Indian Institute of Technology Kharagpur
Centre of Excellence in Artificial Intelligence

AI61003 Linear Algebra for AI and ML

Assignment 3, Due on: Sunday, November 8, 2020

ANSWER ALL THE QUESTIONS

1. A vector $x \in \mathbb{R}^n$ is called as symmetric if $x_k = x_{n-k+1}$ for $k = 1, 2, \ldots, n$. Similarly, a vector $x \in \mathbb{R}^n$ is called as anti-symmetric if $x_k = -x_{n-k+1}$ for $k = 1, 2, \ldots, n$.

   (a) Show that every vector $y \in \mathbb{R}^n$ can be decomposed in a unique way as a sum $y = y_s + y_a$ where $y_s \in \mathbb{R}^n$ is symmetric and $y_a \in \mathbb{R}^n$ is anti-symmetric.

   (b) Show that symmetric and anti-symmetric parts of y are linear functions of y and compute the matrices corresponding to these linear transformations.

2. *Bi-linear interpolation*: We are given scalar value at each of the MN grid points of a grid in $\mathbb{R}^2$ with a typical grid point represented as $P_{ij} = (x_i, y_j)$ where $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$ and $x_1 < x_2 < \cdots < x_M$ and $y_1 < y_2 < \cdots < y_N$. Let the scalar value at the grid point P_{ij} be referred to as F_{ij} for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$. A *bi-linear interpolation* is a function of the form

   $$f(u, v) = \theta_1 + \theta_2 u + \theta_3 v + \theta_4 uv$$

   where $\theta_1, \theta_2, \theta_3, \theta_4$ are the coefficients. This function further satisfies $f(P_{ij}) = F_{ij}$ for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$.

   (a) Express these interpolation conditions as a system linear equations of the form $A\theta = b$ where b is an MN vector consisting of F_{ij} values. Write clearly all the entries of A, θ and b and their sizes.

   (b) What are the minimum values of M and N so that you may expect a unique solution to the system of equations $A\theta = b$?

3. Let $A, B \in \mathbb{R}^{n\times n}$. Prove that $\|AB\|_2 \leqslant \|A\|_2\|B\|_2$. This property of 2-norm is called as sub-multiplicativity property. Does this property hold true for Frobenius norm?

4. A matrix-vector multiplication Ax where $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$ takes $2n^2$ number of operations. Formulate a faster (with less number of operations than $2n^2$) method to compute matrix-vector multiplication if the matrix A is of the form $A = I_n + ab^\top$ where $a, b \in \mathbb{R}^n$.

5. Let $0 \neq x \in \mathbb{R}^n$ with $n > 1$.

   (a) Does x have a left inverse?

(b) Does x have a right inverse?

(c) In either of the cases, if the answer is YES, give at least one example.

6. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ be such that $I_m + AB$ is invertible. Show that the matrix $I_n + BA$ is also invertible. Further show that $B(I + AB)^{-1} = (I + BA)^{-1}B$.

7. Let $A, B \in \mathbb{R}^n$ be invertible matrices. Then in each of the following cases, determine whether the matrix is invertible. Justify your answer.

   (a) $A + B$

   (b) $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

   (c) $\begin{pmatrix} A & A - B \\ 0 & B \end{pmatrix}$

   (d) $ABABA$

8. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Define $\max\operatorname{mag}(A)$ and $\min\operatorname{mag}(A)$ and $\operatorname{cond}(A)$. Show that

   (a) $\max\operatorname{mag}(A) = \dfrac{1}{\min\operatorname{mag}(A^{-1})}$

   (b) $\operatorname{cond}(A) = \dfrac{\max\operatorname{mag}(A)}{\min\operatorname{mag}(A)}$

9. In each of the following cases, consider the matrix $A \in \mathbb{R}^{m\times n}$ as a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Plot the unit sphere in $\mathbb{R}^n$. Plot the ellipsoid obtained in $\mathbb{R}^m$ as image of the unit sphere in $\mathbb{R}^n$. Compute the condition number of A (using inbuilt command). Further, if $m = n$, check whether the matrix is invertible. Compute the determinant of A as well. Is there any relationship between determinant and condition number?

   (a) $A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \\ -1 & 1 \end{pmatrix}$

   (b) $A = \begin{pmatrix} -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 0.8 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 1 & 0 \\ 0 & -10 \end{pmatrix}$

   (e) $A = \begin{pmatrix} 1 & 1 \\ 1 & \varepsilon \end{pmatrix}$, where $\varepsilon = 10, 5, 1, 10^{-1}, 10^{-2}, 10^{-4}, 0$.

****************** THE END ******************