End-Semester Exam: Math-I (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

13th April, 2022

Duration: 120 minutes **Maximum Marks:** 80

Instructions:

1. Solve any 8 of the given 9 questions.
2. Commence each answer to a question on a fresh page. If some part of a question is done later, it should also be commenced on a fresh page, and this should be clearly mentioned in the main question.
3. You may use without proof any result covered in the course (either in lecture or tutorial). However, it should be clearly identified. Results taken from other sources must be proved.

Question 1.

(a) (5 marks) Find an LU-factorization of the following matrix:

$$A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 2 & -7 & -7 & -6 \\ -1 & 2 & 6 & 4 \end{bmatrix}$$

(b) (5 marks) Use the LU-factorization method to solve the linear system $A\mathbf{x} = \mathbf{b}$ where A is the matrix given in part (a) and $\mathbf{b}$ is the vector given below. Write down all the calculations involved.

$$\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Answer.

(a) We reduce A to an echelon form U:

$$\begin{bmatrix} 1 & -2 & -4 & -3 \\ 2 & -7 & -7 & -6 \\ -1 & 2 & 6 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 2 & -7 & -7 & -6 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

We perform the corresponding column operations on the identity matrix to obtain L:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_1 \to C_1 - C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{C_1 \to C_1 + 2C_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Thus

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(b) We first solve $L\mathbf{y} = \mathbf{b}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ -1 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus

$$\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Next we solve $U\mathbf{x} = \mathbf{y}$.

$$\left[\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to -1/3 R_2} \left[\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & 1 & -1/3 & 0 & -1/3 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -14/3 & -3 & 1/3 \\ 0 & 1 & -1/3 & 0 & -1/3 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 1/2R_3} \left[\begin{array}{cccc|c} 1 & 0 & -14/3 & -3 & 1/3 \\ 0 & 1 & -1/3 & 0 & -1/3 \\ 0 & 0 & 1 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 1/3R_3} \left[\begin{array}{cccc|c} 1 & 0 & -14/3 & -3 & 1/3 \\ 0 & 1 & 0 & 1/6 & -1/3 \\ 0 & 0 & 1 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 14/3R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2/3 & 1/3 \\ 0 & 1 & 0 & 1/6 & -1/3 \\ 0 & 0 & 1 & 1/2 & 0 \end{array}\right]$$

Therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2/3 \\ -1/6 \\ -1/2 \\ 1 \end{bmatrix}$$

Rubric.

Part (a):

- 2 marks for reducing A to an echelon form U - please deduct 1/2 mark for each calculation error
- 2 marks for finding L - 1 mark for each operation on I
- 1 mark for writing the LU factorization

Part (b):

- 2 marks for solving Ly=b. Deduct 1/2 mark for each calculation error.
- 3 marks for solving Ux=y. Deduct 1/2 mark for each calculation error.

Question 2.

Find an SVD for the following matrix:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$$

Answer. Let the given matrix be called A. Thus

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

Characteristic polynomial of $A^T A$:

$$\det \begin{bmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{bmatrix} = (2-\lambda)(8-\lambda) - 16 = \lambda(\lambda - 10)$$

Eigenvalues of $A^T A$ in decreasing order: $\lambda_1 = 10, \lambda_2 = 0$

Construction of V:

$$\mathrm{Null}(A^T A - \lambda_1 I) = \mathrm{Null}\left(\begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \right) = \mathrm{Span}\{(1,2)\},$$

by inspection, and by virtue of the fact that $\dim \mathrm{col}(A^T A - 10I) = 1$ (which is also obvious because the columns are linear multiples of each other). Similarly,

$$\mathrm{Null}(A^T A - \lambda_2 I) = \mathrm{Null}\left(\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \right) = \mathrm{Span}\{(2,-1)\}.$$

Since $A^T A$ has distinct eigenvalues, the normalized eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

are orthogonal to each other. Put $V = [\mathbf{v}_1 \quad \mathbf{v_2}]$.

