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MANAGEMENT SCIENCE

CHAPTER 9 POWERPOINT

LINEAR OPTIMIZATION

The Art of Modeling with Spreadsheets

Compatible with Analytic Solver Platform

FOURTH EDITION

WILEY

MODEL CLASSIFICATION

- Linear optimization or linear programming
  - Objective and *all* constraints are linear functions of the decision variables.
- Nonlinear optimization or nonlinear programming
  - Either objective or a constraint (or both) are nonlinear functions of the decision variables.
- Techniques for solving linear models are more powerful.
  - Use wherever possible.

PROPERTIES OF LINEAR FUNCTIONS

- Term “linear” refers to a feature of the objective function and the constraints.
- Linear function exhibits:
  – Additivity
  – Proportionality
  – Divisibility

EXCEL MINI-LESSON: THE SUMPRODUCT FUNCTION

- The SUMPRODUCT function in Excel takes the pairwise products of two sets of numbers and sums the products.
- SUMPRODUCT(*Array1*,*Array2*)
  - *Array1* references the first set of numbers.
  - *Array2* references the second set of numbers.
- The two arrays must have identical layouts and be the same size.

WHAT'S A LINEAR PROGRAM

- Objective function
  - Maximize Profit
  - Minimize Cost
- Constraints
  - On materials
  - On time
  - On money
  - On combinations

EXAMPLE
PRODUCT MIX EXAMPLE 1

- Your company makes pallets in four styles
- Each brings different profit
- Profit = $450T + $1150P + $800S + $200A

	Panel Type				
	Tahoe	Pacific	Savannah	Aspen	
Pallets	**0**	**0**	**0**	**0**	**Total Profit**
Profit	$450	$1,150	$800	$200	**$0**

Note: This is a slight modification of the problem presented in the Excel File "LP Introduction". The profit on Aspen has been reduced to $200 per unit vice $400.

CONSTRAINTS

- Constraints are placed on the materials that are used to make a pallet
- The first constraint (on Glue)

$$50T + 50P + 100S + 50A <= 5800$$

	Tahoe	Pacific	Savannah	Aspen	Used	Available	
Glue	50	50	100	50	0	5,800	**quarts**
Pressing	5	15	10	5	0	730	**hours**
Pine Chips	500	400	300	200	0	29,200	**pounds**
Oak Chips	500	750	250	500	0	60,500	**pounds**

REDUCE THIS PROBLEM TO JUST TWO CHOICES: SAVANNAH AND ASPEN

- Assume you only have two products to choose from
- This allows us to graph how many Aspen and Savannah we make

	Panel Type				
	Tahoe	Pacific	Savannah	Aspen	
Pallets	0	0	0	0	Total Profit
Profit	$450	$1,150	$800	$200	$0

