Page :
Date :

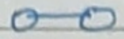
Deepanshu 2019CS50427

Homework 17.1

Ans ⇒ Proof by induction

$P(n)$: Tree of n vertices has $n-1$ edges.

Base Case

$n=2$ 

Tree has only one edge. Hence $P(2)$ is true

Induction Hypothesis

Assume that $P(n)$ is true ~~for all~~ $n=k$ [weak induction].

for $n=k+1$

since tree does not contain a cycle, we know that it will always have a leaf. Call it v.

~~Go~~ Remove the leaf v. The remaining ~~the~~ subgraph is again connected (as we are removing leaf). Also since ~~we did not~~ original tree had no cycle, removing a vertex cannot create a cycle.

Hence it is also a tree having $(k+1)-1 = k$ vertices.

By applying induction hypothesis, this tree has $k-1$ edges.
Now add the leaf back. Note that since degree of leaf is one, adding a leaf would increase count of edges by 1.

Now, this tree has $k+1$ vertices and k edges.

$P(k+1)$ is true whenever $P(k)$ is true.

By principle of mathematical induction, $P(n)$ is true for all n.

Hence Proved

Page :
Date :

Deepanshu 2019CS50427
Homework 17.2

Ans ⇒ Proof by implication
We will first argue that in a tree, there is a unique path between two vertices.

Proof by contradiction
Let there are some $u, v \in E$ such that there is more than one path between them.

Path 1 = $u\, v_1\, v_2 \text{——} v_n\, v$
Path 2 = $u\, w_1\, w_2 \text{ -- } w_m\, v$
The walk $v_1 v_2 \text{——} v_n w_m \text{---} w_1$ is a closed walk that contains a cycle.

Consider smallest i such that $v_i \neq w_i$
So, $v_i \text{---} v_j$ and $w_i \text{---} w_k$ are distinct for some j, k such that $v_{j+1} = w_{k+1}$.
Note that $v_i \text{-----} w_{k+1} w_k \text{----} v_i$ is a cycle which contradicts with property.
Hence our assumption is wrong and there is always a unique path between any two vertices of a tree.

Now we know that tree of n vertices has $n-1$ edges.
consider any edge $e = (u, v) \in E$.
Note that uev is the unique path between u & v and removing e will disconnected the tree.

Hence, if a graph on n vertices has less than $n-1$ edges, it cannot be connected.

Hence Proved

Note :- In other words, a tree is minimally connected graph on n vertices.

Page :
Date :

Deepanshu 2019CS50427

Homework 17.3

Ans ⇒ It will have at least k components

Proof by implication

Note that on adding an edge to a graph, following cases are possible.

Case-1: Number of component remain the same.
If the end vertices of the edge are already connected, the number of connected component would remain the same.

Case-2: Number of components reduces by 1.
If the end vertices ~~are~~ were disjoint i.e there was no path between them, then total number of components would reduce by 1 as now ~~one of~~ the two components merge ~~through th~~ into 1 through this edge.

Proof by construction
Start with n vertices and no edge. The graph currently has n components.
- Add an edge. The ~~com~~ number of component would change.
- Keep adding ~~all ed~~ all $n-k$ edges.

In extreme case, every time we add an edge, it corresponds to case 2 of above argument (which is possible as number of vertices are greater than number of edges).
After insertion, we will be left with at least $n-(n-k)=k$ components.

Hence Proved