Rubric for End-Semester Exam

List of Common Errors and Marks Deductions:

1. Using an undefined symbol. Please deduct 1/2 mark each time this is done. More marks may be deducted if the undefined symbol is used in the argument.
2. Writing an equation in which the LHS and RHS are not comparable, for example, if the LHS is an $m \times n$ matrix and the RHS is a real number. Please deduct 1/2 mark each time this is done.
3. Writing a meaningless or completely illogical statement. Please deduct 1 mark for every meaningless statement.
4. Please deduct 1/2 mark for every calculation mistake.

Question 1 (10 marks)**.** Suppose that V be a vector space over a field F. Let $T \in \mathcal{L}(V, V)$ (i.e. T is a linear transformation from V to V).

Let $v \in V$ be a vector such that $T^m v \neq 0$ but $T^{m+1}(v) = 0$ for some $m \geq 0$.

Show that $v, Tv, \ldots, T^m v$ are linearly independent.

Solution. Let

$$c_1 v + c_2 Tv + \cdots + c_{m+1} T^m v = 0,$$

where $c_1, \ldots, c_{m+1} \in \mathbb{R}$. Suppose if possible that the c_js are not all zero. Let k be the smallest index for which $c_k \neq 0$. Then

$$T^{m-k+1}(c_1 v + c_2 Tv + \cdots + c_{m+1} T^m v) = T^{m-k+1}(0) = 0.$$

Therefore

$$c_1 T^{m-k+1} v + c_2 T^{m-k+2} v + \cdots + c_{m+1} T^{2m-k+1} v = 0 \implies c_k T^m v = 0.$$

As $T^m v \neq 0$, it follows that $c_k = 0$, which is a contradiction. Therefore

$$c_1 = c_2 = \cdots = c_{m+1} = 0.$$

Rubric.

- Award 1 mark for the step: $v, Tv, \ldots, T^m v$ are l.i. iff $c_1 v + c_2 Tv + \cdots + c_{m+1} T^m v = 0 \implies c_1 = \cdots = c_{m+1} = 0$.
- Award 1 mark for **applying** T^m to the linear combination $c_1 v + c_2 Tv + \cdots + c_{m+1} T^m v$
- Award 1 mark for deriving the conclusion from the above step that the coefficient of v is zero, correctly
- Award 1 mark for **applying** T^{m-1} to the linear combination $c_2 Tv + \cdots + c_{m+1} T^m v$
- Award 1 mark for deriving the conclusion from the above step that the coefficient of Tv is zero, correctly.
- Award 5 marks for showing that the remaining coefficients are zero.
- Deduct 1 mark if proof by pattern technique is used, but the proof is otherwise correct.
- Deduct 1 mark if the operation of the function T is described using the word "multiplication".
- Deduct 2 marks for proving that the coefficient of $T^{m+1} v$ is zero, in an arbitrary linear combination.

Question 2 (10 marks)**.** Let $V = F^n$ and consider the operator $T : V \to V$ given by

$$T(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i, \ldots, \sum_{i=1}^{n} x_i \right)$$

(a) (1 mark) Construct the matrix A of T relative to any suitable basis of V.

(b) (6 marks) Determine the eigenvalues and corresponding eigenvectors of T.

(c) (3 marks) Is T diagonalizable (YES/NO)? Justify your answer briefly, by referring to a suitable result.

Solution. (a) Let $A = (a_{ij})$ be the matrix of T with respect to the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V. Then

$$\begin{aligned} A &= [[T(\mathbf{e}_1)]_{\mathcal{B}} \quad \ldots \quad [T(\mathbf{e}_n)]_{\mathcal{B}}] \\ &= [T(\mathbf{e}_1) \quad \ldots \quad T(\mathbf{e}_n)] \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \end{aligned}$$

i.e. $a_{ij} = 1$ for $i, j = 1, \ldots, n$.

(b) Clearly, the rank of A is 1. Hence $\det A = 0$. Thus zero is an eigenvalue of A.

