EDUCATALYSTS

Class(12th)

Introduction to Relations and Functions

CHAPTER – 2

RELATIONS AND FUNCTIONS

CONCEPT MAP

★ **Ordered Pair:** An ordered pair consists of two objects or elements in a given fixed order.

Remarks: An ordered pair is not a set consisting of two elements. The ordering of two elements in on ordered pair is important and the two elements need not be distinct.

★ **Equality of Ordered Pair:** Two ordered pairs (x_1, y_1) & (x_2, y_2) are equal if $x_1 = x_2$ and $y_1 = y_2$.

i.e. $(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$

★ **Cartesian product of two sets:** Cartesian product of two non-empty sets A and B is given by $A \times B$ and $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$.

★ **Cartesian product of three sets:** Let A, B and C be three sets, then $A \times B \times C$ is the set of all ordered triplet having first element from set A, 2nd element from set B and 3rd element from set C.

i.e., $A \times B \times C = \{(x, y, z) : x \in A, y \in B \text{ and } z \in c\}$.

★ **Number of elements in the Cartesian product of two sets:** If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

★ **Relation:** Let A and B be two non-empty sets. Then a relation from set A to set B is a subset of $A \times B$.

- ★ **No. of relations:** If n(A) = p, n(B) = q then no. of relations from set A to set B is given by 2^{pq}.
- ★ **Domain of a relation:** Domain of R = {a : (a, b) ∈ R}
- ★ **Range of a relation:** Range of R = { b : (a, b) ∈ R}
- ★ Co-domain of R from set A to set B = set B.
- ★ Range ⊆ Co-domain
- ★ **Relation an a set:** Let A be non-empty set. Then a relation from A to B itself. i.e., a subset of A × A, is called a relation on a set.
- ★ **Inverse of a relation:** Let A, B be two sets and Let R be a relations from set A to set B.

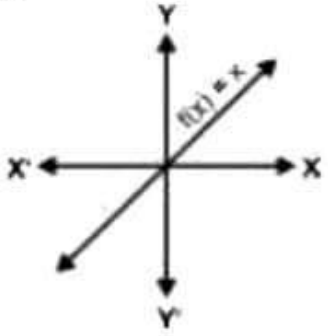
  Then the inverse of R denoted R^{-1} is a relation from set B to A and is defined by R^{-1} = {(b, a) : (a, b) ∈ R}
- ★ **Function:** Let A and B be two non-empty sets. A relation from set A to set B is called a function (or a mapping or a map). If each element of set A has a unique image in set B.

  Remark: If (a, b) ∈ f then 'b' is called the image of 'a' under f and 'a' is called reimage of 'b'.
- ★ **Domain of range of a function:** If a function 'f' is expressed as the set of ordered pairs, the domain of 'f' is the set of all the first components of members of f and range of 'f' is the set of second components of member of 'f'.

  i.e., D_f = {a : (a, b) ∈ f} and R_f = {b : (a, b) ∈ D_f}
- ★ **No. of functions:** Let A and B be two non-empty finite sets such that n(A) = p and n(B) = q then number of functions from A to B = q^p.
- ★ **Real valued function:** A function f : A → B is called a real valued function if B is a subset of R (real numbers).

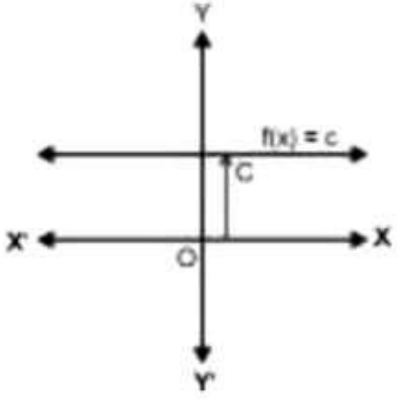
★ **Identity function:** $f : R \to R$ given by $f(x) = x \ \forall \ x \in R$ (real number)

Here, $D_f = R$ and $R_f = R$



★ **Constant function:** $f : R \to R$ given by $f(x) = c$ for all $x \in R$ where c is any constant

