Q1

Let $\mathbb{P}$ denote the vector space of all polynomials in the variable x having real coefficients.

Let

$$V = \{a_0 + a_0 x + a_1 x^2 + a_2 x^3 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

Let $T : V \to \mathbb{P}$ be the mapping defined by

$$T(p(x)) = p'(x), \quad \forall p(x) \in V.$$

Correct answers:

- $\dim \ker T = 0$
- $\dim \operatorname{range} T = 3$

Justification:

$p'(x) = 0 \iff a_0 + 2a_1x + 3a_2x^2 = 0 \iff a_0 = a_1 = a_2 = 0$

Therefore $\ker T = \{0\} \implies \dim\ker T = 0$.

$V = \text{Span}\{1 + x, x^2, x^3\} \implies \dim V = 3$.

Therefore $\dim \text{range } T = \dim V - \dim\ker T = 3 - 0 = 0$.

Q2

Let $\mathcal{C}(\mathbb{R})$ be the vector space of all continuous real valued functions defined on $\mathbb{R}$.

Let $W = \text{Span}\{1, \sin x, \sin^2 x, \cos x, \cos^2 x\} \subset V$.

Let $T : W \to \mathbb{R}$ be the linear transformation defined by

$$T(f) = f(0), \quad \forall f \in W.$$

Correct statement:

- $\dim \ker T = 3, \dim \operatorname{range} T = 1$

Justification:

As W contains all the constant functions, range $T = \mathbb{R}$. Hence $\dim \operatorname{range} T = 1$.

It should be intuitively obvious that $1, \sin x, \sin^2 x$ and $\cos x$ are linearly independent, but let us prove it.

Suppose $f(x) = a + b\sin x + c\cos x + d\sin^2 x = 0$, where $a, b, c, d \in \mathbb{R}$.

As f is the zero function, $f(0) = 0$. Hence $a + c = 0$. Similarly $f(\pi) = 0 \implies a - c = 0$. Therefore $a = c = 0$.

Therefore $f(x) = b\sin x + d\sin^2 x$. Since $f(\pi/2) = 1$, we obtain $b + d = 0$. Similarly $f(-\pi/2) = 0 \implies d - b = 0$. Therefore $b = d = 0$.

Q3

Let $V = M_{m\times n}(\mathbb{R})$ be the vector space of all $m \times n$ matrices having real entries, where $m, n \in \mathbb{N}$ and $n > 1$.

Let $T : V \to \mathbb{R}^m$ be defined by

$$T(A) = \text{ the sum of the first and last columns of } A, \quad \forall A \in V.$$

For example, if $m = 3$, $n = 4$, and

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 7 & 6 & 2 & 2 \\ 1 & 2 & 1 & 7 \end{bmatrix}, \quad \text{then } T(A) = \begin{bmatrix} 3 \\ 9 \\ 8 \end{bmatrix}$$

Let B be the matrix of T with respect to some bases $\mathcal{B}$ and $\mathcal{C}$ respectively.

Correct statement: B is an element of $M_{m\times mn}(\mathbb{R})$

Justification:

We know that $\dim V = mn$ and $\dim \mathbb{R}^m = m$.

It should be intuitively obvious that the matrix of T is of size $m \times mn$, but let us prove it.

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_{mn}\}$ be any basis of V and let $\mathcal{C} = \{c_1, \ldots, c_m\}$ be a basis of $\mathbb{R}^m$. Then

$$T_{\mathcal{B},\mathcal{C}} = [[T(b_1)]_{\mathcal{C}} \quad [T(b_2)]_{\mathcal{C}} \quad \ldots \quad [T(b_{mn})]_{\mathcal{C}}]$$

Clearly, $T_{\mathcal{B},\mathcal{C}}$ has mn columns. For each $j = 1, \ldots, mn$, $[T(b_j)]_{\mathcal{C}} \in \mathbb{R}^m$. Therefore $T_{\mathcal{B},\mathcal{C}}$ has m rows.

Q4

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $B = A^2$.

Correct statement:

- $\dim \operatorname{col} B = 1, \dim \operatorname{null} B = 2, \dim \operatorname{col} A = 2, \dim \operatorname{null} A = 1$

Justification:

As the last two columns of A are non-zero and are not scalar multiples of each other, $\dim \operatorname{col} A = 2$. By the rank-nullity theorem for matrices

$$\dim \operatorname{null} A = 3 - \dim \operatorname{col} A = 1$$

Even without explicit calculation, just by recalling the definition of the matrix product $A\mathbf{x}$, it is easy to see that A^2 has only one non-zero column, and therefore

$$\dim \operatorname{col} A^2 = 1, \dim \operatorname{null} A^2 = 3 - 1 = 2$$