EDUCATALYSTS

Class(12th)

Introduction to Matrices

Matrices

1.1 Definition of a Matrix

Definition 1.1.1 (Matrix) A rectangular array of numbers is called a matrix.

VVe shall mostly be concerned with matrices having real munbers as entries.

The horizontal arrays of a matrix are called its ROWS and the vertical arrays are called its COLUMNS.

A matrix having *rn* rows and *n* columns is said to have the order *m* x *n.*

A matrix *A* of ORDER *in* x *n* can be reprcsentcxl in the following form:

«12 • «In

$A = \Big| a_{21}$ «22 • • A2n

:

«ml "m2 • • a,,in_

where a_{tJ} is the entry at the intersection of th< row and colninn.

In a more concise manner, we also denote t [e matrix *A* by [~]by suppressing its order.

Remark 1.1.2 *Some books nlso use*

' ail ai2 • • a In

«21 «22 • • «2n

a,"2 • • a'"”

to represent a *matrix.*

A matrix having only one colunni is called a COLUMN VECTOR: and a matrix with only one row is called a ROW VECTOR.

WHENEV'ER A VECTOR IS USED, IT SHOULD BE UNDERSTOOD FROM THE CONTEXT WHETHER IT IS A ROW VECTOR OR A COLUMN VECTOR.

Definition 1.1.3 (Equality of two Matrices) Two matrices *A* = [a_M] and *B* = [fe_M] having the same order *HI* x *n* are equal if

for each *i* = 1,2,...,"i and *j* = 1,2,... ,n.

In other words, two matrices are said to l)e eepud if they have the same order and their corresponding entries are equal.

Example 1.1.4 The linear system of equations 2r 4- 3i/ = 5 and 3r + = 5 can be identified with the matrix

1.1.1 Special Matrices

Definition 1.1.5 1. A matrix in which each entry is zero is called a zero-matrix, denoted by 0. For example,

$$0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } 0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. A matrix having the number of rows equal to the number of columns is called a square matrix. Thus, its order *is m x m* (for some *m)* and is represented by *m* only.

3. In a square matrix, *A* = [ay], of order n, the entries an,«22» • • •,arm are called the diagonal entries and form the principal diagonal of *A,*

4. A square matrix *A* = *[atJ]* is said to be a diagonal matrix if = 0 for / *j.* In other words, the . ... [4 0 non-zero entries appear only on the principal diagonal. For example, the zero matrix 0_r, and are a few diagonal matrices.

A diagonal matrix D of order n with the diagonal entriesrfi,r/2," .,dH is denoted by D = diag(di,".,r/„). If J, =(/ for all 1 = 1,2,..., *11* then the diagonal matrix *D* is called a scalar matrix.

5. A square matrix *A* = [a,j] with *aij* = is called the identity matrix, denoted by

if *i=j*
if *i^j*

The subscript *n* is suppressed in case the order is clear from the context or if no confusion arises.

6. A square matrix *A* = [ay] is said to be an upper triangular matrix if = 0 for i > *j.*

A square matrix *A* = [aI7] is said to be an lower triangular matrix if a,j = 0 for i < *j.*

A square matrix *A* is said to be triangular if it is an upper or a lower triangular matrix

For example -1 -2 is an upper triangular matrix. An upper triangular matrix will be represented by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

1.2 Operations on Matrices

Definition 1.2.1 (Transpose of a Matrix) The transpose of an *m* x *n* matrix *A* = [a_{ij}] is defined as the n x *m* matrix *B* = [6^], with for 1 < i < m and 1 < j < n. The transpose of *A* is denoted by *A'.*

That is. by the transpose of an rn x n matrix A, we mean a matrix of order n x m having the rows of -4 as its columns and

For example, if J4 = $\begin{vmatrix} '1 & 4 & 5 \\ 0 & 1 & 2 \end{vmatrix}$ then A^1 = $\begin{vmatrix} \bullet 1 & o' \\ 4 & 1 \\ 5 & 2 \end{vmatrix}$

Tims, the transpose of a row vector is a column vector and vice-versa.

the colmnns of /4 as its rows.

