INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
HYDERABAD

OPTIMIZATION METHODS

ASSIGNMENT 3

REPORT

Name: Deepti Rawat

Roll No.[redacted]

Date: 07 May 2024

1. **Derive the KKT conditions from Dual Ascent.**
2. **For dual ascent, verify analytically that the KKT conditions are satisfied for the final points that your algorithm produces.**

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+-----------+----------------+
| Test case |   Dual ascent  |
+-----------+----------------+
|     0     |     [2. 2.]    |
|     1     |  [1.899 2.146] |
|     2     |  [1.008 1.006] |
|     3     |     [2. 2.]    |
|     4     |  [3.003 3.001] |
+-----------+----------------+
|     5     |  [0.178 0.178] |
|     6     |  [0.962 0.962] |
|     7     | [-0.535 -0.447] |
+-----------+----------------+
```

Let's analytically derive the KKT for each Test Case and see if it is satisfied for the points obtained as shown in the terminal output above.

Trid Function

$$f(\bar{x}) = \sum_{i=1}^{d}(x_i - 1)^2 - \sum_{i=2}^{d} x_{i-1} \cdot x_i$$

Step 1: Generalizing for 'd'

- For **d = 1**: $f(\bar{x}) = (x_1 - 1)^2 - 0$
- For **d = 2**: $f(\bar{x}) = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 \cdot x_2$
- For **d = 3**: $f(\bar{x}) = (x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 - x_1 \cdot x_2 - x_2 \cdot x_3$
- For **d**: $f(\bar{x}) = (x_1 - 1)^2 + \cdots + (x_d - 1)^2 - x_1 \cdot x_2 - \cdots - x_{d-2} \cdot x_{d-1} - x_{d-1}x_d$

Step 2: Computing Jacobian

- $\frac{\partial f(\bar{x})}{\partial x_1} = 2(x_1 - 1) - x_0 - x_2$ $\quad [x_0 = 0]$
- $\frac{\partial f(\bar{x})}{\partial x_2} = 2(x_2 - 1) - x_1 - x_3$
- $\frac{\partial f(\bar{x})}{\partial x_3} = 2(x_3 - 1) - x_2 - x_4$
- $\frac{\partial f(\bar{x})}{\partial x_d} = 2(x_d - 1) - x_{d-1} - x_{d+1}$ $\quad [x_{d+1} = 0]$

$$\therefore \nabla f(\bar{x}) = \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \\ \vdots \\ \partial f(\bar{x})/\partial x_d \end{bmatrix} = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 - x_3 \\ \vdots \\ 2(x_d - 1) - x_{d-1} - x_{d+1} \end{bmatrix}$$

Test Case: 0

Subject to:

- $x_1^2 - 2x_2 \leq 0$ ... c_{25} $(h\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda h = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 \cdot x_2 + \lambda(x_1^2 - 2x_2)$

KKT Conditions:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda \nabla h = \bar{0} \Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda \begin{bmatrix} \partial h(\bar{x})/\partial x_1 \\ \partial h(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda \begin{bmatrix} 2x_1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $2x_1(1 + \lambda) - x_2 - 2 = 0$ ... (i)
- $2x_2 - x_1 - 2(1 + \lambda) = 0$ ... (ii)

```
For trid_function with [-0.7  0. ] as starting point using Dual Ascent:
Lambda value λ*: [0.]
Optimal Value x*: [1.99991643 1.99991643]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
        Grad of Lagrange function wrt optimal value --> [-8.36539038e-05 -8.36539029e-05]
        Norm of Grad of Lagrange function wrt optimal value --> 0.00011830448462858676
```

Substituting the final points $\boldsymbol{x}^* = [2, 2]$ and lambdas $\boldsymbol{\lambda}^* = [0]$ in the equations (i) and (ii), we get,

- $2 \times 2(1 + 0) - 2 - 2 = 0$ ... (i)
- $2 \times 2 - 2 - 2(1 + 0) = 0$ ... (ii)

Hence, the $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ satisfies KKT conditions.

Test Case: 1

Subject to:

- $x_1^2 - x_2^2 + 1 \leq 0$ ... c_{26} $(h\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda h = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 \cdot x_2 + \lambda(x_1^2 - x_2^2 + 1)$

KKT Conditions:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda \nabla h = \bar{0} \Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda \begin{bmatrix} \partial h(\bar{x})/\partial x_1 \\ \partial h(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $2x_1(1 + \lambda) - x_2 - 2 = 0 \qquad \ldots (i)$
- $2x_2(1 - \lambda) - x_1 - 2 = 0 \qquad \ldots (ii)$

