Deepank Agrawal $\quad$ LA for AI/ML

17CS30011 $\quad$ Assignment - 3

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1.) Let for any vector $x \in R^n$:

$$x = [x_1 \; x_2 \; \dots \; x_n]^T$$

(a) here, $y = y_s + y_a$

or,

$$y = [y_{s,1} + y_{a1} \;\; y_{s,2} + y_{a,2} \; \dots \; y_{s,n} + y_{an}]^T$$

or, $y_i = y_{s,i} + y_{a,i} \quad \forall \; i \in [1, n]$

and, $y_{n-i+1} = y_{s,n-i+1} + y_{a,n-i+1}$

Since, $y_{s,i} = y_{s,n-i+1}$ and $y_{a,n-i+1} = -y_{a,i}$

$\therefore \; y_i = y_{s,i} + y_{a,i}$

and, $y_{n-i+1} = y_{s,i} - y_{a,i}$

or, $y_{s,i} = \dfrac{y_i + y_{n-i+1}}{2}$, $\quad y_{a,i} = \dfrac{y_i - y_{n-i+1}}{2}$

So, for any $y \in R^n$,

$$y_s = \left[\frac{y_1 + y_n}{2} \;\; \frac{y_2 + y_{n-1}}{2} \; \dots \; \frac{y_n + y_1}{2}\right]^T \text{(symmetric)}$$

$$y_a = \left[\frac{y_1 - y_n}{2} \;\; \frac{y_2 - y_{n-1}}{2} \; \dots \; \frac{y_n - y_1}{2}\right]^T \text{(assymmetric)}$$

Such that $y = y_s + y_a$

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(b) let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for $y \in \mathbb{R}^n$

$f(y) = y_s$, where y_s is the symmetric part

also, let $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for $y \in \mathbb{R}^n$

$g(y) = y_a$, where y_a is the asymmetric part.

Now, f is linear iff:

$f(\alpha y + \beta x) = \alpha f(y) + \beta f(x)$ for $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

<u>LHS</u>

$$f(\alpha y + \beta x) = (\alpha y + \beta x)_s$$

$$= \left[\frac{(\alpha y+\beta x)_1 + (\alpha y+\beta x)_n}{2} \quad \frac{(\alpha y+\beta x)_2 + (\alpha y+\beta x)_{n-1}}{2} \cdots \frac{(\alpha y+\beta x)_n + (\alpha y+\beta x)_1}{2}\right]^T$$

$$= \left[\frac{\alpha y_1 + \beta x_1 + \alpha y_n + \beta x_n}{2}, \frac{\alpha y_2 + \beta x_2 + \alpha y_{n-1} + \beta x_{n-1}}{2} \cdots \frac{\alpha y_n + \beta x_n + \alpha y_1 + \beta x_1}{2}\right]^T$$

$$= \left[\frac{\beta(x_1 + x_n)}{2} + \frac{\alpha(y_1 + y_n)}{2} \cdots \frac{\alpha(y_n + y_1)}{2} + \frac{\beta(x_n + x_1)}{2}\right]^T$$

$$= \beta\left[\frac{x_1 + x_n}{2} \quad \frac{x_2 + x_{n-1}}{2} \cdots \frac{x_n + x_1}{2}\right]^T +$$

$$\alpha\left[\frac{y_1 + y_n}{2} \quad \frac{y_2 + y_{n-1}}{2} \cdots \frac{y_n + y_1}{2}\right]^T$$

$$= \beta x_s + \alpha y_s = \beta f(x) + \alpha f(y)$$

$$= \text{RHS}$$

Hence, f is a linear function

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Similarly, we can show that g is a linear function.

For a linear function, we know

$$f(y) = A_f \cdot y \quad \text{where, } y \in \mathbb{R}^n, \; A_f \in \mathbb{R}^{n\times n}$$

and $g(y) = A_g \cdot y$

now, $A_f \cdot y = y_s$

$$A_f = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \cdots & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & \frac{1}{2} & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}$$

now, $A_g \cdot y = y_a$

$$A_g = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \cdots & -\frac{1}{2} & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -\frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ -\frac{1}{2} & 0 & 0 & \cdots & & 0 & \frac{1}{2} \end{bmatrix}$$

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2.) We have

$$f(u,v) = \theta_1 + \theta_2 u + \theta_3 v + \theta_4 uv$$

(a) now, $f(P_{ij}) = f(x_i, y_j)$

$$= \theta_1 + \theta_2 x_i + \theta_3 y_j + \theta_4 x_i y_j$$

$$A\theta = b$$

here, $A \Rightarrow \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ \vdots & & \vdots & \vdots \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & y_n & x_m y_n \end{bmatrix} \in \mathbb{R}^{mn \times 4}$

$$\theta = [\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4]^T \in \mathbb{R}^{4 \times 1}$$

$b = \begin{bmatrix} F_{11} \\ F_{12} \\ \vdots \\ F_{21} \\ F_{22} \\ \vdots \\ F_{mn} \end{bmatrix} \in \mathbb{R}^{mn \times 1}$

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(b) For a unique solution to $A\theta = b$, the A matrix should be a tall matrix

i.e., $MN \geq 4$

For $m=1$ or $n=1$, it is easy to verify that the columns of A are not linearly independent.

hence, $M \geq 2$ and $N \geq 2$

minimum values of $M, N = (2, 2)$ to expect an unique solution to $A\theta = b$.

