Q) Given the matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

Find the eigenvalues $\lambda_1$ & $\lambda_2$ and the associated normalised eigenvectors $e_1$ & $e_2$. Determine the spectral decomposition of $A$.

(a) find $A^{-1}$.

(b) compute the eigenvectors & eigenvalues of $A^{-1}$.

(c) Write the spectral decomposition of $A^{-1}$, and compare it with that of $A$.

Solⁿ (b) $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ $\quad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(-2-\lambda) - (2)(2) = \lambda^2 + \lambda - 6 = 0.$$

$$\therefore (\lambda + 3)(\lambda - 2) = 0.$$

$\lambda_1 = -3$ ; $\lambda_2 = 2$. (~~Eig~~ Eigenvalues)

Solⁿ (a) $$A^{-1} = \frac{\text{adj.}(A)}{\det(A)}$$

$$\det(A) = 1(-2) - 2(2) = -2 - 4 = -6.$$

$$\text{adj}(A) = \begin{bmatrix} -2 & -2 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \left(\frac{1}{-6}\right) \times \begin{bmatrix} -2 & -2 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 1/6 \end{bmatrix}$$

(b). $\det(A^{-1} - \lambda I) = \left[(1/3 - \lambda)(1/6 - \lambda) - (1/3 \times 1/3)\right]$

$= (1/18 - \lambda/3 - \lambda/6 + \lambda^2) - (1/9)$

$= 1/18 - \lambda/3 - \lambda/6 + \lambda^2 - 1/9$

$= \dfrac{2 - 12\lambda - 6\lambda + 36\lambda^2 - 4}{36}$

$= \dfrac{36\lambda^2 - 24\lambda - 2 - 4}{36}$

$= \dfrac{36\lambda^2 - 24\lambda - 6}{36} = 0$

$\therefore (36\lambda^2 - 24\lambda - 6) = 0.$

$\Rightarrow 6\lambda^2 - 4\lambda - 1 = 0$

$\Rightarrow \lambda = \left[\dfrac{-b \pm \sqrt{b^2 - 4ac}}{(2a)}\right]$

Here, $a = 6$, $b = -4$, $c = 1$.

$\therefore \lambda = \left[\dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(6)(-1)}}{2(6)}\right]$

$= \dfrac{4 \pm \sqrt{16+24}}{12}$

$= \dfrac{4 \pm \sqrt{40}}{12}$

$= \dfrac{4 \pm 2\sqrt{10}}{12} = \dfrac{2 \pm \sqrt{10}}{6}$

$\left.\begin{aligned} \therefore \lambda_1 &= \dfrac{2+\sqrt{10}}{6} \\ \therefore \lambda_2 &= \dfrac{2-\sqrt{10}}{6} \end{aligned}\right\}$ eigen values.

$\therefore \frac{\sqrt{10}}{6} \times e_2 + (1/3) \times 0 = 0.$

$\therefore \frac{\sqrt{10}}{6} \times e_2 = 0.$ ——— (iii)

$\therefore (1/3) \times e_2 - \left(1/6 + \frac{\sqrt{10}}{6}\right) \times 0 = 0$

$\Rightarrow 1/3 \times e_2 = 0.$ ——— (iv)

$\therefore e_2 = [0, 0]$ [from (iii) & (iv)]

$\therefore$ Eigen ~~values~~ vectors of $A^{-1}$ are :-

$e_1 = [0, 0]$ & $e_2 = [0, 0]$

(c). For matrix A :

Eigenvalues :- $\lambda_1 = \frac{2+\sqrt{10}}{6}$ ; $\lambda_2 = \frac{(2-\sqrt{10})}{6}.$

Eigenvectors: For $\lambda_1$, $e_1 = [0, 0]$

For $\lambda_2$, $e_2 = [0, 0]$.

Now, spectral decomposition of matrix A,

$A = P \times D \times P^{-1}$

where P is the matrix whose columns are the normalised eigenvectors of A (which are [0, 0] for both $\lambda_1$ & $\lambda_2$).

and D is the diagonal matrix with the eigenvalues of A ($\lambda_1$ & $\lambda_2$) as its diagonal elements.

