A coroot calculation

March 6, 2023

Abstract

An informal account of the proof of the lemma `is_root_system.coroot_symmetry_apply_eq`.

1 The coroot of the reflection of a root

Recall our definition of the (pre)symmetry associated to a pair:

Definition 1.1. *Let k be a field of characteristic zero and V a vector space over k. Given a vector $x \in V$ and a linear form $f \in V^*$ the **pre-symmetry** associated to the pair (x, f) is the linear endomorphism of V:*

$$s_{x,f} : y \mapsto y - f(y)x.$$

*If the condition $f(x) = 2$ holds then $s_{x,f}$ is invertible, satisfies $s_{x,f}^{-1} = s_{x,f}$, and we call it a **symmetry**.*

Recall the uniqueness lemma:

Lemma 1.2. *Let k be a field of characteristic zero, V a vector space over k, and $\Phi \subseteq V$ a finite subset which spans V. Given a vector $x \in V$ and two linear forms $f, g \in V^*$ such that:*

- *$f(x) = 2$ and $s_{x,f}(\Phi) \subseteq \Phi$,*
- *$g(x) = 2$ and $s_{x,g}(\Phi) \subseteq \Phi$,*

then $f = g$.

Recall the definition of a root system:

Definition 1.3. *Let k be a field of characteristic zero, V and vector space over k, and $\Phi \subseteq V$. Then we say Φ is a **root system** for V over k if:*

- *Φ is finite,*
- *Φ spans V,*
- *for all $\alpha \in \Phi$, there exists $f \in V^*$ such that $f(\alpha) = 2$ and $s_{\alpha,f}(\Phi) \subseteq \Phi$,*
- *for all $\alpha \in \Phi$ and $f \in V^*$ such that $f(\alpha) = 2$ and $s_{\alpha,f}(\Phi) \subseteq \Phi$, we have $f(\Phi) \subseteq \mathbb{Z} \subseteq k$.*

*We call the elements of $\alpha \in \Phi$ **roots**.*

Recall the definition of the coroot and symmetry of a root:

Definition 1.4. *Let Φ be a root system for V over k and let $\alpha \in \Phi$ be a root. We define the **coroot** $\alpha^* \in V^*$ to be the unique linear form such that:*

- $\alpha^*(\alpha) = 2$,
- $s_{\alpha,\alpha^*}(\Phi) \subseteq \Phi$.

We emphasise that uniqueness follows from lemma 1.2. Furthermore we write:

$$s_\alpha = s_{\alpha,\alpha^*},$$

*and speak of the **symmetry** of a root.*

Now if α and β are two roots of some root system then $s_\alpha(\beta) \in \Phi$ is another root and thus has a coroot $(s_\alpha(\beta))^*$. In order to show that the set of coroots form a root system in V^* we need to calculate this coroot in terms of the coroots α^* and β^*. The following lemma gives the answer:

Lemma 1.5 (`is_root_system.coroot_symmetry_apply_eq`)**.** *Let Φ be a root system for V over k and let $\alpha, \beta \in \Phi$ be a roots, then:*

$$(s_\alpha(\beta))^* = \beta^* - (\beta^*(\alpha))\alpha^*.$$

Proof. Let $\gamma = s_\alpha(\beta)$ and $g = \beta^* - (\beta^*(\alpha))\alpha^*$. By the uniqueness lemma 1.2 it is sufficient to show that:

(i) $g(\gamma) = 2$,

(ii) $s_{\gamma,g}(\Phi) \subseteq \Phi$.

We did the proof of (i) together on Wednesday: you just unfold all definitions, expand brackets, and use $\alpha^*(\alpha) = \beta^*(\beta) = 2$.

To prove (ii), since s_α and s_β both preserve Φ, it is sufficient to show that:

$$s_{\gamma,g} = s_\alpha \circ s_\beta \circ s_\alpha.$$

To prove this we just pick any vector $v \in V$ and unfold the left and right hand sides applied to v and observe that they are equal. $\square$