CONTENTS

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Here are the files:

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/coeffs.
    h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig.1

```
wget https://github.com/Deepshikha11004/
    Random_variable.git/codes/cdf_plot.py
```

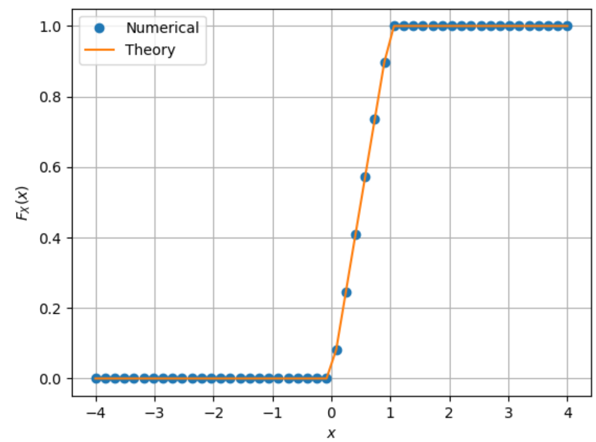

Fig. 1

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$f_u(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{1.2}$$

For $x \le 0$,

$$F_u(x) = \int_{-\infty}^{0} 0dx \tag{1.3}$$
$$= 0 \tag{1.4}$$

For $0 < x < 1$,

$$F_u(x) = \int_{0}^{x} 1dx \tag{1.5}$$
$$= x \tag{1.6}$$

For $x \ge 1$,

$$F_u(x) = \int_{1}^{\infty} 0dx \tag{1.7}$$
$$= 0 \tag{1.8}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.9}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.10}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/coeffs.
    h
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.11}$$

Solution:

$$E[U^k] = 0 + \int_0^1 x^k dF_U(x) + 0 \tag{1.12}$$
$$= \frac{1}{k+1} \tag{1.13}$$
$$\implies E[U] = \frac{1}{2} \tag{1.14}$$
$$= 0.50 \tag{1.15}$$
$$E[U^2] = \frac{1}{3} \tag{1.16}$$

Hence,

$$\text{Variance} = E[U^2] - E[U]^2 \tag{1.17}$$
$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.18}$$
$$= \frac{1}{12} \tag{1.19}$$
$$= 0.0833 \tag{1.20}$$

For $x \geq 1$,

$$E[U^k] = \int_0^{\infty} x^k dF_u(x) \tag{1.21}$$
$$= 0 \tag{1.22}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution:

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/coeffs.
    h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of a random variable U has the following properties:

a) $F_U(x)$ is a non-decreasing function of x where $-\infty < x < \infty$
b) $F_U(x)$ ranges from 0 to 1
c) $F_U(x)$ = 0 as x→ -∞
d) $F_U(x)$ = 1 as x→ ∞

The CDF of X is plotted in Fig.2using the code below

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/
    x_cdf_plot.py
```

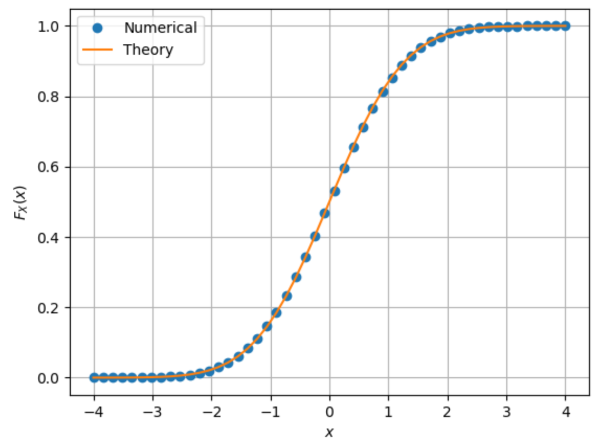

Fig. 2

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

Solution: The PDF of a random variable X has the following properties:

a) The probability density function is non-negative for all the possible values.
b) $\int_{-\infty}^{\infty} f(x)\, dx = 1$
c) $f(x) = 0$ as $x \to -\infty$
d) $f(x) = 0$ as $x \to \infty$

The PDF of X is plotted in Fig.3 using the code below

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/
    pdf_plot.py
```