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3.) We have,

$A, B \in \mathbb{R}^{n \times n}$

$\|\cdot\|_2$ – induced 2-norm of matrix

To prove: $\|AB\|_2 \leq \|A\|_2 \, \|B\|_2$

We know that

for $P \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$

$$\|Px\|_2 \leq \|P\|_2 \, \|x\|_2 \quad \text{— (1)}$$

Put, $P = A$, $x = Bx$

So, $\|ABx\|_2 \leq \|A\|_2 \cdot \|Bx\|_2$

$\leq \|A\|_2 \cdot \|B\|_2 \, \|x\|_2$ [From (1)]

Dividing by $\|x\|_2$:

$$\frac{\|ABx\|_2}{\|x\|_2} \leq \|A\|_2 \cdot \|B\|_2$$

Taking max both sides:

or, $\|AB\|_2 \leq \|A\|_2 \cdot \|B\|_2$

= proved

Yes, Frobenius norm also follows this property but the proof should follow from Cauchy-Schwartz inequality.

4.) We have,

$A \in \mathbb{R}^{n\times n}, \quad x \in \mathbb{R}^n$

$A = I_n + ab^T \,;\; a, b \in \mathbb{R}^n$

now, suppose we are given a and b vectors.

# of operations for $b^T x \Rightarrow n + n - 1$
$\Rightarrow (2n-1)$

and $(ab^T) \in \mathbb{R}^{n\times n}$

now, $Ax = (I_n + (ab^T))x$
$= I_n \cdot x + (ab^T)x$
$= x + (ab^T)x$
$= x + a(b^Tx)$

~~or, $Ax = (1+ab^T)x$~~

# of operations for $(b^Tx) = (2n-1)$
$= 2n-1$ operations
# of operations for $a(b^Tx) = 3n-1$ operations
$\therefore$ # of operations for $(I+ab^T)x$
$= 3n-1+n$
$= 4n-1$ operations

$\therefore$ Ax can be computed in $4n-1$ operations if $A = I_n + ab^T$.

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5.) we have,

$x \in R^n$, $n > 1$

(a) left inverse:

the columns of the matrix should be linearly independent.

Since, $n > 1$ and x has 1 column

$\Rightarrow$ x has a left inverse

For ex: $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

now, for $c = [-1 \quad 1]$,

$cx = I_1 = 1$

(b) Right Inverse:

the rows of the matrix should be linearly independent.

Since, $n > 1$ and x has 1 column

$\Rightarrow$ rows of x are linearly dependent

$\Rightarrow$ right inverse of x doesnot exist.

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6) Given: $I_m + AB$ is Invertible
$A \in R^{m\times n}$, $B \in R^{n\times m}$

now, $(I_n + BA)B = B + BAB = B(I_m + AB)$

or, $(I_n + BA)B(I_m + AB)^{-1} = B$
post multiply by A and add I_n both sides:

$I_n + (I_n + BA)B(I_m + AB)^{-1}A = BA + I_n$

or, $(I_n + BA)B(I_m + AB)^{-1}A - (I_n + BA) = -I_n$
or, $(I_n + BA)[I_n - B(I_m + AB)^{-1}A] = I_n$

hence, inverse exists for $(I_n + BA)$.
$\therefore$ $I_n + BA$ is invertible

now, $(I_n + BA)^{-1} = I_n - B(I_m + AB)^{-1}A$
pre multiply by $(I_n + BA)$:

$\Rightarrow \cancel{I_n} = (\cancel{I_n} + BA) - (I_n + BA)B(I_m + AB)^{-1}A$

or, $BA = (I_n + BA)B(I_m + AB)^{-1}A$

or, $B = (I_n + BA)B(I_m + AB)^{-1}$ or $A = 0$
or, $(I_n + BA)^{-1}B = B(I_m + AB)^{-1}$

now, if $A = 0$, then $(I_n + BA)^{-1}B = B$
and $B(I_m + AB)^{-1} = B$
$\therefore (I_n + BA)^{-1}B = B(I_m + AB)^{-1}$

proved

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7) We have
$A, B \in \mathbb{R}^{n \times n}$

(a) $A+B$:

$A+B$ may not be invertible

Consider, $B = -A$
or, $[B_{ij}] = -[A]_{ij} \quad \forall\, i,j \in [1..n]$

So, $A+B = \mathbf{0}$.
which is not invertible.

(b) $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$

Since, A, B are invertible
$\Rightarrow$ columns of A, B are linearly independent
$\Rightarrow$ columns of $\begin{pmatrix} A \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ B \end{pmatrix}$ will also be linearly independent

$\Rightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ has linearly independent columns.