Theorem 1.2.2 For any matrix A, we have $(A'Y = A$.

PROOF. Let A = [a^], A' = [6,j] and $(4^1)'$ = [cjj]. Then, the definition of transpose gives

$$Cij = bji = aij \text{ for all } i,j$$

and the result follows.

Definition 1.2.3 (Addition of Matrices) let A = [a,j] and B = [6U] be are two // i x n matrices. Then the sum $A + B$ *is* defined to be the matrix C = [c^] with = $a,j + b_{tJ}$.

Note that, we define the sum of two matrices only when the order of the two matrices are stune.

Definition 1.2.4 (Multiplying a Scalar to a Matrix) Let A = [a,j] be an m x n matrix. Then for any element Ar € **R.** we define

For example, if 4 = anti $k = 5$, the» $5A$ = $\begin{matrix} 20 & 25 \\ 5 & 10 \end{matrix}$

kA = [frajj].

Theorem 1.2.5 Let A. B and C be matrices of order in x n. and let fr. € R. Then

1. A -i- $B = B + A$ (commutativity).
2. $(A + B) + C = A + (B + C)$ (associativity).
3. $k(tA) = (kt)A$.
4. $(k + t)A = kA + fA$.

PROOF. PartH
Let A = [a,j] and B = [/>u]. Then

$$A + B = [a_{l?}] + [fcij] = [a\textasciicircum + bij] = [6,j + a_{tj}] = [6_{tj}] + [\sim j = B + A$$

as real mmilM*rs commute.

The reader is required to prove the other parts as all the results follow from the properties of real numbers. □

Exercise 1.2.6 1. Suppose $A + B = A$. Then show that $B = 0$.

2. Suppose >1 + B = 0. Then show that B = (-1)4 =[-~]•

Definition 1.2.7 (Additive Inverse) Let A be an in x n matrix.

1. Then there exists a matrix B with $A + B = 0$. This matrix B is called the additive inverse of A. and is denoted by $-A$ = (-1)/1.
2. Also, for the matrix 0,"x„,J4 + 0 = 0 + >1 = >1. Hence, the matrix $0_{„lxn}$ is called the additive identity.

1.2.1 Multiplication of Matrices

Definition 1.2.8 (Matrix Multiplication / Product) Let A = [a^] be an m x n matrix and B = [6y] be an n x r matrix. The product

AB is a matrix C = [c^] of order *ni* x r, with n *Ci J* = = *ai*、*b*、*j* +a_{i2}*b2j* + ••• +*a,„b„J.*

k=l

Observe that the product AB is defined if and only if

THE NUMBER OF COLUMNS OF

For example, if J4 = and B = then

1 + 6+12 4 2 19

2+12 + 4 3 4 18

A — THE NUMBER OF ROWS OF B. *[l*

Note that in tins example, while AB is defintxl, the product BA is not defined. However, for square matrices A and B of the same order, both the product AB and BA are defined.

Definition 1.2.9 Two square matrices A and B are said to commute if AB — BA.

Remark 1.2.10 *1. Note that if A is a* square *matrix of order n then* $Af_n = I_nA$. *Also for any* d € R. *the matrix dl„ commutes nith* every square matrix *of order n. The matrices df„ for any d E R* are *called* SCALAR

2. In general, the matrix product is not commutative. For example, consider the following two

matrices A = and B = Then check that the matrix product

matrices.

Theorem 1.2.11 Suppose that the matrices *A. B* and *C* are so chosen that the matrix multiplications are defined.

1. Then $(AB)C = A(BC)$. That is, the matrix multiplication is associative.

2. For any A: e R. *{kA)B = k(AB) = A(kB).*

3. Then $A(B + C) = AB + AC$. That is, multiplication distributes over addition.

4. If / is an n x n matrix then *Al„ = f„A = A.*

5. For any square matrix A of order n and *D — diag((li,d2,* ■ ■ ■ *.*d_u*),* we have

   - the first row of *DA* is *d* times the first row of *At*
   - for 1 < » < n. the '•* row of *DA* is *di* times the row of *A.*

A similar statement holds for the columns of A when A is multiplied on the right by *D.* PROOF. PartD Let A = [a,j]mx",B

= [$b_o$lnxP and C = Then

P n

(BC)kj = ᵇktctj an(i (AB)u = ^a,ₖbₖₗ.