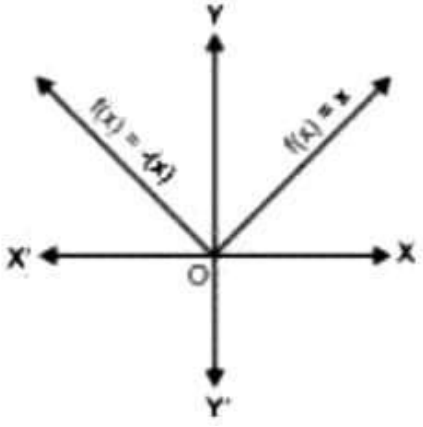
Here, $D_f = R$ and $R_f = \{c\}$



★ **Modulus function:** $f : R \to R$ given by $f(x) = |x| \ \forall \ x \in R$

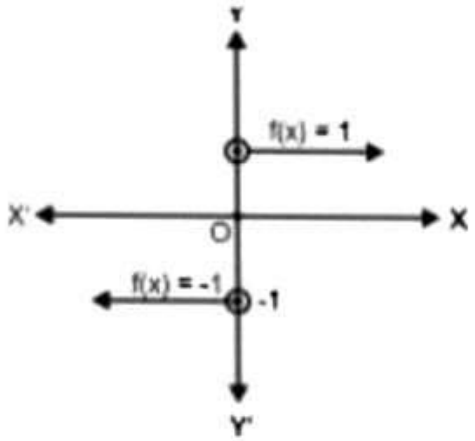
Here, $D_f = R$ and $R_f = [0, \infty)$

Remarks : $\sqrt{x^2} = |x|$



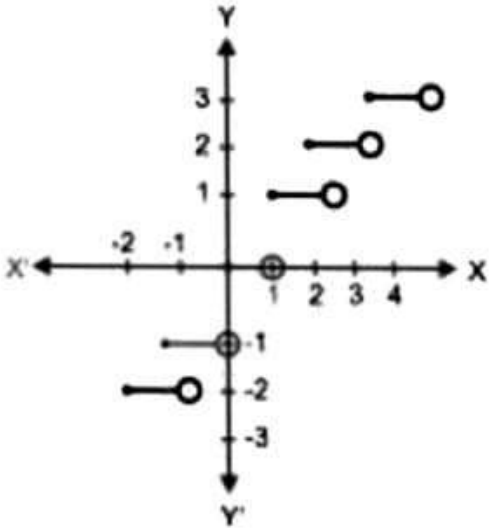
★ **Signum function:** $f : R \to R$ defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

or $f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x > 0 \end{cases}$



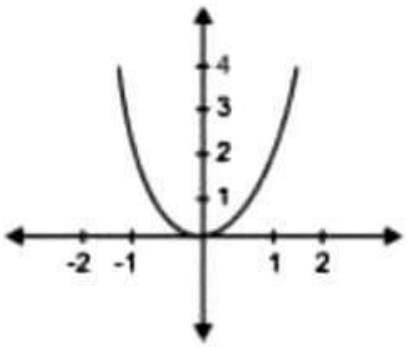
★ **Greatest Integer function:** $f : R \to R$ defined by $f(x) = [x]$, $x \in R$ assumes the value of the greatest integer, less than or equal to x.

