Current Problems in Theoretical Hadron Physics
Chiral Anomalies

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Contents

Introduction

Chirality : Inherent property of a particle defined by the spin (handedness) $\equiv$ **helicity** for massless particles – projection of spin onto momentum

Chiral Symmetry : Invariance under chiral rotations

Anomaly : Disagreement with expected classical predictions

Chiral Symmetry of the QCD Lagrangian

In the chiral limit, i.e., $m_q = 0$,

$$\mathcal{L}_{QCD}^{m_q=0} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R$$

where $\not{D} = \partial_\mu - igA^a_\mu \frac{\lambda_a}{2}$, the QCD covarient derivative.
The left and right handed components are invariant independently under the transformation:

$$\psi_L \to e^{-i\boldsymbol{\theta}_L \cdot \boldsymbol{\tau}} \psi_L \qquad \psi_R \to e^{-i\boldsymbol{\theta}_R \cdot \boldsymbol{\tau}} \psi_R$$

Including mass of quarks, i.e., for $m_q \neq 0$,

$$\mathcal{L}_{mass} = \sum_q -m_q(\bar{q}_L q_R + \bar{q}_R q_L)$$

The mass terms are not invariant under the above transformation for $m_u, m_d \neq 0$; the left and right handed components are mixed.

The U(1) Axial Anomaly

Axial U(1) symmetry : If quarks (u, d, s) were massless,

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \Rightarrow \psi' = e^{-i\theta\gamma_5}\psi$$

$$\implies \mathcal{L}_{QCD} = \mathcal{L}'_{QCD}$$

The axial current,

$$J^{(0)}_{5\mu} = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s,$$

would be conserved according to Noether's theorem

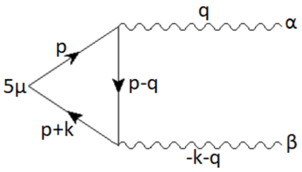
$$\partial^\mu J^{(0)}_{5\mu} = 0$$

But, it is not actually conserved, i.e., $\partial^\mu J^{(0)}_{5\mu} \neq 0$
Two ways of showing the U(1) axial anomaly will be discussed here: the ABJ method and the path integral analysis.

Adler-Bell-Jackiw Anomaly

S. Adler, P.R. 177, 2426 (1969); J.S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969)

— Diagrammatic approach



The matrix element is given by:

$$T_{\mu\alpha\beta} = i \int d^4x \; d^4y \; e^{ik\cdot x} e^{iq\cdot y} \langle 0 | T(J_{5\mu}(x) J_\alpha(y) J_\beta(0)) | 0 \rangle$$

where $J_\alpha = \sum_{q=u,d,s} \bar{q}\gamma_\alpha q$ is the vector current

$J_{5\mu} = \sum_{q=u,d,s} \bar{q}\gamma_\mu\gamma_5 q$ is the axial current

k, q and p are momenta

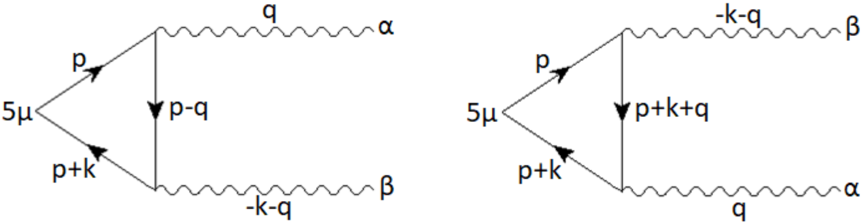
The Ward identities are:

– vector Ward identity (current conservation)

$$q^\alpha T_{\mu\alpha\beta} = 0$$

– axial Ward identity:

$$k^\mu T_{\mu\alpha\beta} = 0$$



$$T_{\mu\alpha\beta} = -3 \int \frac{d^4 p}{(2\pi)^4} \left[\mathrm{tr}\left(\gamma_\mu \gamma_5 \frac{1}{\not{p} + \not{k}} \gamma_\beta \frac{1}{\not{p} - \not{q}} \gamma_\alpha \frac{1}{\not{p}} \right) + \mathrm{tr}\left(\gamma_\mu \gamma_5 \frac{1}{\not{p} + \not{k}} \gamma_\alpha \frac{1}{\not{p} + \not{k} + \not{q}} \gamma_\beta \frac{1}{\not{p}} \right) \right]$$

This is linearly divergent.

