CA2 EXAM

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CA2 Sub:- Computational Statistics

Q. Let X have covariance matrix

$$\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

(a) Determine ρ an $V^{1/2}$

(b) multiply your matrices to check the relation $V^{1/2} \rho V^{1/2} = \Sigma$

(1) Assume that $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ have a covariance matrix given by

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

Therefore,

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix} = \begin{bmatrix} \sqrt{25} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{9} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Therefore the inverse of the matrix is given by:-

$$(V^{1/2})^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Therefore,

$$\rho = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\therefore \rho = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 5 & -1 & \frac{4}{3} \\ -\frac{2}{5} & 2 & \frac{1}{3} \\ \frac{4}{5} & \frac{1}{2} & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{5} & \frac{4}{15} \\ -\frac{1}{5} & 1 & \frac{1}{6} \\ \frac{4}{15} & \frac{1}{6} & 1 \end{bmatrix}$$ Ans

(ii) $$V^{1/2} \rho V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{5} & \frac{4}{15} \\ -\frac{1}{5} & 1 & \frac{1}{6} \\ \frac{4}{15} & \frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & -\frac{2}{5} & \frac{4}{5} \\ -1 & 2 & \frac{1}{2} \\ \frac{4}{3} & \frac{1}{3} & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \Sigma$$ (proved)

1

2. (a) Find ρ_{13}

(b) Find the correlation between x_1 and $\frac{1}{2}x_2 + \frac{1}{2}x_3$

$\Rightarrow \rho_{13} = \frac{4}{15}$

(b) let, $y = \frac{1}{2}x_2 + \frac{1}{2}x_3$

we need to find the correlation between x_1 and y

$$\text{cov}(x_1, y) = \frac{1}{2}(\text{cov}(x_1, x_2) + \frac{1}{2}(\text{cov}(x_1, x_3))$$
$$= \frac{1}{2}(-2) + \frac{1}{2}(4)$$
$$= 1$$

$$\text{var}(y) = \frac{1}{4}(\text{var}\, x_2) + \frac{1}{4}(\text{var}\, x_3) + \frac{1}{2}(\text{cov}(x_2, x_3))$$
$$= \frac{1}{4}(4) + \frac{1}{4}(9) + \frac{1}{2}(1)$$
$$= 1 + \frac{9}{4} + \frac{1}{2} = \frac{13}{4}$$

$$\sigma_y = \sqrt{13/4} = \frac{\sqrt{13}}{2}$$

$\therefore$ Correlation between x_1 and y —

$$\text{cor}(x_1, y) = \frac{\text{cov}(x_1, y)}{\sigma_1 \times \sigma_y} = \frac{1}{5 \times \sqrt{13}/2} = \frac{2}{5\sqrt{13}}$$

Q. You are given the random vector $x' = [x_1, x_2, x_3, x_4]$ with mean vector $\mu'_x = [4, 3, 2, 1]$ and variance - covariance matrix

$$\Sigma_x = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

partition x as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \hdashline x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x^{(1)} \\ \hdashline x^{(2)} \end{bmatrix}$$

let, $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$

Given $\tilde{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\tilde{x}^{(2)} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$

also $\tilde{\mu}_{\tilde{x}} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ Partitioning the mean vector

$$\tilde{\mu}_{\tilde{x}} = \begin{bmatrix} 4 \\ 3 \\ \hdashline 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{\mu}_{\tilde{x}}^{(1)} \\ \hdashline \tilde{\mu}_{(\tilde{x})}^{(2)} \end{bmatrix}$$

(a) $E(x^{(1)}) = \tilde{\mu}_{\tilde{x}}^{(1)} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(b) $E(\tilde{A}\tilde{x}^{(1)}) = \tilde{A}\, E(\tilde{x}^{(1)}) = \tilde{A} \cdot \tilde{\mu}_{\tilde{x}}^{(1)}$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = [10]$$

