A coroot calculation

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Abstract

An informal account of the proof of the lemmas `is_root_system.coroot_symmetry_apply_eq` and `is_root_system.coroot_span_eq_top`.

1 The coroot of the reflection of a root

Recall our definition of the (pre)symmetry associated to a pair:

Definition 1.1. *Let k be a field of characteristic zero and V a vector space over k. Given a vector $x \in V$ and a linear form $f \in V^*$ the* ***pre-symmetry*** *associated to the pair (x, f) is the linear endomorphism of V:*

$$s_{x,f} : y \mapsto y - f(y)x.$$

If the condition $f(x) = 2$ holds then $s_{x,f}$ is invertible, satisfies $s_{x,f}^{-1} = s_{x,f}$, and we call it a ***symmetry****.*

Recall the uniqueness lemma:

Lemma 1.2. *Let k be a field of characteristic zero, V a vector space over k, and $\Phi \subseteq V$ a finite subset which spans V. Given a vector $x \in V$ and two linear forms $f, g \in V^*$ such that:*

- *$f(x) = 2$ and $s_{x,f}(\Phi) \subseteq \Phi$,*
- *$g(x) = 2$ and $s_{x,g}(\Phi) \subseteq \Phi$,*

then $f = g$.

Recall the definition of a root system:

Definition 1.3. *Let k be a field of characteristic zero, V and vector space over k, and $\Phi \subseteq V$. Then we say Φ is a* ***root system*** *in V over k if:*

- *Φ is finite,*
- *Φ spans V,*
- *for all $\alpha \in \Phi$, there exists $f \in V^*$ such that $f(\alpha) = 2$ and $s_{\alpha,f}(\Phi) \subseteq \Phi$,*
- *for all $\alpha \in \Phi$ and $f \in V^*$ such that $f(\alpha) = 2$ and $s_{\alpha,f}(\Phi) \subseteq \Phi$, we have $f(\Phi) \subseteq \mathbb{Z} \subseteq k$.*

We call the elements of $\alpha \in \Phi$ ***roots****.*

Recall the definition of the coroot and symmetry of a root:

Definition 1.4. *Let Φ be a root system in V over k and let $\alpha \in \Phi$ be a root. We define the* ***coroot*** *$\alpha^* \in V^*$ to be the unique linear form such that:*

- *$\alpha^*(\alpha) = 2$,*
- *$s_{\alpha,\alpha^*}(\Phi) \subseteq \Phi$.*

We emphasise that uniqueness follows from lemma 1.2. Furthermore we write:

$$s_\alpha = s_{\alpha,\alpha^*},$$

and speak of the ***symmetry*** *of a root.*

Now if α and β are two roots of some root system then $s_\alpha(\beta) \in \Phi$ is another root and thus has a coroot $(s_\alpha(\beta))^*$. In order to show that the set of coroots form a root system in V^* we need to calculate this coroot in terms of the coroots α^* and β^*. The following lemma gives the answer:

Lemma 1.5 (`is_root_system.coroot_symmetry_apply_eq`)**.** *Let Φ be a root system for V over k and let $\alpha, \beta \in \Phi$ be a roots, then:*

$$(s_\alpha(\beta))^* = \beta^* - (\beta^*(\alpha))\alpha^*.$$

Proof. Let $\gamma = s_\alpha(\beta)$ and $g = \beta^* - (\beta^*(\alpha))\alpha^*$. By the uniqueness lemma 1.2 it is sufficient to show that:

(i) $g(\gamma) = 2$,

(ii) $s_{\gamma,g}(\Phi) \subseteq \Phi$.

We did the proof of (i) together on Wednesday: you just unfold all definitions, expand brackets, and use $\alpha^*(\alpha) = \beta^*(\beta) = 2$.

To prove (ii), since s_α and s_β both preserve Φ, it is sufficient to show that:

$$s_{\gamma,g} = s_\alpha \circ s_\beta \circ s_\alpha.$$

To prove this we just pick any vector $v \in V$ and unfold the left and right hand sides applied to v and observe that they are equal. $\square$

2 The span of the coroots

Definition 2.1. *Let V a vector space over a field k, G a finite group, and:*

$$\rho : G \to GL(k, V)$$

a group homomorphism (aka a representation of G on V). Given a bilinear form:

$$B : V \times V \to k,$$

we define a new bilinear form B_ρ as follows:

$$\begin{aligned} B_\rho : V \times V &\to k \\ (v, w) &\mapsto \sum_g B(g \cdot v, g \cdot w) \end{aligned} \tag{1}$$

where the notation $g \cdot v$ means $\rho(g)(v)$.

Lemma 2.2. *In the notation of definition 2.1, the form B_ρ is G-invariant, i.e.,*

$$B_\rho(g \cdot v, g \cdot w) = B_\rho(v, w),$$

for all v, w in V and g in G. Furthermore if k is an ordered field and B is symmetric and positive definite, then so is B_ρ.

Proof. These are just calculations using formula (1). □

Corollary 2.1. *Let ρ be a representation of a finite group G on a finite-dimensional vector space V over an ordered field k. There exists a G-invariant symmetric positive definite bilinear form on V.*

Proof. Pick any symmetric positive definite bilinear form[1] and apply lemma 2.2 to obtain an invariant form. □

Corollary 2.2. *Let Φ be a root system in a vector space V over an ordered field k and let:*

$$\rho : W \to GL(k, V)$$

be the corresponding representation of the Weyl group. There exists a W-invariant symmetric positive definite bilinear form on V. Note that the map ρ is an inclusion map because W is a subgroup of $GL(k, V)$.

[1] E.g., choose a basis and define the form to be the dot product of coordinates.

Proof. This follows from lemma 2.1 because the Weyl group is finite. $\square$

Lemma 2.3 (is_root_system.coroot_span_eq_top)**.** *Let Φ be a root system in a vector space V over an ordered field k. The coroots span V^*.*

Proof. It is sufficient to show that for any v in V:

$$(\alpha^*(v) = 0 \text{ for all } \alpha \in \Phi) \implies v = 0. \tag{2}$$

(Since a non-zero v satisfying (2) would define a non-zero linear form vanishing on the span of the α^*.)

Thus let v be a vector satisfying the hypothesis of (2). Note that we have:

$$s_\alpha(v) = v \text{ for all } \alpha \in \Phi.$$

Using corollary 2.2 let B be a Weyl-group-invariant non-singular bilinear form on V. Let $\alpha \in \Phi$ and calculate:

$$\begin{aligned} B(v, \alpha) &= B(s_\alpha(v), s_\alpha(\alpha)) \\ &= B(v, -\alpha) \\ &= -B(v, \alpha). \end{aligned}$$

and so:

$$B(v, \alpha) = 0,$$

for all $\alpha \in \Phi$.

Since the roots span V and B is non-singular, we must have $v = 0$ as required. $\square$