	Resources Required per Pallet Type						
	Tahoe	Pacific	Savannah	Aspen	Used	Available	
Glue	50	50	100	50	0	5,800	quarts
Pressing	5	15	10	5	0	730	hours
Pine Chips	500	400	300	200	0	29,200	pounds
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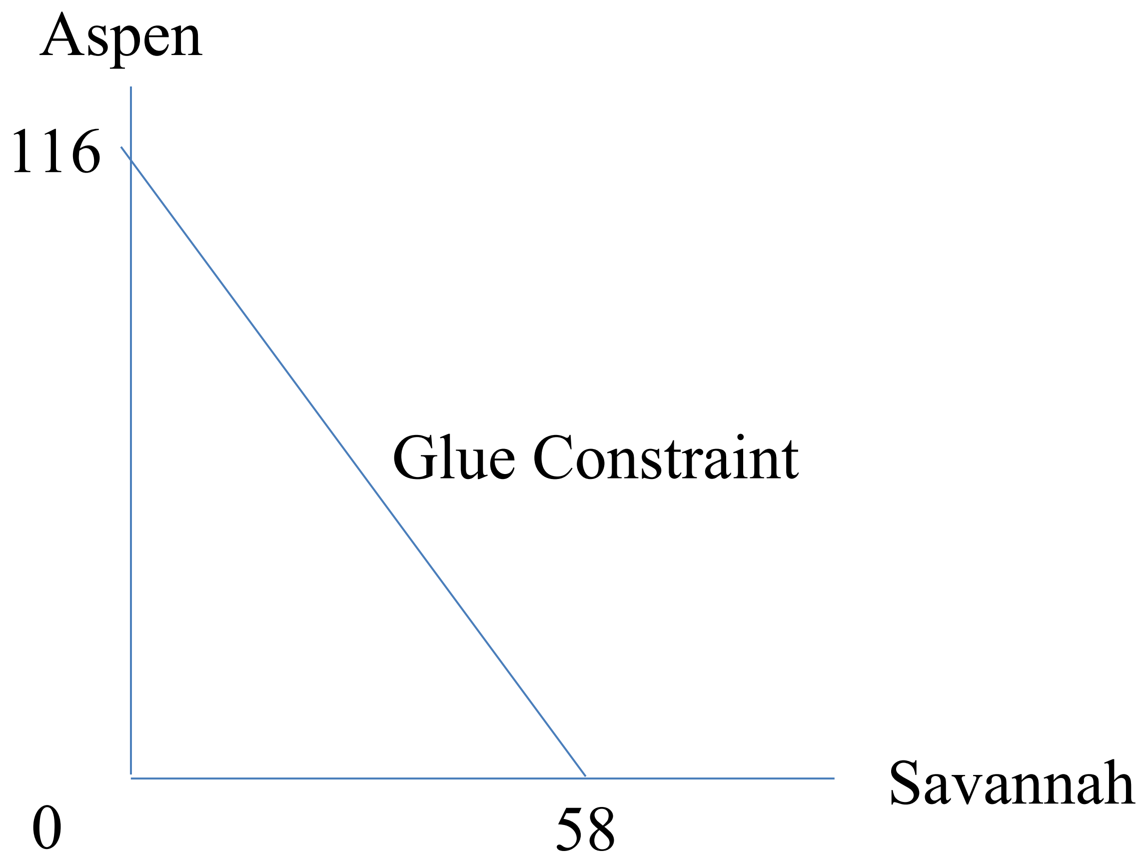
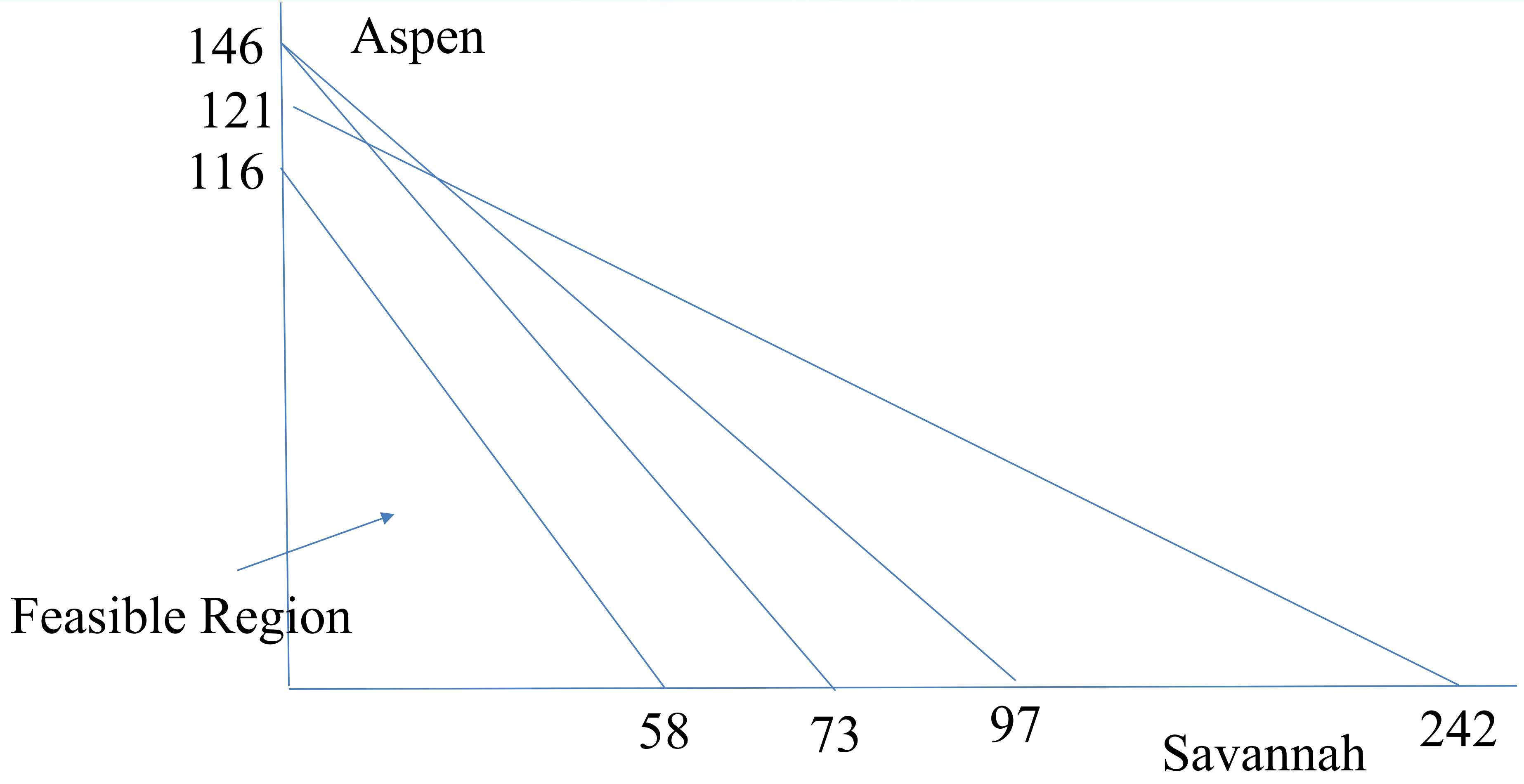
REDUCED PROBLEM TO SEE THE INSIDES



