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Linear Algebra Workshop
- Lecture Notes
by

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References:

1. Down with Determinants — Sheldon Axler
2. Linear Algebra Done Right — Sheldon Axler
3. Linear Algebra : online — Paul Dawkins

Fundamental Question:

$$Av = \lambda v \; ; \qquad A \text{ is an } \underline{\underline{n \times n}} \text{ matrix}$$

then

$\det(\lambda I_n - A) = 0$ computes all eigenvalues & eigenvectors ← WHY?

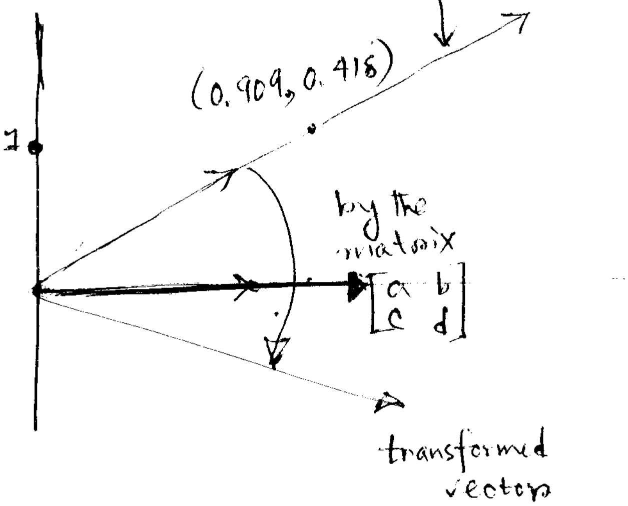
We would endeavour to answer this question through this lecture!!

Snehanshu Saha
28/5/14

Lecture (What is an eigenvalue?) (Dr. Saha) 1

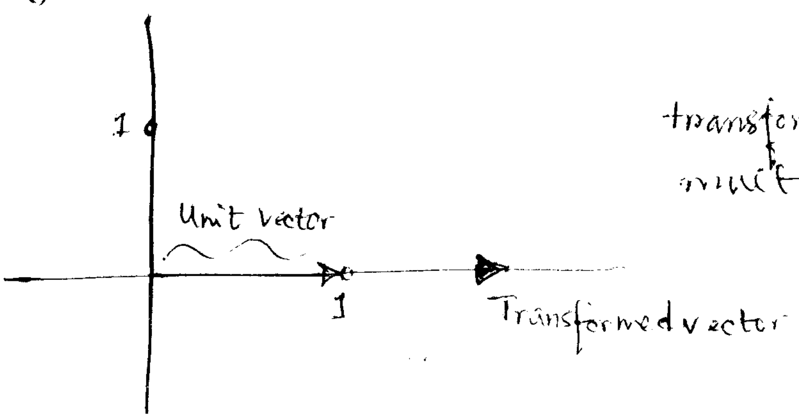
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \approx \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^2$ ~ Geometric Perspective of the "Matrix" structure.

$\uparrow$ 2-D space of Real numbers



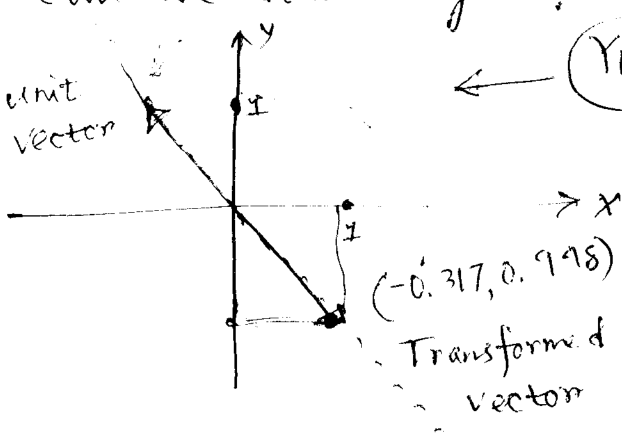
Now, play around w/ these two vectors; unit vector & the transformed vector to see;

When is the transformed vector a "scalar multiple" of the original unit vector? When does it happen & WHY?



transformed vector is a scalar multiple of the unit vector & Length of the new vector $= \boxed{2.0064\,u}$

" Can we do this again? Can we align the two vectors again?