(c) $\operatorname{cov}(x^{(1)}) = \tilde{\Sigma}_{11}$

Σ_x can be partitioned as

$$\left[\begin{array}{cc:cc} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \hdashline \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{array}\right] = \left[\begin{array}{c:c} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \hdashline \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{array}\right]$$

$\therefore \ cov\left(x^{(1)}\right) = \Sigma_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $cov\left(\underset{\sim}{A}\, x^{(1)}\right) = \underset{\sim}{A} \underset{\sim}{\Sigma}\, x^{(1)} A'$

$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$

(e) $E\left(x^{(2)}\right) = \tilde{\mu}\, \tilde{x}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(f) $E\left(B x^{(2)}\right) = B\, \tilde{\mu}\, \tilde{x}^{(2)} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(g) $cov\left(x^{(2)}\right) = \tilde{\Sigma}_{22} = \begin{bmatrix} \sigma_{33} & \sigma_{34} \\ \sigma_{43} & \sigma_{44} \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$

(h) $cov\left(B x^{(2)}\right)$

$= \tilde{B}\, \tilde{\Sigma}_{\tilde{x}^{(2)}}\, \tilde{B}' = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

$\therefore \text{cov}(\tilde{B}x^{(2)}) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 13 & 20 \\ -10 & -8 \end{bmatrix}$

$= \begin{bmatrix} 33 & 36 \\ 36 & 48 \end{bmatrix}$

(i) $\text{cov}(x^{(1)}, x^{(2)}) = \tilde{\Sigma}_{12} = \begin{bmatrix} \sigma_{13} & \sigma_{14} \\ \sigma_{23} & \sigma_{24} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$

(j) $\text{cov}(Ax^{(1)}, Bx^{(2)}) = \tilde{A}\tilde{x}^{(1)} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} x_1 + 2x_2 \end{bmatrix}$

$\tilde{B}\tilde{x}^{(2)} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_4 \\ 2x_3 - x_4 \end{bmatrix}$

let, $\tilde{Y} = \begin{bmatrix} x_1 + 2x_2 \\ \hline x_3 - 2x_4 \\ 2x_3 - x_4 \end{bmatrix}$ be partitioned as

$\begin{bmatrix} \tilde{Y}^{(1)} \\ \hline \tilde{Y}^{(2)} \end{bmatrix}$ where, $\tilde{Y}^{(1)} = \tilde{A}\tilde{x}^{(1)}$

$\tilde{Y}^{(2)} = \tilde{B}\tilde{x}^{(2)}$

Also, $\tilde{Y} = \tilde{C}\tilde{x}$, where, $\tilde{C} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

NOW, $\begin{bmatrix} x_1 + 2x_2 \\ x_3 - 2x_4 \\ 2x_3 - x_4 \end{bmatrix} = \tilde{y} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \tilde{C}\tilde{x}$

The covariance matrix of $\tilde{y}$ is given by

$$\tilde{\Sigma}_{\tilde{y}} = \tilde{C} \tilde{\Sigma}_{\tilde{x}} \tilde{C}'$$

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 & 6 \\ 0 & 33 & 36 \\ 6 & 36 & 48 \end{bmatrix}$$

$\therefore \tilde{\Sigma}_{\tilde{y}}$ can be partitioned as $\left[\begin{array}{c|cc} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \hline \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{11\,1\times1} & \Sigma_{12\,1\times2} \\ \hline \Sigma_{21\,2\times1} & \Sigma_{22\,2\times2} \end{array}\right]$

NOW, $\operatorname{cov}(\tilde{y}^{(1)}, \tilde{y}^{(2)})$

$$= \tilde{\Sigma}_{\tilde{y}_{12}} = [0 \quad 6]$$

$\tilde{y}^{(1)} = \tilde{A}\tilde{x}^{(1)}$ & $\tilde{y}^{(2)} = \tilde{B}\tilde{x}^{(2)}$

$\therefore \operatorname{cov}(\tilde{A}\tilde{x}^{(1)}, \tilde{B}\tilde{x}^{(2)}) = [0, 6]$