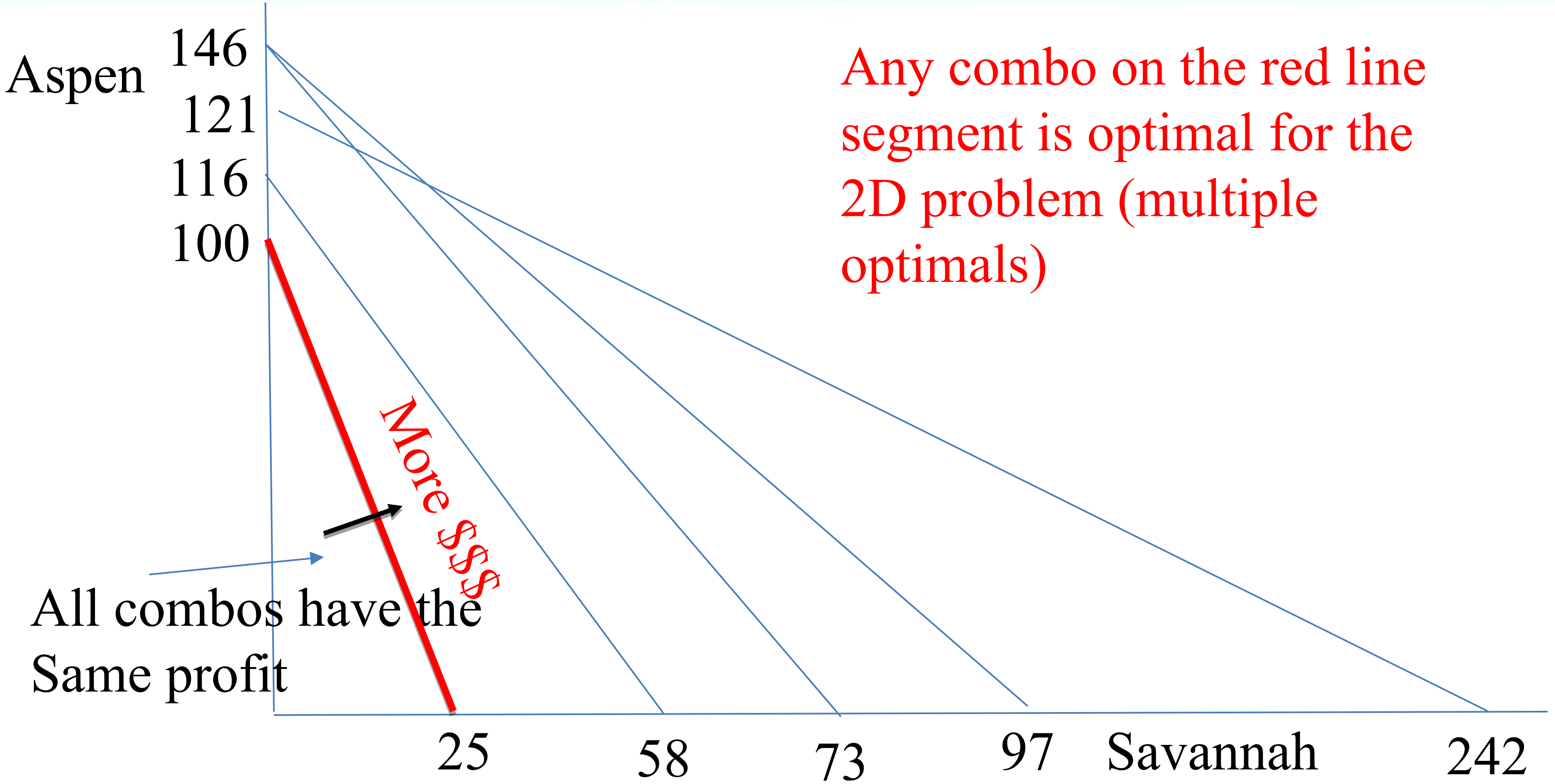
ALL THE CONSTRAINTS



OBJECTIVE FUNCTION

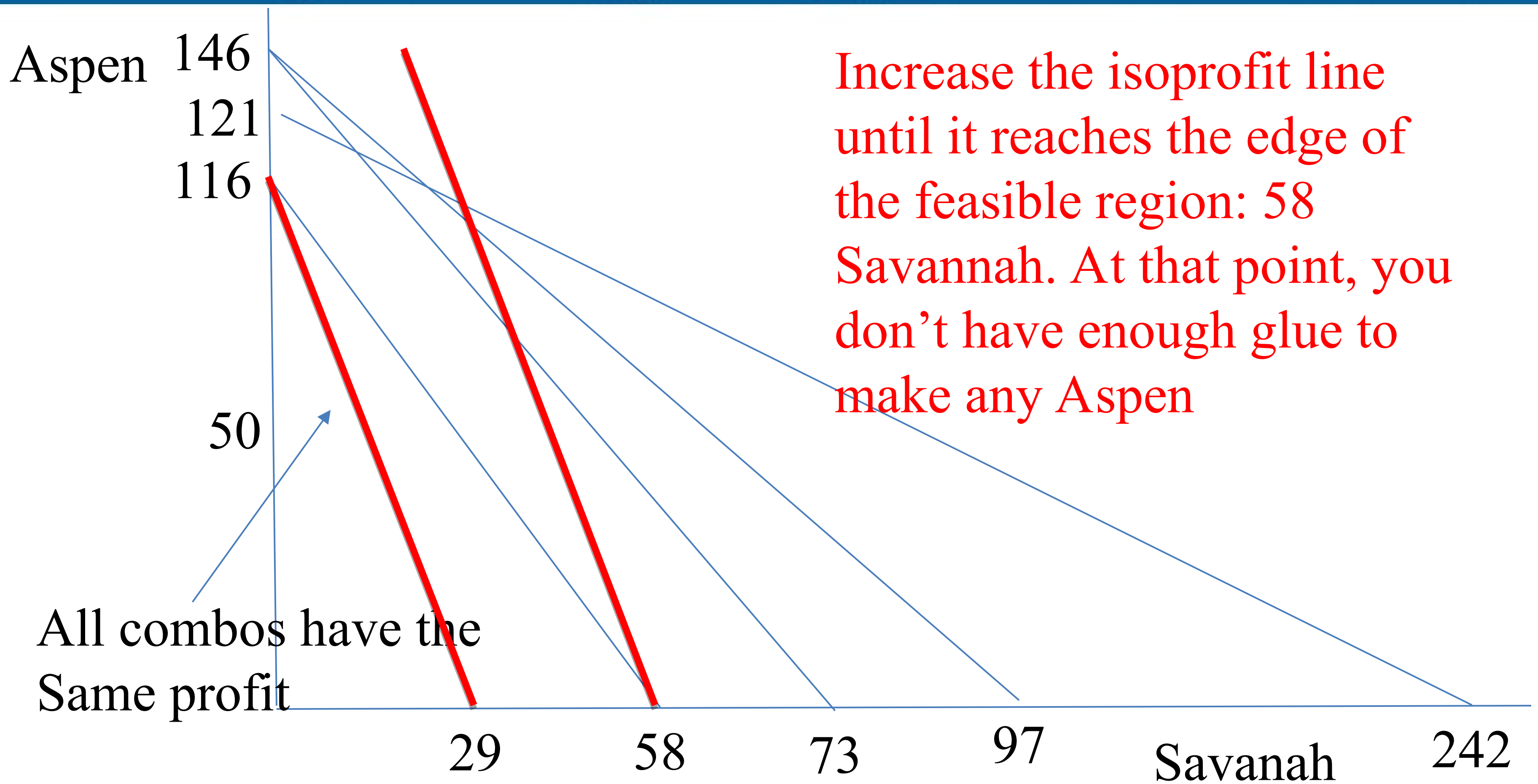
- From the full model
  - Profit = \$450T + \$1150P + \$800S + \$200A
- Reduced to 2D
  - Profit = \$800S + \$200A
  - The profit line has slope 1S = 4A
  - Or, you can need four Aspen to make the same profit as one Savannah
- Plot the “iso-profit” line on the constraint plot…