$\Rightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is invertible

$\Rightarrow \left|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right| \neq 0 \quad - (1)$

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(c) $\begin{pmatrix} A & A-B \\ O & B \end{pmatrix}$

now, determinant $\left(\begin{pmatrix} A & A-B \\ O & B \end{pmatrix}\right)$

$\Rightarrow \left|\begin{pmatrix} A & A-B \\ O & B \end{pmatrix}\right|$

we can do matrix transformations

$\Rightarrow \left|\begin{pmatrix} A & A-B \\ O & B \end{pmatrix}\right| \longrightarrow \left|\begin{pmatrix} A & -B \\ O & B \end{pmatrix}\right|$

$[C_2 \to C_2 - C_1]$

$\Rightarrow \left|\begin{pmatrix} A & -B \\ O & B \end{pmatrix}\right| \rightarrow \left|\begin{pmatrix} A & O \\ O & B \end{pmatrix}\right|$

$[R_1 \to R_1 + R_2]$

From part (b) $\left|\begin{pmatrix} A & O \\ O & B \end{pmatrix}\right| \neq 0$

$\Rightarrow \left|\begin{pmatrix} A & A-B \\ O & B \end{pmatrix}\right| \neq 0$

$\Rightarrow$ invertible

(d) ABABA

we know that, $|A| \neq 0$, $|B| \neq 0$

so, $|ABABA| = |A|\cdot|B|\cdot|A|\cdot|B|\cdot|A|$ [Property of determinants]

$= |A|^3 \cdot |B|^2$

$\neq 0$

$\therefore$ invertible

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8) We have
$A \in \mathbb{R}^{n\times n}$, Invertible matrix.

$$\text{max mag}(A) = \max_{\|x\|_2 = 1} \|Ax\|_2$$

$$\text{min mag}(A) = \min_{\|x_2\|=1} \|Ax\|_2$$

$$\text{Cond}(A) = \|A\|_2 \|A^{-1}\|_2$$

(a) A is invertible

so, $x \in \mathbb{R}^n \longrightarrow Ax = y$, $y \in \mathbb{R}^n$
or, $x = A^{-1}y \in \mathbb{R}^n$

now, $$\text{max mag}(A) = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$= \max_{y\neq 0} \frac{\|y\|_2}{\|A^{-1}y\|_2}$$

$$= \frac{1}{\min_{y\neq 0} \frac{\|A^{-1}y\|_2}{\|y\|_2}}$$

$$= \frac{1}{\text{min mag}(A^{-1})}$$

So, $$\text{max mag}(A) = \frac{1}{\text{min mag}(A^{-1})}$$

proved

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(b)

