\right) \bmod 23 = 6$$

The last step in the preceding equation involves taking the multiplicative inverse of 4 in Z_{23}. This can be done using the extended Euclidean algorithm defined in Section 4.4. To confirm, note that $(6 \times 4) \bmod 23 = 24 \bmod 23 = 1$.

$$x_R = (6^2 - 3 - 3) \bmod 23 = 30 \bmod 23 = 7$$

$$y_R = (6(3 - 7) - 10) \bmod 23 = (-34) \bmod 23 = 12$$

and $2P = (7, 12)$.

For determining the security of various elliptic curve ciphers, it is of some interest to know the number of points in a finite abelian group defined over an elliptic curve. In the case of the finite group $E_P(a, b)$, the number of points N is bounded by

$$p + 1 - 2\sqrt{p} \leq N \leq p + 1 + 2\sqrt{p}$$

Note that the number of points in $E_p(a, b)$ is approximately equal to the number of elements in Z_p, namely p elements.

Elliptic Curves over GF(2^m)

Recall from Chapter 4 that a **finite field** GF(2^m) consists of 2^m elements, together with addition and multiplication operations that can be defined over polynomials. For elliptic curves over GF(2^m), we use a cubic equation in which the variables and coefficients all take on values in GF(2^m) for some number m and in which calculations are performed using the rules of arithmetic in GF(2^m).

It turns out that the form of cubic equation appropriate for cryptographic applications for elliptic curves is somewhat different for GF(2^m) than for Z_p. The form is

$$y^2 + xy = x^3 + ax^2 + b \qquad \textbf{(10.7)}$$

Table 10.2 Points (other than O) on the Elliptic Curve $E_{2^4}(g^4, 1)$

$(0, 1)$	(g^5, g^3)	(g^9, g^{13})
$(1, g^6)$	(g^5, g^{11})	(g^{10}, g)
$(1, g^{13})$	(g^6, g^8)	(g^{10}, g^8)
(g^3, g^8)	(g^6, g^{14})	$(g^{12}, 0)$
(g^3, g^{13})	(g^9, g^{10})	(g^{12}, g^{12})

where it is understood that the variables x and y and the coefficients a and b are elements of GF(2^m) and that calculations are performed in GF(2^m).

Now consider the set $E_{2^m}(a, b)$ consisting of all pairs of integers (x, y) that satisfy Equation (10.7), together with a point at infinity O.

For example, let us use the finite field GF(2^4) with the irreducible polynomial $f(x) = x^4 + x + 1$. This yields a generator g that satisfies $f(g) = 0$ with a value of $g^4 = g + 1$, or in binary, $g = 0010$. We can develop the powers of g as follows.

$g^0 = 0001$	$g^4 = 0011$	$g^8 = 0101$	$g^{12} = 1111$
$g^1 = 0010$	$g^5 = 0110$	$g^9 = 1010$	$g^{13} = 1101$
$g^2 = 0100$	$g^6 = 1100$	$g^{10} = 0111$	$g^{14} = 1001$
$g^3 = 1000$	$g^7 = 1011$	$g^{11} = 1110$	$g^{15} = 0001$

For example, $g^5 = (g^4)(g) = (g + 1)(g) = g^2 + g = 0110$.

Now consider the elliptic curve $y^2 + xy = x^3 + g^4x^2 + 1$. In this case, $a = g^4$ and $b = g^0 = 1$. One point that satisfies this equation is (g^5, g^3):

$$\begin{aligned}
&(g^3)^2 + (g^5)(g^3) = (g^5)^3 + (g^4)(g^5)^2 + 1\\
&g^6 + g^8 = g^{15} + g^{14} + 1\\
&1100 + 0101 = 0001 + 1001 + 0001\\
&1001 = 1001
\end{aligned}$$

Table 10.2 lists the points (other than O) that are part of $E_{2^4}(g^4, 1)$. Figure 10.6 plots the points of $E_{2^4}(g^4, 1)$.

