Rubric for Mid-Semester Exam

List of Common Errors and Marks Deductions:

1. Using an undefined symbol. Please deduct 1/2 mark each time this is done.
2. Writing an equation in which the LHS and RHS are not comparable, for example, if the LHS is an $m \times n$ matrix and the RHS is a real number. Please deduct 1/2 mark each time this is done.
3. Writing a meaningless or completely illogical statement. Please deduct 1 mark for every meaningless statement.
4. Please deduct 1/2 mark for every calculation mistake.

Question 1.

(a) (5 marks) Find the values of x for which the following is an augmented matrix corresponding to a consistent system:

$$\begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix}$$

(b) (5 marks) Determine the RREF of the matrix formed by substituting x with π in the matrix in part (a).

Solution.

(a) We reduce the given augmented matrix to echelon form:

$$\begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & -15 & 6 & x^3 - 4x \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & 0 & 0 & x^3 - 4x + 3x^2 \end{bmatrix}$$

The corresponding linear system is consistent if and only if the augmented column is not a pivot column. This condition holds if is x is a root of the polynomial $x^3 + 3x^2 - 4x$. Now

$$\begin{aligned} x^3 + 3x^2 - 4x &= x(x^2 + 3x - 4) \\ &= x(x+4)(x-1) \end{aligned}$$

Therefore the given matrix is an augmented matrix corresponding to a consistent linear system when $x = 0$, $x = -4$ or $x = 1$.

Rubric:

- Reducing the given matrix to echelon form: 2 marks
- Stating that the system is consistent if the augmented column is not a pivot column: 1 mark
- Solving the equation $x^3 + 3x^2 - 4x = 0$: 2 marks

(b) We find the RREF of the matrix

$$\begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 4 & -23 & 10 & \pi^3 \end{bmatrix}$$

Using part (a), we know that

$$\begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 4 & -23 & 10 & \pi^3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 0 & 0 & 0 & \pi^3 + 3\pi^2 - 4\pi \end{bmatrix}$$

We continue with the row reduction process to reduce the matrix to RREF. This can be done in any of the following ways:

Process 1.

$$\begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 0 & 0 & 0 & \pi^3 + 3\pi^2 - 4\pi \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{5}R_2} \begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 1 & -\frac{2}{5} & \frac{\pi^2}{5} \\ 0 & 0 & 0 & \pi^3 + 3\pi^2 - 4\pi \end{bmatrix}$$

$$\xrightarrow{R_1\to R_1+2R_2}\begin{bmatrix}1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5}\\ 0 & 1 & -\frac{2}{5} & \frac{\pi^2}{5}\\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi\end{bmatrix}$$

$$\xrightarrow{R_3\to\dfrac{1}{\pi^3+3\pi^2-4\pi}R_3}\begin{bmatrix}1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5}\\ 0 & 1 & -\frac{2}{5} & \frac{\pi^2}{5}\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-\dfrac{\pi^2}{5}R_3}\begin{bmatrix}1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5}\\ 0 & 1 & -\frac{2}{5} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-\left(\pi+\dfrac{2\pi^2}{5}\right)R_3}\begin{bmatrix}1 & 0 & \frac{1}{5} & 0\\ 0 & 1 & -\frac{2}{5} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Process 2.

$$\begin{bmatrix}1 & -2 & 1 & \pi\\ 0 & 5 & -2 & \pi^2\\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi\end{bmatrix}\xrightarrow{R_3\to\frac{1}{\pi^3+3\pi^2-4\pi}R_2}\begin{bmatrix}1 & -2 & 1 & \pi\\ 0 & 5 & -2 & \pi^2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-\pi^2R_3}\begin{bmatrix}1 & -2 & 1 & \pi\\ 0 & 5 & -2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-\pi R_3}\begin{bmatrix}1 & -2 & 1 & 0\\ 0 & 5 & -2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_2\to\frac{1}{5}R_2}\begin{bmatrix}1 & -2 & 1 & 0\\ 0 & 1 & -\frac{2}{5} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rubric:

- For using part (a) to reduce the matrix to echelon form, OR for doing the calculations again to achieve the same result: 1 mark
- Having reduced the matrix to echelon form, award 1 mark for every correct row operation performed to reduce to RREF, except the last two row operations, which carry 1/2 mark each.

