Vectorized implementation of RTRL

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Abstract

Vectorized implementation can significantly accelerate Real Time Recurrent Learning (RTRL) algorithm [1].

1 RTRL Vectorization

The dynamics of p:

$$p_{ij}^{k}(t+1) = f_k'[s_k(t)] \left[\sum_{l \in U} w_{kl} * p_{ij}^{l}(t) + \delta_{ik} * z_j(t) \right] \tag{1}$$

Assuming the dimensionalities of hidden and input layers are 2 and 3, respectively. Thus the shapes of p and w are $(2, 2, 6)$ and $(2, 6)$, respectively. Then we have expanded form of Eq. (1):

$$\begin{aligned}
p_{00}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{00}^{0}(t) + w_{01} * p_{00}^{1}(t) + \delta_{00} * z_0(t) \right] \\
p_{01}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{01}^{0}(t) + w_{01} * p_{01}^{1}(t) + \delta_{00} * z_1(t) \right] \\
p_{02}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{02}^{0}(t) + w_{01} * p_{02}^{1}(t) + \delta_{00} * z_2(t) \right] \\
p_{03}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{03}^{0}(t) + w_{01} * p_{03}^{1}(t) + \delta_{00} * z_3(t) \right] \\
p_{04}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{04}^{0}(t) + w_{01} * p_{04}^{1}(t) + \delta_{00} * z_4(t) \right] \\
p_{05}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{05}^{0}(t) + w_{01} * p_{05}^{1}(t) + \delta_{00} * z_5(t) \right] \\
p_{10}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{10}^{0}(t) + w_{01} * p_{10}^{1}(t) + \delta_{10} * z_0(t) \right] \\
p_{11}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{11}^{0}(t) + w_{01} * p_{11}^{1}(t) + \delta_{10} * z_1(t) \right] \\
p_{12}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{12}^{0}(t) + w_{01} * p_{12}^{1}(t) + \delta_{10} * z_2(t) \right] \\
p_{13}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{13}^{0}(t) + w_{01} * p_{13}^{1}(t) + \delta_{10} * z_3(t) \right] \\
p_{14}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{14}^{0}(t) + w_{01} * p_{14}^{1}(t) + \delta_{10} * z_4(t) \right] \\
p_{15}^{0}(t+1) &= f_0'[s_0(t)] \left[w_{00} * p_{15}^{0}(t) + w_{01} * p_{15}^{1}(t) + \delta_{10} * z_5(t) \right]
\end{aligned}$$

$$\begin{pmatrix} p^0_{00}(t+1) \\ p^0_{01}(t+1) \\ p^0_{02}(t+1) \\ p^0_{03}(t+1) \\ p^0_{04}(t+1) \\ p^0_{05}(t+1) \\ p^0_{10}(t+1) \\ p^0_{11}(t+1) \\ p^0_{12}(t+1) \\ p^0_{13}(t+1) \\ p^0_{14}(t+1) \\ p^0_{15}(t+1) \end{pmatrix}^T = f'_0[s_0(t)] * \left[\begin{pmatrix} w_{00} & w_{01} \end{pmatrix} \cdot \begin{pmatrix} p^0_{00}(t) & p^1_{00}(t) \\ p^0_{01}(t) & p^1_{01}(t) \\ p^0_{02}(t) & p^1_{02}(t) \\ p^0_{03}(t) & p^1_{03}(t) \\ p^0_{04}(t) & p^1_{04}(t) \\ p^0_{05}(t) & p^1_{05}(t) \\ p^0_{10}(t) & p^1_{10}(t) \\ p^0_{11}(t) & p^1_{11}(t) \\ p^0_{12}(t) & p^1_{12}(t) \\ p^0_{13}(t) & p^1_{13}(t) \\ p^0_{14}(t) & p^1_{14}(t) \\ p^0_{15}(t) & p^1_{15}(t) \end{pmatrix}^T + \begin{pmatrix} z_0(t) \\ z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ z_5(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \right]$$

$$\begin{pmatrix} p^1_{00}(t+1) \\ p^1_{01}(t+1) \\ p^1_{02}(t+1) \\ p^1_{03}(t+1) \\ p^1_{04}(t+1) \\ p^1_{05}(t+1) \\ p^1_{10}(t+1) \\ p^1_{11}(t+1) \\ p^1_{12}(t+1) \\ p^1_{13}(t+1) \\ p^1_{14}(t+1) \\ p^1_{15}(t+1) \end{pmatrix}^T = f'_1[s_1(t)] * \left[\begin{pmatrix} w_{10} & w_{11} \end{pmatrix} \cdot \begin{pmatrix} p^0_{00}(t) & p^1_{00}(t) \\ p^0_{01}(t) & p^1_{01}(t) \\ p^0_{02}(t) & p^1_{02}(t) \\ p^0_{03}(t) & p^1_{03}(t) \\ p^0_{04}(t) & p^1_{04}(t) \\ p^0_{05}(t) & p^1_{05}(t) \\ p^0_{10}(t) & p^1_{10}(t) \\ p^0_{11}(t) & p^1_{11}(t) \\ p^0_{12}(t) & p^1_{12}(t) \\ p^0_{13}(t) & p^1_{13}(t) \\ p^0_{14}(t) & p^1_{14}(t) \\ p^0_{15}(t) & p^1_{15}(t) \end{pmatrix}^T + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ z_0(t) \\ z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ z_5(t) \end{pmatrix}^T \right]$$

$$\begin{pmatrix} p^0_{00}(t+1) \cdots p^0_{15}(t+1) \\ p^1_{00}(t+1) \cdots p^1_{15}(t+1) \end{pmatrix} = \begin{pmatrix} f'_0[s_0(t)] \\ f'_1[s_1(t)] \end{pmatrix} * \left[\begin{pmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{pmatrix} \cdot \begin{pmatrix} p^0_{00}(t) \cdots p^0_{15}(t) \\ p^1_{00}(t) \cdots p^1_{15}(t) \end{pmatrix} + \begin{pmatrix} z_{0:5}(t) \ 0 \cdots \\ 0 \cdots \ z_{0:5}(t) \end{pmatrix} \right]$$

A general vectorized form:

$$P_{(:)}(t+1) = f'[s(t)] * \left[w_{(:)(0:2)} \cdot P_{(:)}(t) + block\_diag(z) \right]$$

References

[1] Williams R J, Zipser D. Experimental analysis of the real-time recurrent learning algorithm[J]. Connection science, 1989, 1(1): 87-111.