YES → Length of the new vector $= \boxed{0.9949\,u}$

Back to Algebra

Eigen Vectors of $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ &

& Eigen Values: 2 & -1

$\begin{bmatrix} -0.316229 \\ \ldots \end{bmatrix}$

A little bit more... Eigenvalues & eigenvectors are the most critical components in understanding what "matrices" are & actually are useful; not fancy mathematical traps (even though a mathematical trap is fascinating)

So, let's look at a little bit of theory:

Let $T: R^n \longrightarrow R^n$ be a linear transformation from R^n to R^n ($R^n \longrightarrow$ a n-dimensional vector space over real numbers). So, is there a linear transformation (we have already seen, what a transformation can do!) which produces a scaled up version of the original vector (i.e input vector) ??

$$\boxed{T: R^n \rightarrow R^n \qquad T(v) = \lambda v \; ; \; v \in R^n}$$

OR gets reflected around a line, co-ordinate axes or rotated ??

KEYWORDS

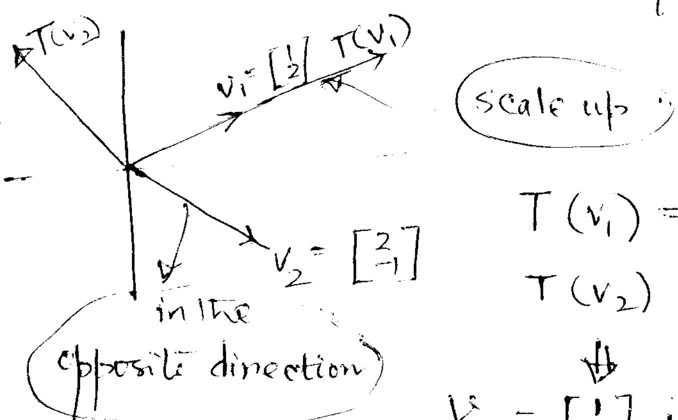
Linear Transformations
Vector Spaces (U, V)
R^n
Span
Linear Independence
Basis
Eigen Values
Eigen Vectors
Orthogonality &
EigenValue - Eigen Vector dynamics
Nullity (Null space)
Rank ; Subspace

$\vec{v} \equiv v$ (throughout the discussion)

M OR A $\equiv$ Matrix induced by the linear transformation

Let's take a 2-D example!

$T: R^2 \longrightarrow R^2$



$T(v_1) = v_1 = 1 . v_1$

$T(v_2) = -v_2 = -1 . v_2$

$\Downarrow$

$v_1 = \begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector

& $\lambda_1 = 1$ is an eigenvalue

$v_2 = \begin{bmatrix}2\\-1\end{bmatrix}$ is an eigenvector

& eigenvalue $= -1$

NOTE: An eigenvalue, in this example is

We may now define, $T(x) = \underset{\sim}{A}x$ ← Matrix of a Linear transformation

Matrix-vector product

Therefore, $T(v) = \boxed{Av = \lambda v}$ $\Longrightarrow$ ($\lambda \rightarrow$ e.val of A; $v \rightarrow$ e.vector of A)

Eigen-value / Eigenvector equation

→ This leads us to a situation where large matrices are characterized by eigenvalues & eigenvectors & these eigenvectors form "Basic vectors" which are computationally simpler & hence interesting !!

Now, what the heck is a "Basic vector"?

↓

we would take a break from "these guys" for a while & try to understand the required concepts & come back to study eigen values & eigenvectors !!

: Fundamentals of Linear Algebra :

(1) We would assume a vector is a directed line segment, defined in a co-ordinated frame of reference i.e $v = (v_1, v_2)$ is a vector in 2-D plane. (R^2 precisely).

(2) We would also assume, for most things, the elements of the vectors are real numbers, unless otherwise specified

(3) A scalar multiple "c" is just a number which when multiplied with a vector, produces $cv = (cv_1, cv_2)$

§. Operations on vectors: What we can do with vectors:

Addition: $u+v = v+u$

Associativity: $u+(v+w) = (u+v)+w$

** Zero vector: (i) $u+0 = 0+u = u$

(ii) $u+(-u) = u-u = 0$

(iii)

to be used later

Identity: $Iu = u$

Associativity wrt scalar multiplication: $(ck)u = c(ku) = k(cu)$

Distributivity: (i) $(c+k)u = cu+ku$

(ii) $c(u+v) = cu+cv$

§. Length of a vector: $v = (v_1, v_2)$; Norm (length) of a vector is defined by $\|v\| = \sqrt{v_1^2+v_2^2}$; (in R^2)

$= \sqrt{v_1^2+v_2^2+\cdots v_n^2}$ (in R^n)