$$\text{Cond}(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

$$= \max \text{mag}(A) \cdot \max \text{mag}(A^{-1})$$

$$= \frac{\max \text{mag}(A)}{\min \text{mag}(A)} \quad [\text{From (a)}]$$

proved

AI61003 Linear Algebra for AI and ML

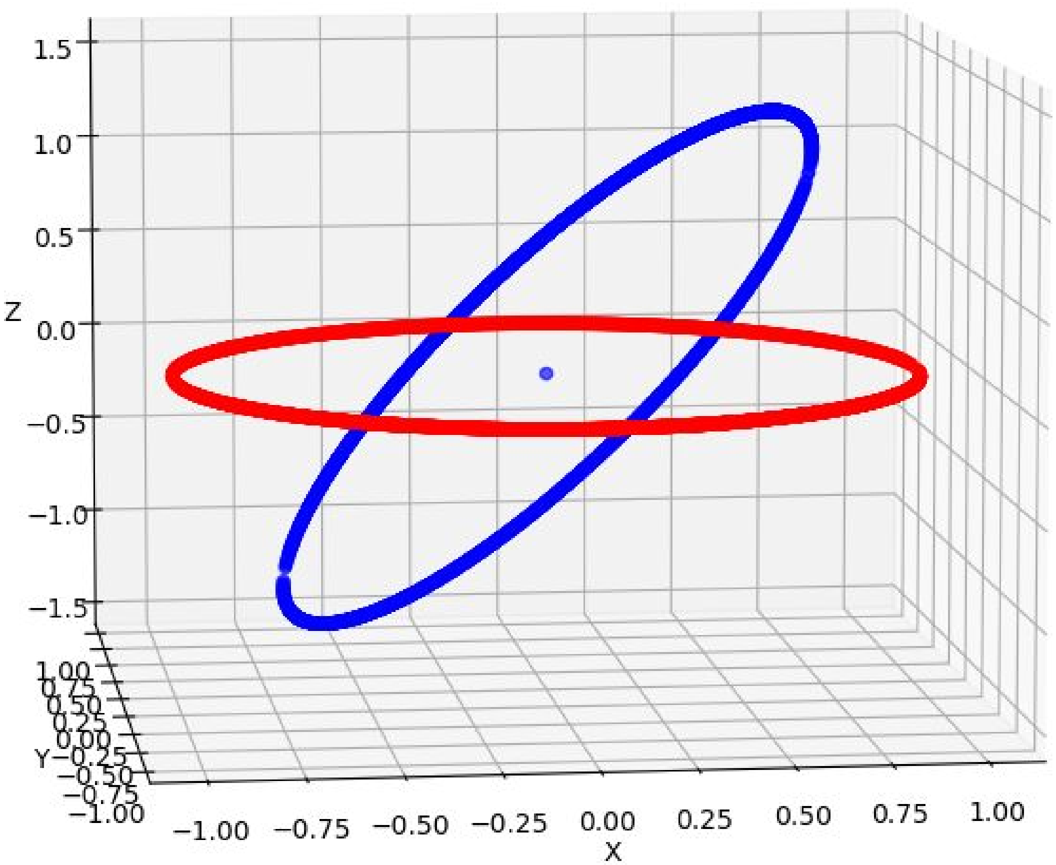
Deepank Agrawal

17CS30011 Assignment 3

9. In the plots, red - original, blue - transformed

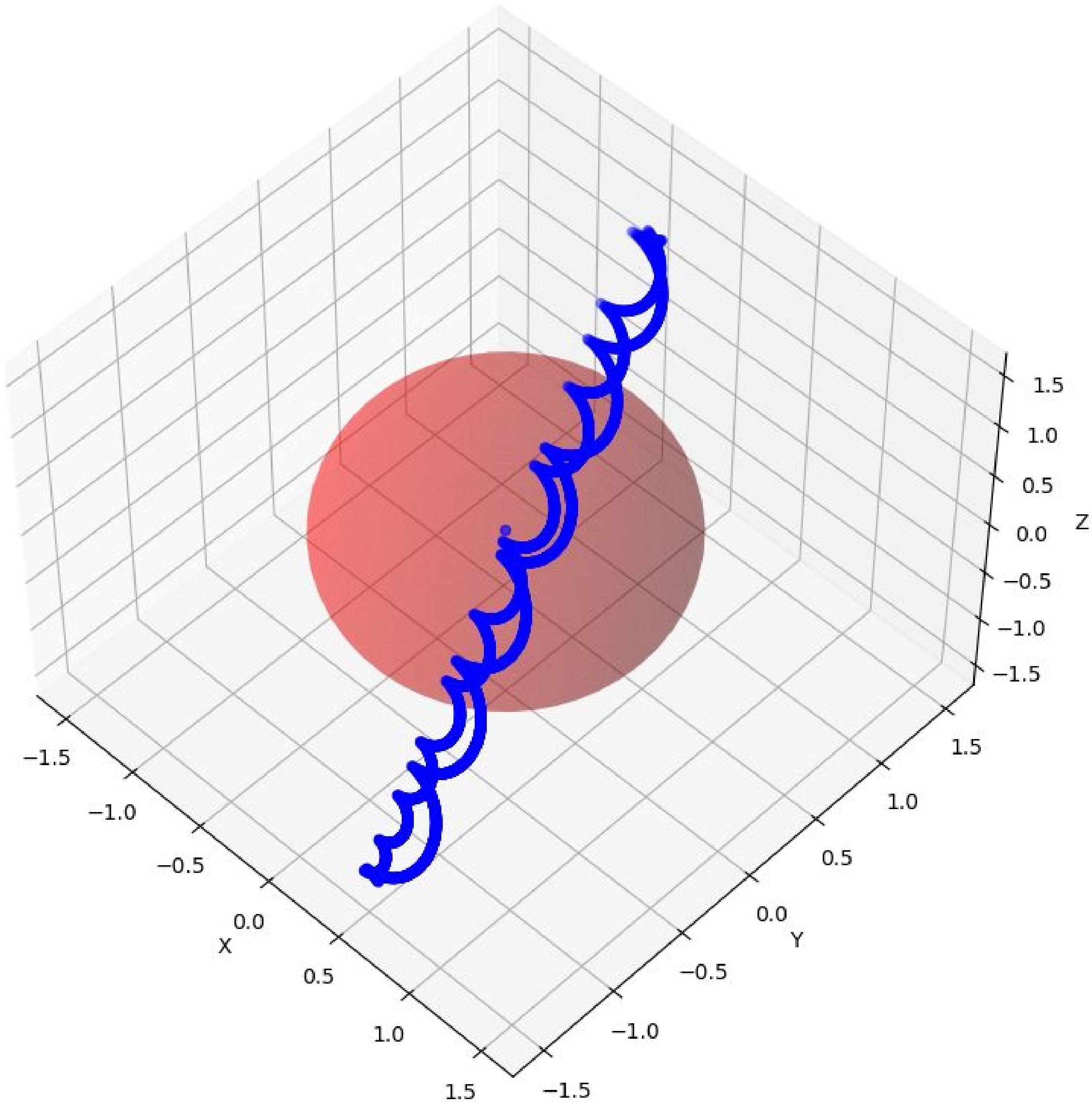
Code: https://github.com/Deepank308/LA-for-AI/blob/master/assgn3/q9.py

(a)