Then **vector Ward identity** is

$$q^\alpha T_{\mu\alpha\beta} = \frac{-3}{2}\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma_\mu\gamma_5\frac{1}{\not p+\not k}\gamma_\beta\frac{1}{\not p-\not q} - \gamma_\mu\gamma_5\frac{1}{\not p+\not k+\not q}\gamma_\beta\frac{1}{\not p}\right]$$
$$= -6i\epsilon_{\mu\beta\rho\sigma}\int \frac{d^4p}{(2\pi)^4}\left[\frac{(p+k)^\rho(p-q)^\sigma}{(p+k)^2(p-q)^2} - \frac{(p+k+q)^\rho p^\sigma}{(p+k+q)^2p^2}\right]$$

and **axial Ward identity** is

$$k^\mu T_{\mu\alpha\beta} = \frac{3}{2}\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma_5\gamma_\beta\frac{1}{\not p-\not q}\gamma_\alpha\frac{1}{\not p} + \gamma_5\frac{1}{\not p+\not k}\gamma_\beta\frac{1}{\not p-\not q}\gamma_\alpha\right.$$
$$\left.\gamma_5\gamma_\alpha\frac{1}{\not p+\not k+\not q}\gamma_\beta\frac{1}{\not p} + \gamma_5\frac{1}{\not p+\not k}\gamma_\alpha\frac{1}{\not p+\not k+\not q}\gamma_\beta\right]$$

$$= 6i\epsilon_{\mu\beta\rho\sigma}\int \frac{d^4p}{(2\pi)^4}\left[\frac{(p-q)^\rho p^\sigma}{(p-q)^2p^2} + \frac{(p+k)^\rho(p-q)^\sigma}{(p+k)^2(p-q)^2}\right.$$
$$\left. - \frac{(p+k+q)^\rho p^\sigma}{(p+k+q)^2p^2} - \frac{(p+k)^\rho(p+k+q)^\sigma}{(p+k)^2(p+k+q)^2}\right]$$

If we could shift the integration variable freely, both identities vanish separately.

Shifting integration variable before applying Ward identity has different results.

Shift $p \to p + b_1 q + b_2(-k - q)$ in 1st term of $T_{\mu\alpha\beta}$. To maintain Bose symmetry ($\alpha \leftrightarrow \beta$), shift $p \to p + b_1(-k - q) + b_2 q$ in 2nd term. Then,

$$\Delta T_{\mu\alpha\beta} = -\frac{3}{16\pi^2}(b_1 - b_2)\epsilon_{\mu\alpha\beta\rho}(2q + k)^\rho$$

The **Ward identities** are:

$$q^\alpha T_{\mu\alpha\beta} = -\frac{3}{16\pi^2}\epsilon_{\mu\alpha\beta\rho}(1 + (b_1 - b_2))k^\rho q^\sigma$$

$$k^\mu T_{\mu\alpha\beta} = \frac{3}{8\pi^2}\epsilon_{\mu\alpha\beta\rho}(1 - (b_1 - b_2))k^\rho q^\sigma$$

Only one identity is conserved for particular choice of $(b_1 - b_2)$.

Vector current (electric charge) is conserved $\implies$ Axial current is not conserved, contrary to Noether theorem.

Path Integral Analysis

– Fujikawa method, Phys. Rev. Lett. 42, 1195 (1979)
– Variable change not always trivial.