Here, $D_f = R$ and $R_f = Z$

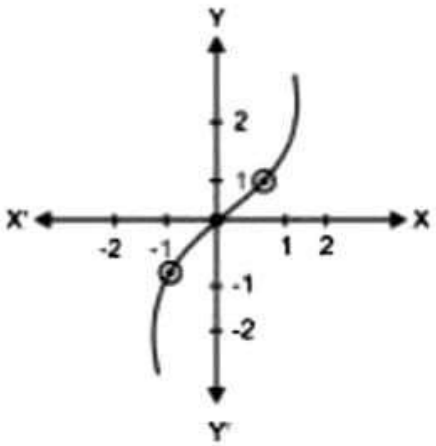


★ Graph for $f : R \to R$, defined by $f(x) = x^2$

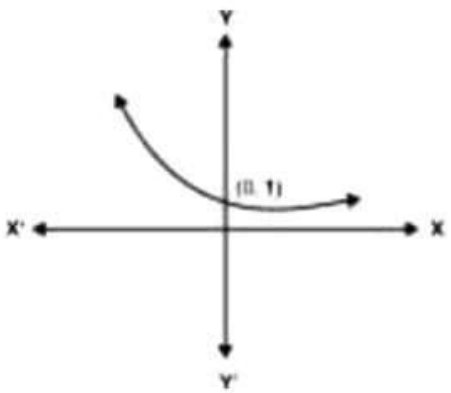
Here, $D_f = R$ and $R_f = [0, \infty)$



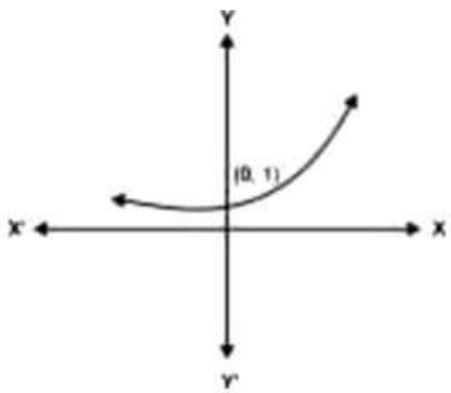
★ Graph for $f : R \to R$, defined by $f(x) = x^3$



★ **Exponential function:** $f : R \to R$, defined by $f(x) = a^x, a > 0, a \neq 1$



When $0 < a < 1$



When $a > 1$

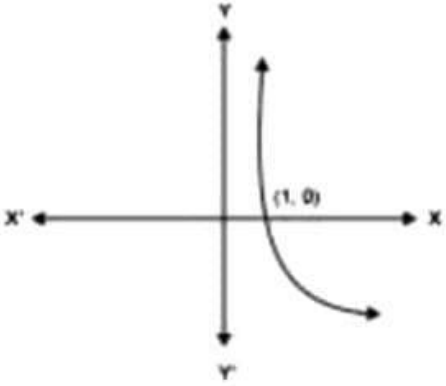
$$f(x) = a^x \begin{cases} > 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ < 1 & \text{for } x > 0 \end{cases}$$

$$f(x) = 0^x \begin{cases} < 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ > 1 & \text{for } x > 0 \end{cases}$$

★ Natural exponential function, $f(x) = e^x$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty, \quad 2 < e < 3$$

★ Logarithmic functions, $f : (0, \infty) \to R$; $f(x) \log_a x$, $a > 0$, $a \neq 1$

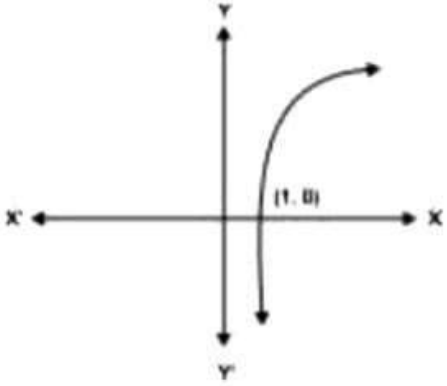


$f(x) = \log_a x$, $0 < a < 1$

$D_f = (0, \infty)$

$R_f = R$

Case I When $0 < a < 1$



$f(x) = \log_a x$, for $a > 1$

$D_f = (0, \infty)$

$R_f = R$

Case II When $a > 1$

★ **Natural logarithm function:** $f(x) = \log_e x$ or $\ln(x)$.

★ Let $f : X \to R$ and $g : X \to R$ be any two real functions where $x \subset R$ then

$(f \pm g)(x) = f(x) \pm g(x) \; \forall \; x \in X$

$(fg)(x) = f(x)\, g(x) \; \forall \; x \in X$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \forall \; x \in X$ provided $g(x) \neq 0$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. If $A = \{1, 2, 4\}$, $B = \{2, 4, 5\}$, $C = \{2, 5\}$ then $(A - B) \times (B - C)$

   (a) $\{(1, 2), (1, 5), (2, 5)\}$ (b) $\{1, 4\}$

   (c) $\{1, 4\}$ (d) None of these.