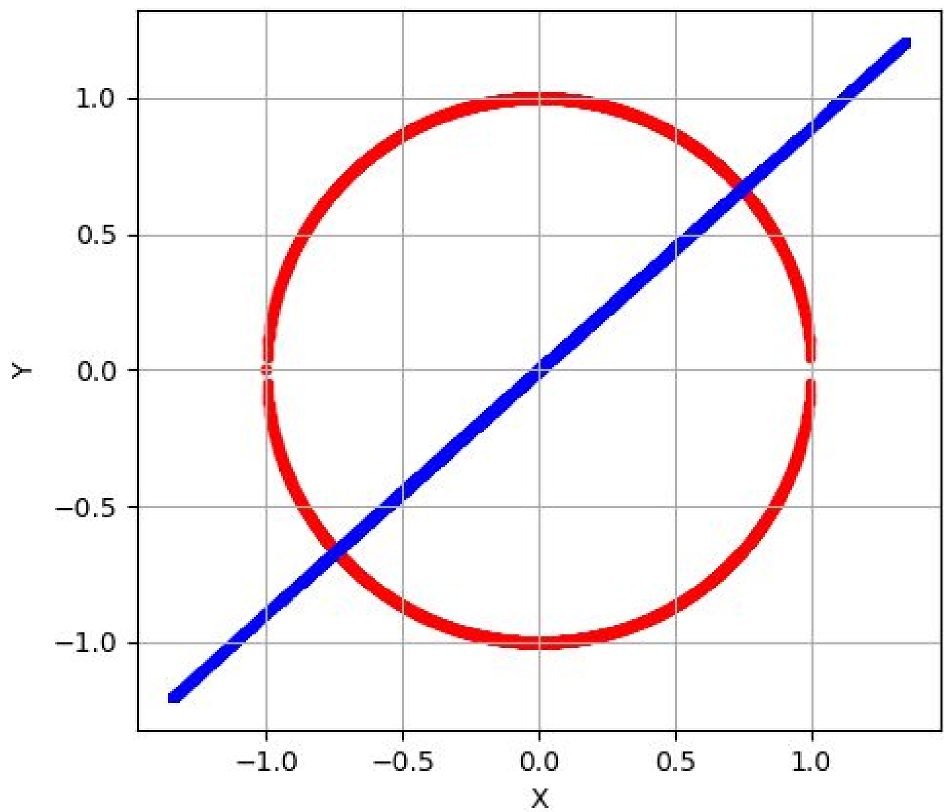
Condition number: 2.23606797749979

(b)



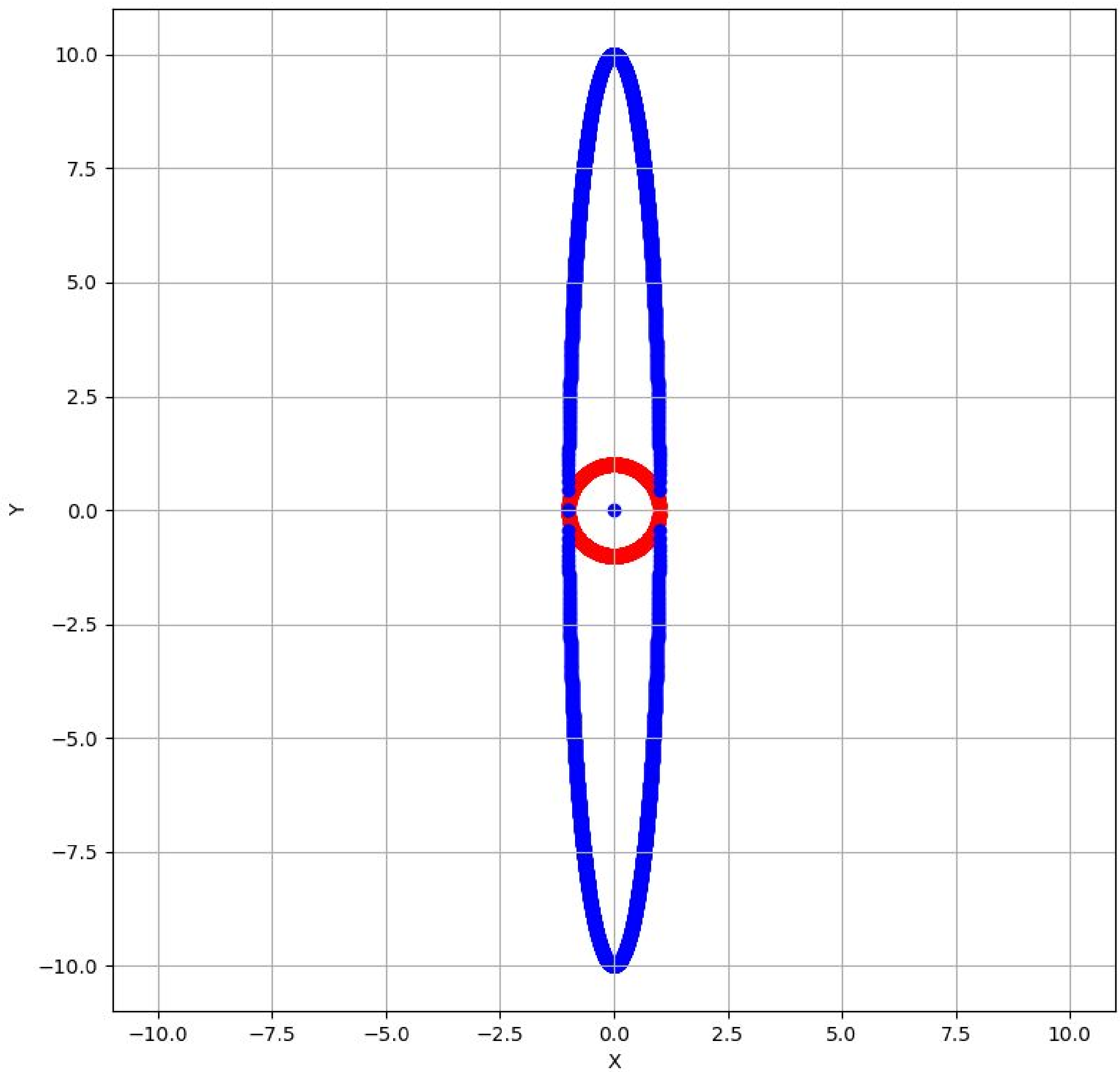
Condition number: 1.715010090561728

(c)