Construction of Σ:

$$\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Construction of U:

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{5} \\ 0 \\ 3/\sqrt{5} \end{bmatrix}$$

Thus

$$\mathbf{u}_1 = \frac{A\mathbf{v}_1}{\|A\mathbf{v}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

We extend $\{\mathbf{u}_1\}$ to an orthonormal basis of $\mathbb{R}^3$. Clearly

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is orthogonal to $\mathbf{u}_1$. We can choose $\mathbf{u}_3$ to be the cross product

$$\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Put $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$.

The SVD:

$$A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

Rubric.

- 1 mark for finding characteristic polynomial of A^TA
- 1 mark for finding the eigenvalues of A^TA
- 1 mark for the construction of Σ
- 2 marks for finding normalized eigenvectors of A^TA (1 mark for each eigenvector)
- 1 mark for normalizing the singular vector ($A\mathbf{v}_1$) and thus finding $\mathbf{u}_1$
- 3 marks for extending $\mathbf{u}_1$ to an orthonormal basis using any approach that works - you can split this as 1 for each of the two vectors and 1/2 mark for normalization, if required.
- 1 mark for writing the final SVD

Question 3.

Let $p(x) = x^2 + 7x + 9$.

(a) (2 marks) Find the coordinate vectors of $p(x), (p(x))^2, p'(x)$ and $(p'(x))^2$ with respect to the basis $\{1, x, x^2, x^3, x^4\}$ of $\mathbb{P}_4$ (Section B: $\mathbb{P}_4$ is the same as $\mathbb{R}_4[t]$).

(b) (6 marks) Consider the equation

$$x_1 p(x) + x_2 p'(x) + x_3 (p'(x))^2 + x_4 (p(x))^2 = 7$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Let

$$A = [[p(x)]_{\mathcal{B}} \quad [(p(x))^2]_{\mathcal{B}} \quad [p'(x)]_{\mathcal{B}} \quad [(p'(x))^2]_{\mathcal{B}}],$$

the matrix formed by the coordinate vectors from part (a), and let $\mathbf{b} = [7]_{\mathcal{B}}$. Solve the system of equations $A\mathbf{x} = \mathbf{b}$.

(c) (2 marks) Prove or disprove: $1 \in \operatorname{Span}\{p(x), (p(x))^2, p'(x), (p'(x))^2\}$

Answer.

Part (a): $\mathcal{B} = \{1, x, x^2, x^3, x^4\}$.

$$(p(x))^2 = (x^2 + 7x + 9)^2 = x^4 + 14x^3 + 67x^2 + 126x + 81$$

$$p'(x) = 2x + 7$$
$$(p'(x))^2 = (2x + 7)^2 = 4x^2 + 28x + 49$$

Thus

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 9 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [(p(x))^2]_{\mathcal{B}} = \begin{bmatrix} 81 \\ 126 \\ 67 \\ 14 \\ 1 \end{bmatrix}, \quad [p'(x)]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [(p'(x))^2]_{\mathcal{B}} = \begin{bmatrix} 49 \\ 28 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Part (b):

$$A = \begin{bmatrix} 9 & 81 & 7 & 49 \\ 7 & 126 & 2 & 28 \\ 1 & 67 & 0 & 4 \\ 0 & 14 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We solve $A\mathbf{x} = \mathbf{b}$ by reducing $[A \quad \mathbf{b}]$ to RREF:

$$\begin{bmatrix} 9 & 81 & 7 & 49 & 7 \\ 7 & 126 & 2 & 28 & 0 \\ 1 & 67 & 0 & 4 & 0 \\ 0 & 14 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 14R_5} \begin{bmatrix} 9 & 81 & 7 & 49 & 7 \\ 7 & 126 & 2 & 28 & 0 \\ 1 & 67 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_5} \begin{bmatrix} 9 & 81 & 7 & 49 & 7 \\ 7 & 126 & 2 & 28 & 0 \\ 1 & 67 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-67R_4}\begin{bmatrix} 9 & 81 & 7 & 49 & 7 \\ 7 & 126 & 2 & 28 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2\to R_2-126R_4}\begin{bmatrix} 9 & 81 & 7 & 49 & 7 \\ 7 & 0 & 2 & 28 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-81R_4}\begin{bmatrix} 9 & 0 & 7 & 49 & 7 \\ 7 & 0 & 2 & 28 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1\to R_1-9R_3}\begin{bmatrix} 0 & 0 & 7 & 13 & 7 \\ 7 & 0 & 2 & 28 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-7R_3}\begin{bmatrix} 0 & 0 & 7 & 13 & 7 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_3}\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 13 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2\leftrightarrow R_4}\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 7 & 13 & 7 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4\to 1/2R_4}\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 7 & 13 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3\to R_3-7R_4}\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3\leftrightarrow R_4}\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 13 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4\to 1/13R_4}\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7/13 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution:

$$x_4 = \frac{7}{13}, x_3 = 0, x_2 = 0, x_1 = -4x_4 = -\frac{28}{13}$$

Part (c):

We prove the given assertion.

The system $A\mathbf{x} = \mathbf{b}$ in part (b) has a solution. Since the coordinate transformation is an invertible linear transformation, it follows that the equation

$$x_1p(x) + x_2(p(x)^2) + x_3p'(x) + x_4(p'(x))^2 = 7$$

has a solution. Hence

$$7 \in \mathrm{Span}\{p(x), (p(x))^2, p'(x), (p'(x))^2\}.$$

As vector spaces are closed under scalar muliplication,

$$1 \in \mathrm{Span}\{p(x), (p(x))^2, p'(x), (p'(x))^2\}.$$

Rubric.

Part (a): 1/2 mark for each coordinate vector.

Part (b):

- 1 mark for writing the correct values of A and $\mathbf{b}$.
- 4 marks for solving the equation $A\mathbf{x} = \mathbf{b}$ by row reduction or any other method that works. You can deduct 1/2 mark for every calculation mistake if the solution is correct upto some errors.
- 1 mark for writing the correct solution.

Part (c):

- 1 mark for choosing to prove the given statement (please do award one mark if this intention is clearly obvious from the student's argument)
- 1 mark for a valid proof

Question 4.

Let $V = M_{2\times 2}(\mathbb{R})$ (Section B: $V = \mathbb{R}^{2\times 2}$), the set of all 2×2 matrices having real entries. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Define $\langle .,. \rangle : V \times V \to \mathbb{R}$ by

$$\langle v, w \rangle = [v]_{\mathcal{B}} \cdot [w]_{\mathcal{B}}, \quad \forall v, w \in V,$$

where $[v]_{\mathcal{B}}$ denotes the coordinates of v with respect to the basis $\mathcal{B}$, of V.

You may use the fact that $\langle .,. \rangle$ is an inner product on V without proof.

(a) (3 marks) Prove or disprove: $\mathcal{B}$ is an orthonormal basis of V with respect to the inner product $\langle .,. \rangle$.

(b) (7 marks) Let

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Apply the Gram-Schmidt process to $\mathcal{A}$ to find an orthogonal basis of V, with respect to the inner product $\langle .,. \rangle$.

Answer.

Part (a): We prove that $\mathcal{B}$ is a an orthonormal basis of V.

Let

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $[v_j]_{\mathcal{B}} = \mathbf{e}_j$ for $j = 1, 2, 3, 4$, where $\mathbf{e}_1, \ldots, \mathbf{e}_4$ is the standard basis of $\mathbb{R}^4$. Hence for any $i, j \in \{1, \ldots, 4\}$,

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Hence $\mathcal{B}$ is an orthonormal basis of V.

Part (b):

Let

$$w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

We apply the Gram-Schmidt process.