PROFIT



Any combo on the red line segment is optimal for the 2D problem (multiple optimals)

INCREASE PLANNED PRODUCTION UNTIL YOU REACH THE FIRST CONSTRAINT



Increase the isoprofit line until it reaches the edge of the feasible region: 58 Savannah. At that point, you don’t have enough glue to make any Aspen

CONSIDER THIS PROBLEM IN THREE DIMENSIONS

- Increase from two decision variables to three DVs
- Now a graph in two dimensions is inadequate for the problem

	Panel Type				
	Tahoe	Pacific	Savannah	Aspen	
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Profit	$450	$1,150	$800	$200	$0

	Tahoe	Pacific	Savannah	Aspen	Used	Available	
Glue	50	50	100	50	0	5,800	quarts
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Pine Chips	500	400	300	200	0	29,200	pounds
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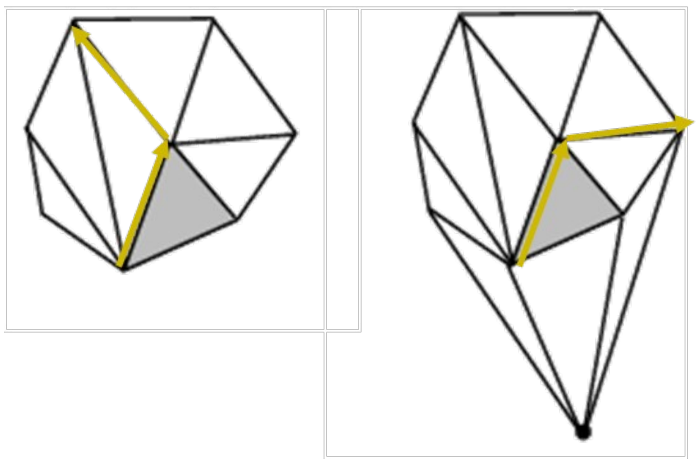
GENERALIZING FINDING THE EDGE OF THE FEASIBLE REGION

- Three decision variables creates a gem-shaped feasible region
  - The constraints represent “facets”
  - Two facets join along a line, three or more at a “vertex”
- A principle of linear algebra is that in *n*-dimensions, with a linear objective function, the optimal point is at a vertex
- Solution method is to move from one vertex to the best neighboring vertex until where you are is better than all your neighbors

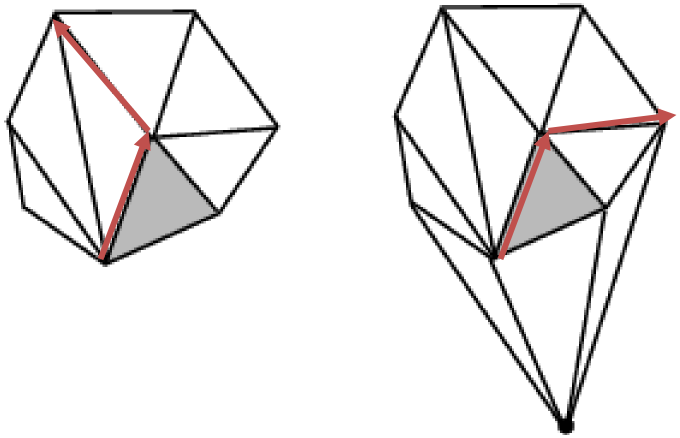


THE SIMPLEX ALGORITHM FOR LINEAR OPTIMIZATION

The *simplex algorithm* works with *n*-dimensional linear problems

1. Start with a feasible set of decision variables that corresponds to a corner on a diamond, a *vertex*.
2. Check to see if a feasible neighboring corner point is better.
3. If not, stop; otherwise move to that better neighbor and return to step 2.

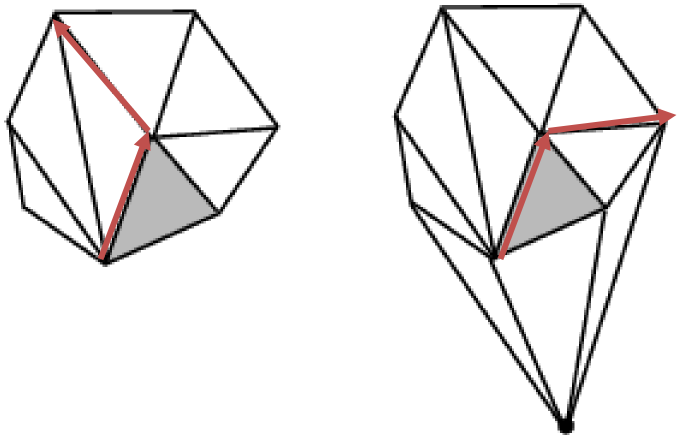
Guaranteed to converge
to the global optimal solution



THE SIMPLEX ALGORITHM FOR LINEAR OPTIMIZATION

- Exploits special properties of linearity to find optimal solutions.
- Imagine the surface of a diamond which represents feasible decision variables:
  - Starts with a feasible set of decision variables that corresponds to a corner on a diamond.
  - Checks to see if a feasible neighboring corner point is better.
  - If not, stops; otherwise moves to that better neighbor and return to step 2.