```
For trid_function with [-0.7  0. ] as starting point using Dual Ascent:
Lambda value λ*: [0.09166402]
Optimal Value x*: [1.89902547 2.14622895]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
        Grad of Lagrange function wrt optimal value --> [-3.34586851e-05 -3.15194196e-05]
        Norm of Grad of Lagrange function wrt optimal value --> 4.5966916628901213e-05
```

Substituting the final points $\boldsymbol{x^*} = [1.899, 2.146]$ and lambdas $\boldsymbol{\lambda^*} = [0.09166402]$ in the equations (i) and (ii), we get,

- $2 \times 1.899(1 + 0.09166402) - 2.146 - 2 = 0.0001399 \cong 0 \qquad \ldots (i)$
- $2 \times 2.146(1 - 0.09166402) - 1.899 - 2 = -0.0004219 \cong 0 \qquad \ldots (ii)$

The output is very close to zero. Hence, the $\boldsymbol{x^*}$ and $\boldsymbol{\lambda^*}$ satisfies KKT conditions.

Test Case: 2

Subject to:

- $-1 - x_1 \leq 0 \qquad \ldots c_1\ (h_1\ constraint)$
- $x_1 - 1 \leq 0 \qquad \ldots c_2\ (h_2\ constraint)$
- $-1 - x_2 \leq 0 \qquad \ldots c_3\ (h_3\ constraint)$
- $x_2 - 1 \leq 0 \qquad \ldots c_4\ (h_4\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 h_4$

$$\Rightarrow (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 \cdot x_2 + \lambda_1(-1 - x_1) + \lambda_2(x_1 - 1) + \lambda_3(-1 - x_2) + \lambda_4(x_2 - 1)$$

KKT Conditions:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3 + \lambda_4 \nabla h_4 = \bar{0}$

$$\Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} \partial h_1(\bar{x})/\partial x_1 \\ \partial h_1(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} \partial h_2(\bar{x})/\partial x_1 \\ \partial h_2(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} \partial h_3(\bar{x})/\partial x_1 \\ \partial h_3(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_4 \begin{bmatrix} \partial h_4(\bar{x})/\partial x_1 \\ \partial h_4(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $2x_1 - 2 - x_2 - \lambda_1 + \lambda_2 = 0 \qquad \ldots (i)$
- $2x_2 - 2 - x_1 - \lambda_3 + \lambda_4 = 0 \qquad \ldots (ii)$