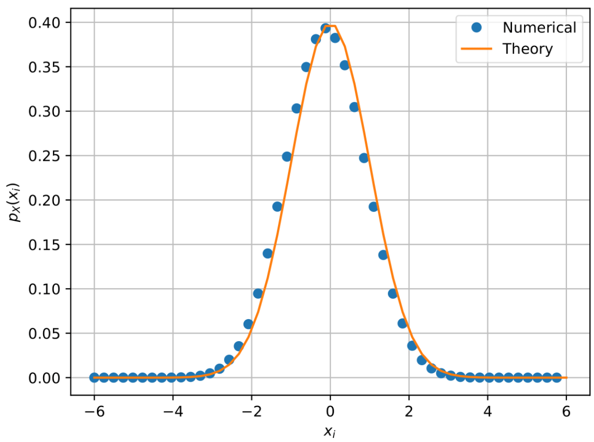

Fig. 2

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
wget https://github.com/Deepshikha11004/
    Random_variable.git/codes/main.c
wget https://github.com/Deepshikha11004/
    Random_variable.git/codes/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution:

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x)dx \tag{2.4}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})dx \tag{2.5}$$
$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi}\Big|_{-\infty}^{\infty} \tag{2.6}$$
$$= 1 \tag{2.7}$$

By definition,

$$p_X(x)dx = dF_U(x) \tag{2.8}$$

Hence,

$$E[X] = \int_{-\infty}^{\infty} x p_X(x)dx \tag{2.9}$$
$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})dx \tag{2.10}$$
$$= 0 \tag{2.11}$$

Also,

$$\text{Variance} = E[X^2] - E[X]^2 \tag{2.12}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x)dx \tag{2.13}$$
$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})dx \tag{2.14}$$
$$= \int_{-\infty}^{\infty} xx \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})dx \tag{2.15}$$
$$= \frac{1}{\sqrt{2\pi}}(-xexp(\frac{-x^2}{2})\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} exp(\frac{-x^2}{2})dx) \tag{2.16}$$
$$= \frac{1}{\sqrt{2\pi}}(0 + \sqrt{2\pi}) \tag{2.17}$$
$$= 1 \implies \text{Variance} = 1 \tag{2.18}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: The following code generates samples of V in vdis.dat:

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/coeffs.
    h
```

The following code plots CDF of V:

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/
    v_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Let, $V = g(U)$

$$V = -2\ln(1 - U) \tag{3.2}$$
$$\implies U = 1 - e^{-\frac{V}{2}} \tag{3.3}$$

Now,

$$F_V(x) = P(g(U) \le x) \tag{3.4}$$
$$= P\left(X < g^{-1}(V)\right) \tag{3.5}$$
$$= F_U\left(g^{-1}(V)\right) \tag{3.6}$$
$$= F_U\left(1 - e^{-\frac{V}{2}}\right) \tag{3.7}$$

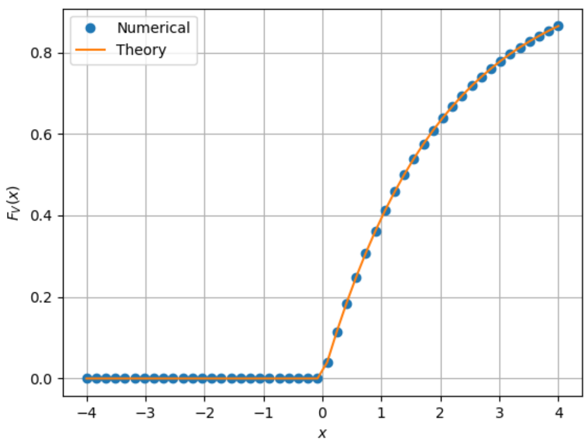

Fig. 3

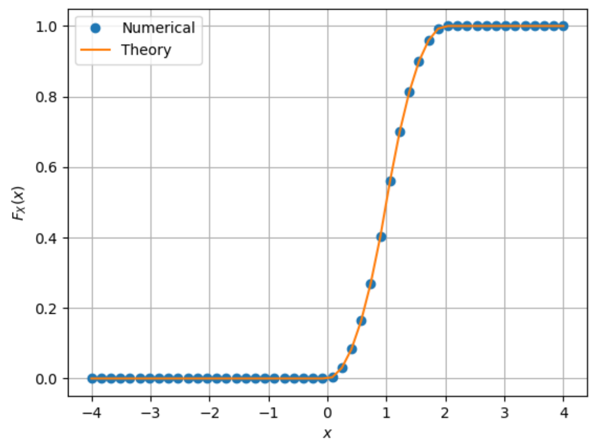

Fig. 4

$$F_U\left(1-e^{-\frac{V}{2}}\right) = \begin{cases} 1-e^{-\frac{V}{2}}, & V \in (0,\infty) \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download and run the code to generate tri.dat file.

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/coeffs.
    h
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig4 using the code given below:

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/
    t_cdf_plot2.py
```