§. Inner product: If $u, v \in R^2$ & θ is the angle b/w them, then the inner product (dot product) is defined as $\langle u, v\rangle = \|u\|\|v\|\cos\theta \Leftrightarrow u_1v_1+u_2v_2$

§. Orthogonality: $u, v \in R^2$ are orthogonal to each other if $\langle u, v\rangle = 0$

Properties: $\langle u, v\rangle = \langle v, u\rangle$

$\langle u, u\rangle = \|u\|^2$

$\langle u, u\rangle > 0$ if $u \neq 0$

$\langle u, u\rangle = 0$ iff $u = 0$

$\ldots = \langle u, v\rangle + \langle u, w\rangle$

Standard Basis vectors : $i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$, $k = \langle 0,0,1 \rangle$ in R^3

(We would come back to this later)

eg: Any vector $u \in R^3$ can be expressed as

$$(u_1, u_2, u_3) = u_1 (1,0,0) + u_2 (0,1,0) + u_3 (0,0,1)$$
$$= u_1 i + u_2 j + u_3 k$$

Let's Move Up!

The n-space: Given a positive integer n, an ordered n-tuple is a sequence of n real numbers denoted by $(v_1, v_2, \dots v_n)$. This complete set of all ordered n-tuples is called n-space & is denoted by R^n.

eg: (i) $\langle u,v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$; where

$$u = (u_1, u_2, \dots, u_n) \in R^n$$
$$v = (v_1, v_2, \dots v_n)$$

(ii) $\|u\| = (u_1^2 + u_2^2 + \dots u_n^2)^{1/2}$

(iii) $\underbrace{d(u,v)}_{\text{Euclidean Distance}} = \{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots (u_n - v_n)^2\}^{1/2}$

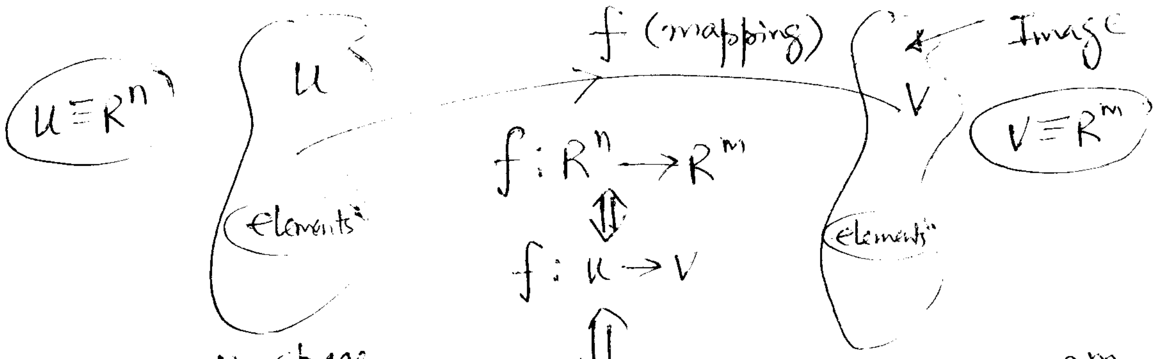
(iv) $|\langle u,v \rangle| \leq \|u\| \|v\|$ (Cauchy - Schwarz Inequality)

(v) $\|u+v\| \leq \|u\| + \|v\|$ (Triangle Inequality)

(vi) $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ (Pythagorean theorem)

⋮

WELCOME to LINEAR TRANSFORMATIONS ..... ⟶



$\Updownarrow$

n-space
R^n
dimension = n

m-space, R^m
dimension = m

$\boxed{\text{The function "f" takes elements from the set (space) } U = R^n \text{ \& maps onto elements in the set (space) } V = R^m}$

NOW, think of elements as vectors, R^n & R^m as spaces (or sets) that "house" those vectors & f as a function OR transformation T that maps R^n into R^m

Therefore, $f: R^n \longrightarrow R^m$ can be alternatively represented as $T: R^n \longrightarrow R^m$ or more generally