```
For trid_function with [-0.7  0. ] as starting point using Dual Ascent:
Lambda value λ*: [0.         0.99310162 0.         0.99882089]
Optimal Value x*: [1.00791732 1.0057329 ]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
        Grad of Lagrange function wrt optimal value --> [0.00319948 0.00236516]
        Norm of Grad of Lagrange function wrt optimal value --> 0.003978780781882498
```

Substituting the final points $\boldsymbol{x}^* = [1.008,\ 1.006]$ and lambdas $\boldsymbol{\lambda}^* = [0,\ 0.99310162,\ 0,\ 0.99882089]$ in the equations (i) and (ii), we get,

- $2 \times 1.008 - 2 - 1.006 - 0 + 0.99310162 = 0.0031 \cong 0$ ... (i)
- $2 \times 1.006 - 2 - 1.008 - 0 + 0.99882089 = 0.0028 \cong 0$ ... (ii)

The output is very close to zero. Hence, the $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ satisfies KKT conditions.

<u>Test Case: 3</u>

Subject to:

- $0 - x_1 \leq 0$ ... $c_5\ (h_1\ constraint)$
- $x_1 - 3 \leq 0$ ... $c_6\ (h_2\ constraint)$
- $0 - x_2 \leq 0$ ... $c_7\ (h_3\ constraint)$
- $x_2 - 3 \leq 0$ ... $c_8\ (h_4\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 h_4$

$$\Rightarrow (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 \cdot x_2 + \lambda_1(0 - x_1) + \lambda_2(x_1 - 3) + \lambda_3(0 - x_2) + \lambda_4(x_2 - 3)$$

KKT Conditions:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3 + \lambda_4 \nabla h_4 = \bar{0}$

$$\Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} \partial h_1(\bar{x})/\partial x_1 \\ \partial h_1(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} \partial h_2(\bar{x})/\partial x_1 \\ \partial h_2(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} \partial h_3(\bar{x})/\partial x_1 \\ \partial h_3(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_4 \begin{bmatrix} \partial h_4(\bar{x})/\partial x_1 \\ \partial h_4(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $2x_1 - 2 - x_2 - \lambda_1 + \lambda_2 = 0$ ... (i)
- $2x_2 - 2 - x_1 - \lambda_3 + \lambda_4 = 0$ ... (ii)

```
For trid_function with [0. 1.] as starting point using Dual Ascent:
Lambda value λ*: [0. 0. 0. 0.]
Optimal Value x*: [1.99995383 1.99995383]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
        Grad of Lagrange function wrt optimal value --> [-4.62132521e-05 -4.62132521e-05]
        Norm of Grad of Lagrange function wrt optimal value --> 6.5355407858377e-05
```

Substituting the final points $\boldsymbol{x^*} = [2,\ 2]$ and lambdas $\boldsymbol{\lambda^*} = [0, 0, 0, 0]$ in the equations (i) and (ii), we get,

- $2 \times 2 - 2 - 2 - 0 + 0 = 0$ ... (i)
- $2 \times 2 - 2 - 2 - 0 + 0 = 0$ ... (ii)

Hence $\boldsymbol{x^*}$ and $\boldsymbol{\lambda^*}$ satisfies the KKT conditions.

Test Case: 4

Subject to:

- $3 - x_1 \leq 0$ ... $c_9\ (h_1\ constraint)$
- $x_1 - 4 \leq 0$ ... $c_{10}\ (h_2\ constraint)$
- $3 - x_2 \leq 0$ ... $c_{11}\ (h_3\ constraint)$
- $x_2 - 4 \leq 0$ ... $c_{12}\ (h_4\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 h_4$

$$\Rightarrow (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 \cdot x_2 + \lambda_1(3 - x_1) + \lambda_2(x_1 - 4) + \lambda_3(3 - x_2) + \lambda_4(x_2 - 4)$$

KKT Conditions:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3 + \lambda_4 \nabla h_4 = \bar{0}$

$$\Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} \partial h_1(\bar{x})/\partial x_1 \\ \partial h_1(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} \partial h_2(\bar{x})/\partial x_1 \\ \partial h_2(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} \partial h_3(\bar{x})/\partial x_1 \\ \partial h_3(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_4 \begin{bmatrix} \partial h_4(\bar{x})/\partial x_1 \\ \partial h_4(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $2x_1 - 2 - x_2 - \lambda_1 + \lambda_2 = 0$ ... (i)
- $2x_2 - 2 - x_1 - \lambda_3 + \lambda_4 = 0$ ... (ii)