Guaranteed to converge
to the global optimal solution

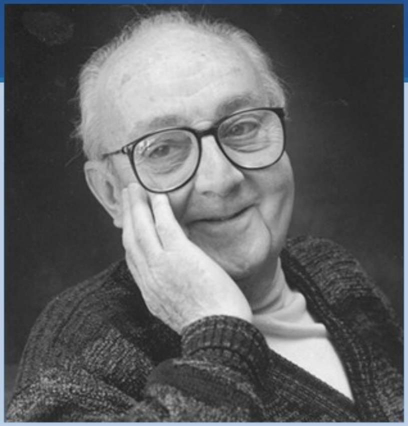


LINEAR PROGRAMMING PROBLEMS: WHERE WE WILL APPLY THIS

- Allocation models
  - Maximize objective (e.g., profit) subject to LT constraints on capacity
- Covering models
  - Minimize objective (e.g., cost) subject to GT constraints on required coverage
- Blending models
  - E.g., in determining product mix; mix materials with different properties to find best blend
- Network models
  - Describe patterns of flow in a connected system
  - Covered in Chapter 10

MOVE TO SOLVING EXAMPLE PROBLEMS

SENSITIVITY ANALYSIS FOR LINEAR PROGRAMS: WHAT IF THINGS CHANGE?



- "All models are wrong. Some are useful"
  - George Box
- Determine what we should pay for more of things we don't have enough of?
  - Called *binding constraint sensitivity*
- Determining the proportional change in the optimal solution when varying a *coefficient in the objective function*
- This will be valid for some interval around the base case
  - No change in optimal decisions (what to make vice how much)
  - Objective value will change if decision variable is positive
- Outside this interval a different set of values is optimal for decision variables

SENSITIVITY ANALYSIS FOR BINDING CAPACITY CONSTRAINTS

- The search for the pattern in decision variables and objective function when *varying availability of scarce resource*
- In some interval around the base case:
  - Marginal value (shadow price) of capacity remains constant
  - Some variables change linearly with capacity
  - Others remain the same
- Below this interval the value decreases and eventually reaches zero.

SOLVER TIP: OPTIMIZATION SENSITIVITY AND SHADOW PRICES

- The **shadow price** is based on the break-even price for a scarce resource where it would be attractive to acquire more
- It is calculated as the improvement in objective function from a unit increase (or decrease) in RHS of constraint of the resource in question
- In linear programs, shadow price is constant for some range of changes to RHS.
- Shadow price is how much *more* you would pay for that resource over what you pay in the base solution
  - Thus, if base solution price is $31 and shadow price is $4, you would be willing to pay $35 for another unit of the scarce resource

SENSITIVITY ANALYSIS FOR REDUCED PRODUCTION PROBLEM

- Let's go back to the Savannah-Aspen 2D problem and do sensitivity analysis

	Panel Type						
	Tahoe	Pacific	Savannah	Aspen			
Pallets	**0**	**0**	**0**	**0**	**Total Profit**		
Profit	$450	$1,150	$800	$200	**$0**		
	Tahoe	**Pacific**	**Savannah**	**Aspen**	**Used**	**Available**	
Glue	50	50	100	50	0	5,800	**quarts**
Pressing	5	15	10	5	0	730	**hours**
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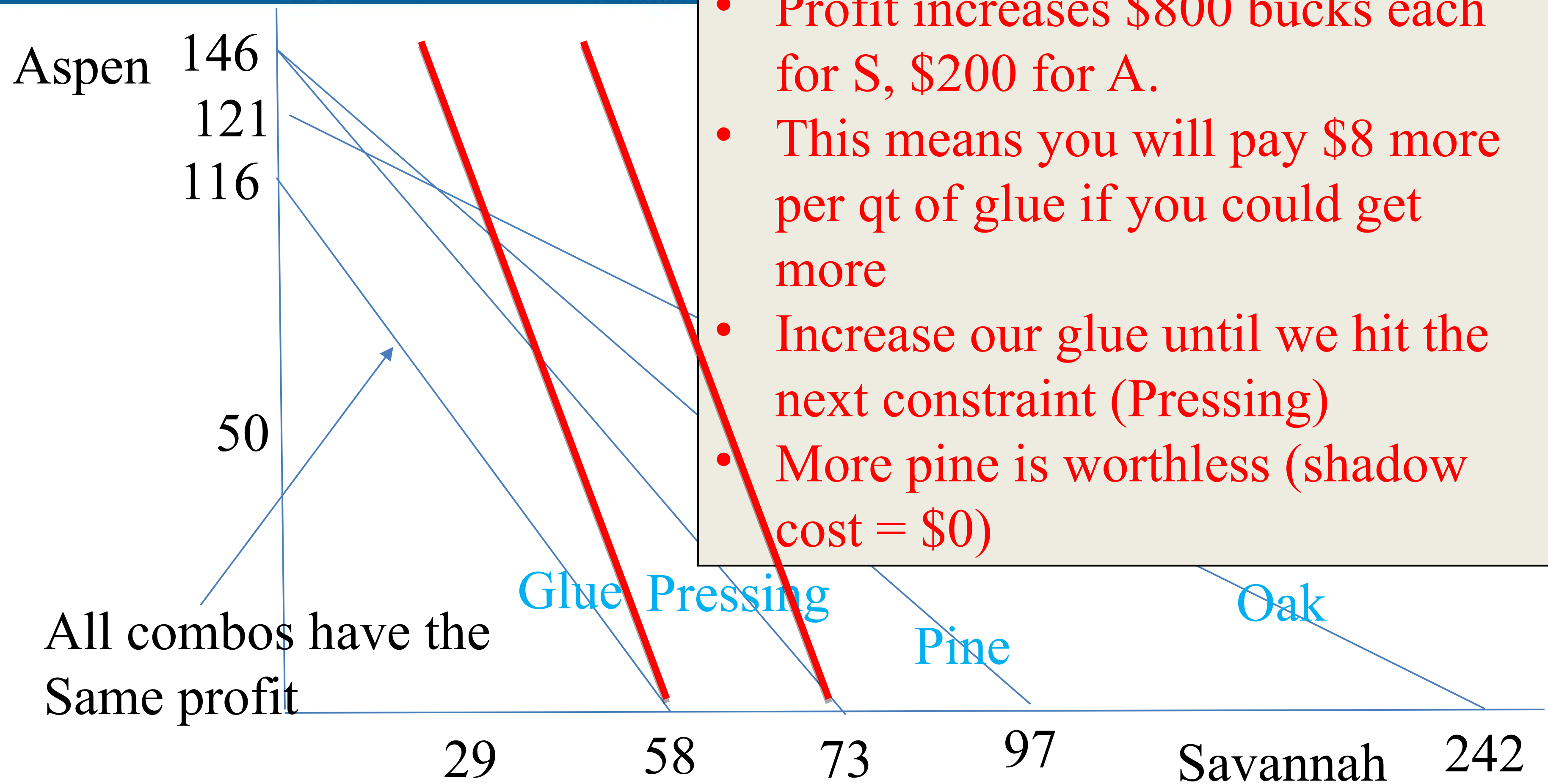
SOLVED: SHADOW COSTS