1=1 k=l

Therefore,

$Y,a_{,k}(BC)_{kj}$ = t

"):, '1

PartS For all J = 1,2....,n, we have

$$(DA)ij = \hat{}2 \sum_{k=l}^{n} = {}^{d}»^{a}ij$$

as $d,k = 0$ whenever $i \neq k$. Hence, the required result follows. The reader is required to prove the other parts.

Exercise 1.2.12 1. Let *A* and *B* be two matrices. If the matrix addition >1 + B is defined, then prove

bi

b2

2. Compute the matrix products *AB* and *BA.*

3. Let n be a positive integer. Compute *A"*

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

that *(A + B)'* = A^1 + *B'.* Also, if the matrix product *AB* is defined then prove that $(AB)^1 = B'A^1$.

for the following matrices:

,a few subinatrices of *A* are

Can you guess a formula for A'' and prove it by induction?

4. Find examples for the following statements.

(a) Suppose that the matrix product AB is defined. Then the product BA need not be defined.

(b) Suppose that the matrix products AB and BA are defined. Then the matrices AB and BA can have different orders.

(c) Suppose that the matrices A and B are square matrices of order n. Then AB and BA may or may not be equal.

1.3 Some More Special Matrices

Definition 1.3.1 1. A matrix A over R is called symmetric *if* $A^1 = A$ and skew-symmetric if $A^1 = —A$.

2. A matrix A is said to be orthogonal if $AA' = A'A = I$.

Example 1.3.2 **1. Let** $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$. Then >1 is a symmetric matrix and B is a skew-symmetric matrix.

2. Let ?1 = ★ I ★ 0 ൠ6 .Then A is an orthogonal matrix.

3. Let $A = [a_M]$ be an n x n matrix with . Then .4" = 0 and /1' / 0 for 1 < f <
10 otherwise
n - 1. The matrices A for which a positive integer k exists such that $A^k = 0$ are called NILPOTENT matrices. The

4. Let A = .Then $A^2 = A$. The matrices that satisfy the condition that $A^2 = A$ are called
least positive integer k for which $A^k = 0$ is called the ORDER OF NILPOTENCY.

IDEMPOTENT matrices.

Exercise 1.3.3 1. Show that for any square matrix A, S = ^(A + A^) is symmetric, T = ^(A — .4') is skew-symmetric, and $A = S + T$.

2. Show that the product of two lower triangular matrices is *a* lower triangular matrix. A similar statement holds for upper triangular matrices.
3. Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.
4. Show that the diagonal entries of a skew-symmetric matrix are zero.
5. Let A. B be skew-symmetric matrices with $AB = BA$. Is the matrix AB symmetric or skew-symmetric?
6. Let >1 be a symmetric matrix of order » with $A^2 = 0$. Is it necessarily true that 4 = 0?
7. Let >1 be a nilpotent matrix. Show that there exists a matrix B such that $B(1 + A) = I = (I + A)B$.

1.3.1 Submatrix of a Matrix

Definition 1.3.4 A matrix obtained by deleting some of the rows and/or columns of a matrix is said to be a submatrix of the given matrix. *

For example, if =

[O] ,[1 5]

But the matrices and are not submatriees of A. (The reader is advised to give reasons.)

Miscellaneous Exercises

Exercise 1.3.5 1. Complete the proofs of Theorems IL2^l and IL2A11

-sill ൠ COS0 .Geometrically interpret y = <4x

3. Consider the two coordinate transformations

=anyi +ai2y2 and !/i — + ^12²2
J/2 = ^»2121 + 622Z2
12 = «21J/1 + «22</2

(a) Compose the two transformations to express *ri,X2* in terms of zi, 22-

(b) If x' = [j-j, J:2],y' = [t/i, t/2] and z' = [z>, 22] then find matrices *A. B* and *C* such that x = *Ay.* y = *Bz* and x = Cz.