```
For trid_function with [3.  3.5] as starting point using Dual Ascent:
Lambda value λ*: [1.0044302  0.         1.00064367 0.        ]
Optimal Value x*: [3.00266499 3.00121877]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
        Grad of Lagrange function wrt optimal value --> [-0.00032141 -0.00087378]
        Norm of Grad of Lagrange function wrt optimal value --> 0.0009310196893038386
```

Substituting the final points $\boldsymbol{x^*} = [3.003,\ 3.001]$ and lambdas $\boldsymbol{\lambda^*} = [1.0044302,\ 0,\ 1.00064367,\ 0]$ in the equations (i) and (ii), we get,

- $2 \times 3.003 - 2 - 3.001 - 1.0044302 + 0 = 0.0005698 \cong 0$ ... (i)
- $2 \times 3.001 - 2 - 3.003 - 1.00064367 + 0 = -0.00164 \cong 0$ ... (ii)

The output is very close to zero. Hence $\boldsymbol{x^*}$ and $\boldsymbol{\lambda^*}$ satisfies the KKT conditions.

Matyas Function

$$f(\bar{x}) = 0.26({x_1}^2 + {x_2}^2) - 0.48x_1 \times x_2$$

Test Case: 5

Subject to:

- $0 - x_1 \le 0$ ... $c_{13}\ (h_1\ constraint)$
- $x_1 - 1 \le 0$ ... $c_{14}\ (h_2\ constraint)$
- $0 - x_2 \le 0$ ... $c_{15}\ (h_3\ constraint)$
- $x_2 - 1 \le 0$ ... $c_{16}\ (h_4\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 h_4$

$$\Rightarrow 0.26({x_1}^2 + {x_2}^2) - 0.48x_1 \times x_2 + \lambda_1(0 - x_1) + \lambda_2(x_1 - 1) + \lambda_3(0 - x_2) + \lambda_4(x_2 - 1)$$

KKT Conditions:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3 + \lambda_4 \nabla h_4 = \bar{0}$

$$\Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} \partial h_1(\bar{x})/\partial x_1 \\ \partial h_1(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} \partial h_2(\bar{x})/\partial x_1 \\ \partial h_2(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} \partial h_3(\bar{x})/\partial x_1 \\ \partial h_3(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_4 \begin{bmatrix} \partial h_4(\bar{x})/\partial x_1 \\ \partial h_4(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $0.52x_1 - 0.48x_2 - \lambda_1 + \lambda_2 = 0$ ... (i)
- $0.52x_2 - 0.48x_1 - \lambda_3 + \lambda_4 = 0$ ... (ii)

```
For matyas_function with [1.5 1.5] as starting point using Dual Ascent:
Lambda value λ*: [0. 0. 0. 0.]
Optimal Value x*: [0.17803582 0.17803582]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
	Grad of Lagrange function wrt optimal value --> [0.00712172 0.00712172]
	Norm of Grad of Lagrange function wrt optimal value --> 0.010071629830485675
```

Substituting the final points $\boldsymbol{x^*} = [0.178,\ 0.178]$ and lambdas $\boldsymbol{\lambda^*} = [0,\ 0,\ 0,\ 0]$ in the equations (i) and (ii), we get,

- $0.52 \times 0.178 - 0.48 \times 0.178 - 0 + 0 = 0.00712 \cong 0$ ... (i)
- $0.52 \times 0.178 - 0.48 \times 0.178 - 0 + 0 = 0.00712 \cong 0$ ... (ii)

The output is close to zero. Hence $\boldsymbol{x^*}$ and $\boldsymbol{\lambda^*}$ satisfies the KKT conditions.

Test Case: 6

Subject to:

- $1 - x_1 \leq 0$ ... c_{17} $(h_1\ constraint)$
- $x_1 - 2 \leq 0$ ... c_{18} $(h_2\ constraint)$
- $1 - x_2 \leq 0$ ... c_{19} $(h_3\ constraint)$
- $x_2 - 2 \leq 0$ ... c_{20} $(h_4\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 h_4$

$$\Rightarrow 0.26({x_1}^2 + {x_2}^2) - 0.48x_1 \times x_2 + \lambda_1(1 - x_1) + \lambda_2(x_1 - 2) + \lambda_3(1 - x_2) + \lambda_4(x_2 - 2)$$

KKT Conditions:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3 + \lambda_4 \nabla h_4 = \bar{0}$
$$\Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} \partial h_1(\bar{x})/\partial x_1 \\ \partial h_1(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} \partial h_2(\bar{x})/\partial x_1 \\ \partial h_2(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} \partial h_3(\bar{x})/\partial x_1 \\ \partial h_3(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_4 \begin{bmatrix} \partial h_4(\bar{x})/\partial x_1 \\ \partial h_4(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $0.52x_1 - 0.48x_2 - \lambda_1 + \lambda_2 = 0$ ... (i)
- $0.52x_2 - 0.48x_1 - \lambda_3 + \lambda_4 = 0$ ... (ii)