Now the eigenspace corresponding to the zero eigenvalue is simply $\operatorname{Nul} A$. Further,

$$\dim \operatorname{Nul} A = n - \operatorname{rank}(A) = n - 1.$$

Thus the multiplicity of the zero eigenvalue is at least $n - 1$. In fact, the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix}$$

are a basis for the eigenspace corresponding to the zero eigenvalue Also

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ n \\ \vdots \\ n \end{bmatrix}$$

Thus n is an eigenvalue of A and $(1, 1, \ldots, 1)$ is a corresponding eigenvector.

Hence the multiplicity of the zero eigenvalue equals $n - 1$.

(c) A is a symmetric matrix, therefore diagonalizable. Thus the matrix of T with respect to the standard basis is diagonalizable. Therefore T is diagonalizable.

Rubric.

(a) 1 mark for fixing a basis and finding matrix of T relative to that basis

(b)
- 1 mark for each correctly found eigenvalue
- 1 mark for finding the eigenvector corresponding to n
- 3 marks for finding the remaining eigenvectors

(c) 3 marks for the correct justification

Question 3. Let $(V, \langle ., . \rangle)$ be a finite dimensional inner product space, and let U and W be subspaces of V.

(a) (5 marks) Let W be a subspace of V. Show that

$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$$

(b) (5 marks) Show that

$$(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$$

Solution. (a) Let $x \in (U + W)^{\perp}$. Then

$$\langle x, y \rangle = 0, \ \forall y \in U + W.$$

For every $u \in U, u = u + 0$. Therefore $U \subset U + W$. Similarly, $W \subset U + W$. Hence

$$\langle x, y \rangle = 0, \ \forall y \in U \implies x \in U^{\perp}.$$

Similarly, $x \in W^{\perp}$. Therefore $x \in U^{\perp} \cap W^{\perp}$. As the choice of x was arbitrary, $(U + W)^{\perp} \subset U^{\perp} \cap W^{\perp}$.

Next, suppose $x \in U^{\perp} \cap W^{\perp}$. Let $y \in U + W$. Then $y = u + w$ for some $u \in U$ and $w \in W$. Therefore

$$\langle x, y \rangle = \langle x, u + w \rangle = \langle x, u \rangle + \langle x, w \rangle = 0 + 0 = 0.$$

Hence $x \in (U + W)^{\perp}$. As the choice of x was arbitrary,

$$U^{\perp} \cap W^{\perp} \subset (U + W)^{\perp}.$$

Thus

$$U^{\perp} \cap W^{\perp} = (U + W)^{\perp}$$

(b) Claim: If X is any subspace of V, then $(X^{\perp})^{\perp} = X$.

The proof of the above claim can be found on page 195 of [6].

Using part (a) and the above claim, we obtain

$$U \cap W = (U^{\perp})^{\perp} \cap (W^{\perp})^{\perp} = (U^{\perp} + W^{\perp})^{\perp}$$

Using the stated claim again, we obtain

$$(U \cap W)^{\perp} = ((U^{\perp} + W^{\perp})^{\perp})^{\perp} = U^{\perp} + W^{\perp}$$

Rubric.

(a)
- 2.5 marks for showing that $(U+W)^{\perp} \subset U^{\perp} \cap W^{\perp}$.
- 2.5 marks for showing that $U^{\perp} \cap W^{\perp} \subset (U+W)^{\perp}$

(b) 5 marks for the applying the relation $(X^{\perp})^{\perp} = X$ to derive the required result.

Question 4 (10 marks)**.** Let $a, b \in \mathbb{R}$, and let $b \neq 0$. Orthogonally diagonalize

$$A = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix}$$

Solution. Let $p(\lambda)$ be the characteristic polynomial of A. Then

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= (a - \lambda)((a - \lambda)^2 - b^2) \\ &= (a - \lambda)(a - \lambda + b)(a - \lambda - b) \end{aligned}$$

As $b \neq 0$, the eigenvalues $\lambda_1 = a, \lambda_2 = a + b, \lambda_3 = a - b$ are distinct.