2. If R is a relation on set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ given by $xRy \Leftrightarrow y = 3x$, then R = ?

   (a) $\{(3, 1), (6, 2), (8, 2), (9, 3)\}$ (b) $\{(3, 1), (6, 2), (9, 3)\}$

   (c) $\{(3, 1), (2, 6), (3, 9)\}$ (d) None of these.

3. Let $A = \{1, 2, 3\}$, $B = \{4, 6, 9\}$ if relation R from A to B defined by x is greater then y. the range of R is -

   (a) $\{1, 4, 6, 9\}$ (b) $\{4, 6, 9\}$

   (c) $\{1\}$ (d) None of these.

4. If R be a relation from a set A to a set B then -

   (a) $R = A \cup B$ (b) $R = A \cap B$

   (c) $R \subseteq A \times B$ (d) $R \subseteq B \times A$.

5. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ $(x \neq 0)$, then f(2) is equal to -

   (a) $\frac{-7}{4}$ (b) $\frac{5}{2}$

   (c) -1 (d) None of these.

6. Range of the function $f(x) = \cos[x]$ for $\frac{-\pi}{2} < x < \frac{\pi}{2}$ is -

   (a) $\{-1, 1, 0\}$ (b) $\{\cos 1, \cos 2, 1\}$

   (c) $\{\cos 1, -\cos 1, 1\}$ (d) $\{-1, 1\}$.

7. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{3x+x^3}{1+3x^2}$ then $f\{g(x)\}$ is equal to -

(a) $f(3x)$ (b) $\{f(x)\}^3$

(c) $3f(x)$ (d) $-(f(x)$.

8. If $f(x) = \cos(\log x)$ then value of $f(x).f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\}$ is -

(a) 1 (b) -1

(c) 0 (d) ± 1.

9. Doman of $f(x) = \sqrt{4x - x^2}$ is -

(a) $R - [0, 4]$ (b) $R - (0, 4)$

(c) $(0, 4)$ (d) $[0, 4]$.

10. If $[x]^2 - 5[x] + 6 = 0$, where $[\ .\ ]$ denote the greater integer function then -

(a) $x \in [3, 4]$ (b) $x \in (2, 3]$

(c) $x \in [2, 3]$ (d) $x \in [2, 4)$.

11. Find a and b if $(a - 1, b + 5) = (2, 3)$

If $A = \{1,3,5\}$, $B = \{2,3\}$, find : (Question - 12, 13)

12. $A \times B$

13. $B \times A$

Let $A = \{1,2\}$, $B = \{2,3,4\}$, $C = \{4,5\}$, find (Question - 14, 15)

14. $A \times (B \cap C)$

15. $A \times (B \cup C)$

16. If P = {1,3}, Q = {2,3,5}, find the number of relations from P to Q

17. If R = {(x,y): x,y $\in$ Z, $x^2 + y^2 = 64$}, then,

    Write R in roster form

    Which of the following relations are functions? Give reason. (Questions 18 to 20)

18. R = { (1,1), (2,2), (3,3), (4,4), (4,5)}

19. R = { (2,1), (2,2), (2,3), (2,4)}

20. R = { (1,2), (2,5), (3,8), (4,10), (5,12), (6,12)}

SHORT ANSWER TYPE QUESTIONS (2 MARKS)

21. If A and B are finite sets such that n(A) = 5 and n(B) = 7, then find the number of functions from A to B.

22. If $f(x) = x^2 - 3x + 1$ find $x \in R$ such that f (2x) = f (x)
    Let f and g be two real valued functions, defined by, f(x) = x, g(x) = |x|.