- Glue is the binding constraint
- 100 more quarts enables one more Savannah or two more Aspen
- Profit increases $800 bucks each for S, $200 for A.
- This means you will pay $8 more per qt of glue if you could get more
- Increase our glue until we hit the next constraint (Pressing)
- More pine is worthless (shadow cost = $0)



OBJECTIVE FUNCTION SENSITIVITY: REDUCED COSTS

- How much can the objective function change before the optimal combination changes?
- In our current case, Savannah profit per pallet cannot change without changing the answer
  - Aspen can change quite a lot

LETS CHANGE THE 2D MODEL SOME MORE

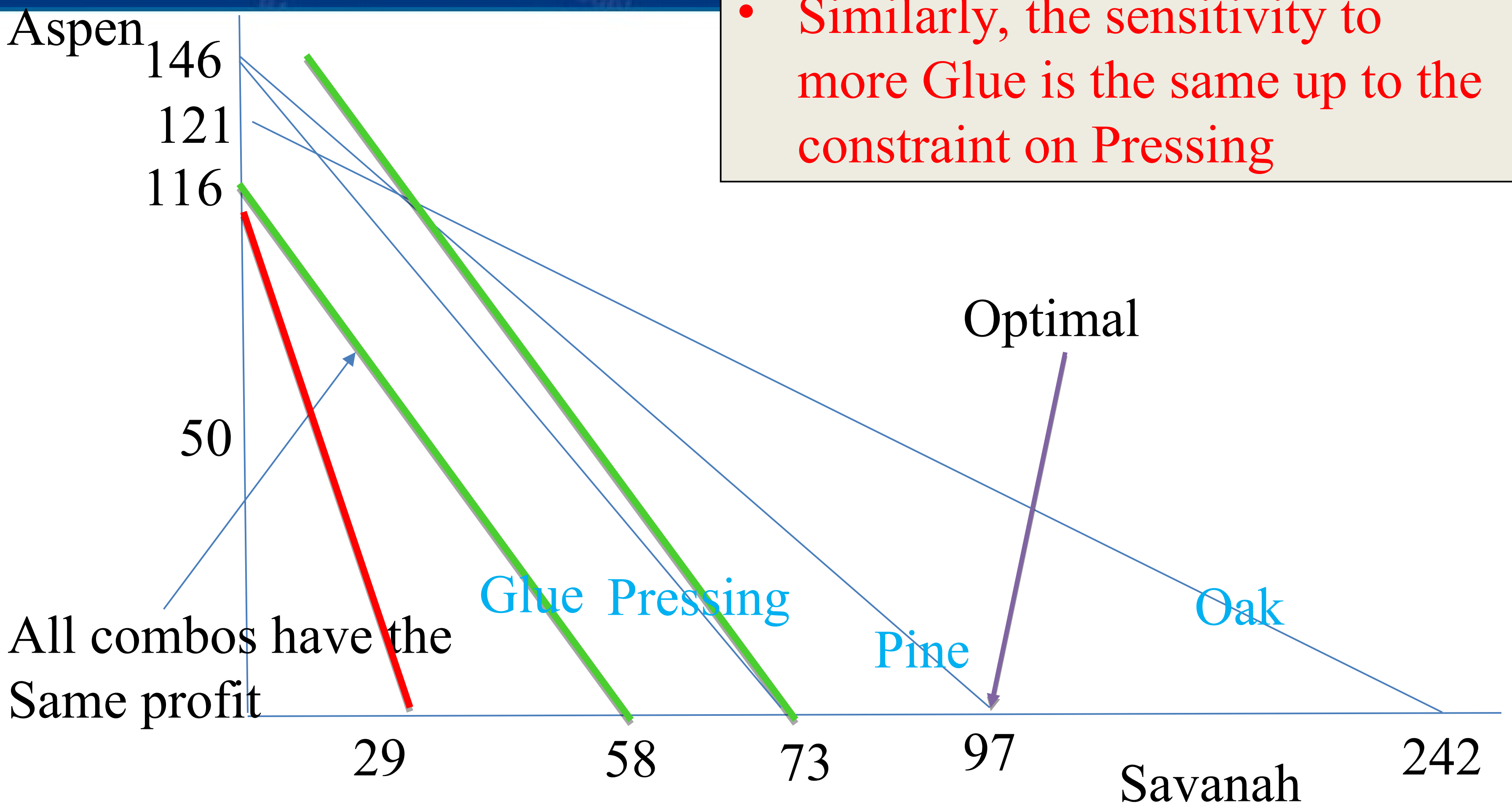
- Change Aspen from $200 to $400 per pallet

	Tahoe	Pacific	Savannah	Aspen	
Pallets	**0**	**0**	**0**	**0**	**Total Profit**
Profit	$450	$1,150	$800	$400	**$0**

	Resources Required per Pallet Type				Used	Available	
~~**Glue**~~	~~50~~	~~50~~	~~100~~	~~50~~	~~0~~	~~5,800~~	~~**quarts**~~
Pressing	5	15	10	5	0	730	**hours**
Pine Chips	500	400	300	200	0	29,200	**pounds**
~~**Oak Chips**~~	~~500~~	~~750~~	~~250~~	~~500~~	~~0~~	~~60,500~~	~~**pounds**~~