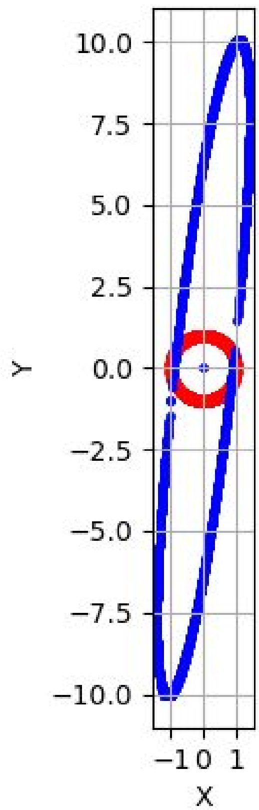
Condition number: 325.99693248647975
Invertible
Determinant: -0.009999999999999995

(d)



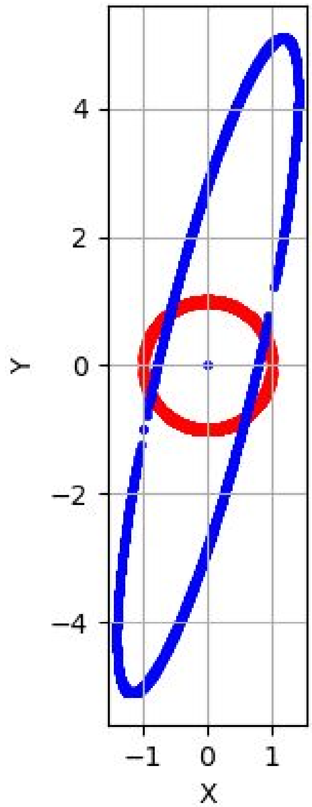
Condition number: 10.0
Invertible
Determinant: -10.000000000000002

(e)
ε = 10



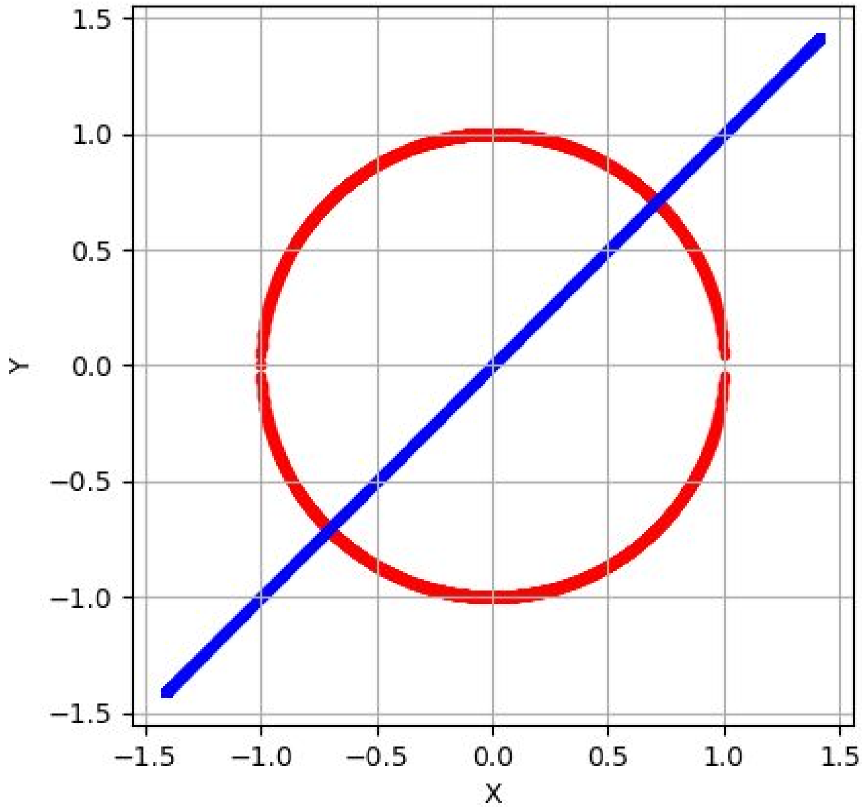
Condition number: 11.35638827945676
Invertible
Determinant: 9.000000000000002