```
For matyas_function with [1.  2.5] as starting point using Dual Ascent:
Lambda value λ*: [0.04379274 0.         0.04427943 0.        ]
Optimal Value x*: [0.96152541 0.96172559]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
        Grad of Lagrange function wrt optimal value --> [-0.00538941 -0.00567641]
        Norm of Grad of Lagrange function wrt optimal value --> 0.007827349711763352
```

Substituting the final points $\boldsymbol{x^*} = [0.962,\ 0.962]$ and lambdas $\boldsymbol{\lambda^*} = [0.04379274,\ 0,\ 0.04427943,\ 0]$ in the equations (i) and (ii), we get,

- $0.52 \times 0.962 - 0.48 \times 0.962 - 0.04379274 + 0 = -0.005312 \cong 0$ ... (i)
- $0.52 \times 0.962 - 0.48 \times 0.962 - 0.04427943 + 0 = -0.005799 \cong 0$ ... (ii)

The output is close to zero. Hence $\boldsymbol{x^*}$ and $\boldsymbol{\lambda^*}$ satisfies the KKT conditions.

Test Case: 7

Subject to:

- $-1 - x_1 \leq 0$ ... c_{21} $(h_1\ constraint)$

- $x_1 + 0.5 \le 0$ ... c_{22} $(h_2\ constraint)$
- $-0.5 - x_2 \le 0$ ... c_{23} $(h_3\ constraint)$
- $x_2 - 0.5 \le 0$ ... c_{24} $(h_4\ constraint)$

Lagrangian:

- $\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 h_4$

$$\Rightarrow 0.26({x_1}^2 + {x_2}^2) - 0.48x_1 \times x_2 + \lambda_1(-1 - x_1) + \lambda_2(x_1 + 0.5) + \lambda_3(-0.5 - x_2) + \lambda_4(x_2 - 0.5)$$

KKT Condition:

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) + \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3 + \lambda_4 \nabla h_4 = \bar{0}$
$$\Rightarrow \begin{bmatrix} \partial f(\bar{x})/\partial x_1 \\ \partial f(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} \partial h_1(\bar{x})/\partial x_1 \\ \partial h_1(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} \partial h_2(\bar{x})/\partial x_1 \\ \partial h_2(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} \partial h_3(\bar{x})/\partial x_1 \\ \partial h_3(\bar{x})/\partial x_2 \end{bmatrix} + \lambda_4 \begin{bmatrix} \partial h_4(\bar{x})/\partial x_1 \\ \partial h_4(\bar{x})/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $0.52x_1 - 0.48x_2 - \lambda_1 + \lambda_2 = 0$ ... (i)
- $0.52x_2 - 0.48x_1 - \lambda_3 + \lambda_4 = 0$ ... (ii)

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For matyas_function with [-0.5  0.5] as starting point using Dual Ascent:
Lambda value λ*: [0.         0.07699913 0.         0.        ]
Optimal Value x*: [-0.53528401 -0.44697899]
Verifying if obtained optimal value satisfies 1st Order KKT conditions:
        Grad of Lagrange function wrt optimal value --> [0.01323176 0.02451365]
        Norm of Grad of Lagrange function wrt optimal value --> 0.027856745626975796
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Substituting the final points $\boldsymbol{x^*} = [-0.535, -0.447]$ and lambdas $\boldsymbol{\lambda^*} = [0,\ 0.7699913,\ 0,\ 0]$ in the equations (i) and (ii), we get,

- $0.52 \times -0.535 - 0.48 \times -0.447 - 0 + 0.7699913 = 0.70635$ ... (i)
- $0.52 \times -0.447 - 0.48 \times -0.53 - 0 + 0 = 0.02196$ ... (ii)

The output is close to zero. Hence $\boldsymbol{x^*}$ and $\boldsymbol{\lambda^*}$ satisfies the KKT conditions.