Therefore the eigenvectors corresponding to λ_1, λ_2 and λ_3 are mutually orthogonal. Let us find them and normalize them, to obtain an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

$$A - \lambda_1 I = A - aI = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix}$$

As $\dim \mathrm{Nul}(A - aI) = 1$, a basis for the eigenspace corresponding to λ_1 is $\mathbf{e}_2$. Next,

$$A - \lambda_2 I = A - (a + b)I = \begin{bmatrix} -b & 0 & b \\ 0 & -b & 0 \\ b & 0 & -b \end{bmatrix}$$

As $\dim \mathrm{Nul}(A - (a + b)I) = 1$, a basis for the eigenspace corresponding to λ_2 is the eigenvector $(1, 0, 1)$. Next,

$$A - \lambda_3 I = A - (a - b)I = \begin{bmatrix} b & 0 & b \\ 0 & b & 0 \\ b & 0 & b \end{bmatrix}$$

As $\dim \mathrm{Nul}(A - (a - b)I) = 1$, a basis for the eigenspace corresponding to λ_1 is the eigenvector $(1, 0, -1)$.

After normalizing the eigenvectors corresponding to λ_1, λ_2 and λ_3 we construct them to form the orthogonal matrix

$$P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Thus

$$A = PDP^T,$$

where

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

Please note: The matrices P and D are only unique upto a permutation of the columns of P and a corresponding permutation of the diagonal entries of D.

Rubric.

- Finding characteristic polynomial: 2 marks
- Finding eigenvalues : 1 mark
- Finding eigenvectors: 3 marks
- Normalizing the eigenvectors: 1 mark
- Justifying the fact that the eigenvectors obtained are mutually orthogonal: 1 mark
- Writing P and D in the proper order (eigenvectors should correspond to eigenvalues in the same order): 2 marks

Question 5 (10 marks)**.** Let

$$A = \begin{bmatrix} -49\pi & 20\pi \\ -136\pi & 55\pi \end{bmatrix}$$

Find an invertible matrix P and a matrix B of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that

$$A = PBP^{-1}.$$

Solution. Let $p(\lambda)$ be the characteristic polynomial of A. Then

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} -49\pi - \lambda & 20\pi \\ -136\pi & 55\pi - \lambda \end{bmatrix} \\ &= \lambda^2 - 6\pi\lambda + 25\pi^2 \end{aligned}$$

The eigenvalues are $3\pi \pm 4\pi i$.

Let us find the eigenspace corresponding to the eigenvalue $\lambda = 3\pi - 4\pi i$.

$$\operatorname{Nul}(A - \lambda I) = \operatorname{Nul} \begin{bmatrix} -49\pi - \lambda & 20\pi \\ -136\pi & 55\pi - \lambda \end{bmatrix} = \operatorname{Nul} \begin{bmatrix} -52\pi + 4\pi i & 20\pi \\ -136\pi & 52\pi + 4\pi i \end{bmatrix}$$

We solve the simultaneous system

$$\begin{aligned} (-52\pi + 4\pi i)z_1 + 20\pi z_2 &= 0 \\ -136\pi z_1 + (52\pi + 4\pi i)z_2 &= 0 \end{aligned}$$

to obtain the general solution

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mu \begin{bmatrix} 5 \\ 13 - i \end{bmatrix}$$

where $\mu \in \mathbb{C}$. Let

$$\mathbf{v}_1 = \operatorname{Re} \begin{bmatrix} 5 \\ 13 - i \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}, \quad \mathbf{v}_2 = \operatorname{Im} \begin{bmatrix} 5 \\ 13 - i \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Put $P = [\mathbf{v}_1 \quad \mathbf{v}_2]$. Then

$$P = \begin{bmatrix} 5 & 0 \\ 13 & -1 \end{bmatrix}$$

Then

$$P^{-1}AP = \begin{bmatrix} 1/5 & 0 \\ 13/5 & -1 \end{bmatrix} \begin{bmatrix} -49\pi & 20\pi \\ -136\pi & 55\pi \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 13 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3\pi & -4\pi \\ 4\pi & 3\pi \end{bmatrix} = B$$

Rubric.

