AI1110 Assignment 9

DEEPSHIKHA-CS21BTECH11016

June 2, 2022

Outline

Abstract

- This document contains the solution to Question of Chapter 6 of Papoulis book.

Question

Exercise 6.58

The random variables x and y are jointly distributed over the region $0 <$ x $<$ y < 1 as

$$f_{xy}(x,y) = \begin{cases} kx, & 0 < x < y < 1 \\ 0. & \text{otherwise} \end{cases} \tag{1}$$

for some k. Determine k. Find the variances of x and y. What is the covariance between x and y?

Theory

Expression	Formula
Joint P.d.f	$\iint f_{XY}(x, y)\; dx\, dy$
$E[X]$	$\int\limits_{-\infty}^{+\infty} xf_X(x)\; dx$
$E[X^2]$	$\int\limits_{-\infty}^{+\infty} x^2 f_X(x)\; dx$
$E[XY]$	$\iint xyf_{XY}(xy)\; dy\, dx$
$Var(X)$	$E[X^2] - (E^2[X]^2)$
$Cov(X, Y)$	$E[XY] - E[X].E[Y]$

Table 1: Two random variables

Value of k

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}\left(x,y\right)\,dx\,dy = 1 \tag{2}$$

$$\implies \int_{0}^{1}\int_{x}^{1} kx\,dy\,dx = k\int_0^1 x\left(1-x\right)\,dx = 1 \tag{3}$$

$$\implies \frac{k}{6} = 1 \implies \boxed{k=1} \tag{4}$$

PDF

$$f_X(x) = \int_x^1 6x\, dy = 6x(1-x) \qquad , 0 < x < 1 \qquad (5)$$

$$f_Y(y) = \int_0^1 6x\, dx = 3y^2 \qquad , 0 < y < 1 \qquad (6)$$

Expectation

$$E[X] = \int_0^1 x f_X(x)\ dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\bigg|_0^1 = \frac{1}{2} \tag{7}$$

$$E[X^2] = \int_0^1 y^2 f_Y\, dy = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\bigg|_0^1 = \frac{3}{10} \tag{8}$$

$$E[Y] = \int_0^1 y f_Y(y)\ dy = 3\frac{y^4}{4}\bigg|_0^1 = \frac{3}{4} \tag{9}$$

$$E[Y^2] = \int_0^1 y^2 f_Y(y)\ dy = 3\frac{y^5}{5}\bigg|_0^1 = \frac{3}{5} \tag{10}$$

LaTeX

Expectation

$$E[XY] = \int_0^1 \int_0^x xyf_{XY}\left(xy\right)\, dy\, dx \tag{11}$$

$$= \int_0^1 3x\left(1 - x^2\right)\, dx \tag{12}$$

$$= 3\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 \tag{13}$$

$$= 3\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{5} \tag{14}$$

Variance

1.

$$Var\left(X\right) = \frac{3}{10} - \frac{1}{4} = \boxed{\frac{1}{20}} \quad (15)$$

2.

$$Var\left(Y\right) = \frac{3}{5} - \frac{9}{16} = \boxed{\frac{3}{80}} \quad (16)$$

Covariance

$$Cov\left(X,Y\right) = \frac{2}{5} - \frac{1}{2}.\frac{3}{4} = \boxed{\frac{1}{40}} \tag{17}$$