First member of orthogonal set:

$$u_1 = w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Second member of orthogonal set:

$$u_2 = w_2 - \frac{\langle w_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$$

Computation:

$$u_1 = v_3 - v_4 \implies [u_1]_{\mathcal{B}} = (0,0,1,-1)$$

$$w_2 = v_4 \implies [w_2]_{\mathcal{B}} = (0,0,0,1)$$

Hence

$$\langle u_1, u_1\rangle = 2, \langle w_2, u_1\rangle = -1$$

Therefore

$$u_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ 0 & 0 \end{bmatrix}$$

Third member of orthogonal set:

$$u_3 = w_3 - \frac{\langle w_3, u_1\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle w_3, u_2\rangle}{\langle u_2, u_2\rangle}u_2$$

Computation:

$$w_3 = v_2 - v_4 \implies [w_3]_{\mathcal{B}} = (0,1,0,-1)$$

$$u_2 = \frac{1}{2}v_3 + \frac{1}{2}v_4 \implies [u_2]_{\mathcal{B}} = (0,0,1/2,1/2)$$

$$\langle w_3, u_1\rangle = 1, \langle w_3, u_2\rangle = -1/2, \langle u_2, u_2\rangle = 1/\sqrt{2}$$

Therefore

$$u_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1/2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (1-\sqrt{2})/2\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 \end{bmatrix}$$

Fourth member of orthogonal set:

$$u_4 = w_4 - \frac{\langle w_4, u_1\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle w_4, u_2\rangle}{\langle u_2, u_2\rangle}u_2 - \frac{\langle w_4, u_3\rangle}{\langle u_3, u_3\rangle}u_3$$

Computation:

$$w_4 = v_1 - v_3 + v_4 \implies [w_4]_{\mathcal{B}} = (1,0,-1,1)$$

$$u_3 = v_2 + \frac{1-\sqrt{2}}{2\sqrt{2}}v_3 - \frac{1-\sqrt{2}}{2\sqrt{2}}v_4 \implies [u_3]_{\mathcal{B}} = \left(0, 1, \frac{1-\sqrt{2}}{2\sqrt{2}}, \frac{1-\sqrt{2}}{2\sqrt{2}}\right)$$

$$\langle w_4, u_1\rangle = -2, \langle w_4, u_2\rangle = 0, \langle w_4, u_3\rangle = 0$$

Therefore

$$u_4 = w_4 + u_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rubric.

Part (a):

- 1 mark for choosing to prove the given statement (please do award one mark if this intention is clearly obvious from the student's argument)
- 1 mark for writing that the coordinate vector of each basis element is a standard basis vector in $\mathbb{R}^4$, or for stating the same in words - for example"the coordinate map takes each basis vector to a standard basis vector"

- 1 mark for the conclusion, i.e. the remainder of the argument

Part (b):

- 3 marks for computing coordinate vectors (1/2 mark for each vector) of $u_1, w_2, w_3, u_2, w_4, u_3$
- 2 marks for computing (1/2 mark for every two correct) inner products $\langle u_1, u_1\rangle, \langle w_2, u_1\rangle, \langle u_2, u_2\rangle, \langle w_3, u_1\rangle, \langle w_3, u_2\rangle, \langle w_4, u_1\rangle, \langle w_4, u_2\rangle, \langle w_4, u_3\rangle$
- 2 marks for finding the four members of the orthogonal basis (1/2 mark each for u_1, u_2, u_3, u_4)

Question 5 (10 marks)**.**

Let

$$A = \begin{bmatrix} 9 & -8 & 3 \\ 11 & 10 & 3 \\ 1 & -1 & -2 \end{bmatrix}$$

Is A diagonalizable? Justify your answer.

Answer. Characteristic polynomial of A:

$$p(\lambda) = \det(A - \lambda I) = -\lambda^3 + 17\lambda^2 - 140\lambda - 416$$

Now, $p'(\lambda) = -3\lambda^2 + 38\lambda - 140$.

Suppose if possible that $p(\lambda)$ has more than one real root. Then by Rolle's theorem, $p'(\lambda)$ must vanish between any two consecutive roots.