- Finding characteristic polynomial: 1 mark
- Finding eigenvalues: 1 mark
- Finding a complex eigenvector: 3 marks
- Constructing P using real and imaginary parts of this complex eigenvector: 3 marks
- Finding B: 2 marks

Please note: The answers for P and B are not unique. They will vary according to which complex eigenvector is chosen in order to construct P. Please check the calculations so that the equation $PBP^{-1} = A$ (or $P^{-1}AP = B$) holds.

Question 6.

(a) (4 marks) Find an LU factorization of the following matrix:

$$A = \begin{bmatrix} 2 & -4 & 2\pi^2 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

(b) (6 marks) Solve the equation $A\mathbf{x} = \mathbf{b}$ by using the LU-factorization of A that you obtained in part (a). **Do not use any other method.**

$$\mathbf{b} = \begin{bmatrix} -4\pi^2 \\ -12\pi^2 + 7 \\ 8 - 4\pi^2 \end{bmatrix}$$

Solution.

(a) Step 1. Divide the first column of A by 2 to obtain the first column of L. Perform the corresponding inverse row operations clear out the entries in the first column of A below the first entry. Resultant matrices:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}, A \sim \begin{bmatrix} 2 & -4 & 2\pi^2 & -2 \\ 0 & 3 & 7 - 6\pi^2 & 3 \\ 0 & -6 & 8 + \pi^2 & -1 \end{bmatrix}$$

Step 2. Divide the entries below the first entry of the second column of the matrix obtained in Step 1 (row equivalent to A) by 3 to obtain the second column of L. Perform the corresponding inverse row operations on the matrix obtained in Step 1 (row equivalent to A). Resultant matrices:

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}, A \sim \begin{bmatrix} 2 & -4 & 2\pi^2 & -2 \\ 0 & 3 & 7 - 6\pi^2 & 3 \\ 0 & 0 & 22 - 11\pi^2 & 5 \end{bmatrix}$$

Step 3. As the resultant matrix (row equivalent to A) is in echelon form, the required factorization is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -4 & 2\pi^2 & -2 \\ 0 & 3 & 7 - 6\pi^2 & 3 \\ 0 & 0 & 22 - 11\pi^2 & 5 \end{bmatrix}$$

(b) We first solve the system $L\mathbf{y} = \mathbf{b}$, where $\mathbf{y} = (y_1, y_2, y_3)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ 3y_1 + y_2 \\ -y_1/2 - 2y_2 + y_3 \end{bmatrix} = \begin{bmatrix} -4\pi^2 \\ -12\pi^2 + 7 \\ 8 - 4\pi^2 \end{bmatrix}$$

to obtain

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4\pi^2 \\ -12\pi^2 + 7 - 3y_1 \\ 8 - 4\pi^2 + y_1/2 + 2y_2 \end{bmatrix} = \begin{bmatrix} -4\pi^2 \\ 7 \\ 22 - 6\pi^2 \end{bmatrix}$$

We next solve the system $U\mathbf{x} = \mathbf{y}$, using back substitution.

$$\begin{bmatrix} 2 & -4 & 2\pi^2 & -2 \\ 0 & 3 & 7 - 6\pi^2 & 3 \\ 0 & 0 & 22 - 11\pi^2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4\pi^2 \\ 7 \\ 22 - 6\pi^2 \end{bmatrix}$$

As the fourth column of U is not a pivot column, x_4 is a free variable. So

$$(22 - 11\pi^2)x_3 + 5x_4 = 22 - 6\pi^2 \implies x_3 = \frac{22 - 6\pi^2 - 5x_4}{22 - 11\pi^2}$$