It can be shown that a finite abelian group can be defined based on the set $E_{2^m}(a, b)$, provided that $b \neq 0$. The rules for addition can be stated as follows. For all points $P, Q \in E_{2^m}(a, b)$:

1. $P + O = P$.
2. If $P = (x_P, y_P)$, then $P + (x_P, x_P + y_P) = O$. The point $(x_P, x_P + y_P)$ is the negative of P, which is denoted as $-P$.
3. If $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ with $P \neq -Q$ and $P \neq Q$, then $R = P + Q = (x_R, y_R)$ is determined by the following rules:

$$\begin{aligned}
x_R &= \lambda^2 + \lambda + x_P + x_Q + a\\
y_R &= \lambda(x_P + x_R) + x_R + y_P
\end{aligned}$$

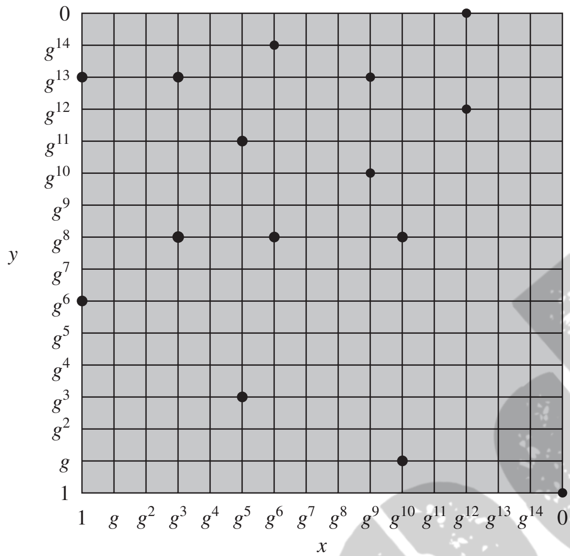


Figure 10.6 The Elliptic Curve $E_{2^4}(g^4, 1)$

where

$$\lambda = \frac{y_Q + y_P}{x_Q + x_P}$$

4. If $P = (x_P, y_P)$ then $R = 2P = (x_R, y_R)$ is determined by the following rules:

$$x_R = \lambda^2 + \lambda + a$$
$$y_R = x_P^2 + (\lambda + 1)x_R$$

where

$$\lambda = x_P + \frac{y_P}{x_P}$$

10.4 ELLIPTIC CURVE CRYPTOGRAPHY

The addition operation in ECC is the counterpart of modular multiplication in RSA, and multiple addition is the counterpart of modular exponentiation. To form a cryptographic system using elliptic curves, we need to find a "hard problem" corresponding to factoring the product of two primes or taking the discrete logarithm.

Consider the equation $Q = kP$ where $Q, P \in E_P(a, b)$ and $k < p$. It is relatively easy to calculate Q given k and P, but it is hard to determine k given Q and P. This is called the discrete logarithm problem for elliptic curves.

We give an example taken from the Certicom Web site (www.certicom.com). Consider the group $E_{23}(9,17)$. This is the group defined by the equation $y^2 \bmod 23 = (x^3 + 9x + 17) \bmod 23$. What is the discrete logarithm k of $Q = (4, 5)$ to the base $P = (16, 5)$? The brute-force method is to compute multiples of P until

Q is found. Thus,

$$P = (16, 5); 2P = (20, 20); 3P = (14, 14); 4P = (19, 20); 5P = (13, 10);$$
$$6P = (7, 3); 7P = (8, 7); 8P = (12, 17); 9P = (4, 5)$$

Because $9P = (4, 5) = Q$, the discrete logarithm $Q = (4, 5)$ to the base $P = (16, 5)$ is $k = 9$. In a real application, k would be so large as to make the brute-force approach infeasible.

In the remainder of this section, we show two approaches to ECC that give the flavor of this technique.

Analog of Diffie-Hellman Key Exchange

Key exchange using elliptic curves can be done in the following manner. First pick a large integer q, which is either a prime number p or an integer of the form 2^m, and elliptic curve parameters a and b for Equation (10.5) or Equation (10.7). This defines the elliptic group of points $E_q(a, b)$. Next, pick a *base point* $G = (x_1, y_1)$ in $E_p(a, b)$ whose order is a very large value n. The **order** n of a point G on an elliptic curve is the smallest positive integer n such that $nG = 0$ and G are parameters of the cryptosystem known to all participants.