$T: U \to V$; where $u = (u_1, u_2, \dots u_n) \in U$ &
$v = (v_1, v_2, \dots v_m) \in V$

i.e

$$T(u_1, u_2, \dots u_n) = (v_1, v_2, \dots v_m)$$

Note: $n \neq m$, necessarily ; if $n = m$,
$T: R^n \to R^m \Rightarrow T: R^n \to R^n$ & T represents a "square" relationship $u = (u_1, \dots u_n)$ & $v = (v_1, \dots v_n)$ (Keep in mind)

ex: $v_1 = 3u_1 - 4u_2$; $v_2 = u_1 + 2u_2$; $v_3 = 6u_1 - u_2$; $v_4 = 10u_2$

$u = (u_1, u_2)$ $\Downarrow$ $\Rightarrow T: R^2 \to R^4 \Leftrightarrow T(u_1, u_2) = (v_1, v_2, v_3, v_4)$

$\Rightarrow \quad \underbrace{T(u_1, u_2)}_{(v_1, v_2, v_3, v_4)} = (3u_1 - 4u_2, u_1 + 2u_2, 6u_1 - u_2, 10u_2)$

$$\Updownarrow$$

$$\underset{\substack{\text{OR} \\ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}_{4\times 1}}}{T(u_1, u_2)} = \begin{bmatrix} 3 & -4 \\ 1 & 2 \\ 6 & -1 \\ 0 & 10 \end{bmatrix}_{4\times 2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times 1}$$

(What does this tell you? $T: R^n \to R^m$ induces a matrix of dimension "$m \times n$" & any vector $\in R^n$ is an $n \times 1$ column vector & any vector $\in R^m$ is an $m \times 1$ column vector & the "$m \times n$" matrix is denoted as $T_A \to$ a matrix induced by the linear transformation T, $\boxed{T_A(v) = Av}$

If $\dim(R^n) = \dim(R^m)$ i.e $n = m$, then T_A becomes a square matrix of dimension "$n \times n$" & is thus useful for multiple types of computations)

Now, consider any two matrices T_A & T_B. Do you believe that

(i) $(T_A + T_B)u = T_A u + T_B u$

(ii) $K T_A(v) = T_A(Kv)$

$\Leftarrow$ What do these properties imply?

(Very simple: matrix addition & scalar multiplication)

$\Downarrow$ YES

Therefore, these properties (Known as "linearity properties") do apply to transformations, T as well. Hence, such

(NOTE: Not all transformations are "linear"; only some are; We would focus on Linear Transformation ONLY)

Thus, A function $T: R^n \longrightarrow R^m$ is called a linear transformation if $\forall$ all $u \in R^n$, $\forall$ $v \in R^m$

$$T(u+v) = T(u) + T(v)$$
$$T(cu) = cT(u)$$

$T: U \rightarrow V$, $u \in U$, $v \in V$;
$T(u+v) = T(u) + T(v)$
$T(cu) = cT(u)$

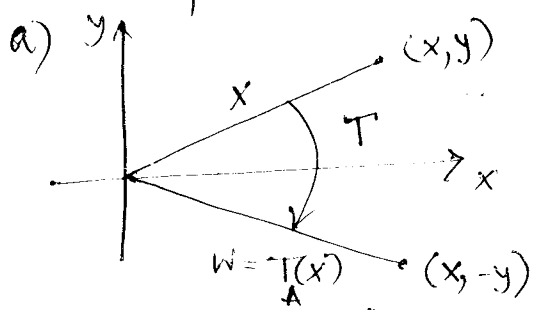
Special cases: ① zero transformation
② Identity transformation ; $T(x) = v$

: A few examples (Back to geometry) :

Determine the matrix induced by the following:

a) Reflection about the x-axis b) Reflection about the y-axis

$\Rightarrow$ It's a 2-D co-ordinate axis system, hence $T: R^2 \rightarrow R^2 \Rightarrow$ we expect to obtain square matrix.

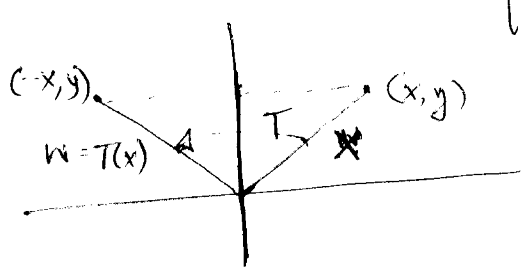
a) 

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in R^2$; $\Rightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