(c) Is C = *AB?*

4. For a square matrix *A* of order n. we define trace of *A,* denoted by tr *(A)* as

$$\text{tr}\,(A) = an + a_22 + \cdot a,,,,.$$

Then for two square matrices, *A* and *B* of the same order, show the following:

(a) tr *(A + B)* = tr M) + tr(B).

(b) tr *(AB)* = tr *(BA).*

5. Show that, there do not exist matrices *A* and *B* such that *AB - BA* = cI_n for any c / 0.

6. Let *A* and *B* be two rn x n matrices and let x be an n x 1 column vector.

(a) Prove that if 4x = 0 for all x. then *A* is the zero matrix.

7. Let *A* be an *n x n* matrix such that *AB* = *BA* for all *n* x n matrices *B*. Show that *A* = *al* for some a € R.

8. Let *A* = .Show that there exist infinitely many matrices *B* such that *BA* = /2. Also, show
3 j
that there does not exist any matrix *C* such that *AC* = /3.

1.3.1 Block Matrices

Let *A* be an n x *ni* matrix and B be an m x p matrix. Suppose r < ni. Tlien, we can decompose the matrices *A* and B as >1 = [P Q] aiid *B* = ;where *P* has order n x r and *H* has order r x p. That is, the matrices *P* aiid *Q* are submatrices o *A* and *P* consists of the first r coluiuiis of *A* and *Q* consists of the last *m — r* columns of *A.* Similarly, *H* and *K* are suhmatrices of *B* and *H* consists of the first r rows of *B* and *K* consists of the last *m — r* rows of *B.* We now prove the following important theorem.

Theorem 1.3.6 Let *A* — [aM] — [P Q] and *B* —[~)]— be defined as above. Then

$$AB\^{} PH + QK.$$

PROOF. First note that the matrices *PH* and *QK* arc each of order n x *p.* The matrix products *PH* and *QK* are valid as the order of the matrices *P, H, Q* and *K* are respectiwly, *n* x r, r x p, n x *(w - r)* and *(m — r)* x *p.* Let *P* = [P,j], *Q* = [Q,j], *H* = and *K* = Then, for 1 < i < n and 1 < j < p, we have

$$(AB),, \quad \hat{}\sum_{k=1} a_{ik}b_{kj} = \sum_{k=l} \quad + 52 \sum_{k=r+l} a_{ik}b_{kj}$$

$$y?\ p_{ik}H_{kj} + 52\ Q_{ik}K_{kj}$$

$$(PH)_{tj} + (QK)_{tJ} = (PH + QK)_{tj}.$$

(b) Prove that if *Ax.* = *Bx* for all x, then *A* = *B.*

Theorem 11.3.61 is very useful due to tlie following reasons:

1. The order of the matrices *P. Q. H* and *K* are smaller than that of *A* or *B.*
2. It may be possible to block the matrix in such a way that a few blocks are either identitj' matrices or zero matrices. In this case, it may be easy to handle the matrix product iismg the block fonii.
3. Or when we want to prove results using hiduction, then we may assume the result for r x r submatrices and then look

For example, *if A* = and *B* = ,Then
for (r + 1) x (r + 1) submatrices, etc.

$$\begin{bmatrix} b \\ d \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e\ f] = \quad a + 2c\ 2a + 5c \quad b + 2d \quad 26 + 5J$$

If A = $\begin{vmatrix} 0 & -1 & 2 \\ 3 & 1 & 4 \\ -2 & 5 & -3 \end{vmatrix}$,then *A* can be decoinposed as follows:

$A = \left[\begin{array}{c|cc} 0 & -1 & 2 \\ 3 & 1 & 4 \\ \hline -2 & 5 & -3 \end{array}\right]$,or $A = \left[\begin{array}{cc|c} 0 & -1 & 2 \\ 3 & 1 & 4 \\ \hline -2 & 5 & -3 \end{array}\right]$,or

$\left[\begin{array}{cc|c} 0 & -1 & 2 \\ \hline 3 & 1 & 4 \\ -2 & 5 & -3 \end{array}\right]$ and so on.