Note: Two ways of reducing to RREF have been given here. Please note that only one method needs to be used. In case the student has used any other method, please inform your course instructor.

Question 2.

(a) (5 marks) Let

$$A = \begin{bmatrix} 1 & 2 & x \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Express the inverse of A as a function of x (i.e. a matrix whose entries are functions of x), without calculating the determinant or using Cramer's rule.

(b) Is the span of the columns of A^{-1} all of $\mathbb{R}^3$ (YES/NO) ? Justify your answer briefly.

(Note for Section A students: You may assume that x is a fixed scalar in Part (b).)

Solution.

(a) We reduce $[A \quad I]$ to RREF:

$$\begin{bmatrix} 1 & 2 & x & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & x & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & x - \frac{8}{3} & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{4}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{5} R_3} \begin{bmatrix} 1 & 0 & x - \frac{8}{3} & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{4}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - \frac{4}{3} R_3} \begin{bmatrix} 1 & 0 & x - \frac{8}{3} & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{4}{15} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(x - \frac{8}{3}\right) R_3} \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{2}{3} & \frac{8-3x}{15} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{4}{15} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & -\frac{2}{3} & \frac{8-3x}{15} \\ 0 & \frac{1}{3} & -\frac{4}{15} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

Rubric:

- Identifying the method correctly, i.e. for stating something like "we apply the row reduction algorithm to $[A \quad I]$" - 1 mark
- Substituting values into the expression $[A \quad I]$ correctly - 1/2 mark
- Note: If both of the above steps are written as a single step, award 1.5 marks.
- Award 1/2 mark for every correct row operation performed to reduce to RREF.
- Award 1 mark for writing the correct answer.

(b) We first note that A^{-1} exists for every choice of $x \in \mathbb{R}$. Having fixed $x \in \mathbb{R}$, the span of the columns of A^{-1} is all of $\mathbb{R}^3$. This may be justified in any of the following ways:

Method 1. For any vector $\mathbf{b} \in \mathbb{R}^{\mathbf{3}}$,

$$A^{-1}(A\mathbf{b}) = \mathbf{b}.$$

Therefore $\mathbf{b}$ can be expressed as a linear combination of the columns of A^{-1} using the entries of $A\mathbf{b}$ as coefficients. As the choice of $\mathbf{b}$ was arbitrary, the span of the columns of A^{-1} is $\mathbb{R}^3$.

Method 2. Since A^{-1} is invertible, it follows by the Invertible Matrix Theorem that for any vector $\mathbf{b} \in \mathbb{R}^{\mathbf{3}}$, the equation

$$A^{-1}\mathbf{x} = \mathbf{b}$$

has a solution. Therefore every $\mathbf{b} \in \mathbb{R}^3$ can be expressed as a linear combination of the columns of A^{-1}. Hence the span of the columns of A^{-1} is $\mathbb{R}^3$.

Method 3. For any vector $\mathbf{b} \in \mathbb{R}^{\mathbf{3}}$, the equation

$$A^{-1}\mathbf{x} = \mathbf{b}$$

has a solution because the last columns of the augmented matrix $[A^{-1} \quad \mathbf{b}]$ is not a pivot column. Therefore every $\mathbf{b} \in \mathbb{R}^3$ can be expressed as a linear combination of the columns of A^{-1}. Hence the span of the columns of A^{-1} is $\mathbb{R}^3$.

Method 4. Since A^{-1} has a pivot position in every row, it follows by Theorem 4 on p. 43, Section 1.4 of the course textbook (3rd edition), that the columns of A^{-1} span $\mathbb{R}^3$.

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- For answering YES, award 1 mark.
- Methods 1, 2 & 3:
  - For stating that any arbitrary vector in $\mathbb{R}^3$ can be expressed as a linear combination of columns of A^{-1}, award 1 mark.

  - For giving a correct justification for the statement above, award 2 marks.
  - For arriving at the conclusion that the span of the columns of A^{-1} is $\mathbb{R}^3$, award 1 mark.
- Method 4:
  - For stating that A^{-1} has a pivot position in every row, award 2 marks.
  - For citing the theorem correctly, award 1 mark.
  - For arriving at the conclusion that the span of the columns of A^{-1} is $\mathbb{R}^3$, award 1 mark.

