# A coroot calculation

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**Abstract**

An informal account of the proof of the lemmas `is_root_system.coroot_symmetry_apply_eq` and `is_root_system.coroot_span_eq_top`.

## 1 The coroot of the reflection of a root

Recall our definition of the (pre)symmetry associated to a pair:

**Definition 1.1.** *Let $k$ be a field of characteristic zero and $V$ a vector space over $k$. Given a vector $x \in V$ and a linear form $f \in V^*$ the* ***pre-symmetry*** *associated to the pair $(x, f)$ is the linear endomorphism of $V$:*

$$s_{x,f} : y \mapsto y - f(y)x.$$

*If the condition $f(x) = 2$ holds then $s_{x,f}$ is invertible, satisfies $s_{x,f}^{-1} = s_{x,f}$, and we call it a* ***symmetry****.*

Recall the uniqueness lemma:

**Lemma 1.2.** *Let $k$ be a field of characteristic zero, $V$ a vector space over $k$, and $\Phi \subseteq V$ a finite subset which spans $V$. Given a vector $x \in V$ and two linear forms $f, g \in V^*$ such that:*

- *$f(x) = 2$ and $s_{x,f}(\Phi) \subseteq \Phi$,*
- *$g(x) = 2$ and $s_{x,g}(\Phi) \subseteq \Phi$,*

*then $f = g$.*

Recall the definition of a root system:

**Definition 1.3.** *Let $k$ be a field of characteristic zero, $V$ and vector space over $k$, and $\Phi \subseteq V$. Then we say $\Phi$ is a* ***root system*** *in $V$ over $k$ if:*

- *$\Phi$ is finite,*
- *$\Phi$ spans $V$,*
- *for all $\alpha \in \Phi$, there exists $f \in V^*$ such that $f(\alpha) = 2$ and $s_{\alpha,f}(\Phi) \subseteq \Phi$,*
- *for all $\alpha \in \Phi$ and $f \in V^*$ such that $f(\alpha) = 2$ and $s_{\alpha,f}(\Phi) \subseteq \Phi$, we have $f(\Phi) \subseteq \mathbb{Z} \subseteq k$.*

*We call the elements of $\alpha \in \Phi$ **roots**.*

Recall the definition of the coroot and symmetry of a root:

**Definition 1.4.** *Let $\Phi$ be a root system in $V$ over $k$ and let $\alpha \in \Phi$ be a root. We define the **coroot** $\alpha^* \in V^*$ to be the unique linear form such that:*

- *$\alpha^*(\alpha) = 2$,*
- *$s_{\alpha,\alpha^*}(\Phi) \subseteq \Phi$.*

*We emphasise that uniqueness follows from lemma 1.2. Furthermore we write:*

$$s_\alpha = s_{\alpha,\alpha^*},$$

*and speak of the **symmetry** of a root.*

Now if $\alpha$ and $\beta$ are two roots of some root system then $s_\alpha(\beta) \in \Phi$ is another root and thus has a coroot $(s_\alpha(\beta))^*$. In order to show that the set of coroots form a root system in $V^*$ we need to calculate this coroot in terms of the coroots $\alpha^*$ and $\beta^*$. The following lemma gives the answer:

**Lemma 1.5** (`is_root_system.coroot_symmetry_apply_eq`)**.** *Let $\Phi$ be a root system for $V$ over $k$ and let $\alpha, \beta \in \Phi$ be a roots, then:*

$$(s_\alpha(\beta))^* = \beta^* - (\beta^*(\alpha))\alpha^*.$$

*Proof.* Let $\gamma = s_\alpha(\beta)$ and $g = \beta^* - (\beta^*(\alpha))\alpha^*$. By the uniqueness lemma 1.2 it is sufficient to show that:

(i) $g(\gamma) = 2$,

(ii) $s_{\gamma,g}(\Phi) \subseteq \Phi$.

We did the proof of (i) together on Wednesday: you just unfold all definitions, expand brackets, and use $\alpha^*(\alpha) = \beta^*(\beta) = 2$.

To prove (ii), since $s_\alpha$ and $s_\beta$ both preserve $\Phi$, it is sufficient to show that:

$$s_{\gamma,g} = s_\alpha \circ s_\beta \circ s_\alpha.$$

To prove this we just pick any vector $v \in V$ and unfold the left and right hand sides applied to $v$ and observe that they are equal. □

## 2 The span of the coroots

**Definition 2.1.** *Let $V$ a vector space over a field $k$, $G$ a finite group, and:*

$$\rho : G \to GL(k, V)$$

*a group homomorphism (aka a representation of $G$ on $V$). Given a bilinear form:*

$$B : V \times V \to k,$$

*we define a new bilinear form $B_\rho$ as follows:*

$$\begin{aligned} B_\rho : V \times V &\to k \\ (v, w) &\mapsto \sum_g B(g \cdot v, g \cdot w) \end{aligned} \tag{1}$$

*where the notation $g \cdot v$ means $\rho(g)(v)$.*

**Lemma 2.2.** *In the notation of definition 2.1, the form $B_\rho$ is $G$-invariant, i.e.,*

$$B_\rho(g \cdot v, g \cdot w) = B_\rho(v, w),$$

*for all $v$, $w$ in $V$ and $g$ in $G$. Furthermore if $k$ is an ordered field and $B$ is symmetric and positive definite, then so is $B_\rho$.*

*Proof.* These are just calculations using formula (1). □

**Corollary 2.1.** *Let $\rho$ be a representation of a finite group $G$ on a finite-dimensional vector space $V$ over an ordered field $k$. There exists a $G$-invariant symmetric positive definite bilinear form on $V$.*

*Proof.* Pick any symmetric positive definite bilinear form[1] and apply lemma 2.2 to obtain an invariant form. □

**Corollary 2.2.** *Let $\Phi$ be a root system in a vector space $V$ over an ordered field $k$ and let:*

$$\rho : W \to GL(k, V)$$

*be the corresponding representation of the Weyl group. There exists a $W$-invariant symmetric positive definite bilinear form on $V$.*

[1] E.g., choose a basis and define the form to be the dot product of coordinates.

*Proof.* This follows from lemma 2.1 because the Weyl group is finite. □

**Lemma 2.3** (is_root_system.coroot_span_eq_top)**.** *Let $\Phi$ be a root system in a vector space $V$ over an ordered field $k$. The coroots span $V^*$.*

*Proof.* It is sufficient to show that for any $v$ in $V$:

$$(\alpha^*(v) = 0 \text{ for all } \alpha \in \Phi) \implies v = 0. \tag{2}$$

(Since a non-zero $v$ satisfying (2) would define a non-zero linear form vanishing on the span of the $\alpha^*$.)

Thus let $v$ be a vector satisfying the hypothesis of (2). Note that we have:

$$s_\alpha(v) = v \text{ for all } \alpha \in \Phi.$$

Using corollary 2.2 let $B$ be a Weyl-group-invariant non-singular bilinear form on $V$. Let $\alpha \in \Phi$ and calculate:

$$\begin{aligned} B(v, \alpha) &= B(s_\alpha(v), s_\alpha(\alpha)) \\ &= B(v, -\alpha) \\ &= -B(v, \alpha). \end{aligned}$$

and so:

$$B(v, \alpha) = 0,$$

for all $\alpha \in \Phi$.

Since the roots span $V$ and $B$ is non-singular, we must have $v = 0$ as required. □