Consider *generating function* of field (A_μ) and axial current source (a_μ):

$$W[a_\mu, A_\lambda] = \int [d\psi][d\bar{\psi}] e^{i \int d^4x (\mathcal{L}_{QCD}(\psi, \bar{\psi}, A_\lambda) - a_\mu J_5^\mu)}$$

Changing the integration variables ψ and $\bar{\psi}$ to

$$\psi = e^{i\beta(x)\gamma_5} \psi'$$
$$\bar{\psi} = e^{i\beta(x)\gamma_5} \bar{\psi}'$$

introduces **Jacobian**:

$$\int [d\psi][d\bar{\psi}] = \int [d\psi'][d\bar{\psi}'] \mathcal{J}$$

with $\mathcal{J} = e^{-i \int d^4x \beta(x) \partial^\mu \bar{J}_{5\mu}(x)}$

$$\mathcal{J} = [\det(e^{i\beta(x)\gamma_5})]^{-2}$$

Regularizing the trace,

$$\mathcal{J} = \lim_{M\to\infty} e^{-2i\mathrm{tr}(\beta\gamma_5 e^{-(\not{D}/M)^2})}$$

where $D^\mu = \partial^\mu - igA^\mu$

Using identity

$$\begin{aligned}\not{D}\not{D} &= \gamma_\mu D^\mu \gamma_\nu D^\nu = \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} D^\mu D^\nu + \frac{1}{2}[\gamma_\mu, \gamma_\nu] D^\mu D^\nu \\ &= D_\mu D^\mu + \frac{1}{4}\{\gamma_\mu, \gamma_\nu\}[D^\mu, D^\nu] \\ &= D_\mu D^\mu + \frac{g}{4}\sigma^{\mu\nu} F_{\mu\nu}\end{aligned}$$

the Jacobian is

$$\begin{aligned}\mathcal{J} &= e^{-i\int d^4x\ \beta(x)\frac{g^2}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}} \\ &\neq 1\end{aligned}$$

$$\implies \int [d\psi][d\bar{\psi}] \neq \int [d\psi'][d\bar{\psi}']$$

$\Rightarrow$ U(1) is not a symmetry of the theory.

$$\mathcal{J} = e^{-i\int d^4x\beta(x)\partial^\mu \bar{J}_{5\mu}(x)} = e^{-i\int d^4x\beta(x)\frac{3g^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}}$$

$$\implies \partial^\mu \bar{J}_{5\mu}(x) = \frac{3g^2}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \neq 0$$

For $m_q \neq 0$,

$$\partial^\mu J_{5\mu} = \frac{3g^2}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + 2(m_u\bar{u}i\gamma_5 u + m_d\bar{d}i\gamma_5 d + m_s\bar{s}i\gamma_5 s)$$

— There are no conserved currents here as per Noether's theorem

— No Goldstone bosons in the U(1) theory as there is no symmetry to be broken

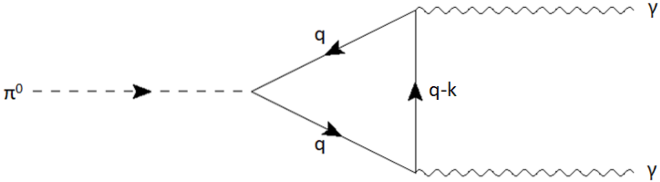
Consequences of Axial Anomaly

– $\pi^0 \to \gamma\gamma$ decay

– Goldstone bosons in large N_c limit

$\pi^0 \to \gamma\gamma$

Historically where axial anomaly was detected.
Also used as a test for value of N_c



$$\Gamma_{\pi^0 \to \gamma\gamma} = \frac{m_\pi^3}{64\pi} |A_{\gamma\gamma}|^2$$

Peskin and Schroeder

In case there is no anomaly, using the PCAC technique:

$$\langle 0|j^{5\mu}(x)|\pi(q)\rangle = -iq^\mu F_\pi e^{-iq\cdot x}$$

$$\implies \langle 0|\partial_\mu j^{5\mu}(x)|\pi(q)\rangle = m_\pi^2 F_\pi e^{-iq\cdot x}$$

$$\therefore \partial_\mu j^{5\mu} = m_\pi^2 F_\pi \phi_\pi$$

where F_π is the pion decay constant.