From

$$3x_2 + (7 - 6\pi^2)x_3 + 3x_4 = 7$$

we get

$$\begin{aligned} x_2 &= \frac{1}{3}(7 - 3x_4 - (7 - 6\pi^2)x_3) \\ &= \frac{1}{3}(7 - 3x_4 - (7 - 6\pi^2)\left(\frac{22 - 6\pi^2 - 5x_4}{22 - 11\pi^2}\right) \\ &= \frac{\pi^2(97 - 36\pi^2) + (3\pi^2 - 31)x_4}{66 - 33\pi^2}. \end{aligned}$$

From

$$2x_1 - 4x_2 + 2\pi^2 x_3 - 2x_4 = -4\pi^2,$$

we get

$$\begin{aligned} x_1 &= 2x_2 - \pi^2 x_3 + x_4 - 2\pi^2 \\ &= \frac{2\pi^2(97 - 36\pi^2) + (6\pi^2 - 62)x_4}{66 - 33\pi^2} - \frac{22\pi^2 - 6\pi^4 - 5\pi^2 x_4}{22 - 11\pi^2} + x_4 - 2\pi^2 \\ &= \frac{(4 - 12\pi^2)(x_4 - \pi^2)}{66 - 33\pi^2} \end{aligned}$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \dfrac{4-12\pi^2}{66-33\pi^2} \\[2ex] \dfrac{3\pi^2-31}{66-33\pi^2} \\[2ex] \dfrac{-5}{22-11\pi^2} \\[2ex] 1 \end{bmatrix} + \begin{bmatrix} \dfrac{-\pi^2(4-12\pi^2)}{66-33\pi^2} \\[2ex] \dfrac{\pi^2(97-36\pi^2)}{66-33\pi^2} \\[2ex] \dfrac{22-6\pi^2}{22-11\pi^2} \\[2ex] 0 \end{bmatrix}$$

Rubric.

Part (a)
- 2 marks for Step 1
- 2 marks for Step 2

Pbrt (b)
- 2 marks for solving $L\mathbf{y} = \mathbf{b}$
- 1 mark for recognizing that x_4 is a free variable in the system $U\mathbf{x} = \mathbf{y}$
- 1 mark for computing x_3 as a function of x_4
- 1 mark each for computing x_2 as a function of x_4
- 1 mark each for computing x_1 as a function of x_4

Question 7. Let $V = C^1[-\pi, \pi]$, the set of all continuously differentiable functions defined on the interval $[-\pi, \pi]$.

(A function f is said to be continuously differentiable on $[-\pi, \pi]$, if f is differentiable at every point in $[-\pi, \pi]$ and its derivative f' is continuous on $[-\pi, \pi]$.)

(a) (2 marks) Show that V is a vector space over $\mathbb{R}$, under the usual operations of pointwise addition of functions and pointwise multiplication of a function by a scalar.

(You may assume without proof that every differentiable function is continuous, i.e. $C^1[-\pi, \pi] \subset C[-\pi, \pi]$.)

(b) (2 marks) Show that the mapping $\langle ., . \rangle : V \times V \to \mathbb{R}$, defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} f'(t)g'(t)\,\mathrm{dt}$$

is an inner product on V.

(c) (6 marks) Find an orthogonal basis for the subspace

$$W = \mathrm{Span}\{1, \cos t, \sin t, \cos^2 t\},$$

of V, with respect to the inner product defined in part (b).

Solution. (a) Let $f, g \in V$. Then

$$(f + g)' = f' + g' \in C[-\pi, \pi].$$

Therefore $f + g$ is continuously differentiable. Hence V is closed under addition.

Next, if $f \in V, c \in \mathbb{R}$, then

$$(cf)' = cf' \in C[-\pi, \pi].$$

Therefore cf is continuously differentiable. Hence V is closed under scalar multiplication. Thus V is a subspace of $C[-\pi, \pi]$. Therefore V is a vector space.

