Indian Institute of Technology Kharagpur
Centre of Excellence in Artificial Intelligence

AI61003 Linear Algebra for AI and ML
Assignment 2, Due on: Thursday, October 15, 2020

ANSWER ALL THE QUESTIONS

1. Let $A = \left\{ a_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} a_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, a_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, a_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\}$.

   Perform Gram Schmidt orthogonalization on the vectors listed in the set A with the following order and comment on the results obtained in each of these instances.

   (a) a_1, a_2, a_3, a_4, a_5

   (b) a_1, a_3, a_5, a_2, a_4

   (c) a_5, a_4, a_3, a_2, a_1

2. Let $Q \in \mathbb{R}^{n \times m}$ where $n \geqslant m$ be such that the columns of Q form an orthonormal collection. Such a matrix Q is called an isometry of size $n \times m$. For any vectors $x, y \in \mathbb{R}^m$ show the following.

   - Angle between x and y is same as the angle between Qx and Qy. (Isometry preserves angle!)
   - $\|x\|_2 = \|Qx\|_2$ (Isometry preserves the norm!)

3. Let $A = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{R}^n$ be a linearly independent set. Obtain $B = \{q_1, q_2, \ldots, q_m\}$ from the set A using Gram Schmidt orthogonalization algorithm. Discuss the computational complexity of this algorithm.

4. For $t_1, t_2, \ldots, t_n \in \mathbb{R}$, consider the matrix of the form

$$V_{n,m} = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^m \\ 1 & t_2 & t_2^2 & \cdots & t_2^m \\ \vdots & \vdots & & & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^m \end{pmatrix} \in \mathbb{R}^{n \times (m+1)}$$

   (a) Explain how this matrix may be used to evaluate a polynomial of degree m at n different real numbers $t_1, t_2, \ldots, t_n$.

   (b) For distinct n real numbers $t_1, t_2, \ldots, t_n$ and $n \geqslant m + 1$, show that the the columns of $V_{n,m}$ are linearly independent.

5. Let $A, B \in \mathbb{R}^{m \times n}$ and $x, y \in \mathbb{R}^n$. Define $z = (A + B)(x + y)$. Discuss the number of floating point operations required to perform the computation of z in the following two ways.

(a) Compute $A+B$ and $x+y$ and then compute $z=(A+B)(x+y)$.

(b) Compute Ax, Ay, Bx, By and then compute $z=Ax+Ay+Bx+By$.

Which of the above two ways is computationally beneficial?

6. In each of the following case, an action by a function on n-vectors is specified. Check whether these actions give rise to linear functions. In case, the function is linear, find the equivalent matrix representation. Specify clearly the size of matrix and all the entries in the matrix.

   (a) $f:\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top = \begin{pmatrix} x_2 & x_3 & \cdots & x_{n-1} \end{pmatrix}^\top$

   (b) $g:\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top = \left(x_1, \frac{x_1+x_2}{2}, x_2, \frac{x_2+x_3}{2}, x_3, \ldots, x_{n-1}, \frac{x_{n-1}+x_n}{2}, x_n\right)^\top$

   (c) $h:\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top = \begin{pmatrix} x_1 & x_2 & \cdots & x_n & 1 \end{pmatrix}^\top$

7. Let $a, b \in \mathbb{R}^n$ and let $\mathbf{1}_n \in \mathbb{R}^n$ be the n-vector of all ones. Let $a \star b$ denote the convolution of a with b. Answer the following questions.

   (a) Compute $\mathbf{1}_n \star a$.

   (b) Compute $e_j \star a$ where e_j is the j^{th} unit vector in $\mathbb{R}^n$.

   (c) Prove $\mathbf{1}_{2n-1}^\top (a \star b) = (\mathbf{1}_n^\top a)(\mathbf{1}_n^\top b)$

8. **2D convolution, image blurring** Let $X \in \mathbb{R}^{m\times n}$ represent an image. For a matrix $B \in \mathbb{R}^{p\times q}$, 2D convolution of X and B represented as $Y = X \star B$ is the effect of blurring the image by the point spread function (PSF) given by the entries of the matrix B. If X and Y are represented as vectors as x and y, then $y = T(B)x$ where $T(B) \in \mathbb{R}^{(m+p-1)(n+q-1)\times mn}$ is the corresponding Toeplitz matrix.

   Download any image of a cat and a dog from Kaggle data set. These are coloured images which are to be converted to grey-scale images. (Tutorial video on downloading the data and converting a colour image to grey-scale image is uploaded in the course Moodle page.) Denote by $\mathcal{C}$ the matrix representing grey-scale image of the cat and $\mathcal{D}$ is the matrix representing grey-scale image of the dog. For each of the following blurring matrices B, compute $\mathcal{C} \star B$ and $\mathcal{D} \star B$ and plot corresponding blurred images.

   (a) Let $B \in \mathbb{R}^{p\times p}$ for $p = 5, 10, 20$. Here $B_{ij} = \frac{1}{p^2}$ for $1 \leqslant i, j \leqslant p$.

   (b) Let $B = \begin{pmatrix} 1 & -1 \end{pmatrix} \in \mathbb{R}^{1\times 2}$.

   (c) Let $B = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^{2\times 1}$.

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