However, $p'(\lambda)$ can never be zero, because the quadratic polynomial $-3\lambda^2 + 38\lambda - 140$ has discriminant $-236 < 0$.

Therefore $p(\lambda)$ has two complex roots. As roots of polynomials occur in complex conjugate pairs, $p(\lambda)$ must have one real root and two complex roots.

Therefore A has distinct eigenvalues. Therefore A is diagonalizable.

Rubric.

- 1 mark for choosing to show that A is diagonalizable.
- 1 mark for finding the characteristic polynomial of A.
- 2 marks for finding the derivative of the characteristic polynomial.
- 1 mark for finding the discriminant of the derivative and showing that it is negative.
- 1 mark for concluding that the derivative can never vanish.
- 2 marks for using Rolle's theorem to conclude that the characteristic polynomial has at most one real root.
- 1 marks for concluding that the characteristic polynomial has 2 complex roots.
- 1 marks for concluding that A is diagonalizable, as its eigenvalues are distinct.

Question 6.

(a) (6 marks) Find $\operatorname{col} A, \operatorname{null} A$ and $\operatorname{row} A$ for the matrix

$$A = \begin{bmatrix} 7 & 4 & -1 \\ 3 & 2 & -1 \\ 5 & 1 & 3 \end{bmatrix}$$

and a basis for each of the three.

(b) (4 marks) Find an orthonormal basis for $(\operatorname{null}(A))^{\perp}$.

Answer.

Part (a):

We row reduce A to RREF:

$$A = \begin{bmatrix} 7 & 4 & -1 \\ 3 & 2 & -1 \\ 5 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 \to 1/7 R_1} \begin{bmatrix} 1 & 4/7 & -1/7 \\ 3 & 2 & -1 \\ 5 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 4/7 & -1/7 \\ 0 & 2/7 & -4/7 \\ 5 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 5R_1} \begin{bmatrix} 1 & 4/7 & -1/7 \\ 0 & 2/7 & -4/7 \\ 0 & -13/7 & 26/7 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 13/2 R_2} \begin{bmatrix} 1 & 4/7 & -1/7 \\ 0 & 2/7 & -4/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2/7 & -4/7 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to 7/2 R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\operatorname{null} A = \operatorname{Span}\{(-1, 2, 1)\}$. So a basis of $\operatorname{null} A$ is $\{(-1, 2, 1)\}$.

Since $\dim \operatorname{null} A$, it follows from the Rank Nullity Theorem that

$$\dim \operatorname{col} A = \dim \operatorname{row} A = 2$$

Hence any two rows or columns of A which are not linear multiples of each other constitute a basis for $\operatorname{row} A$ and $\operatorname{col} A$ respectively.

For example, a basis of colA is $\{(7, 3, 5), (4, 2, 1)\}$ and a basis of $\operatorname{row} A$ is $\{(3, 2, -1), (5, 1, 3)\}$.

Part (b):

Since the orthogonal complement of $\operatorname{null} A$ is $\operatorname{row} A$, we apply the Gram-Schmidt process to the basis in part (a) and normalize the vectors to obtain an orthonormal basis of $\operatorname{row} A$.

Let $\mathbf{w}_1 = (3, 2, -1)$.

Let

$$\mathbf{w}_2 = (5, 1, 3) - \frac{(3, 2, -1) \cdot (5, 1, 3)}{(3, 2, -1) \cdot (3, 2, -1)}(3, 2, -1) = (5, 1, 3) - (3, 2, -1) = (2, -1, 4)$$

Thus an orthonormal basis of $\operatorname{row} A$ is

$$\left\{ \begin{bmatrix} 3/\sqrt{14} \\ 2/\sqrt{14} \\ -1/\sqrt{14} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{21} \\ -1/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix} \right\}$$

Rubric. Part (a):

- 2 marks for basis of null A: 1 mark for solving Ax=0 and 1 mark for writing the basis. (Please note that this basis is unique upto multiplication by scalars).
- 2 marks for basis of col A: 1 mark for finding a basis, 1 mark for justification - either by identifying pivot columns of A, or using the argument mentioned above.
- : 2 marks for basis of row A: 1 mark for finding a basis, 1 mark for justification.