4.3 Find the PDF of T.
Solution:
Solution: The PDF of T is plotted in Fig4 using the code given below:

```
wget https://github.com/Deepshikha11004/
    Random_variable/blob/main/codes/
    t_pdf_plot.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T.

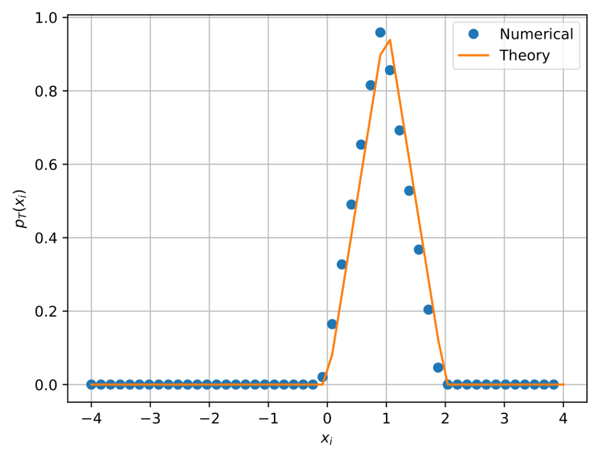

Fig. 4

Solution:

$$p_T(T) = p_{U_1+U_2}(T) = p_{U_1}(T) * p_{U_2}(T) \tag{4.2}$$

$$T = U_1 + U_2 \tag{4.3}$$

$$\implies p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(U_2)p_{U_2}(T-U_1)dU_1 \tag{4.4}$$

$$p_T(T) = \int_0^1 p_{U_1}(T-U_1)dU_1 \tag{4.5}$$

For $T < 0$,

$$p_T(T) = 0 \tag{4.6}$$

For $0 \geq T < 1$,

$$p_T(T) = \int_0^T dU_2 \tag{4.7}$$
$$= T \tag{4.8}$$

For $1 \geq T < 2$,

$$p_T(T) = \int_{T-1}^2 dU_2 \tag{4.9}$$
$$= 2 - T \tag{4.10}$$

For $T > 2$,

$$p_T(T) = 0 \tag{4.11}$$

Therefore,

$$p_T(T) = \begin{cases} 0 & otherwise \\ T & 0 < x < 1 \\ 2 - T & 1 \leq x < 2 \end{cases} \tag{4.12}$$

Now,

$$F_T(T) = \int p_T(T)dx \tag{4.13}$$

Therefore,

$$F_T(x) = \begin{cases} 0 & otherwise \\ \frac{T^2}{2} & 0 < T < 1 \\ -\frac{T^2}{2} + 2T - 1 & 1 \leq T < 2 \end{cases} \tag{4.14}$$

4.5 Verify your results through a plot.
Solution: Done in 4.2 and 4.3.

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Run the main.c file to generate 'ber.dat' file.

