AI1110 Assignment 8

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Outline

Abstract

- This document contains the solution to Question of Chapter 6 of Papoulis book.

Question

Exercise 6.23

The random variables x and y are independent with respective densities $\mathcal{X}^2(m)$ and $\mathcal{X}^2(n)$. Show that if

$$z = \frac{\frac{x}{m}}{\frac{y}{n}}$$

then

$$f_z(z) = \gamma \frac{z^{\frac{m}{2}-1}}{\sqrt{(1+\frac{mz}{n})^{m+n}}} U(z)$$

Theory

Densities

1. The density of x/m is given by

$$f_1(x) = \frac{m(mx)^{(\frac{m}{2}-1)}e^{\frac{-mx}{2}}}{\Gamma(\frac{m}{2})2^{\frac{m}{2}}} \tag{1}$$

2. The density of y/n is given by

$$f_2(y) = \frac{n(nx)^{(\frac{n}{2}-1)}e^{\frac{-nx}{2}}}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}} \tag{2}$$

3. The density of z is given by

$$f_z(z) = \int_0^\infty y f_{xy}(yz, y)\, dy \tag{3}$$

Solution

$$
\begin{aligned}
f_z(z) &= \int_0^\infty y \left(\frac{m(mzy)^{(\frac{m}{2}-1)} e^{\frac{-mzy}{2}}}{\Gamma(\frac{m}{2})} \right) \left(\frac{n(nx)^{(\frac{n}{2}-1)} e^{\frac{-nx}{2}}}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}}} \right) dy && (4) \\
&= \frac{(\frac{m}{2})^{\frac{m}{2}} (\frac{n}{2})^{\frac{n}{2}}}{\Gamma(\frac{m}{2})} z^{(\frac{m}{2}-1)} \int_0^\infty y^{(\frac{m+n}{2}-1)} e^{y(\frac{n+mz}{2})}\, dy && (5) \\
&= \frac{(\frac{m}{2})^{\frac{m}{2}} (\frac{n}{2})^{\frac{n}{2}}}{\Gamma(\frac{m}{2})} z^{(\frac{m}{2}-1)} \Gamma\left(\frac{m+n}{2}\right) \left(\frac{2}{n+mz}\right)^{(\frac{m+n}{2})} && (6)
\end{aligned}
$$

LaTeX

Solution

$$= \frac{\Gamma(\frac{m+n}{2})(m)^{\frac{m}{2}}(n)^{\frac{n}{2}}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{z^{(\frac{m}{2}-1)}}{(n+mz)^{\frac{m+n}{2}}} \tag{7}$$

$$= \frac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{\frac{m}{2}}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{z^{(\frac{m}{2}-1)}}{\sqrt{(1+\frac{mz}{n})^{m+n}}} \tag{8}$$

$$= \gamma \frac{z^{\frac{m}{2}-1}}{\sqrt{(1+\frac{mz}{n})^{m+n}}} U(z) \tag{9}$$

Proved.