Part (b):

- 2 marks for finding two linearly independent vectors orthogonal to null A, either by using the argument outlined above, or just by inspection. (You need to check that the vectors are orthogonal to $(-1, 2, 1)$.)
- 2 marks for normalizing the vectors - 1 for each.

Question 7.

(a) (5 marks) Evaluate $\det(A)$, where A is the $(n-1)\times(n-1)$ matrix given below (answer in terms of a formula for $n \geq 2$).

$$\begin{bmatrix} (n-1) & -1 & \dots & \dots & -1 \\ -1 & (n-1) & \dots & \dots & -1 \\ -1 & -1 & \ddots & & -1 \\ \vdots & \vdots & & \ddots & -1 \\ -1 & -1 & \dots & & (n-1) \end{bmatrix}$$

(In case it's not obvious, the diagonal entries of A are $n-1$ and the rest of the entries are -1.)

(b) (5 marks) Determine the eigenvalues and corresponding eigenvectors of A.

Answer.

First Solution:

We will first do (b). (a) will then follow directly.

Note that A is a symmetric matrix. Therefore all eigenvalues of A are real and A is orthogonally diagonalizable. Furthermore each row sum of A is equals $(n-1)+(n-2)(-1)=1$. Hence $\lambda_1 = 1$ is an eigenvalue of A, with the corresponding eigenvector $\mathbf{v}_1 = [1]$, the vector with all entries equal to 1. It also follows by inspection that if we put $\lambda_2 = n$, then

$$B = A - \lambda_2 I = \begin{bmatrix} -1 & -1 & \dots & \dots & -1 \\ -1 & -1 & \dots & \dots & -1 \\ -1 & -1 & \ddots & & -1 \\ \vdots & \vdots & & \ddots & -1 \\ -1 & -1 & \dots & & -1 \end{bmatrix}$$

has RREF

$$\begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & & 0 \end{bmatrix}$$

Therefore $\lambda_2 = n$ is an eigenvalue with geometric multiplicity = nullity of $B = n-2$.

Hence $\lambda_1 = 1$ and $\lambda_2 = n$ are the only eigenvalues of A with (geometric) multiplicities 1 and $n-2$ respectively, and corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$.

For $i = 2, \dots, n-1$, $\mathbf{v}_i$ is the vector which has 1 in its first position, -1 in its i-th position and zeros elsewhere. So, if

$$P = [\mathbf{v}_1 \quad \dots \quad \mathbf{v}_{n-1}],$$

then $A = PDP^{-1}$ where $D = \mathrm{diag}(1, n, \dots, n)$.

(Note: $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ is not an orthonormal set, but this is not required for solving the problem.)

(a):

From (b) it follows that

$$\det A = \det D = n^{n-2}$$

Second Solution:

It is possible to determine $\det A$ directly using row reduction, without using part (b).

Illustrated below:-(Steps have been combined in a valid way to save space and time.)

$$\begin{bmatrix} (n-1) & -1 & \dots & \dots & -1 \\ -1 & (n-1) & \dots & \dots & -1 \\ -1 & -1 & \ddots & & -1 \\ \vdots & \vdots & & \ddots & -1 \\ -1 & -1 & \dots & & (n-1) \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2 + \dots + R_{n-1}} \begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ -1 & (n-1) & \dots & \dots & -1 \\ -1 & -1 & \ddots & & -1 \\ \vdots & \vdots & & \ddots & -1 \\ -1 & -1 & \dots & & (n-1) \end{bmatrix}$$

For each $i = 2, \dots, n-1$ do the row operation $R_i \to R_i + R_1$, and obtain

$$\begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ 0 & n & \dots & \dots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & & n \end{bmatrix}$$

We now have an upper triangular matrix.