$\therefore A = PDP^{-1} = [0, 0; 0, 0] \times \left[\left(\frac{2+\sqrt{10}}{6}\right), 0; 0, \left(\frac{2-\sqrt{10}}{6}\right)\right]$

$\times [0, 0; 0, 0].$

Now, $A^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 1/6 \end{bmatrix}$

Soln $A = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\therefore \det(A) = ad - bc$

$\Rightarrow \det(A) = (4)(4.002) - (4.001)(4.001)$

$\Rightarrow \det(A) = 16.008 - 16.008001$

$\Rightarrow \det(A) \approx -0.000001.$

$$A^{-1} = \left(\frac{1}{\det(A)}\right) \times \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$\approx (1/-0.000001) \times \begin{vmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{vmatrix}$$

$$B = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

$$B^{-1} = \left(\frac{1}{\det(B)}\right) \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{aligned} \therefore \det(B) &= ad - bc \\ &= (4.002001)(4) - (4.001)(4.001) \\ &\approx 16.008004 - 16.008001 \\ &\approx 0.000003 \end{aligned}$$

$$\therefore B^{-1} = \frac{1}{\det(B)} \times \begin{vmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{vmatrix}$$

$$B^{-1} \approx \left(\frac{1}{0.000003}\right) \times \begin{vmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{vmatrix}$$

$$\approx 333333.333 \times \begin{vmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{vmatrix}$$

Now, we can see that $A^{-1} \approx (-1,000,000) \times B^{-1}$

$$\approx -3 \times (333,333.333) \times B^{-1}.$$

So, $A^{-1}$ is approximately equal to $(-3)\, B^{-1}$, as requested.

This demonstrates that even small differences in the matrix elements can lead to significantly different inverses due to numerical precision limitations in floating-point arithmetic.

For $\lambda_1$ here, ~~eig~~ $e_1 = [0, 0]$

For $\lambda_2$, $e_2 = [0, 0]$.

$$A^{-1} = PDP^{-1} = [0, 0; 0, 0] \times \left[\left(\frac{2+\sqrt{10}}{6}\right), 0; 0, \left(\frac{2-\sqrt{10}}{6}\right)\right] \times [0, 0; 0, 0]$$

As you can see, the spectral decompositions of matrices $A$ & $A^{-1}$ are the same. This is because matrix $A^{-1}$ is the inverse of matrix $A$, and their eigenvalues are the same. However, in both cases, the eigenvectors associated with these eigenvalues are $[0, 0]$, indicating that the matrices have repeated eigenvalues and no linearly independent eigenvectors.

$\lambda_1 = \frac{(2+\sqrt{10})}{6}$.

$(A^{-1} - \lambda_1) \times e1 = 0$.

$$\begin{bmatrix} (1/3 - \lambda 1) & 1/3 \\ (1/3) & (1/6 - \lambda 1) \end{bmatrix} \begin{bmatrix} e1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1/3 - \left(\frac{(2+\sqrt{10})}{6}\right) & 1/3 \\ 1/3 & \left(1/6 - \left(\frac{2+\sqrt{10}}{6}\right)\right) \end{vmatrix} \begin{vmatrix} e1 \\ \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\Rightarrow 1/3 - \frac{(2+\sqrt{10})}{6} = \frac{2}{6} - \frac{(2+\sqrt{10})}{6} = (-\sqrt{10}/6)$$

$$\Rightarrow 1/6 - \frac{2+\sqrt{10}}{6} = (-1/6) - \left(\frac{\sqrt{10}}{6}\right)$$

$$\Rightarrow \begin{vmatrix} \sqrt{10}/6 & 1/3 \\ 1/3 & -\sqrt{10}/6 \end{vmatrix} \begin{vmatrix} e1 \\ 0 \end{vmatrix} = |0|$$

$\therefore -\sqrt{10}/6 \times e_1 + (1/3) \times 0 = 0$.

$\Rightarrow -\frac{\sqrt{10}}{6} \times e_1 = 0$ — (i)

$\therefore \frac{1}{3} \times e_1 - \left(-\frac{\sqrt{10}}{6}\right) \times 0 = 0$

$\Rightarrow \frac{1}{3} \times e_1 = 0$. — (ii)

$\therefore$ from (i) & (ii), we get,

$e_1 = [0, 0]$.

therefore, for $\lambda_2 = \frac{2-\sqrt{10}}{6}$, we have,

$$\begin{bmatrix} \sqrt{10}/6 & 1/3 \\ 1/3 & (1/6 + \sqrt{10}/6) \end{bmatrix} \begin{bmatrix} e_2 \\ 0 \end{bmatrix} = [0]$$