A key exchange between users A and B can be accomplished as follows (Figure 10.7).

1. A selects an integer n_A less than n. This is A's private key. A then generates a public key $P_A = n_A \times G$; the public key is a point in $E_q(a, b)$.
2. B similarly selects a private key n_B and computes a public key P_B.
3. A generates the secret key $k = n_A \times P_B$. B generates the secret key $k = n_B \times P_A$.

The two calculations in step 3 produce the same result because

$$n_{\mathrm{A}} \times P_B = n_A \times (n_B \times G) = n_B \times (n_A \times G) = n_B \times P_A$$

To break this scheme, an attacker would need to be able to compute k given G and kG, which is assumed to be hard.

As an example,[6] take $p = 211$; $\mathrm{E}_p(0,-4)$, which is equivalent to the curve $y^2 = x^3 - 4$; and $G = (2, 2)$. One can calculate that $240G = O$. A's private key is $n_A = 121$, so A's public key is $P_A = 121(2, 2) = (115, 48)$. B's private key is $n_B = 203$, so B's public key is $203(2, 3) = (130, 203)$. The shared secret key is $121(130, 203) = 203(115, 48) = (161, 69)$.

Note that the secret key is a pair of numbers. If this key is to be used as a session key for conventional encryption, then a single number must be generated. We could simply use the x coordinates or some simple function of the x coordinate.

Elliptic Curve Encryption/Decryption

Several approaches to encryption/decryption using elliptic curves have been analyzed in the literature. In this subsection, we look at perhaps the simplest. The first task in this system is to encode the plaintext message m to be sent as an (x, y) point P_m.

Global Public Elements

$E_q(a, b)$	elliptic curve with parameters a, b, and q, where q is a prime or an integer of the form 2^m
G	point on elliptic curve whose order is large value n

User A Key Generation

Select private n_A	$n_A < n$
Calculate public P_A	$P_A = n_A \times G$

User B Key Generation

Select private n_B	$n_B < n$
Calculate public P_B	$P_B = n_B \times G$

Calculation of Secret Key by User A

$K = n_A \times P_B$

Calculation of Secret Key by User B

$K = n_B \times P_A$

Figure 10.7 ECC Diffie-Hellman Key Exchange

It is the point P_m that will be encrypted as a ciphertext and subsequently decrypted. Note that we cannot simply encode the message as the x or y coordinate of a point, because not all such coordinates are in $E_q(a, b)$; for example, see Table 10.1. Again, there are several approaches to this encoding, which we will not address here, but suffice it to say that there are relatively straightforward techniques that can be used.

As with the key exchange system, an encryption/decryption system requires a point G and an elliptic group $E_q(a, b)$ as parameters. Each user A selects a private key n_A and generates a public key $P_A = n_A \times G$.

To encrypt and send a message P_m to B, A chooses a random positive integer k and produces the ciphertext C_m consisting of the pair of points:

$$C_m = \{kG, P_m + kP_B\}$$

Note that A has used B's public key P_B. To decrypt the ciphertext, B multiplies the first point in the pair by B's private key and subtracts the result from the second point:

$$P_m + kP_B - n_B(kG) = P_m + k(n_BG) - n_B(kG) = P_m$$

A has masked the message P_m by adding kP_B to it. Nobody but A knows the value of k, so even though P_b is a public key, nobody can remove the mask kP_B. However, A also includes a "clue," which is enough to remove the mask if one

Table 10.3 Comparable Key Sizes in Terms of Computational Effort for Cryptanalysis (NIST SP-800-57)

Symmetric Key Algorithms	Diffie-Hellman, Digital Signature Algorithm	RSA (size of *n* in bits)	ECC (modulus size in bits)
80	$L = 1024$ $N = 160$	1024	160–223
112	$L = 2048$ $N = 224$	2048	224–255
128	$L = 3072$ $N = 256$	3072	256–383
192	$L = 7680$ $N = 384$	7680	384–511
256	$L = 15{,}360$ $N = 512$	15,360	512+

Note: L = size of public key, N = size of private key

knows the private key n_B. For an attacker to recover the message, the attacker would have to compute k given G and kG, which is assumed to be hard.