$\therefore \det(A) =$ product of diagonal elements $= n^{n-2}$.

(Remark: It is possible to follow another sequence of row operations to get the same answer.)

Rubric.

(a)

- 2 marks for getting the correct formula $\det A = n^{n-2}$
- 3 marks for a valid method:

  Either using the result of part (b) as done in the solution OR any valid row reduction scheme.

  Zero marks for method in case of errors in method even if answer is correct (if answer is wrong, method is not correct obviously).

(b)

- 0.5 marks for showing that $\lambda_1 = 1$ is an eigenvalue
- 0.5 marks for identifying its eigenvector $\mathbf{v}_1$
- 1.5 marks for showing that $\lambda_2 = n$ is an eigenvalue
- 2.5 marks for identifying all its $n-2$ eigenvectors and concluding that 1 and n are all the eigenvalues.

  If less than $n-2$ eigenvectors are identified, award only 0.5 marks instead of 2.5

Question 8 (10 marks)**.**

Let V be a finite-dimensional vector space over F, with $\dim V = n$, and let $T \in L(V, V)$ with $(T - \lambda I)^n = 0$ but $(T - \lambda I)^{n-1} \neq 0$, where $\lambda \in F$. Show that there exists an ordered basis $\mathcal{B}$ of V such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda & 0 & & \dots & 0 \\ 1 & \lambda & 0 & \dots & 0 \\ 0 & 1 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & \lambda \end{bmatrix}$$

Answer.

Choose $v \in V$ such that $(T - \lambda I)^{n-1} v \neq 0$.

Let

$$v_j = \begin{cases} v & \text{if } j = 1 \\ (T - \lambda I)^{j-1} v & \text{if } j = 2, \dots, n \end{cases}$$

Let $\mathcal{B} = \{v_1, \dots, v_n\}$. Since $\dim V = n$ and $\mathcal{B}$ has exactly n elements, it follows that $\mathcal{B}$ is a basis of V if we show that $\mathcal{B}$ is linearly independent.

Suppose

$$\sum_{i=1}^{n} c_i v_i = 0$$

where $c_1, \dots, c_n \in \mathbb{R}$. Let k be the smallest index for which $c_k \neq 0$. Then

$$(T - \lambda I)^{n-k} (\sum_{i=1}^{n} c_i v_i) = 0 \implies (T - \lambda I)^{n-k} (\sum_{i=k}^{n} c_i v_i) = 0$$

Hence

$$(T - \lambda I)^{n-k} (c_k v_k) + (T - \lambda I)^{n-k} \left(\sum_{i=k+1}^{n} c_i (T - \lambda I)^{i-1} v \right) = 0$$

As $(T - \lambda I)^n = 0$, it follows that

$$(T - \lambda I)^{n-k} (c_k v_k) = 0$$

As $c_k \neq 0$, we obtain

$$(T - \lambda I)^{n-k} (v_k) = 0 \implies (T - \lambda I)^{n-1} (v) = 0,$$

which is a contradiction.

For each $j = 1, \dots, n$,

$$T(v_j) = (T - \lambda I) v_j + \lambda v_j = v_{j+1} + \lambda_j v_j$$

Hence the j-th column of $[T]_{\mathcal{B}}$ is

$$[T(v_j)]_{\mathcal{B}} = \mathbf{e}_{j+1} + \lambda_j \mathbf{e}_j,$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ denotes the standard basis of $\mathbb{R}^n$. Hence the matrix $[T]_{\mathcal{B}}$ has the required form.

Rubric.

- 1 mark for a correct guess for the basis $\mathcal{B}$
- 4 marks for showing that $\mathcal{B}$ is linearly independent: 1 mark for induction hypothesis or smallest integer argument, 2 marks for applying the transformation to the linear combination and reaching an expression that involves one scalar, 1 mark for reaching a contradiction
- 1 mark for justifying that $\mathcal{B}$ is a basis, using the dimension equals cardinality argument
- 2 marks for showing that $T(v_j) = v_{j+1} + \lambda_j v_j$
- 2 marks for finding the columns of $[T]_{\mathcal{B}}$

Question 9.