ε = 5



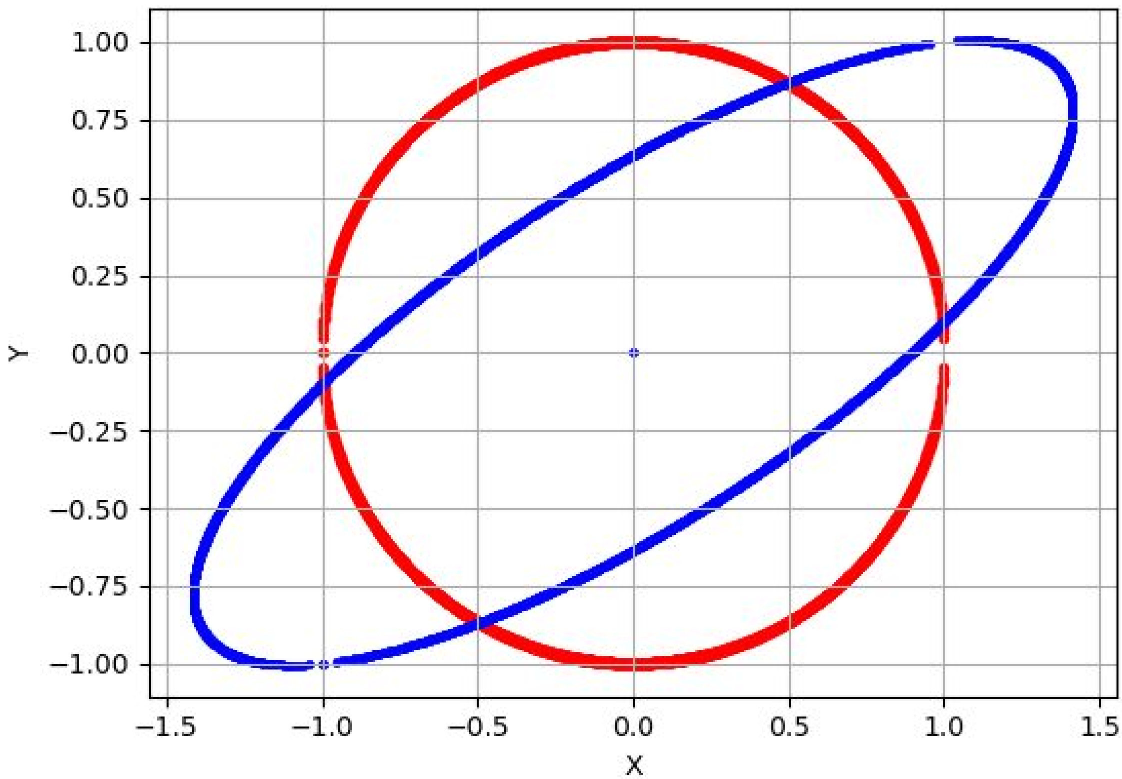
Condition number: 6.854101966249685
Invertible
Determinant: 4.0