Let us consider a simple example. The global public elements are $q = 257$; $E_q(a, b) = E_{257}(0, -4)$, which is equivalent to the curve $y^2 = x^3 - 4$; and $G = (2, 2)$. Bob's private key is $n_B = 101$, and his public key is $P_B = n_B G = 101(2, 2) = (197, 167)$. Alice wishes to send a message to Bob that is encoded in the elliptic point $P_m = (112, 26)$. Alice chooses random integer $k = 41$ and computes $kG = 41(2, 2) = (136, 128)$, $kP_B = 41(197, 167) = (68, 84)$ and $P_m + kP_B = (112, 26) + (68, 84) = (246, 174)$. Alice sends the ciphertext $C_m = (C_1, C_2) = \{(136, 128), (246, 174)\}$ to Bob. Bob receives the ciphertext and computes $C_2 - n_B C_1 = (246, 174) - 101(136, 128) = (246, 174) - (68, 84) = (112, 26)$.

Security of Elliptic Curve Cryptography

The security of ECC depends on how difficult it is to determine k given kP and P. This is referred to as the elliptic curve logarithm problem. The fastest known technique for taking the elliptic curve logarithm is known as the Pollard rho method. Table 10.3, from NIST SP800-57 *(Recommendation for Key Management—Part 1: General*, July 2012), compares various algorithms by showing comparable key sizes in terms of computational effort for cryptanalysis. As can be seen, a considerably smaller key size can be used for ECC compared to RSA. Furthermore, for equal key lengths, the computational effort required for ECC and RSA is comparable [JURI97]. Thus, there is a computational advantage to using ECC with a shorter key length than a comparably secure RSA.

10.5 PSEUDORANDOM NUMBER GENERATION BASED ON AN ASYMMETRIC CIPHER

We noted in Chapter 7 that because a symmetric block cipher produces an apparently random output, it can serve as the basis of a pseudorandom number generator (PRNG). Similarly, an asymmetric encryption algorithm produces apparently random output and can be used to build a PRNG. Because asymmetric algorithms are typically much slower than symmetric algorithms, asymmetric algorithms are not used to generate open-ended PRNG bit streams. Rather, the asymmetric approach is useful for creating a pseudorandom function (PRF) for generating a short pseudorandom bit sequence.

In this section, we examine two PRNG designs based on pseudorandom functions.

PRNG Based on RSA

For a sufficient key length, the RSA algorithm is considered secure and is a good candidate to form the basis of a PRNG. Such a PRNG, known as the Micali-Schnorr PRNG [MICA91], is recommended in the ANSI standard X9.82 (*Random Number Generation*) and in the ISO standard 18031 (*Random Bit Generation*).

The PRNG is illustrated in Figure 10.8. As can be seen, this PRNG has much the same structure as the output feedback (OFB) mode used as a PRNG (see Figure 7.4b and the portion of Figure 6.6a enclosed with a dashed box). In this case, the encryption algorithm is RSA rather than a symmetric block cipher. Also, a portion of the output is fed back to the next iteration of the encryption algorithm and the remainder of the output is used as pseudorandom bits. The motivation for this separation of the output into two distinct parts is so that the pseudorandom bits from one stage do not provide input to the next stage. This separation should contribute to forward unpredictability.

We can define the PRNG as follows.

Setup Select p, q primes; $n = pq; \phi(n) = (p - 1)(q - 1)$. Select e such that $\gcd(e, \phi(n)) = 1$. These are the standard RSA setup selections (see Figure 9.5). In addition, let $N = [\log_2 n] + 1$ (the bitlength of n). Select r, k such that $r + k = N$.

Seed Select a random seed x_0 of bitlength r.

Generate Generate a pseudorandom sequence of length $k \times m$ using the loop
for i **from** 1 to m **do**

$$\begin{aligned} y_i &= x_{i-1}^e \bmod n \\ x_i &= r \text{ most significant bits of } y_i \\ z_i &= k \text{ least significant bits of } y_i \end{aligned}$$

Output The output sequence is $z_1 \| z_2 \| \ldots \| z_m$.