So, for $m_q \to 0, m_\pi \to 0$ and $\partial_\mu j^{5\mu} \to 0$

Then, $A_{\gamma\gamma} \propto m_\pi^2 \approx 0 \implies \Gamma_{\pi^0\to\gamma\gamma} = 0$
The decay is highly suppressed.

But including anomaly, there is an extra term:

$$\partial_\mu j^{5\mu} = m_\pi^2 F_\pi \phi_\pi + \frac{\alpha N_c}{24\pi^2 F_\pi} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. Then,

$$\Gamma_{\pi^0\to\gamma\gamma} = \left(\frac{N_c}{3}\right)^2 \frac{\alpha^2 m_\pi^3}{64\pi^3 F_\pi^2} \overset{N_3\to 3}{=} 7.63\ \mathrm{eV}$$

Agrees with experimental value (JLab, 2011 via Primakoff effect) :

$$\Gamma^{exp}_{\pi^0\to\gamma\gamma} = (7.82 \pm 0.31)\ \mathrm{eV}$$

Goldstone Bosons at Large N_c

Spontaneous symmetry breaking implies existence of Goldstone bosons

For any pseudo-scalar meson P,

$$\langle P_j(q)|A_k^\mu(0)|0\rangle = -iF_j q^\mu \delta_{jk}$$

$$\langle P_j(q)|\partial_\mu A_k^\mu(0)|0\rangle = F_j m_j^2 \delta_{jk}$$

— For octet of gluon current, $\partial_\mu A_{(8)}^\mu = 0 \xrightarrow{m_q=0}$ 8 Goldstone bosons $(\pi^+, \pi^-, \pi^0, K^+, K^-, K^0, \bar{K}^0, \eta_8)$

— For singlet current, $\partial_\mu A_{(0)}^\mu$,
Calculate matrix element for arbitrary N_c in chiral limit,

$$\langle P_j(q)|A_k^\mu(0)|0\rangle \sim \mathcal{O}(N_c^{-1/2} N_c) = \mathcal{O}(N_c^{1/2}) \implies F_j \sim \mathcal{O}(N_c^{1/2})$$

For the mixture of η_0 and η_8 : η and η',

$$m^2_{\eta'} \sim N_c^{-1} \rightarrow 0$$

$\implies$ η' is a Goldstone boson in large N_c limit

But, anomaly applies to singlet current; $\partial_\mu A^\mu_{(0)} \neq 0$

— Diagonalization of quark mass matrix predicts $m_{\eta'} = 0.98$ GeV and $m_\eta = 0.50$ GeV with $\eta - \eta'$ mixing angle , $\theta = 18^\circ$ (Donoghue et. al.)
This is in agreement with experimental values: $m_\eta \approx 0.55$ GeV and $m_{\eta'} \approx 0.96$ GeV [PDG]

Trace Anomaly

Scale Invariance: In chiral limit, QCD Lagrangian has no dimensional parameters and exhibits scale invariance under the transformation:

$$\psi_q(x) \rightarrow \lambda^{3/2}\psi_q(\lambda x) \qquad A^a_\mu(x) \rightarrow \lambda A^a_\mu(\lambda x)$$

The current is then conserved

$$J^\mu = x_\nu \theta^{\mu\nu} \longrightarrow \partial_\mu J^\mu = \theta^\nu{}_\nu = 0$$

For any hadron H,

$$\langle H(k)|\theta^{\mu\nu}|H(k)\rangle = 2k^\mu k^\nu$$

where $\theta^{\mu\nu}$ is the energy-momentum tensor.

$$\theta^\nu{}_\nu = 0 \implies \langle H(k)|\theta^\nu{}_\nu|H(k)\rangle = 2m_H^2 = 0$$

$$\therefore m_H = 0$$

But hadrons have mass – renormalization required $\longrightarrow$ make coupling constant $\alpha_s = \alpha_s(q^2)$

Path integral treatment:

Donoghue et. al.

$$W[h, A^a_\mu] = \int d[\psi] d[\bar{\psi}] e^{i \int d^4x [\mathcal{L}_{QCD}(\psi, A^a_\mu) + h(x) \theta^\mu{}_\mu]}$$

where $\theta^{\mu\nu} = \frac{i}{2} \bar{\psi} \gamma_\mu D^\nu \psi + \frac{i}{2} D^\nu \bar{\psi} \gamma^\mu \psi$