```
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/coeffs.h
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathbb{N}(0, 1)$.
Solution: Run the main.c file to generate 'maxlike.dat' file.

```
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/coeffs.h
```

5.3 Plot Y using a scatter plot.
Solution: The Y scatter plot is generated using

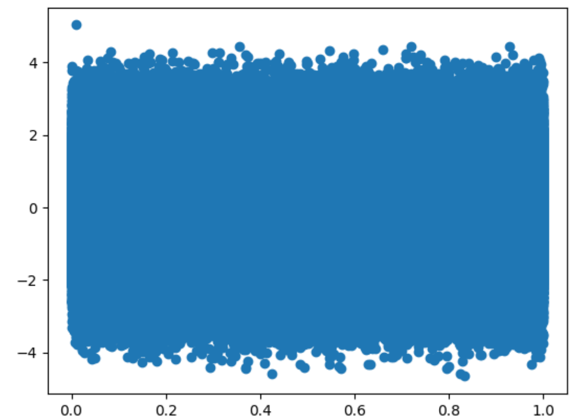

Fig. 5

'maxlike.dat' file using the following code:

```
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/
    scatter_plot.py
```

5.4 Guess how to estimate X from Y.
Solution: We have,

$$Y = AX + N \tag{5.2}$$

Estimation of X from Y :

$$\hat{X} = sgn(Y) \tag{5.3}$$

where,

$$sgn(Y) = \begin{cases} -1 & ; -\infty < y < 0 \\ 1 & ; 0 < y < \infty \end{cases} \tag{5.4}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.5}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.6}$$

Solution: Run the following code and using the function'x_cap' we get the required values.

```
wget https://github.com/Deepshikha11004
    /Random variable/blob/main/codes/
    main.c
```

```
wget https://github.com/Deepshikha11004
    /Random variable/blob/main/codes/
    coeffs.h
```

$$P_{e|0} = 0.310536 \tag{5.7}$$
$$P_{e|1} = 0.310530 \tag{5.8}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = \frac{P_{e|0} + P_{e|1}}{2} \tag{5.9}$$
$$= \frac{0.310536 + 0.310530}{2} \tag{5.10}$$
$$= 0.310533 \tag{5.11}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

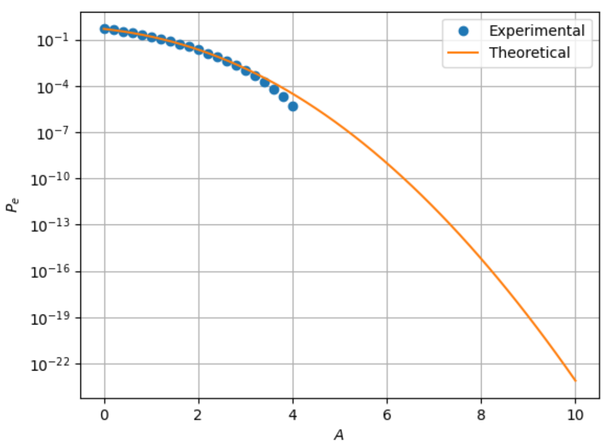

Fig. 5

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution We can define our guess as

$$\hat{X} = \begin{cases} 1 & Y \geq \delta \\ -1 & Y < \delta \end{cases} \tag{5.12}$$

Now,

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.13}$$
$$= \Pr\left(AX + N < \delta | X = 1\right) \tag{5.14}$$
$$= \Pr\left(A + N < \delta\right) \tag{5.15}$$
$$= \Pr\left(N < -A + \delta\right) \tag{5.16}$$
$$= F_N(-(A - \delta)) \tag{5.17}$$
$$= Q(A - \delta) \tag{5.18}$$

Similarly,

$$P_{e|1} = Q(A + \delta) \tag{5.19}$$

So we have,

$$P_e = P_{e|0} \cdot \Pr\left(X = -1\right) + P_{e|1} \cdot \Pr\left(X = 1\right) \tag{5.20}$$
$$= Q(A - \delta) \cdot \frac{1}{2} + Q(A + \delta) \cdot \frac{1}{2} \tag{5.21}$$

To find the minimal P_e, we differentiate Eq (5.21) wrt δ and equate it to 0

$$0 = \frac{d}{d\delta}\left(\frac{1}{2}\left(Q(A - \delta) + Q(A + \delta)\right)\right) \tag{5.22}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(A-\delta)^2}{2}} - e^{-\frac{(A+\delta)^2}{2}}\right) \tag{5.23}$$

So we have

$$(A - \delta)^2 = (A + \delta)^2 \tag{5.24}$$
$$\implies \delta = 0 \tag{5.25}$$

Therefore, P_e is minimal when the threshold is ,$\delta = 0$.