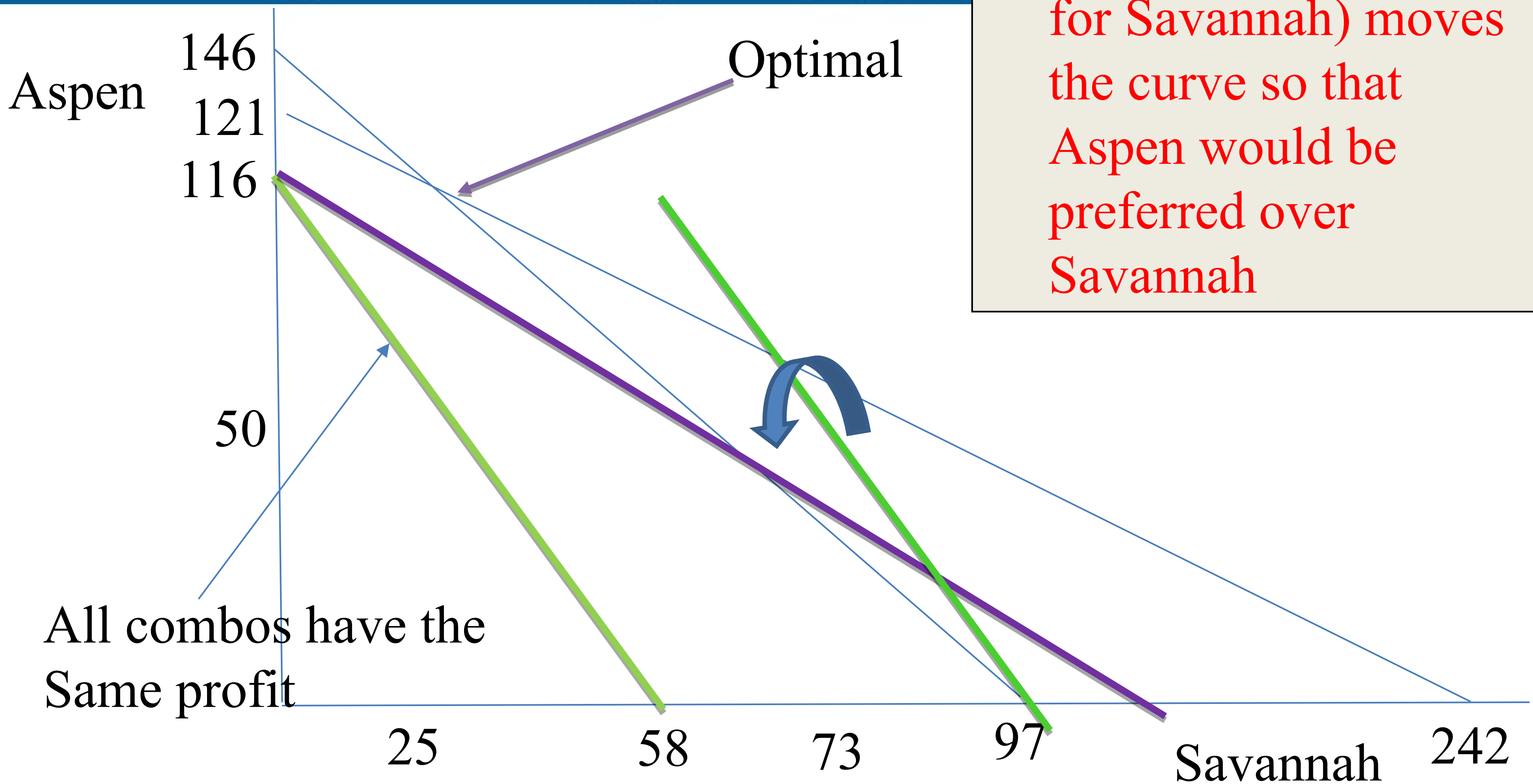
NEW PICTURE

- This change makes us indifferent to whether we make Aspen or Savannah
- Similarly, the sensitivity to more Glue is the same up to the constraint on Pressing