Changing integration variable $\psi = e^{-\alpha(x)/2} \psi'(x)$ for $\alpha \to 0$, we get:

$$W[h + \alpha, A^a_\mu] = \int d\psi' d\bar{\psi}' \mathcal{J} e^{i[\mathcal{L}_{QCD}(\psi', A^a_\mu) + h(x) \theta^\mu{}_\mu + \alpha(x) m \bar{\psi}' \psi']}$$

with $i \int d^4x \ \alpha(x) \theta^\mu{}_\mu = \ln \mathcal{J} + i \int d^4x \ \alpha(x) m \bar{\psi} \psi$

$$\mathcal{J} = [\det(e^{-\alpha/2})]^{-2}$$

Regularization:

$$\mathcal{J} = \lim_{M \to \infty} e^{\mathrm{Tr} \int d^4x \langle x | \alpha e^{-(\not{D}/M)^2} | x \rangle}$$

$$= \lim_{M \to \infty} e^{\left(\frac{i g_s^2}{48\pi^2} F^a_{\mu\nu} F^{\mu\nu}_a \right)}$$

Combining the equations:

$$i \int d^4x \; \alpha(x) \theta^{\mu}{}_{\mu} = i \int d^4x \left[\frac{g_s^2}{48\pi^2} F^a_{\mu\nu} F_a^{\mu\nu} + m\bar{\psi}\psi \right]$$

$$\implies \theta^{\mu}{}_{\mu} = \frac{\alpha_s(x)}{12\pi} F^a_{\mu\nu} F_a^{\mu\nu} + m\bar{\psi}\psi$$

Including mass of quarks and introducing beta function,

$$\theta^{\mu}{}_{\mu} = \frac{\beta_{QCD}}{2g_s} F^a_{\mu\nu} F_a^{\mu\nu} + \sum_q m_q \bar{q} q$$

$$\therefore \partial_\mu J^\mu \neq 0$$

The current is not conserved.

Consequences of Trace Anomaly

Mass of hadrons are determined with the help of the trace anomaly.

Mass can be expressed as a matrix element of energy-momentum trace. For example, the nucleon mass is:

$$
\begin{aligned}
m_N^2 &= \langle N(k)|\theta^{\mu}{}_{\mu}|H(k)\rangle \\
&= \langle H(k)|\frac{\beta_{QCD}}{2g_s}F^a_{\mu\nu}F_a^{\mu\nu} + m_s\bar{s}s + m_u\bar{u}u + m_d\bar{d}d|N(k)\rangle
\end{aligned}
$$

Maximum contribution to the mass is expected from the $F^a_{\mu\nu}F_a^{\mu\nu}$ (gluonic) part.

For proton and neutron, the quark-mass terms contribute to only $\sim$10% of the nucleon's mass.

The QCD Vacuum

— θ-vacuum adds extra phase to to $\mathcal{L}_{QCD}$:

$$\mathcal{L}_{QCD} = \mathcal{L}_{QCD}^{\theta=0} + \theta \frac{g_s^2}{64\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu}$$

— $F\tilde{F}$ is P-odd and T-odd $\overset{\theta\neq 0}{\Longrightarrow}$ PT- violation

— For $m_q = 0$, θ can be rotated away by chiral U(1) transformation.

— For $m_q \neq 0$, CP violation of $\mathcal{L}_{QCD}$ arises (the strong CP problem).

Conclusion

— Basis for determining mass spectrum and decay rates of hadrons, and the number of colors (N_c).

— Facilitate certain interactions otherwise not included in the theory.

— Trace anomaly helps in determining hadronic mass in experiments and sheds light on contributions from the internal degrees of freedom.

— A dynamical scale parameter of mass-dimension is generated at the quantum level, and all hadronic masses are proportional to this scale.

— θ vacuum leads to strong CP problem - an issue of fine tuning of the theory.

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