    Find: (Question 23 to 26)

23. f + g

24. f – g

25. fg

26. $\frac{f}{g}$

27. If $f(x) = x^3$, find the value of, $\frac{f(5)-f(1)}{5-1}$

28. Find the domain of the real function, $f(x) = \sqrt{x^2 - 4}$

29. Find the domain of the function, $f(x) = \frac{x^2 + 2x + 3}{x^2 - 5x + 6}$

Find the range of the following functions. (Question- 30, 31)

30. $f(x) = \frac{1}{4 - x^2}$

31. $f(x) = x^2 + 2$

32. Find the domain of the relation,
$R = \{(x, y): x, y \in Z, xy = 4\}$

Find the range of the following relations: (Question-33, 34)

33. $R = \{(a,b) : a, b \in N \text{ and } 2a + b = 10\}$

34. $R = \left\{\left(x, \frac{1}{x}\right): x \in z, 0 < x < 6\right\}$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

35. Let $A = \{1,2,3,4\}$, $B = \{1,4,9,16,25\}$ and R be a relation defined from A to B as,

$R = \{(x, y): x \in A, y \in B \text{ and } y = x^2\}$

(a) Depict this relation using arrow diagram.
(b) Find domain of R.
(c) Find range of R.
(d) Write co-domain of R.

36. If $A = \{2,4,6,9\}$ $B = \{4,6,18,27,54\}$ and a relation R from A to B is defined by $R = \{(a,b): a \in A, b \in B, a \text{ is a factor of } b \text{ and } a < b\}$, then find in Roster form. Also find its domain and range.

37. Let $f(x) = \begin{cases} x^2, & \text{when } 0 \le x \le 2 \\ 2x, & \text{when } 2 \le x \le 5 \end{cases}$

$g(x) = \begin{cases} x^2, & \text{when } 0 \le x \le 3 \\ 2x, & \text{when } 3 \le x \le 5 \end{cases}$

Show that f is a function while g is not a function.

38. Find the domain and range of,

$f(x) = |2x - 3| - 3$

39. Draw the graph of the Greatest Integer function

40. Draw the graph of the Constant function $f : R \to R$; $f(x) = 2 \ \forall \ x \in R$. Also find its domain and range.

41. Draw the graph of the function $|x - 2|$

Find the domain and range of the following real functions (Question 42 to 47)

42. $f(x) = \sqrt{x^2 + 4}$

43. $f(x) = \dfrac{x+1}{x-2}$

44. $f(x) = \dfrac{|x+1|}{x+1}$

45. $f(x) = \dfrac{x^2 - 9}{x-3}$

46. $f(x) = \dfrac{4-x}{x-4}$

47. $f(x) = 1 - |x - 3|$

48. Determine a quadratic function (f) is defined by $f(x) = ax^2 + bx + c$. If $f(0) = 6$; $f(2) = 11$, $f(-3) = 6$

49. Draw the graph of the function $f(x) = \begin{cases} 1+2x & x<0 \\ 3+5x & x \geq 0 \end{cases}$ also find its range.

50. Draw the graph of following function

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Also find its range.

Find the domain of the following function.

51. $f(x) = \frac{1}{\sqrt{x+|x|}}$

52. $f(x) = \frac{1}{\sqrt{x-|x|}}$

53. $f(x) = \frac{1}{\sqrt{[x]^2-[x]-6}}$

54. $f(x) = \frac{1}{\sqrt{9-x^2}}$

55. $f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$

56. Find the domain for which the followings:

$f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.

57. If $f(x) = x - \frac{1}{x}$ prove that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$

58. If [x] denotes the greatest integer function. Find the solution set

$[x]^2 + 5[x] + 6 = 0$

59. If $f(x) = \dfrac{ax-b}{bx-a} = y$

Find the value of f(y)

60. Draw the graph of following function and find range (R_f) of

$f(x) = |x-2| + |2+x| \quad \forall \quad -3 \le x \le 3$

ANSWERS

1. (b) 2. (d) 3. (c) 4. (c) 5. (a)

6. (b) 7. (c) 8. (c) 9. (d) 10. (d)