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.26}$$

Solution: We have

$$P_e = P_{e|0}p + P_{e|1}(1 - p) \tag{5.27}$$
$$= Q(A - \delta)p + Q(A + \delta)(1 - p) \tag{5.28}$$

Differentiating wrt δ to get minimum,

$$0 = \frac{d}{d\delta}\left(Q(A - \delta)p + Q(A + \delta)(1 - p)\right) \tag{5.29}$$
$$= pe^{-\frac{(A-\delta)^2}{2}} - (1 - p)e^{-\frac{(A+\delta)^2}{2}} \tag{5.30}$$

Taking ln on both sides,

$$\ln p - \frac{(A-\delta)^2}{2} = \ln(1-p) - \frac{(A+\delta)^2}{2} \quad (5.31)$$

$$2A\delta = \ln\frac{1-p}{p} \quad (5.32)$$

$$\delta = \frac{1}{2A}\ln\frac{1-p}{p} \quad (5.33)$$

5.10 Repeat the above exercise using the MAP criterion.
Solution: Assume $\Pr(X=-1) = p$.
According to the MAP criterion, when

$$p_{X|Y}(-1|y) > p_{X|Y}(1|y) \quad (5.34)$$

We should guess -1, else we guess 1. Now,

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} \quad (5.35)$$

Now,

$$p_{Y|X}(y|-1) = p_{(-A+N)}(y) \quad (5.36)$$

Since A is a constant,

$$p_{Y|X}(y|-1) = p_N(y+A) \quad (5.37)$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{(y+A)^2}{2}} \quad (5.38)$$

Substituting into (5.35),

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y|-1)p_X(-1)}{p_Y(y)} \quad (5.39)$$

$$p_{X|Y}(-1|y) = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+A)^2}{2}}p}{p_Y(y)} \quad (5.40)$$

Similarly,

$$p_{X|Y}(1|y) = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-A)^2}{2}}(1-p)}{p_Y(y)} \quad (5.41)$$

Finally, substituting Eqs (5.40) and (5.41) into Eq (5.34), we should guess that $X=-1$ when

$$\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+A)^2}{2}}p}{p_Y(y)} > \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-A)^2}{2}}(1-p)}{p_Y(y)} \quad (5.42)$$

$$\frac{p}{1-p} > e^{2Ay} \quad (5.43)$$

$$y < \frac{1}{2A}\ln\frac{p}{1-p} \quad (5.44)$$

So we have our guess,

$$\hat{X} = \begin{cases} -1 & y < \delta \\ 1 & \text{otherwise} \end{cases} \quad (5.45)$$

where $\delta = \frac{1}{2A}\ln\frac{p}{1-p}$.
Considering the special case when $p = \frac{1}{2}$,