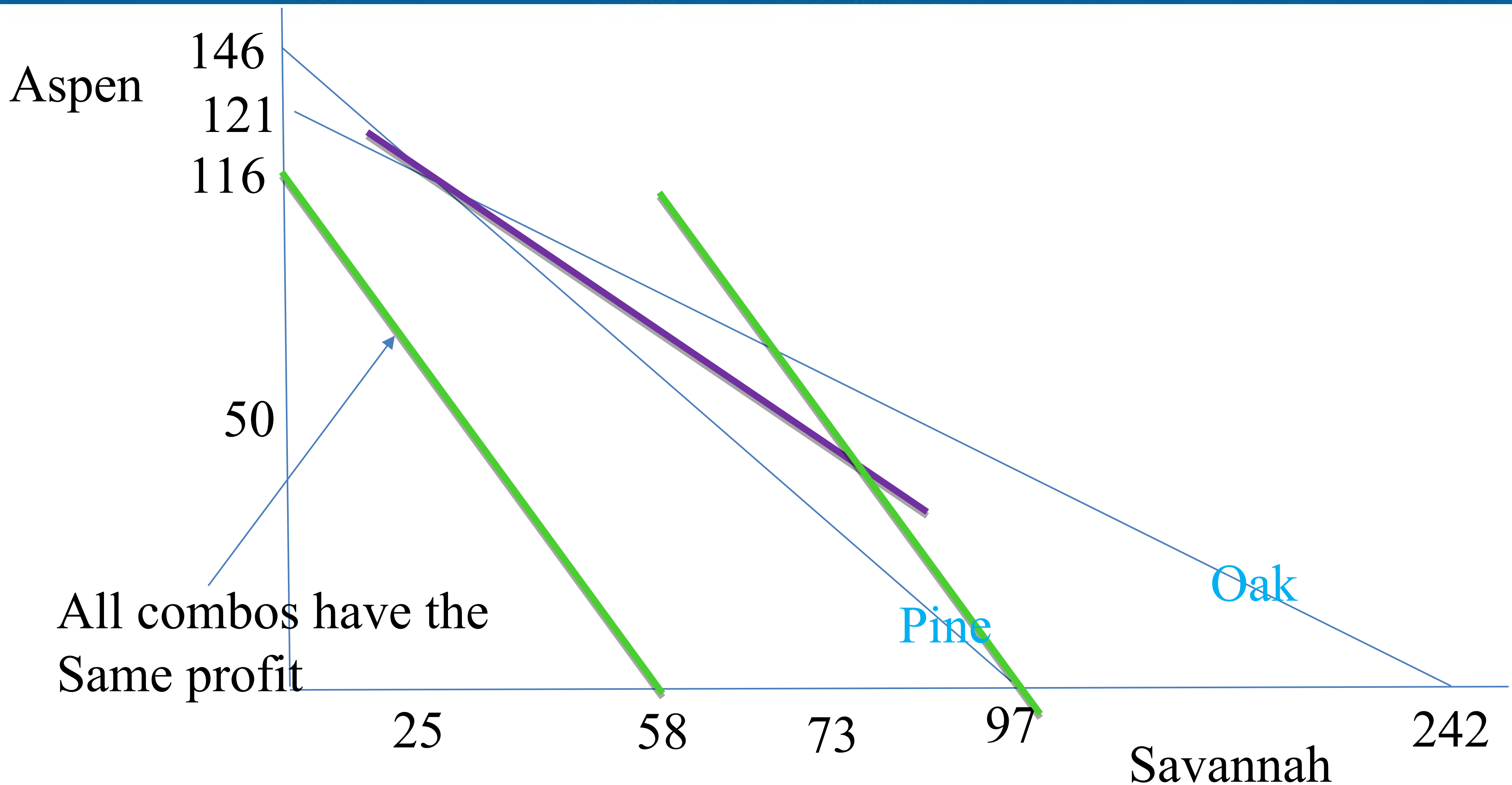


REDUCED COST (INCREASED PROFIT)

- A slightly larger change to the profit on Aspen (or reduction for Savannah) moves the curve so that Aspen would be preferred over Savannah



REDUCED COST CHALK TALK



PATTERNS IN LINEAR PROGRAMMING SOLUTIONS

- The optimal solution tells a “story” about a pattern of economic priorities.
  - Leads to more convincing explanations for solutions
  - Can anticipate answers to “what-if” questions
  - Provides a level of understanding that enhances decision making
- After optimization, should always try to discern the qualitative pattern in the solution.

CONSTRUCTING PATTERNS

- Decision variables
  - Which are positive and which are zero?
- Constraints
  - Which are binding and which are not?
- "Construct" the optimal solution from the given parameters
  - Determine one variable at a time
  - Can be interpreted as a list of priorities which reveal the economic forces at work

DEFINING PATTERNS

- Qualitative description
- Pattern should be complete and unambiguous
  - Leads to full solution
  - Always leads to same solution
- Ask where shadow prices come from
  - Should be able to trace the incremental changes to derive shadow price of constraint

REVIEW: BUILDING THE MODEL

- Determine the **decision variables**
- Determine the **objective function**
- Create a **constraint matrix**
- Enter all three into Analytic Solver
- Consider other constraints
  - The decision variables often must be greater than zero
  - The decision variables sometimes must be integers
- Solve for the base solution
- Interpret the solution
  - Patterns in the decision variables
  - What constraints are binding?
- If the question asks for sensitivity analysis
  - Use optimization sensitivity for validity bounds for objective function or shadow prices
  - Use parametric sensitivity for all others (to be taught later)

EXCEL MINI-LESSON: THE INDEX FUNCTION

- The INDEX function finds a value in a rectangular array according to the row number and column number of its location.
- The basic form of the function, as we use it for DEA models, is the following:
  - INDEX(*Array*, *Row*, *Column*)
- *Array* references a rectangular array.
- *Row* specifies a row number in the array.
- *Column* specifies a column number in the array. If *Array* has just one column, then this argument can be omitted.

SOLVER TIP: RESCALING THE MODEL

- Consider scaling parameters to appear in thousands or millions
- Saves work in data entry – decreases errors
- Spreadsheet looks less crowded
- Helps with Solver algorithms
  - Value of objective, constraints, and decision variables should not differ from each other by more than a factor of 1000, at most 10,000.
- Can always display model output on separate sheet with separate units
- Beware: Scaling can be confusing comparing fixed cost and per-unit costs

AUTOMATIC SCALING

- Use if scaling problems difficult to avoid
- Consider when:
  - Solver claims no feasible solution when user is sure there is one.
- Preferable for model-builder to do the scaling

SUMMARY

- Linear programming represents the most widely used optimization technique in practice.
- The special features of a linear program are a linear objective function and linear constraints.
- Linearity in the optimization model allows us to apply the simplex method as a solution procedure, which in turn guarantees finding a global optimum whenever an optimum of any kind exists.
- Therefore, when we have a choice, we are better off with a linear formulation of a problem than with a nonlinear formulation.

SUMMARY

- While optimization is a powerful technique, we should not assume that a solution that is optimal for a model is also optimal for the real world.
- Often, the realities of the application will force changes in the optimal solution determined by the model.
- One powerful method for making this translation is to look for the pattern, or the economic priorities, in the optimal solution.
- These economic priorities are often more valuable to decision makers than the precise solution to a particular instance of the model.

*DATA ENVELOPMENT ANALYSIS

- DEA is a linear programming application aimed at evaluating the efficiencies of similar organizational departments or *decision-making units* (DMUs).
- DMUs are characterized in terms of inputs and outputs, not in terms of operating details.
- A DMU is considered **efficient** if it gets the most output from its inputs.
- The purpose of DEA is to identify inefficient DMUs when there are multiple outputs and multiple inputs.