(b) Let us verify that the mapping $\langle ., . \rangle : V \times V \to \mathbb{R}$ satisfies the conditions for being an inner product:

(i) Let $f, g, h \in V$. Then

$$\begin{aligned}
\langle f+g, h\rangle &= \int_{-\pi}^{\pi} (f+g)(t)h(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} (f+g)'(t)h'(t)\,\mathrm{dt} \\
&= \int_{-\pi}^{\pi} (f(t)+g(t))h(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} (f'(t)+g'(t))h'(t)\,\mathrm{dt} \\
&= \int_{-\pi}^{\pi} f(t)h(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} g(t)h(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} f'(t)h'(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} g'(t)h'(t)\,\mathrm{dt} \\
&= \int_{-\pi}^{\pi} f(t)h(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} f'(t)h'(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} g(t)h(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} g'(t)h'(t)\,\mathrm{dt} \\
&= \langle f, h\rangle + \langle g, h\rangle
\end{aligned}$$

(ii) Let $f, g \in V, c \in \mathbb{R}$. Then

$$\begin{aligned}
\langle cf, g\rangle &= \int_{-\pi}^{\pi} (cf)(t)g(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} (cf)'(t)g'(t)\,\mathrm{dt} \\
&= \int_{-\pi}^{\pi} cf(t)g(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} cf'(t)g'(t)\,\mathrm{dt} \\
&= c\left(\int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} f'(t)g'(t)\,\mathrm{dt}\right) \\
&= c\langle f, g\rangle
\end{aligned}$$

(iii) Let $f, g \in V$. Then

$$\begin{aligned}
\langle f, g\rangle &= \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} f'(t)g'(t)\,\mathrm{dt} \\
&= \int_{-\pi}^{\pi} g(t)f(t)\,\mathrm{dt} + \int_{-\pi}^{\pi} g'(t)f'(t)\,\mathrm{dt} \\
&= \langle g, f\rangle
\end{aligned}$$

(iv) If $f = 0$, then

$$\langle f, f\rangle = 2\int_{-\pi}^{\pi} 0\,dt = 0.$$

Suppose $f \in V$ and $\langle f, f\rangle = 0$. Then

$$\int_{-\pi}^{\pi} (f(t))^2\,dt + \int_{-\pi}^{\pi} (f'(t))^2\,dt = 0$$

Hence $f = 0$.

(The integral of a non-negative continuous function can only be zero if the function is the zero function. This is a fact which will be covered later on in Calculus/Analysis courses. The students do not need to mention this in the proof.)

(c) Let $f_0 = 1, f_1 = \cos t, f_2 = \sin t, f_3 = \cos^2 t = \dfrac{1+\cos 2t}{2}$. Observe the for any $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \cos nt \,\mathrm{dt} = -\frac{\sin nt}{n}\bigg|_{-\pi}^{\pi} = 0 \tag{1}$$

and

$$\int_{-\pi}^{\pi} \sin nt \,\mathrm{dt} = \frac{\cos nt}{n}\bigg|_{-\pi}^{\pi} = 0. \tag{2}$$

Method 1: We use the Gram-Schmidt algorithm.

$$g_0 = f_0 = 1$$

$$\begin{aligned} g_1 &= f_1 - \frac{\langle f_1, g_0\rangle}{\langle g_0, g_0\rangle} g_0 \\ &= \cos t - \frac{1}{\langle g_0, g_0\rangle}\left(\int_{-\pi}^{\pi} \cos t \,\mathrm{dt} + \int_{-\pi}^{\pi} 0 \,\mathrm{dt}\right) \\ &= \cos t \end{aligned}$$

$$\begin{aligned} g_2 &= f_2 - \frac{\langle f_2, g_0\rangle}{\langle g_0, g_0\rangle} g_0 - \frac{\langle f_2, g_1\rangle}{\langle g_1, g_1\rangle} g_1 \\ &= \sin t - \frac{1}{\langle g_0, g_0\rangle}\left(\int_{-\pi}^{\pi} \sin t \,\mathrm{dt} + \int_{-\pi}^{\pi} 0 \,\mathrm{dt}\right) - \frac{g_1}{\langle g_1, g_1\rangle}\left(\int_{-\pi}^{\pi} \sin t \cos t \,\mathrm{dt} - \int_{-\pi}^{\pi} \sin t \cos t \,\mathrm{dt}\right) \\ &= \sin t \end{aligned}$$

