债券交易系统讲义

货币的时间价值

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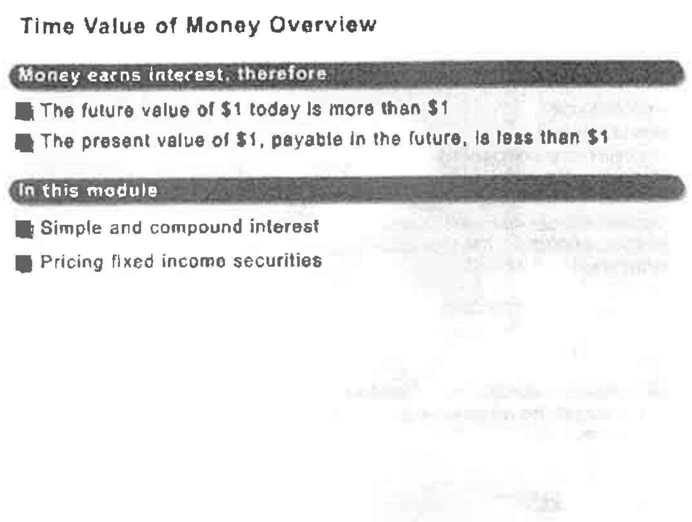
1. Time Value of Money

1.1. Overview



We prefer money today rather than in the future, because money today can be lent to earn interest. The **future value** of \$1 today is therefore more than \$1; the **present value** of \$1, payable in the future, is therefore less than \$1.

In this module we distinguish between simple interest and compound interest and we explain how the time value of money is used to price fixed income instruments such as bonds.

Learning Objectives

By the end of this module, you will be able to:

- Calculate **present** and **future** values of cash flows using **simple** and **compound** interest
- Convert between a **nominal** and an **effective interest rate**
- Price a simple fixed **coupon** bond

1.2. Simple & Compound Interest

Simple Interest

Mankind's two most important inventions: the wheel and compound interest. Nathan M. Rothschild.

The interest amount payable on a loan depends on:

- The rate of interest
- The term of the loan
- The interest compounding period

Example

Suppose the quoted interest rate on a 12 month loan for \$100 is 10% per annum and the interest is payable in one bullet at maturity. Then the total repayment at maturity (i.e. the **future value**) would be the \$100 principal plus interest of \$10:

$$\begin{aligned} \text{Repayment amount} &= 100 + (100 \times 0.10) \\ &= 100 + 10 \\ &= \mathbf{\$110} \end{aligned}$$

This is a **simple** interest calculation: the interest amount is proportional to the term of the loan and is calculated by pro-rating the quoted annual rate by the term of the loan. Each day the loan accrues the same amount of interest.

Compound Interest

Semi-annual Compounding

Now suppose that interest at 10% on a 12 month loan for \$100 is compounded every 6 months (semi-annually). Assuming that 6 months is exactly half a year, then after the first 6 months:

$$\begin{aligned} \text{Repayment amount} &= 100 + (100 \times 0.10 \times 1/2) \\ &= 100 \times (1 + 0.10/2) \\ &= \mathbf{\$105} \end{aligned}$$

The first \$5 of interest is now paid after 6 months and is reinvested (rolled over), together with the principal, at the annual rate of 10% for another 6 months. So after 12 months:

$$\begin{aligned} \text{Repayment amount} &= 105 \times (1 + 0.10/2) \\ &= [\ 100 \times (1 + 0.10/2)\ ] \times (1 + 0.10/2) \\ &= (1 + 0.10/2)^2 \\ &= \mathbf{\$110.25} \end{aligned}$$