Question 3. Let
$$V = \{x \in \mathbb{R} \, : \, x > 0\},$$
and define addition for V by
$$x \oplus y := xy$$
and scalar multiplication by any $\alpha \in \mathbb{R}$ by
$$\alpha * x = x^\alpha.$$

(a) (7 marks) Verify the closure axioms, the commutative, zero and inverse properties for addition, and the property $1 * x = x$ for all $x \in V$.

(Remark: V is in fact a vector space over the field $\mathbb{R}$. You need not verify the other properties of a vector space.)

(b) (3 marks) Is V a subspace of $\mathbb{R}$ regarded as a vector space over itself (YES/NO) ? Justify your answer clearly.

Solution.

(a)
- First closure axiom:
  Let $x, y \in V$. Then $xy \in \mathbb{R}$ and $x > 0, y > 0 \implies xy > 0$. Therefore $xy \in V \implies x \oplus y \in V$.
- Second closure axiom:
  Let $x \in V$ and $\alpha \in \mathbb{R}$. As $x > 0$ and $x \in \mathbb{R}$, it follows that $x^\alpha \in \mathbb{R}$ and $x^\alpha > 0$ (A proof of this fact is outlined in the first chapter of the text "Principles of Mathematical Analysis" by W. Rudin - p. 22, 3rd edition). Hence $x^\alpha \in V \implies \alpha * x \in V$.

- Commutativity of addition:
  Let $x, y \in V$. Then
  $$x \oplus y = xy = yx = y \oplus x.$$
- Existence of additive identity:
  For every $x \in V$,
  $$x \oplus 1 = x \times 1 = x$$
  Hence 1 is a zero vector in V.
- Existence of additive inverse:
  Let $x \in V$. Since $\frac{1}{x} > 0$ it follows that $\frac{1}{x} \in V$. Further
  $$x \oplus \frac{1}{x} = 1.$$
  Hence $\frac{1}{x}$ is an additive inverse of x in V.
- Lastly, let $x \in V$. Then $1 * x = x^1 = x$.

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- 1 mark for each correctly verified axiom.
- Add 1/2 mark, if at least three axioms are verified correctly.
- Add 1/2 mark, if all axioms are verified correctly.

(b) When we consider the question whether V is a subspace of $\mathbb{R}$ (with the usual vector space structure on $\mathbb{R}$), we strip V of the vector space structure defined in part (a) and view it simply as a subset of $\mathbb{R}$. In order for V to be a subspace of $\mathbb{R}$ it must be a vector space under the **usual operations of vector addition and scalar multiplication which are defined on** $\mathbb{R}$.

Thus the answer to our question is the following.

No, V is not a subspace of $\mathbb{R}$, when $\mathbb{R}$ is regarded as a vector space over itself.

One reason is that V is not closed under scalar multiplication. Any of the following counterexamples can be cited.

1. Let $x \in V$. Then $0 \times x = 0$, but $0 \notin V$.
2. Let $x \in V$. Then $-1 \times x = -x$, but $-x \notin V$.
3. Let $x \in V$. Let $c \in \mathbb{R}$ be chosen such that $c \leq 0$. Then

$$x > 0 \implies cx \leq 0.$$

   Hence $cx \notin V$.
4. $13 \in V, -13 \in \mathbb{R}$, but $-169 \notin V$.

Alternatively, the fact that $0 \notin V$ can be used as a standalone justification. In this case, V is not a subspace of $\mathbb{R}$, by the definition listed on p. 220 of the course textbook (3rd edition).

Rubric:

- 1 mark for answering NO. Remaining 2 marks distributed as:
  - 1 mark for stating that V is not closed under scalar multiplication.

- 1 mark for giving any of the above counterexamples. Please note that students may have chosen numbers other than 13 and -13. There are infinitely many correct answers.

OR

- 1 mark for answering NO. 2 marks for the reason that $0 \notin V$.

Question 4 (10 marks)**.** Choose any four of the five sets below. For each set you choose, state whether or not it is a subspace of $M_{3\times 3}$ (the space of all 3×3 matrices having real entries). Justify each answer. All choices carry equal marks.

(a) The set of all invertible 3×3 matrices

(b) The set of all 3×3 matrices whose trace is 0 (The trace of a square matrix A is the sum of its diagonal entries.)

(c) The set of all 3×3 echelon matrices

(d) The set of all symmetric 3×3 matrices (A square matrix A is said to be symmetric if $A^T = A$)

(e) The set of all skew-symmetric 3×3 matrices (A square matrix A is said to be skew-symmetric if $A^T = -A$)

(Note for Section B students: $M_{3\times 3}$ is the same as $\mathbb{R}^{3\times 3}$).