11. $a = 3, b = -2$

12. $A \times B = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\}$

13. $B \times A = \{ (2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$

14. $\{(1,4), (2,4)\}$

15. $\{(1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5)\}$

16. $2^6 = 64$

17. $R = \{ (0,8), (0,-8), (8,0), (-8,0)\}$

18. Not a function because 4 has two images.

19. Not a function because 2 does not have a unique image.

20. Function because every element in the domain has its unique image.

21. 7^5

22. 0,1

23. $f+g = \begin{cases} 2x & x \ge 0 \\ 0 & x < 0 \end{cases}$

24. $f-g = \begin{cases} 0 & x \ge 0 \\ 2x & x < 0 \end{cases}$

25. $fg = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$

26. $\frac{f}{g} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ and Note:- $\frac{f}{g}$ is not defined at x = 0

27. 31

28. $(-\infty, -2] \cup [2, \infty)$

29. $R - \{2,3\}$

30. $(-\infty, 0) \cup [1/4, \infty)$

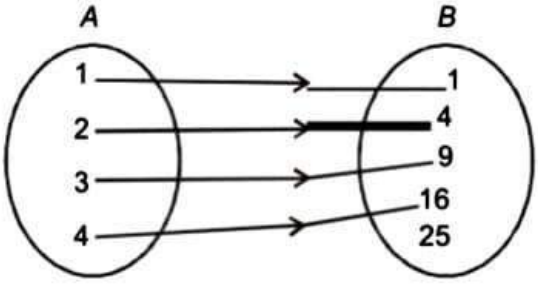
31. $[2,\infty)$

32. $\{-4, -2, -1, 1, 2, 4\}$

33. $\{2, 4, 6, 8\}$

34. $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$

35. (a)



(b) {1, 2, 3, 4}

(c) {1, 4, 9, 16}

(d) {1, 4, 9,16, 25}

36. R = { (2,4) (2,6) (2,18) (2,54) (6,18) (6,54) (9,18) (9,27) (9,54) }

Domain is R = {2,6,9}

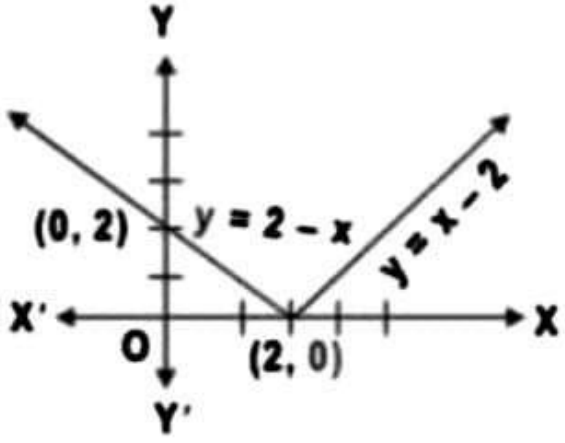
Range of R = { 4, 6, 18, 27, 54}

38. Domain is R

Range is $[-3, \infty)$

40. Domain = R, Range = {2}

41.



42. Domain = R,

Range = $[2, \infty)$

43. Domain = R – {2}

Range = R – {1}

44. Domain = R – {–1}

Range = {1, –1}

45. Domain = R – {3}

Range = R – {6}

46. Domain = R – {4}

Range = {–1}

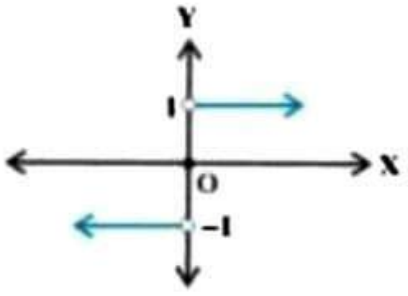
47. Domain = R

Range = $(-\infty, 1]$

48. $\frac{1}{2}x^2 + \frac{3}{2}x + 6$

49. $(-\infty, 1) \cup [3, \infty)$

50. Range of f = {–1,0,1}



51. $(0, \infty)$

52. ϕ (given function is not defined)

53. $(-\infty, -2) \cup (4, \infty)$

54. $(-3, 3)$

55. $(-\infty, -1) \cup (1, 4]$

56. $\left\{-2, \frac{1}{2}\right\}$

58. $[-3, -1)$

59. x

60. $R_f = [4, 6]$ and graph is