χ^2 test

Theory and solved numerical problems

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χ^2 test (Strickberger, 1990)

Actually observed ratios may depart to a greater or lesser extent from those expected. Once the degrees of freedom and a significance level for testing a hypothesis has been decided upon, the actual measurement of the size of the discrepancy between observed and expected results remains to be done. One measure commonly used for this purpose is *chi-square* χ^2. For 1 degrees of freedom (referred to as *df* herein after), the test statistic is calculated with the following relation;

$$\chi^2 = \frac{\left[\sum |observed - expected| - \frac{1}{2}\right]^2}{expected}$$

The term $\frac{1}{2}$ above from the absolute value of the deviation ($observed - expected$) is known as Yates correction term. This adds to the accuracy of χ^2 determinations when the number of either of the expected classes is small.

Example case

Let us suppose that a garden pea plant, heterozygous for the gene pair Tt, produced 30 tall and 20 short offspring. Since the effect of the tall allele (T)

	Tall	Short	Subtotals
Observed	30	20	50
Expected	$\frac{3}{4} \times 50 = 37.5$	$\frac{1}{4} \times 50 = 12.5$	50
Observed - Expected	-7.5	7.5	
$\lvert Observed - Expected \rvert - \frac{1}{2}$	-7	7	
$\left(\lvert Observed - Expected \rvert - \frac{1}{2}\right)^2$	49	49	
$\left(\lvert Observed - Expected \rvert - \frac{1}{2}\right)^2 / expected$	$\frac{49.0}{37.5} = 1.31$	$\frac{49.0}{12.5} = 3.92$	$\chi^2 = 1.31 + 3.92 = 5.23$

Table 1: Chi-square calculations for the hypothesis that the observations of 30 tall and 20 short plants arises from a cross between heterozygotes, Tt x Tt, which would ideally produce a ratio of 3/4 tall:1/4 short plants

is dominant over that for short (t), a 3:1 ratio would have been expected if the plant had been self-fertilized ($Tt \times Tt$), or an exact numerical ratio of 37.5 tall to 12.5 short. Is the observed deviation from the expected ratio sufficient reason to discard the self-fertilization hypothesis and look for another explaination? For instance, must we substitute the hypothesis that the Tt plant was fertilized by a short plant tt which, ideally, would have yielded 25 tall and 25 short offspring?

(Solution)$\rightarrow$ χ^2 for the hypothesis that the tall plant in the above example was self-fertilized would be calculated as shown in Table 1.

$$\left(\lvert Observed - Expected \rvert - \frac{1}{2}\right)^2$$

Note that in calculating χ^2, always the actual numbers observed and expected should be used. Using proportions or percentages will lead to unintended results.

According to statisticians, if the “expected” hypothesis is true, χ^2 values have certain probabilities of occurance, depending on the number of degrees of freedom in the experiment. On the basis of such, calculation tables have been constructed that relate the number of degrees of freedom with the probability of particular groups of χ^2 values will be found. For any given number of degrees of freedom, the probability that chi-square values or discrepancies will be found is, of course, much less than for small discrepancies. This relationship can be observed in Table 2, where, for example, χ^2 value larger than 3.84 will occur 5 percent or less of the time in experiments with 1 degree of freedom. Thus χ^2 values of 6.64 or greater are sufficiently rare that they occur only 1 percent of the time, and values of 10.83 or greater occur only 1 out of 1000 times. On the other hand, χ^2 values less than 3.84 are relatively frequent; i.e., a value of 1.07 or greater would be found in 30 out to 100 experiments if the proposed hypothesis were true expaination for the results.

Since we have agreed on a 5 percent level of significance, the χ^2 value of 5.23 calculated in our example is therefore rare enough to be “significant”, so that we have cause to reject our hypothesis. For convenience, a line has been placed at the 0.5 value in the Table 2, and χ^2 values to the *right* of this line can be considered sufficient cause for rejection of the hypothesis at the 5 percent level of significance.