$$\delta = \frac{1}{2A}\ln 1 \quad (5.46)$$

$$\delta = 0 \quad (5.47)$$

So our guess is,

$$\hat{X} = \begin{cases} -1 & y < 0 \\ 1 & y > 0 \end{cases} \quad (5.48)$$

6 Gaussian to Other

6.1

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: Run the following code to generate 'chi.dat' file.

```
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/main.c
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/coeffs.h
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α.
Solution: Define the random variables as:

$$X_1 = R\cos\theta \quad (6.3)$$

$$X_2 = R\sin\theta \quad (6.4)$$

The Jacobian matrix is defined as:

$$J = \begin{bmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{bmatrix} \quad (6.5)$$

$$J = \begin{bmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{bmatrix} \quad (6.6)$$

$$\implies |J| = R \quad (6.7)$$

Since,X_1 and X_2 are iid,

$$p_{X_1X_2}(x_1,x_2) = p_{X_1}x_1 p_{X_2}x_2 \quad (6.8)$$

$$= \frac{1}{2\pi}exp(\frac{-(x_1^2+x_2^2)}{2}) \quad (6.9)$$

$$= \frac{1}{2\pi}exp(\frac{-r^2}{2}) \quad (6.10)$$

We know that,

$$p_{R\theta}(r,\theta) = |J| p_{X_1X_2}(x_1,x_2) \tag{6.11}$$

Therefore,

$$p_{R\theta}(r,\theta) = \frac{r}{2\pi} exp(\frac{-r^2}{2}) \tag{6.12}$$

$$\implies p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta) d\theta \tag{6.13}$$

$$= re^{-\frac{r^2}{2}} \tag{6.14}$$

We have,

$$F_R(r) = \int_0^r p_R(r) \tag{6.15}$$

$$= \int_0^r re^{-\frac{r^2}{2}} dr \tag{6.16}$$

$$= \left(-e^{-\frac{r^2}{2}}\right)_0^r \tag{6.17}$$

$$= 1 - e^{-\frac{r^2}{2}} \tag{6.18}$$

and,

$$F_V(x) = \Pr(V \le x) \tag{6.19}$$

$$= \Pr\left(X_1^2 + X_2^2 \le x\right) \tag{6.20}$$

$$= \Pr\left(R^2 \le x\right) \quad = \Pr\left(R \le \sqrt{x}\right) \tag{6.21}$$

$$= F_R(\sqrt{x}) \tag{6.22}$$

$$= 1 - e^{-\frac{x}{2}} \tag{6.23}$$

$$= 1 - e^{-\alpha x} \tag{6.24}$$

Hence we get,

$$\alpha = \frac{1}{2} \tag{6.25}$$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.26}$$

Solution: Following is the code:

```
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/
    chiroot_pdf.py
wget https://github.com/Deepshikha11004/
    Random variable/blob/main/codes/
    chiroot_cdf.py
```

These are the PDF and CDf plots:

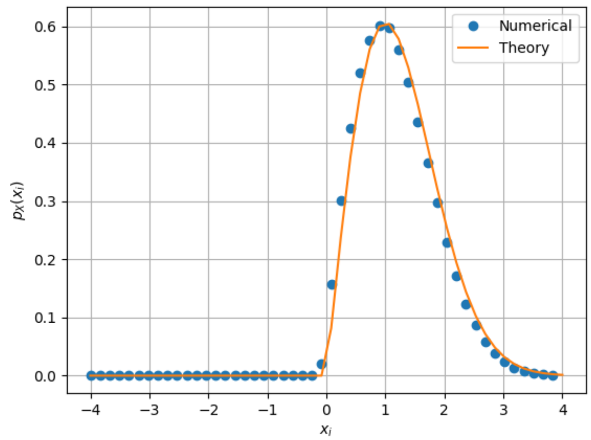

Fig. 6

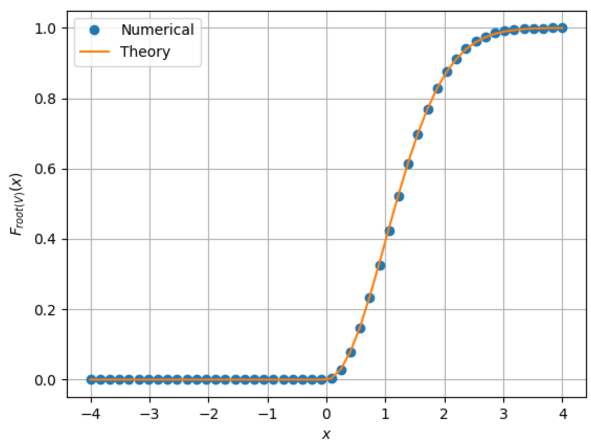

Fig. 6

7 Conditional Probability

7.1

7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E[P_e(N)]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.