$$
\begin{aligned}
g_3 &= f_3 - \frac{\langle f_3, g_0\rangle}{\langle g_0, g_0\rangle} g_0 - \frac{\langle f_3, g_1\rangle}{\langle g_1, g_1\rangle} g_1 - \frac{\langle f_3, g_2\rangle}{\langle g_2, g_2\rangle} g_2 \\
&= \frac{1+\cos 2t}{2} - \frac{1}{2\langle g_0, g_0\rangle} \int_{-\pi}^{\pi} (1+\cos 2t)\,\mathrm{dt} \\
&\quad - \frac{g_1}{2\langle g_1, g_1\rangle} \left(\int_{-\pi}^{\pi} \cos t(1+\cos 2t)\,\mathrm{dt} + 2\int_{-\pi}^{\pi} \sin t \sin 2t\,\mathrm{dt} \right) \\
&\quad - \frac{g_2}{2\langle g_2, g_2\rangle} \left(\int_{-\pi}^{\pi} \sin t(1+\cos 2t)\,\mathrm{dt} - 2\int_{-\pi}^{\pi} \cos t \sin 2t\,\mathrm{dt} \right) \\
&= \frac{1+\cos 2t}{2} - \frac{1}{2} - \frac{g_1}{2\langle g_1, g_1\rangle} \left(\int_{-\pi}^{\pi} \cos t \cos 2t\,\mathrm{dt} + \int_{-\pi}^{\pi} (\cos t - \cos 3t)\,\mathrm{dt} \right) \\
&\quad - \frac{g_2}{2\langle g_2, g_2\rangle} \left(\int_{-\pi}^{\pi} \sin t \cos 2t\,\mathrm{dt} + \int_{-\pi}^{\pi} (\sin 3t - \sin t)\,\mathrm{dt} \right) \\
&= \frac{1+\cos 2t}{2} - \frac{1}{2} - \frac{g_1}{2\langle g_1, g_1\rangle} \int_{-\pi}^{\pi} \frac{\cos 3t + \cos t}{2} - \frac{g_2}{2\langle g_2, g_2\rangle} \int_{-\pi}^{\pi} \frac{\sin 3t - \sin t}{2}\,\mathrm{dt}\,\mathrm{dt} \\
&= \frac{\cos 2t}{2}
\end{aligned}
$$

Method 2: Since $\cos^2 t = \dfrac{1+\cos 2t}{2}$ and $\cos 2t = 2\cos^2 t - 1$, it follows that

$$W = \mathrm{Span}\{1, \cos t, \sin t, \cos^2 t\} = \mathrm{Span}\{1, \cos t, \sin t, \cos 2t\}$$

From (1) and (2) it is clear that

$$\{1, \cos t, \sin t, \cos 2t\}$$

is an orthogonal set of nonzero vectors in V, and is therefore an orthogonal basis for W.

Rubric.

(a) 2 marks for proving that V is a subspace of $C[-\pi, \pi]$

(b)
- 1 marks for linearity in the first variable
- 1/2 mark for symmetric property $\langle f, g\rangle = \langle g, f\rangle$
- 1/2 mark for positive definite property $\langle f, f\rangle = 0 \iff f = 0$

(c) Method 1:

- 1 mark for computing g_0
- 1 mark for computing g_1
- 1 mark for computing g_2
- 3 marks for computing g_3

Method 2:

- 3 marks for showing that $W = \mathrm{Span}\{1, \cos t, \sin t, \cos 2t\}$
- 2 marks for showing that $\{1, \cos t, \sin t, \cos 2t\}$ is an orthogonal set
- 1 mark for concluding that $\{1, \cos t, \sin t, \cos 2t\}$ is a basis

Question 8 (10 marks)**.**

Let $n \geq 2$. Let $V = \mathbb{P}_n$, the vector space of polynomials of degree at most n, with real coefficients.

Let $\{p_1, p_2, p_3\}$ be a linearly independent subset of V. Let $A = (a_{ij})$ be a 3×3 matrix having real entries. Let

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Show that the $\{q_1, q_2, q_3\}$ is a linearly independent subset of V if and only if A is invertible.

