

Singular value decomposition for approximating data, reduced order modelling, de-noise data



$r = 200$, Rel no of stored values = $4.13e-01$ Relative error: $2.30e-03$



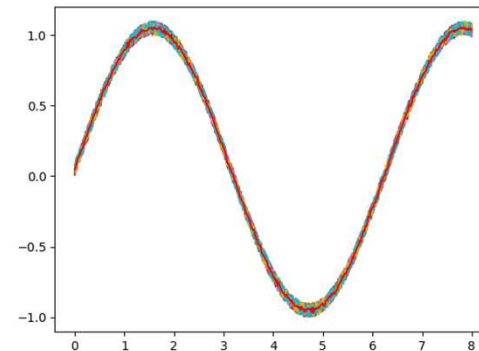
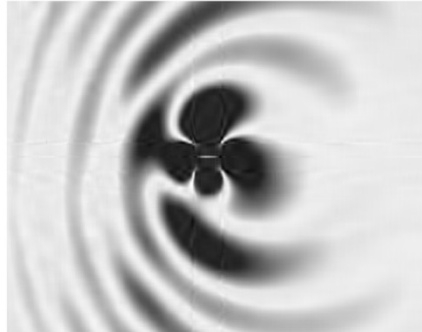
$r = 50$, Rel no of stored values = $1.03e-01$ Relative error: $1.28e-02$



$r = 5$, Rel no of stored values = $1.03e-02$ Relative error: $5.63e-02$



$r = 20$, Rel no of stored values = $5.58e-02$ Relative error: $6.08e-03$



Full SVD

$$\begin{bmatrix} \mathbf{X} \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\mathbf{U}} & \hat{\mathbf{U}}^\perp \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} \hat{\Sigma} \\ \mathbf{0} \end{bmatrix}}_{\Sigma} \begin{bmatrix} \mathbf{V}^* \end{bmatrix}$$

The diagram illustrates the Full SVD decomposition of matrix \mathbf{X} . Matrix \mathbf{X} is represented as a gray rectangle. It is equal to the product of three matrices: \mathbf{U} , Σ , and \mathbf{V}^* . Matrix \mathbf{U} is shown as a gray rectangle divided into two parts: $\hat{\mathbf{U}}$ (gray) and $\hat{\mathbf{U}}^\perp$ (light gray). Matrix Σ is shown as a gray rectangle divided into two parts: $\hat{\Sigma}$ (gray, with diagonal lines indicating non-zero singular values) and a zero block $\mathbf{0}$ (light gray). Matrix \mathbf{V}^* is a gray rectangle. Braces below the matrices indicate the grouping of $\hat{\mathbf{U}}$ and $\hat{\mathbf{U}}^\perp$ into \mathbf{U} , and $\hat{\Sigma}$ and $\mathbf{0}$ into Σ .

Economy SVD

$$\begin{bmatrix} \mathbf{X} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{U}} \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{V}^* \end{bmatrix}$$

The diagram illustrates the Economy SVD decomposition of matrix \mathbf{X} . Matrix \mathbf{X} is represented as a gray rectangle. It is equal to the product of three matrices: $\hat{\mathbf{U}}$, $\hat{\Sigma}$, and \mathbf{V}^* . Matrix $\hat{\mathbf{U}}$ is a gray rectangle. Matrix $\hat{\Sigma}$ is a gray rectangle with diagonal lines indicating non-zero singular values. Matrix \mathbf{V}^* is a gray rectangle. The label $\hat{\mathbf{U}}$ in the first matrix has a small red 'U' above it.

Theorem 1 (Eckart-Young [170]) *The optimal rank- r approximation to \mathbf{X} , in a least-squares sense, is given by the rank- r SVD truncation $\tilde{\mathbf{X}}$:*

$$\underset{\tilde{\mathbf{X}}, \text{ s.t. } \text{rank}(\tilde{\mathbf{X}})=r}{\text{argmin}} \quad \|\mathbf{X} - \tilde{\mathbf{X}}\|_F = \tilde{\mathbf{U}}\tilde{\Sigma}\tilde{\mathbf{V}}^*. \quad (1.4)$$

$$\begin{aligned} \left[\begin{array}{c} \mathbf{X} \end{array} \right] &= \underbrace{\left[\begin{array}{c|c|c} \tilde{\mathbf{U}} & \hat{\mathbf{U}}_{\text{rem}} & \hat{\mathbf{U}}^\perp \end{array} \right]}_{\tilde{\mathbf{U}}} \left[\begin{array}{c|c} \begin{array}{c} \tilde{\Sigma} \\ \hline 0 \end{array} & \begin{array}{c} \hat{\Sigma}_{\text{rem}} \\ \hline 0 \end{array} \end{array} \right] \left[\begin{array}{c} \tilde{\mathbf{V}}^* \\ \hline \mathbf{V}_{\text{rem}} \end{array} \right] \\ &\approx \left[\begin{array}{c} \tilde{\mathbf{U}} \end{array} \right] \left[\begin{array}{c} \tilde{\Sigma} \end{array} \right] \left[\begin{array}{c} \tilde{\mathbf{V}}^* \end{array} \right] \\ &\quad \text{Truncated SVD} \end{aligned}$$