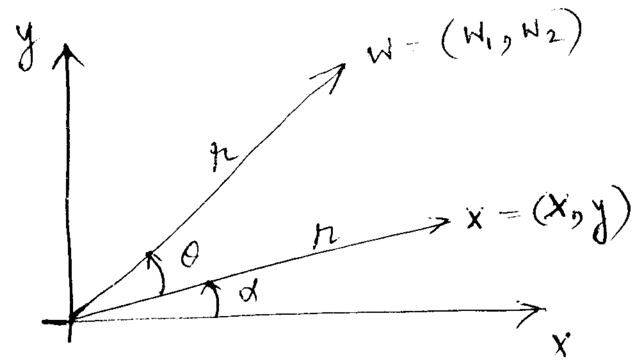
$x = \begin{bmatrix} x \\ y \end{bmatrix} \in R^2$;

T_A

b) y-axis: 

$$\Rightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Contraction / dilation in $\mathbb{R}^2$: $T_A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$



Rotation in $\mathbb{R}^2$:

This involves a little work:

$x = r\cos\alpha,\ y = r\sin\alpha;$

$w_1 = r\cos(\alpha+\theta);\ w_2 = r\sin(\alpha+\theta)$

Now,

$$w_1 = \underbrace{r\cos\alpha}_{x}\cos\theta - \underbrace{r\sin\alpha}_{y}\sin\theta = x\cos\theta - y\sin\theta$$

$$w_2 = \underbrace{r\sin\alpha}_{y}\cos\theta + \underbrace{r\cos\alpha}_{x}\sin\theta = x\sin\theta + y\cos\theta$$

$$\Rightarrow \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{T_A} \begin{bmatrix} x \\ y \end{bmatrix}$$

Some more examples: (Assigned as exercises)

(i) Reflection about the line $y = x$

(ii) Contraction/ dilation in $\mathbb{R}^n$

(Welcome to vector spaces)

Definition: Let V be a set where addition & scalar multiplication are defined with respect to all elements in V satisfying (**) [Page 4]

Ex: An euclidean n-space, R^n is a vector space

NOT every space is a vector space; eg: the set $V = R^3$ w/ the standard vector addition & scalar multiplication defined as

$$c(u_1, u_2, u_3) = (0, 0, cu_3)$$

Subspaces: What is common b/w R^2 & R^n ?

Both are vector spaces & R^2 is a subset of R^n under the definition of standard vector addition & scalar multiplication.

Definition: Suppose that V is a vector space & W is a subset of V. If under the addition and scalar multiplication defined on V, W is also a vectorspace then we call W a subspace of V.

RECALL (**) [Page 4]. If we want to show that W is a vector space as well, all 8 ~~properties~~ axioms mentioned in (**) need to be verified. However, this is not the case. Many of the properties other than addition and scalar multiplication are just variants of the two axioms. Since W inherits the "major" properties from V, the other six axioms follows automatically. These facts lead to the following

Theorem - - -

Theorem 1: Suppose that W is a non-empty (at least one element is in W) subset of V. W is a subspace of V if the following two conditions hold.

(i) $u, v \in W$ then $u+v \in W$ (Closed under addition)

(ii) $u \in W$, $c \in R$ (scalar) then $cu \in W$ (Closed under scalar multiplication)

The definition of addition & scalar multiplication is inherited from V.

Proof: Left as an exercise

Theorem 2: Every vector space, V has at least two subspaces, V & $\{0\}$ (the zero space)

Example: ① Let W be a set of all points $(x, y) \in R^2$ where $x \geq 0$.
Is W a subspace of R^2?

NO ; scalar multiplication doesn't hold. Let c be any scalar $(c<0)$, then $c(x,y) = (cx, cy)$; $cx < 0$ $(\because c<0, x>0)$

the first component is $\not\geq 0$ & doesn't belong to W.

② Let W be the set of all points from R^3 of the form $(0, x_2, x_3)$

YES, W is a subspace of R^3. (Verify yourself!)

Let's now explore a very important subspace of R^m (Recall $T: R^n \rightarrow R^m$, where T_A is an $m \times n$ matrix)
(A)

Null space: Suppose A (the matrix induced by T) is an $m \times n$ matrix. Then, null-space of A is the set of all $x \in R^n \ni$

$$Ax = \theta$$

(For $T: U \to V$, null space $(T) \equiv$ null(T) is the subset of U consisting of those vectors that T maps to 0 i.e.

$$\text{null}(T) = \{v \in U : Tv = \theta\}$$

NOTE: The parity b/w a linear transformation T & the matrix A, induced by T must always be maintained.

Lemma: Null (A) is the subspace of R^n, A is an ~~nx~~ $m \times n$ matrix.