Solution.

(a) Let V be the set of all invertible 3×3 matrices. V is not a subspace of $M_{3\times 3}$. This can be justified in any of the following ways:

1. V is not closed under vector addition. For example, $I \in V, -I \in V$, but $I - I = 0$ and $0 \notin V$.
2. V is not closed under scalar multiplication. If $c = 0$, then $I \in V$ but $cI = 0$ and $cI \notin V$.
3. $0 \notin V$. Thus V is not a subspace of $M_{3\times 3}$, by the definition listed on p. 220 of the course textbook (3rd edition).

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 1 mark for answering NO.
- Award 1.5 mark for any one of the reasons listed above.

  Please note that a counterexample provided by the student may be different from the one which is written here. There are infinitely many possibilities. You may check with your course instructor if you have any doubts about verifying a counterexample from a student.

(b) Let V be the set of all 3×3 matrices whose trace is 0. V is a subspace of $M_{3\times 3}$. This is verified as follows:

Let $A, B \in V$. Let $A = (a_{ij})$ and $B = (b_{ij})$. Then

$$\begin{aligned} trace(A + B) &= \sum_{i=1}^{3} a_{ii} + b_{ii} \\ &= \sum_{i=1}^{3} a_{ii} + \sum_{i=1}^{3} b_{ii} \\ &= trace(A) + trace(B) \\ &= 0 \end{aligned}$$

Hence $A + B \in V$. Thus V is closed under vector addition.

Let $A \in V$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} trace(cA) &= \sum_{i=1}^{3} ca_{ii} \\ &= c \sum_{i=1}^{3} a_{ii} \\ &= c\, trace(A) \\ &= 0 \end{aligned}$$

Hence $cA \in V$. Thus V is closed under scalar multiplication.

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 1 mark for answering YES.

- Award 1.5 marks for verifying that V is closed under vector addition and scalar multiplication.

(c) Let V be the set of all 3×3 echelon matrices. V is not a subspace of $M_{3\times 3}$. This is because V is not closed under vector addition. Counterexamples abound. One is given here for your reference: Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $A \in V$ and $I \in V$. However

$$A + I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin V$$

because $A + I$ has a row of zeros in the middle, which violates the conditions for being an echelon matrix.

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 1 mark for answering NO.
- Award 1/2 mark for stating that V is not closed under vector addition.
- Award 1 mark for providing an appropriate counterexample.

  Please note that a counterexample provided by the student may be different from the one which is written here. There are infinitely many possibilities. You may check with your course instructor if you have any doubts about verifying a counterexample from a student.

(d) Let V be the set of all symmetric 3×3 matrices. V is a subspace of $M_{3\times 3}$. This is verified as follows:

Let $A, B \in V$. Then

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ &= A + B \end{aligned}$$

Hence $A + B \in V$. Thus V is closed under vector addition.

Let $A \in V$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}(cA)^T &= cA^T \\ &= cA\end{aligned}$$

Hence $cA \in V$. Thus V is closed under scalar multiplication.

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 1 mark for answering YES.
- Award 1.5 marks for verifying that V is closed under vector addition and scalar multiplication.

(e) Let V be the set of all skew-symmetric 3×3 matrices. V is a subspace of $M_{3\times 3}$. This is verified as follows:

Let $A, B \in V$. Then

$$\begin{aligned}(A+B)^T &= A^T + B^T \\ &= -A - B \\ &= -(A+B)\end{aligned}$$

Hence $A + B \in V$. Thus V is closed under vector addition.

Let $A \in V$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}(cA)^T &= cA^T \\ &= c(-A) \\ &= -cA\end{aligned}$$

Hence $cA \in V$. Thus V is closed under scalar multiplication.

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 1 mark for answering YES.

- Award 1.5 marks for verifying that V is closed under vector addition and scalar multiplication.

Question 5. (10 marks) Let V be a vector space over a field F. Suppose $v_1, v_2, \dots, v_n$ are linearly independent in V and $w \in V$.

Show that if $v_1 + w, v_2 + w, \dots, v_n + w$ are linearly dependent in V, then $w \in Span\{v_1, v_2, \dots, v_n\}$.

(Note for Section A students: You may assume that $F = \mathbb{R}$.)