ε = 1



Condition number: 5.961777047638983e+16
Not Invertible

$\varepsilon = 10^{-1}$

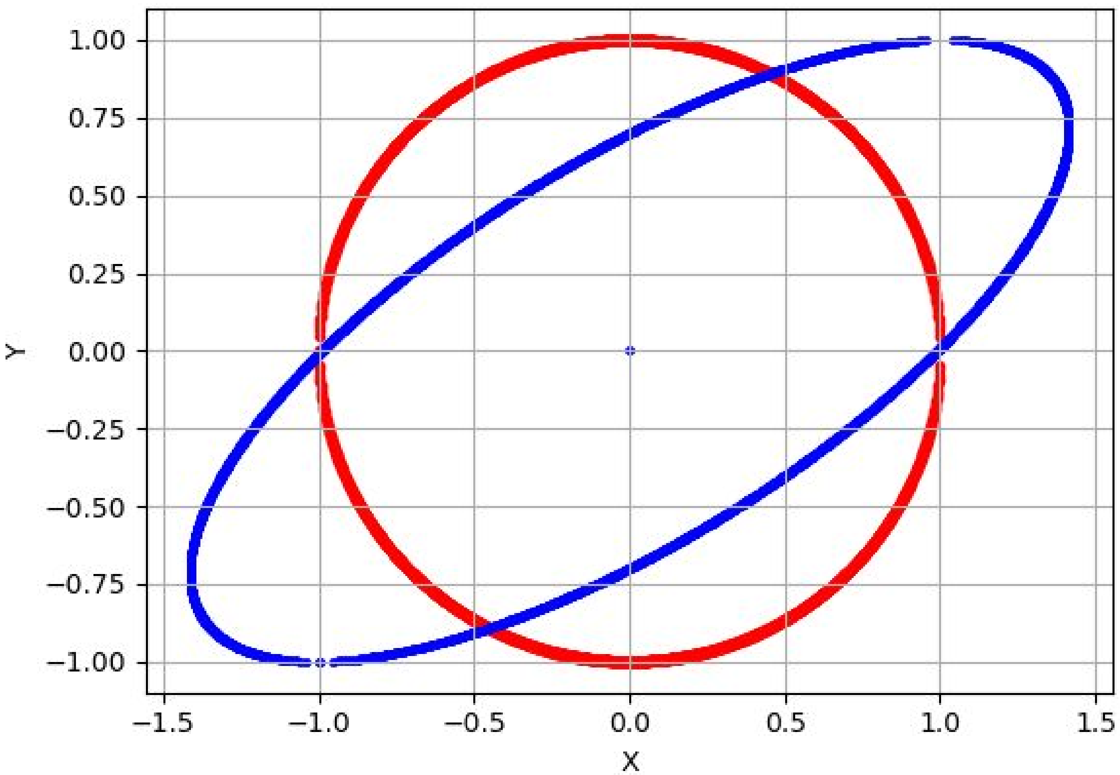


Condition number: 3.0124935233004138

Invertible

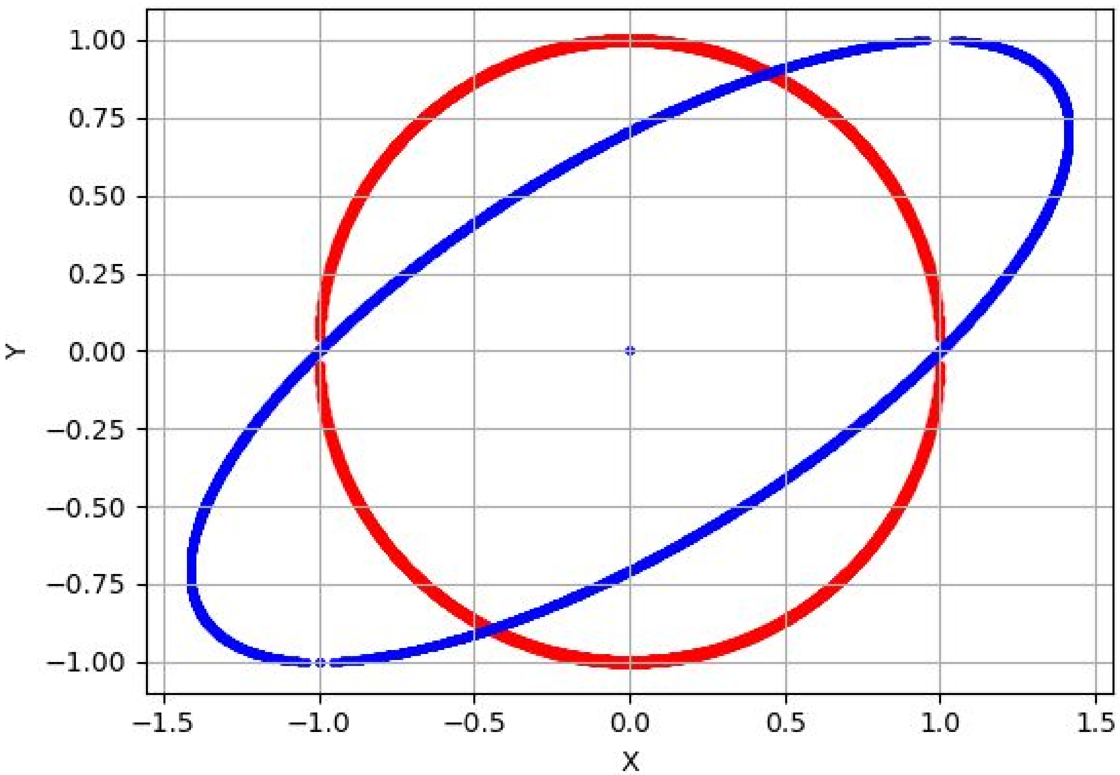
Determinant: -0.9

$\varepsilon = 10^{-2}$



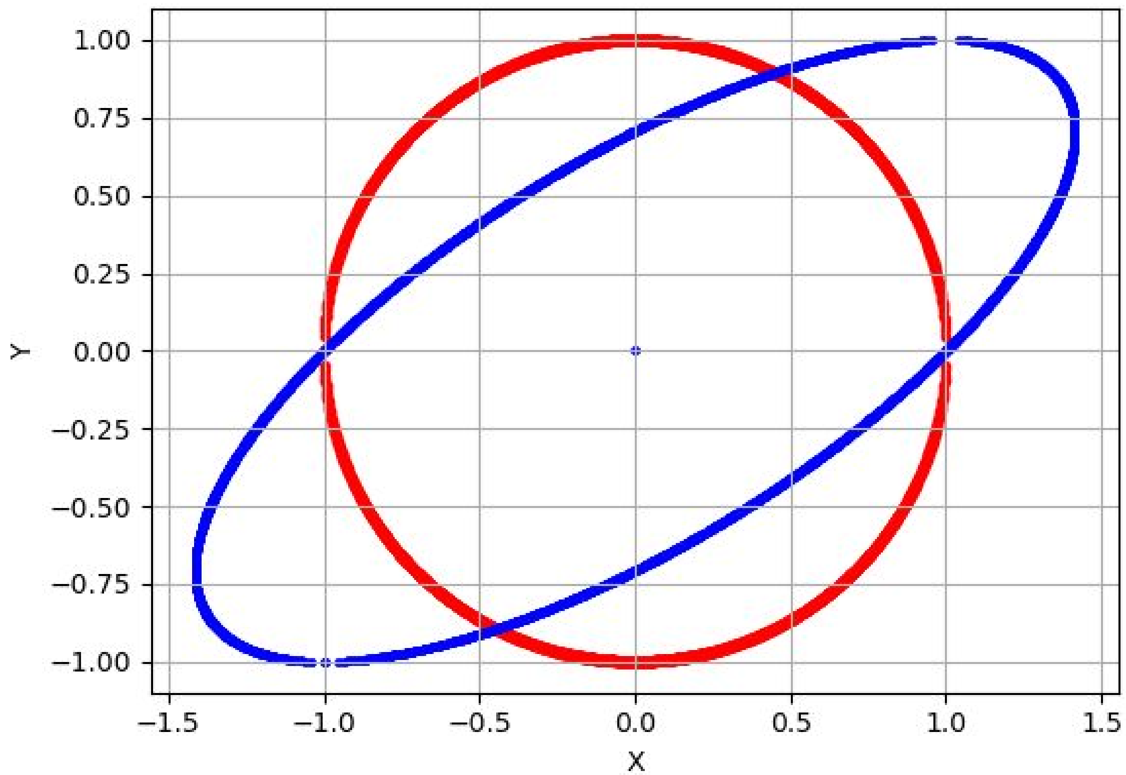
Condition number: 2.6535504563252843
Invertible
Determinant: -0.99

$\varepsilon = 10^{-4}$



Condition number: 2.618385273654826
Invertible
Determinant: -0.9999

ε = 0



Condition number: 2.6180339887498953
Invertible
Determinant: -1.0

As evident from the condition numbers and determinant values, we can say that as condition number becomes larger, determinant approaches 0, i.e., the matrix becomes singular.