Suppose $A = \begin{array}{c} \\ n_1 \\ n_2 \end{array}\begin{array}{cc} m_1 & m_2 \\ \hline P & Q \\ R & S \end{array}$ and $B = \begin{array}{c} \\ r_1 \\ r_2 \end{array}\begin{array}{cc} s_1 & s_2 \\ \hline E & F \\ G & H \end{array}$ .Then the matrices *P. Q. R, S* and *E, F, G. H.* are called the blocks of the matrices *A* and *B,* respectively.

Even if ?! + B is defined, the orders of *P* and *E* may not be same and hence, wc may not lw able to add *A* and *B* in the block form. Bnt, if 4 + B and *P + E is* defined then $A + B = \begin{bmatrix} P+E & Q+F \\ R+G & S+H \end{bmatrix}$.

Similarly, if the product *AB* is defined, the pro<luct *PE* need not be definetl. Therefore, we can talk of matrix product *AB* as block product of matrices, if both the products *AB* and *PE* are defined. And in tliis case, we have $AB = \begin{bmatrix} PE + QG & PF + QH \\ RE + SG & RF + SH \end{bmatrix}$.

That is, once a partition of *A* is fixed, the partition of *B* has to be properly chosen for purposes of block addition or multiplication.

Exercise 1.3.7 1. Compute the matrix product *AB* using the block matrix multiplication for the matrices

$A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$ and $B = \left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ \hline 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{array}\right]$.

2. Let *A* = If *P. Q. R* and *S* are symmetric, what can you say about *A?* Are *P. Q. R* and *S* symmetric, when *A* is symmetric?

3. Let A = [a$_o$] and B = [6$_{tJ}$] be two matrices. Suppose ai, a ,...,a„ are the rows of A and bi, b2. ..., b$_;$, are the columns of B. If the product AB is defined, then show that

AB [^bj, Ab_2, ... Ab_p]=

a_2B

aiB

a„S

[That is, left multiplication by A, is same as multiplying each column of B by A. Similarly, right multiplication by B. is same as multiplying each row of A by B.]

1.4 Matrices over Complex Numbers

Here the entries of the matrix are complex munbcrs. All the deHiiitioiis still hold. Oue just needs to look at the following additional deHnitioiis.

Definition 1.4.1 (Conjugate Transpose of a Matrix) 1. Let A be an rn x n matrix over C. IM = [a$_M$] then the Conjugate of A.

For example, Let A = 4 + 3i i 1 i-2 .Then

denoted by A, is the matrix B = [6,j] with b'' = a^j.

$$\overline{A} = \begin{bmatrix} 1 & 4\text{-}3t & —i \\ 0 & 1 & \text{-i-2}\ 2 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1 & 0 \\ 4-3i & 1\ \text{-i-2} \\ -i & \end{bmatrix}$$

2 Let X be an in x n matrix over C. If 4 =[~]then the Conjugate Transpose of A, denoted by .4', is the matrix B = [6^] with $b_{,}j$ = aJ7,

3. A square matrix A over C is called Hermitian if $A' = A$.
4. A square matrix A over C is called skew-Hermitian if $A' = —A$.
5. A square matrix A over C is called unitary if $A'A — AA^* — I$.
6. A square matrix A over C is called Normal if $AA' = A' A$.

Remark 1.4.2 *If* A = [a ; j] *with* € R. *then* $A' = A^1$.

Exercise 1.4.3 1. Give examples of Hermitian, skew-Hermitian and unitary matrices that have entries with non-zero imaginary parts.

2. Restate the results on transpose in terms of conjugate transpose.

3. Show that for any square matrix .4. $S =$ is Hermitian, $T =$ is skew-Hermitian, and $A = S + T$.

4. Show that if is a complex triangular matrix and $AA' = A'A$ then A is a diagonal matrix.