Solution.

Method 1. As $v_1 + w, v_2 + w, \dots, v_n + w$ are linearly dependent in V, there exist scalars $c_1, \dots, c_n \in F$, not all zero, such that

$$\begin{aligned} 0 &= \sum_{i=1}^{n} c_i(v_i + w) \\ &= \sum_{i=1}^{n} c_i v_i + \left(\sum_{i=1}^{n} c_i \right) w \end{aligned} \tag{1}$$

Put $c = \sum_{i=1}^{n} c_i$. Suppose if possible that $c = 0$. Then (1) gives us

$$\sum_{i=1}^{n} c_i v_i = 0.$$

As $v_1, v_2, \dots, v_n$ are linearly independent in V, it follows that $c_i = 0$ for $i = 1, \dots, n$. Contradiction.

Therefore $c \neq 0$. Now multiply both sides of (1) by $\frac{1}{c}$ to obtain

$$w + \frac{1}{c} \sum_{i=1}^{n} c_i v_i = 0$$

Hence

$$w = \sum_{i=1}^{n} \frac{c_i}{c} v_i \in Span\{v_1, v_2, \dots, v_n\}.$$

□

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 2 marks for using the linear dependence relation between $v_1 + w, v_2 + w, \ldots, v_n + w$ to set up equation (1).
- Award 4 marks for showing that $\sum_{i=1}^{n} c_i \neq 0$.
- Award 4 marks for expressing w as a linear combination of $v_1, v_2, \ldots, v_n$ and concluding the proof.

Method 2. The following result was proved in class for both Section A as well as Section B:

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent set in a vector space V. If $\mathbf{w} \notin Span(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\})$ then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}\}$ is linearly independent.

Using the above result, it suffices to show that the set $\{v_1, \ldots, v_n, w\}$ is not linearly independent in V.

As $v_1 + w, v_2 + w, \ldots, v_n + w$ are linearly dependent in V, there exist scalars $c_1, \ldots, c_n \in F$, not all zero, such that

$$\sum_{i=1}^{n} c_i(v_i + w) = 0$$

Hence

$$\sum_{i=1}^{n} c_i v_i + \left(\sum_{i=1}^{n} c_i\right) w = 0. \tag{2}$$

As $c_1, \ldots, c_n$ are not all zero, (2) is a linear dependence relation between $v_1, \ldots, v_n, w$. □

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 2 marks for citing the result which was covered in class correctly.
- Award 2 marks for recognizing that the result can be used in its contrapositive form.
- Award 2 marks for using the linear dependence relation between $v_1 + w, v_2 + w, \ldots, v_n + w$ to set up equation (2).

- Award 4 marks for recognizing that (2) is the required linear dependence relation between the vectors $v_1, \ldots, v_n, w$.

Method 3. We prove the contrapositive of the given statement, i.e.

$$w \notin Span\{v_1, v_2, \ldots, v_n\} \implies \{v_1 + w, v_2 + w, \ldots, v_n + w\} \text{ is l.i.}$$

So, assume that $w \notin Span\{v_1, v_2, \ldots, v_n\}$. Suppose

$$\sum_{i=1}^{n} c_i(v_i + w) = 0,$$

where $c_1, \ldots, c_n \in \mathbb{R}$. Put $c = \sum_{i=1}^{n} c_i$. Then

$$cw = -\sum_{i=1}^{n} c_i v_i \tag{3}$$

Were $c \neq 0$, we would obtain $w \in Span\{v_1, v_2, \ldots, v_n\}$ by multiplying both sides of (3) by $\frac{1}{c}$, which would then contradict our assumption. Therefore $c = 0$. Therefore

$$\sum_{i=1}^{n} c_i v_i = 0$$

Now, since $v_1, v_2, \ldots, v_n$ are linearly independent in V, it follows that $c_i = 0$ for $i = 1, \ldots, n$. Thus the vectors $v_1 + w, \ldots, v_n + w$ are linearly independent in V. $\square$

Rubric: As this is a proof-type question, please refer to the list of marks deductions for common errors listed at the beginning of this document. In addition, you may award marks as follows.

- Award 2 marks for deriving equation (3).
- Award 4 marks for showing that $\sum_{i=1}^{n} c_i = 0$.
- Award 4 marks for showing that $c_i = 0$ for $i = 1, \ldots, n$ and concluding the proof.