Alternatively hypothesis, that the Tt heterozygous plant had been fertilized by a tt plant, also deserves consideration. According to this hypothesis, the

	0.995	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.005
1	7.879439	6.634897	5.023886	3.841459	2.705544	0.0157908	0.0039321	0.0009821	0.0001571	0.0000393
2	10.596635	9.210340	7.377759	5.991465	4.605170	0.2107210	0.1025866	0.0506356	0.0201007	0.0100251
3	12.838157	11.344867	9.348404	7.814728	6.251389	0.5843744	0.3518463	0.2157953	0.1148318	0.0717218
4	14.860259	13.276704	11.143287	9.487729	7.779440	1.0636232	0.7107230	0.4844186	0.2971095	0.2069891
5	16.749602	15.086272	12.832502	11.070498	9.236357	1.6103080	1.1454762	0.8312116	0.5542981	0.4117419
6	18.547584	16.811894	14.449375	12.591587	10.644641	2.2041307	1.6353829	1.2373442	0.8720903	0.6757268
7	20.277740	18.475307	16.012764	14.067140	12.017037	2.8331069	2.1673499	1.6898692	1.2390423	0.9892557
8	21.954955	20.090235	17.534546	15.507313	13.361566	3.4895391	2.7326368	2.1797307	1.6464974	1.3444131
9	23.589351	21.665994	19.022768	16.918978	14.683657	4.1681590	3.3251128	2.7003895	2.0879007	1.7349329
10	25.188180	23.209251	20.483177	18.307038	15.987179	4.8651821	3.9402991	3.2469728	2.5582122	2.1558565
11	26.756849	24.724970	21.920049	19.675138	17.275008	5.5777848	4.5748131	3.8157483	3.0534841	2.6032219
12	28.299519	26.216967	23.336664	21.026070	18.549348	6.3037961	5.2260295	4.4037885	3.5705690	3.0738236
13	29.819471	27.688250	24.735605	22.362033	19.811929	7.0415046	5.8918643	5.0087505	4.1069155	3.5650346
14	31.319350	29.141238	26.118948	23.684791	21.064144	7.7895336	6.5706314	5.6287261	4.6604251	4.0746750
15	32.801321	30.577914	27.488393	24.995790	22.307130	8.5467562	7.2609439	6.2621378	5.2293489	4.6009156
16	34.267187	31.999927	28.845351	26.296228	23.541829	9.3122364	7.9616456	6.9076644	5.8122125	5.1422054
17	35.718466	33.408664	30.191009	27.587112	24.769035	10.0851863	8.6717602	7.5641864	6.4077598	5.6972171
18	37.156452	34.805306	31.526378	28.869299	25.989423	10.8649361	9.3904551	8.2307462	7.0149109	6.2648047
19	38.582257	36.190869	32.852327	30.143527	27.203571	11.6509100	10.1170131	8.9065165	7.6327296	6.8439714
20	39.996846	37.566235	34.169607	31.410433	28.411981	12.4426092	10.8508114	9.5907774	8.2603983	7.4338443
25	46.927890	44.314105	40.646469	37.652484	34.381587	16.4734080	14.6114076	13.1197200	11.5239754	10.5196521
30	53.671962	50.892181	46.979242	43.772972	40.256024	20.5992346	18.4926610	16.7907723	14.9534565	13.7867199
35	60.274771	57.342073	53.203348	49.801850	46.058788	24.7966548	22.4650152	20.5693766	18.5089262	17.1918203
40	66.765962	63.690740	59.341707	55.758479	51.805057	29.0505229	26.5093032	24.4330392	22.1642613	20.7065353
45	73.166061	69.956832	65.410159	61.656233	57.505305	33.3503809	30.6122591	28.3661523	25.9012692	24.3110142
50	79.489979	76.153891	71.420195	67.504807	63.167121	37.6886484	34.7642517	32.3573637	29.7066827	27.9907489
75	110.285583	106.392923	100.839338	96.216671	91.061460	59.7945570	56.0540723	52.9419398	49.4750289	47.2060477
100	140.169489	135.806723	129.561197	124.342113	118.498004	82.3581358	77.9294652	74.2219275	70.0648949	67.3275633

Table 2: The probabilities of exceeding different chi-square values for degrees of freedom from 1 to 50 when the expected hypothesis is true