Proof: Simple & left as an exercise!
(Non-empty, addition, scalar multiplication)

Ex: Let's evaluate the ~~span~~ null space of $A = \begin{bmatrix} 1 & -7 \\ -3 & 21 \end{bmatrix}$

$A \in R^{2\times 2}$, $T: R^2 \to R^2$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \in R^2 \ni Ax = 0$ (By definition of Null (A))

$$\Rightarrow \begin{bmatrix} 1 & -7 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underbrace{x_1 - 7x_2 = 0 \ \& \ -3x_1 + 21x_2 = 0}_{\text{same eq}^n\text{; use either one}}$$

$x_1 = 7x_2$; set $x_2 = t$ (arbitrary) $\Rightarrow x_1 = 7t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7t \\ t \end{bmatrix} = t\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$\therefore$ Any vector of the form $t\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ is in the null (A)

$\Updownarrow$

(All points lying on the line $x_1 - 7x_2 = 0$)

~~Let's talk about "span" next~~

[Range of T: For $T: U \to V$, Range (T) is the subset of V consisting of those vectors of the form Tv, $v \in U$:

~~[illegible]~~ **Proposition:** $T: U \to V$, Range (T) is a subspace of V.

Proof: Left as an exercise.

Theorem: $T: U \to V$, U is finite dimensional (?) then

$$\dim(V) = \underbrace{\dim(\text{null } T)}_{\text{Nullity}} + \underbrace{\dim(\text{Range } T)}_{\text{Rank}}$$

Proof: Left as an exercise!

— Implications in Matrix Algebra →

$T: R^n \to R^n$, $n < \infty$, then

$n = \text{Nullity}(A) + \text{Rank}(A)$; where A is the matrix induced by T.]

⇒ Now, lets talk about "Span": ⇐

(Span, Linear Combination, Linear Independence, Basis Set & Dimension)

[Notes from this section are not complete, please refer to Linear Algebra online: Paul Dawkins (Lamar University, USA)]

Just a few pointers:

Linear Combination: $w \in V$, is a linear combination of the vectors $\{v_1, v_2, \dots v_n\} \in V$, if $\exists$ $c_1, c_2, \dots c_n$; all $\in R$ $\ni$

$$w = \sum_{i=1}^{n} c_i v_i$$

Ex: Euclidean n-space: Lets take $u = (u_1, u_2, \dots u_n) \in R^n$ & we can write u as

$$u = \sum_{i=1}^{n} u_i e_i = u_1 e_1 + u_2 e_2 + \dots + u_n e_n$$

where $e_1 = (1, 0, \dots 0)$, $e_2 = (0, 1, 0, \dots 0)$ $\dots$ $e_n = (0, 0, \dots, 1)$

Back to the example on Page 12 ; nullspace (A) is the ~~st~~ set of all linear combinations of $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$

Now, let W be the set of all linear combinations of $\{v_1, \cdots v_n\}$ then $W \equiv \text{span}\{v_1, \cdots v_n\}$

Linear Independence & Basis Set

$\Downarrow$

dimension of a vector

$\Downarrow$

reduction in dimensionality of the vector space by using Basis set $\Longrightarrow$ THE BIG PICTURE !!

Hope, we have covered the basics $\Rightarrow$ Let's go back to the fundamental question:

$$Av = \lambda v \iff \det(\lambda I_n - A) = 0$$ yields a set of eigen values & eigen vectors. WHY ??

$\Rightarrow$ A systematic way to compute eigenvalues & eigenvectors

$$Av = \lambda v \quad (\lambda \text{ scaling factor} \iff \text{eigenvalues}, \quad v \text{ eigenvectors})$$

$$\Rightarrow \lambda v - Av = 0$$

$$\Rightarrow \lambda I_n v - Av = 0 \Rightarrow (\lambda I_n - A)v = 0 ;$$ one solution is $v = 0$; the zero vector. But this is not interesting, why ? For any λ, $v = 0$ could be a solution. So, $v \neq 0 \Rightarrow$ a non-trivial vector (solution). Let $\lambda I_n - A = B$, a matrix ; Right ?

$\therefore$ $Bv = 0 \Rightarrow \underset{\neq 0}{v} \in \text{Null}(B)$; (RECALL the def of Nullspace)

Using the "fact" that : B's columns are L. I iff null $(B) = \{0\}$

$\Rightarrow$ But $v \in \text{Null}\{B\} \neq 0 \Rightarrow$ B's columns are Not L. I

$\det(B) = 0 \Rightarrow \boxed{\det(\lambda I_n - A) = 0}$