Let $V = \mathbb{R}^{n\times n}$ (Section A: $\mathbb{R}^{n\times n}$ is the same as $M_{n\times n}(\mathbb{R})$) and let $W = \{A \in V \mid A \text{ is skew-symmetric}\}$. (A square matrix A is said to be *skew-symmetric* if $A^T = -A$.)

(a) (2 marks) Show that W is a subspace of V.

(b) (4 marks) Find a basis of W, and hence its dimension.

(c) (4 marks) Prove or Disprove: Every $A \in V$ satisfies $A = B + C$, where B, C are symmetric and skew-symmetric respectively.

Answer.

Part (a):

Clearly, $0^T = 0$ (whhere 0 is the zero matrix).

$\therefore 0 \in W$.

If $A, B \in W$ then $A^T = -A, B^T = -B \implies A^T + B^T = -A - B \implies (A+B)^T = -(A+B) \implies A + B \in W$. Hence W is closed under addition.

If $A \in W, c \in \mathbb{R}$ then $A^T = -A \implies cA^T = -cA \implies (cA)^T = -(cA) \implies cA \in W$. Hence W is closed under scalar multiplication.

Part (b):

For $i \neq j$, let us denote by V_{ij} the matrix whose i,j-th entry is 1, j,i-th entry is -1 and all other entries are zeros. Let

$$\mathcal{B} = \{V_{ij} \mid 1 \leq i \leq n, j < i\}$$

Claim: $\mathcal{B}$ is a basis of W.

Suppose

$$\sum_{\substack{i,j=1 \\ i>j}}^{n} c_{i,j} V_{i,j} = 0$$

where $c_{i,j} \in \mathbb{R}$. Let $l, m \in \{1, \dots, n\}, l > m$. Then the l,m-th entry of $\sum\limits_{\substack{i,j=1 \\ i>j}}^{n} c_{i,j} V_{i,j}$ is $c_{l,m}$ and therefore $c_{l,m} = 0$. Hence $\mathcal{B}$ is linearly independent.

Each $V_{i,j}$ is skew-symmetric by construction. Therefore $\mathcal{B} \subset W$. Let $A \in W$ be arbitrary. Let us denote the entries of A by a_{ij}. Then

$$a_{ij} = -a_{ji}, \forall i, j \in \{1, \dots, n\}$$

Therefore

$$A = \sum_{\substack{i,j=1 \\ i>j}}^{n} a_{i,j} V_{i,j} \in \operatorname{Span} \mathcal{B}.$$

Hence $\mathcal{B} \subset W \subset \operatorname{Span} \mathcal{B} \implies W = \operatorname{Span} \mathcal{B}$. It follows that $\mathcal{B}$ is a basis of W.

The dimension of W equals the number of elements in $\mathcal{B}$, which equals the number of pairs (i, j) where $1 \leq i \leq n$ and $j < i$. This number is the sum of the first $n - 1$ natural numbers, which is $\dfrac{n(n-1)}{2}$.

Part (c):

Let $B = \dfrac{1}{2}(A + A^T)$ and $C = \dfrac{1}{2}(A - A^T)$. Then

$$B^T = B, \quad C^T = -C$$

and

$$A = B + C$$

Rubric.

Part (a):

- 1 mark for showing that W is non-empty
- 1/2 mark for closure under addition
- 1/2 mark for closure under scalar multiplication

Part (b):

- 1 mark for identifying a basis $\mathcal{B}$
- 1 mark for show $\mathcal{B}$ is linearly independent
- 1 mark for show $\mathcal{B}$ spans W
- 1 mark for finding the dimension of W

Part (c):

- 1 mark for selecting prove (please give marks if the intention is obvious from the argument)
- 1 mark for finding B
- 1 mark for finding C
- 1 mark for the expression $A = B + C$