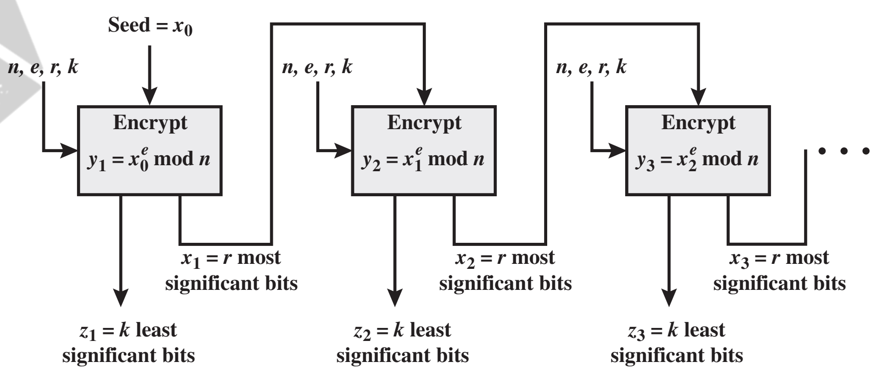


Figure 10.8 Micali-Schnorr Pseudorandom Bit Generator

The parameters n, r, e, and k are selected to satisfy the following six requirements.

1. $n = pq$	n is chosen as the product of two primes to have the cryptographic strength required of RSA.
2. $1 < e < \phi(n)$; $\gcd(e, \phi(n)) = 1$	Ensures that the mapping $s \rightarrow s^e \bmod n$ is 1 to 1.
3. $re \geq 2N$	Ensures that the exponentiation requires a full modular reduction.
4. $r \geq 2$ *strength*	Protects against a cryptographic attacks.
5. k, r are multiples of 8	An implementation convenience.
6. $k \geq 8$; $r + k = N$	All bits are used.

The variable *strength* in requirement 4 is defined in NIST SP 800-90 as follows: A number associated with the amount of work (that is, the number of operations) required to break a cryptographic algorithm or system; a security strength is specified in bits and is a specific value from the set (112, 128, 192, 256) for this Recommendation. The amount of work needed is $2^{strength}$.

There is clearly a tradeoff between r and k. Because RSA is computationally intensive compared to a block cipher, we would like to generate as many pseudorandom bits per iteration as possible and therefore would like a large value of k. However, for cryptographic strength, we would like r to be as large as possible.

For example, if $e = 3$ and $N = 1024$, then we have the inequality $3r > 1024$, yielding a minimum required size for r of 683 bits. For r set to that size, $k = 341$ bits are generated for each exponentiation (each RSA encryption). In this case, each exponentiation requires only one modular squaring of a 683-bit number and one modular multiplication. That is, we need only calculate $(x_i \times (x_i^2 \bmod n)) \bmod n$.

PRNG Based on Elliptic Curve Cryptography

In this subsection, we briefly summarize a technique developed by the U.S. National Security Agency (NSA) known as dual elliptic curve PRNG (DEC PRNG). This technique is recommended in NIST SP 800-90, the ANSI standard X9.82, and the ISO standard 18031. There has been some controversy regarding both the security and efficiency of this algorithm compared to other alternatives (e.g., see [SCHO06], [BROW07]).

[SCHO06] summarizes the algorithm as follows: Let P and Q be two known points on a given elliptic curve. The seed of the DEC PRNG is a random integer $s_0 \in \{0, 1, \ldots, \#E(GF(p)) - 1\}$, where $\#\,E(GF(p))$ denotes the number of points on the curve. Let x denote a function that gives the x-coordinate of a point of the curve. Let $lsb_i(s)$ denote the i least significant bits of an integer s. The DEC PRNG transforms the seed into the pseudorandom sequence of length $240k$, $k > 0$, as follows.

```
for i = 1 to k do
   Set s_i ← x(S_{i−1} P)
   Set r_i ← lsb_240 (x(s_i Q))
end for
   Return r_1,...,r_k
```

Given the security concerns expressed for this PRNG, the only motivation for its use would be that it is used in a system that already implements ECC but does not implement any other symmetric, asymmetric, or hash cryptographic algorithm that could be used to build a PRNG.