Deepanshu 2019CS50427

Page :
Date :

## Homework 19

Ans =>

```
Algorithm connectedComponents (Graph G)
    for each vertex u ∈ V(G)
        colour [v] ← white                 } Init all
        componentNumber [v] ← 0            } arrays.

    count ← 1
    i ← 0
    while (i<n)
        if (componentNumber[vertices[i]] ≠ 0)
            i++
        else
            BFS ( G, vertices[i] , componentNumber, count)
            count ++
    return componentNumber
```

---

```
Algorithm BFS ( G, s , componentNumber, count)
    colour [s] ← gray
    componentNumber [s] ← count
    Q ← [s]
    while Q ≠ ∅ do
        u ← head [Q]
        for each v ∈ adj[u] do
            if ( colour [v] ← white) then
                colour [v] ← gray
                componentNumber [v] ← count
                Enqueue (Q, v)
        Dequeue (Q)
        colour[u] ← black.
```

Deepanshu 2019CS50427

Page :
Date :

Runtime Analysis of the Algorithm

- for doing BFS, every vertex is visited once and ~~we are~~ it is $O(|V|+|E|)$. Now since we are doing BFS in connected component $|E| \geq |V|-1$

$$O(|V|+|E|) = O(|E|).$$

- Also, we are traversing the ~~connectedcom~~ componentNumber array once which has a size of $|V|$.

So, total running time of algorithm is $O(|V|+|E|)$.

Ans-2 ⇒ If G has an odd cycle, it cannot be bipartite

Proof by contradiction
Assume that there exist an odd cycle in a bipartite graph.
Call the vertices of that odd cycle as $v_1, v_2 \text{———} v_{2k+1}$.

Now since graph is bipartite, there exist 2 sets (call them A & B) such that every edge in graph is from vertex in set A to vertex in set B.

Without loss of generality, assume $v_1 \in A$. This implies that $v_2 \in B$ (from definition of bipartite).
This implies that $v_3 \in A$ which implies $v_4 \in B$.
Observe that odd index vertex belongs to A. Thus, $v_{2k+1} \in A$
Now $v_1 \in A \Rightarrow v_{2k+1} \in A$
But $v_1$ & $v_{2k+1}$ are adjacent vertices. And belongs to same set. which contradicts the definition of bipartite. Hence, our assumption is wrong and "If G has an odd cycle, it cannot be bipartite".

Hence Proved.

If all cycles in a graph are of even length, then graph is bipartite

<u>Proof by construction</u>

We have to come up with 2 disjoint sets and each vertex should be present in exactly one of the set satisfying the bipartite condition of graph.

It is enough to prove for connected graph since ~~graph~~ it will be applied to every component. ~~and final set~~.

Start with 2 empty sets, A & B

Pick any vertex. call it $v_1$. Put $v_1$ in A.

- Pick all the vertices at odd distance from $v_1$ and put them in B. (distance here means shortest distance)
- Pick all the vertices at even distance from $v_1$ and put in A.
- By doing this, all vertices of connected graph will be in exactly one of the sets.
- All the edges will be from vertex in A to vertex in B because if there is an edge with the vertices of set A (one) it would lead to odd cycle in graph (by argument in part i) which is not possible.
- For graph having more than 1 component, repeat the process separately for all component and combine all the vertices in $A_i$'s and $B_i$'s ($A_i$'s & $B_i$'s are sets for different components).
- Thus A & B will be required set satisfying all the conditions of bipartite graphs.

Hence, if all cycles in a graph are of even length then graph is bipartite.

<u>Hence Proved</u>.

Deepanshu 2019CS50427

Algorithm checkBipartite ( G, s)

for each vertex $u \in V[G] - [s]$
    colour [u] ← white
    d[u] ← ∞
    π[u] ← NIL

} Init all vertices

colour[s] ← gray
    d[s] ← 0
    π[s] ← NIL
    Q ← [s]

} Init BFS with source s.

while $Q \neq \phi$ do
    u ← head [Q]
    for each $v \in adj[u]$ do
        if colour[v] = white then
            colour[v] ← gray
            d[v] ← d[u] + 1
            π[v] = u
            Enqueue (Q, v)
    for each $v \in adj[u]$ do
        if colour [v] = gray and d[u] = d[v]
            return false [u & v are in same level and are directly connected by an edge]
    Deque [Q]
    colour [u] ← black.

return true

Runtime.

In worst case, we will have to execute whole of BFS (if graph is bipartite). Hence, time complexity is

$O(|V| + |E|)$.

Page :
Date :

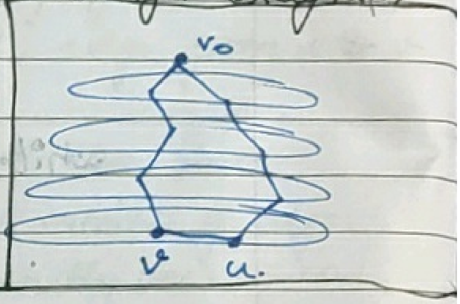
Deepanshu 2019CS50427

We claim that graph is not bipartite if two vertices in same level are directly connected.

Proof by implication.

Let v and u be at same level in BFS Tree.
Let $v_0$ be the nearest common ancestor. Note that they will always have a common ancestor because they are ~~no~~ vertices of ~~BF~~ tree (BFS Tree).

Rough Diagram




Let $v_0$ be k levels above v and u. ~~it is~~

Now since v and u are already connected by an edge, a cycle of length $2k+1$ is formed.

And since a cycle of odd length is found, graph is not bipartite.

If ~~no such~~ this is not the case, then graph formed is bipartite.

Proof by construction
We have to come up with 2 sets (call them A and B). such that $A \cap B = \phi$, $A+B = V$ and there is no edge between vertices in A (same for B). Define A and B as follows –

$A = \{ v : d[v]$ from s is odd$\}$.
$B = \{ v : d[v]$ from s is even$\}$.
Since we came up with 2 such sets, graph is bipartite.