	Tall	Short	Subtotals
Observed	30	20	50
Expected	$\frac{1}{2} \times 50 = 25$	$\frac{1}{2} \times 50 = 25$	50
Observed - Expected	5	-5	
$\lvert Observed - Expected \rvert - \frac{1}{2}$	4.5	-4.5	
$\left(\lvert Observed - Expected \rvert - \frac{1}{2}\right)^2$	20.5	20.5	
$\left(\lvert Observed - Expected \rvert - \frac{1}{2}\right)^2 / expected$	$\frac{20.5}{25} = 0.81$	$\frac{20.5}{25} = 0.81$	$\chi^2 = 0.81 + 0.81 = 1.62$

Table 3: Chi-square calculations for the hypothesis that the observations of 30 tall and 20 short plants arises from a cross between tall heterozygote (Tt) and a short homozygote (tt), which would ideally produce a ratio of 1/2 tall:1/2 short plants

offspring of such mating would be expeted to follow phenotypic ratio of 1 tall: 1 short. χ^2 calculation can therefore be made as in Table 3

According to Table 2, in the experiment with 1 degrees of freedom, the probability of exceeding a chi-square value of 1.62 is between 0.20 and 0.30. This χ^2 value is therefore small enough, and the probability for the occurance of such a small value is therefore sufficiently great that we do not reject our hypothesis.

In χ^2 tests with more than one degrees of freedom, however, for example with 4 classes (3 df), the calculation for χ^2 statistic follows the same but without Yate's correction factor ($\frac{1}{2}$) being subtracted from the absolute deviation.

χ^2 test for independence

It is ocassionally desirable to compare one set of observations taken under particular conditions to those of a similar nature taken under different conditions. In this case there are not definite expected values; the question is wheteher the results are dependent (contingent upon) or independent of the conditions under which they are observed. This test is therefore called a test for independence, or *contingency test*.

In genetics, problems of this kind may be concerned with the effect of

		Phenotype of offspring	
	Condition	Dominant	recessive
Environment	A	80	30
	B	90	25

Table 4: 2x2 contingency table of experimental outcomes different experimental conditions

	Categories of observations		
Condition	1	2	Totals
A	a	b	a+b
B	c	d	c+d
	a+c	b+d	a+b+c+d = N

Table 5: Calculation of chi-square statistic for test of independence

different environment or different genetic constitution on a set of experimental observations. For example, crosses between individuals heterozygous for the same gene difference were performed in two different experiments, A and B, and gave following results as shown in Table 4

For experiment A the dominant:recessive ratio is 2.67:1, whereas for B it is 3.60:1. Because of this difference, we may reasonably ask whether the observed results are independent of the particular experimental conditions. One statistical answer to this question depends upon the calculation of a χ^2 that has only 1 df. If the calculated χ^2 is less than the χ^2 at the particular level of significance desired (i.e., less than 3.84 at the 5 percent level), the hypothesis can be accepted that the observed results are statistically independent of the experimental conditions. If the calculated χ^2 exceeds this value, this would indicate that the results depend upon the conditions, i.e., that there is an *interaction* between the results of a cross and its experimental condition sufficient to cause differences between the two sets of results.

To calculate a contingency χ^2, both the observed numbers and the marginal totals must be used. If we call the total number of observations N, and the individual numerical contributions to this value, a, b, c and d, respectively, then the χ^2 calculations are as follows (Shown in Table 5):

$$\chi^2 = \frac{\left[|ad - bc| - \frac{1}{2}N\right]^2 N}{(a+b)(a+c)(c+d)(b+d)}$$

The vertical lines on the either sides of $|ab - bc|$ mean the absolute, or positive, value of this difference. If this difference is zero or less than the Yates correction factor of $\frac{1}{2}N$ in this equation, the numerator becomes zero and thus χ^2 is consequently zero.

From the data given above (in the Table 6), the χ^2 computations are:

$$\begin{aligned}\chi^2 &= \frac{(|2000 - 2700| - 112.5)^2 \times 225}{(100)(170)(115)(55)} \\ &= 0.66\end{aligned}$$

Condition	Phenotypes		Totals
	Dominant	Recessive	
A	80	30	110
B	90	25	115
	170	55	225

Table 6: Contingency table of phenotypes by test condition

Thus, in spite of the difference in rations between the two experiments, the χ^2 is still small enough (probability between .3 and 0.5) to allow statistical independence of the particular experimental conditions.

Bibliography

Monroe W Strickberger. Genetics. macmillan publishing co. *New York, USA*, 842, 1990.