Solution. Let $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a_3}$ denote the columns of A.

If $\xi = (c_1, c_2, c_3) \in \mathbb{R}^3$, then

$$\begin{aligned} &c_1q_1 + c_2q_2 + c_3q_3 \\ &\quad = c_1(a_{11}p_1 + a_{12}p_2 + a_{13}p_3) \\ &\qquad\quad + c_2(a_{21}p_1 + a_{22}p_2 + a_{23}p_3) \\ &\qquad\quad + c_3(a_{31}p_1 + a_{32}p_2 + a_{33}p_3) \\ &\quad = (c_1a_{11} + c_2a_{21} + c_3a_{31})p_1 \\ &\qquad\quad + (c_1a_{12} + c_2a_{22} + c_3a_{32})p_2 \\ &\qquad\quad + (c_1a_{13} + c_2a_{23} + c_3a_{33})p_3 \\ &\quad = (\mathbf{a}_1^T\xi)p_1 + (\mathbf{a}_2^T\xi)p_2 + (\mathbf{a}_3^T\xi)p_3 \end{aligned} \tag{3}$$

Suppose A is not invertible. Then A^T is also not invertible.

Hence there exists a nontrivial solution $\xi = (c_1, c_2, c_3) \in \mathbb{R}^3$ of the equation $A^T\mathbf{x} = 0$. Then

$$c_1q_1 + c_2q_2 + c_3q_3 = (\mathbf{a}_1^T\xi)p_1 + (\mathbf{a}_2^T\xi)p_2 + (\mathbf{a}_3^T\xi)p_3 = 0$$

Therefore $\{q_1, q_2, q_3\}$ is linearly dependent.

Conversely, suppose $\{q_1, q_2, q_3\}$ is linearly dependent. Then there exist numbers c_1, c_2, c_3 not all zero such that

$$c_1q_1 + c_2q_2 + c_3q_3 = 0.$$

Let $\xi = (c_1, c_2, c_3)$. Then

$$(\mathbf{a}_1^T\xi)p_1 + (\mathbf{a}_2^T\xi)p_2 + (\mathbf{a}_3^T\xi)p_3 = 0$$

As the set $\{p_1, p_2, p_3\}$ is linearly independent, it follows that

$$\mathbf{a}_1^T \xi = \mathbf{a}_2^T \xi = \mathbf{a}_3^T \xi = 0$$

Therefore $A^T \mathbf{x} = 0$ has a nontrivial solution. Therefore A^T is not invertible, so A is not invertible.

Rubric.

- For the method I've used above:
  - For establishing the relationship expressed by equation 3 (1 mark each for the first two subequations, 2 marks for the third): 4 marks
  - For recognizing that A is invertible iff A^T is invertible: 2 marks
  - For showing that $\{q_1, q_2, q_3\}$ is a linearly independent subset of V implies A is invertible, using the two above ideas: 2 marks
  - For showing that A is invertible implies $\{q_1, q_2, q_3\}$ is a linearly independent subset of V, using the two above ideas: 2 marks
- For other methods:
  - 5 marks for showing that A invertible implies $\{q_1, q_2, q_3\}$ linearly independent.
  - 5 marks for showing that $\{q_1, q_2, q_3\}$ linearly independent implies A invertible.

References

[1] David C. Lay, *Linear Algebra and its Applications.* Pearson, 2005.

[2] Strang *Linear Algebra and Its Applications.*

[3] Lipschutz, *Linear Algebra, Schaum's Outline Series.*

[4] Hoffman and Kunze, *Linear Algebra.*

[5] Kumaresan, *Linear Algebra: A Geometric Approach*

[6] Axler, *Linear Algebra Done Right.* Undergraduate Texts in Mathematics, Springer 2015.

[7] Halmos, *Finite-Dimensional Vector Spaces*

[8] Michael Artin, *Algebra.* Prentice-Hall Inc.,New Jersey, 1991.