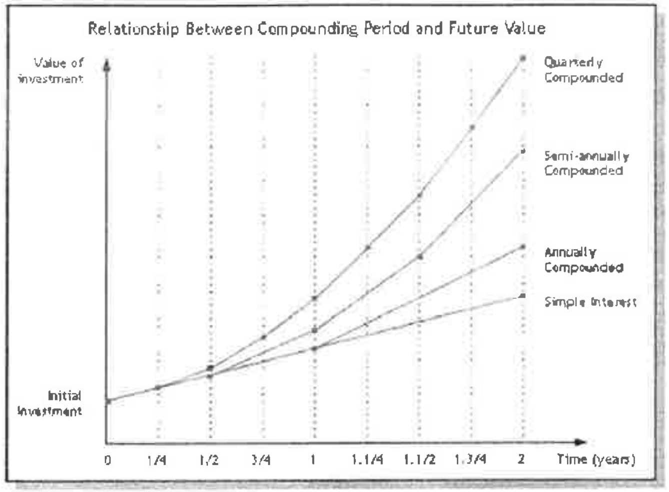
This is \$0.25 more than what is earned if all the interest had been paid at maturity, as in the example on the previous page. The extra \$0.25 represents interest on the \$5 interest paid at the six-month point: interest on interest. Thus, although the nominal interest rate is 10%, the effective rate is 10.25%.

Quarterly Compounding

Had interest been compounded quarterly, then after 12 months:

Repayment amount $= 100 \times (1 + 0.10/4)^4$
$= \mathbf{\$110.38!}$

Quarterly compounded, a nominal rate of 10% is effectively a rate of 10.38%. The more frequent the compounding period, the higher is the effective rate.



Effective Rate & Reinvestment Risk

The examples on the previous page illustrate that, with compounding, the interest rate that you see is not necessarily the interest that, effectively, you get. You can use the formula below to convert a nominal rate into an effective rate.

Effective return annually compounded = Equivalent return compounded t times

$(1 + \text{Effective Rate}) = (1 + \text{Nominal Rate}/t)^t$

Therefore:

Effective Rate $= \mathbf{(1 + \text{Nominal Rate}/t)^t - 1}$

Where t = compounding period (1 = annual, 2 = semi-annual, 4 = quarterly, etc.)

Reinvestment Risk

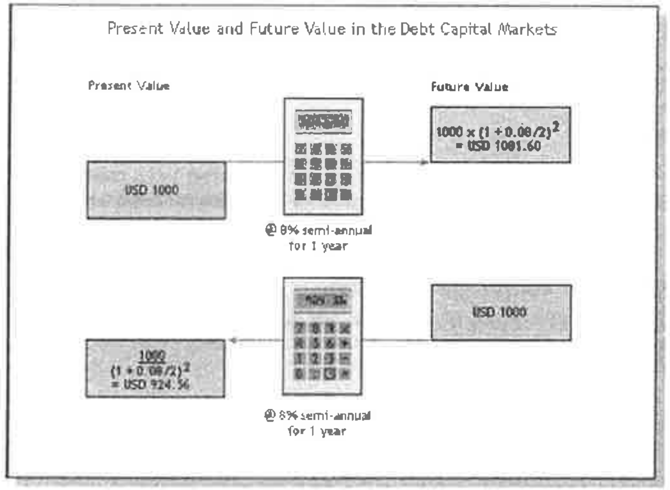
In bullet instruments, the interest due at the end of each compounding period is **capitalised** - in other words, it is added to the balance of the principal outstanding - and therefore continues to accrue at the same rate as the original loan.

In other contracts, such as bonds, the interest due is actually paid out in instalments (known as **coupons**) and there is no guarantee that the lender may be able to reinvest the coupons received at the same rate as the original loan. In such cases the total return on the investment depends on the reinvestment rate achieved. Reinvestment risk is discussed in more detail in the context of bond yields.

1.3. Present & Future Value

The Principles

In section *Simple & Compound Interest* we showed how you calculate the future value of a sum invested today, given rate of interest, maturity and compounding period. The figure below illustrates how the same methodology is used to calculate the present value of a future cash flow in the debt capital markets.



In the example $924.56 is the present value of $1,000 payable in 1 year: the sum of capital which, when invested at 8% per annum semi-annually compounded, accumulates to exactly $1,000 after 1 year.

Notice the logic between present and future values:

- When we go forward in time we compound the present value by a factor (1+ ...)
- When we go back we divide, or **discount**, the future value by the same factor

Compounding and discounting are fundamental techniques in financial analysis.

The Formulas

Compounding and Discounting Capital Market Style

Future value = Present value x $(1 + \text{Rate} / t)^{nxt}$

$$\text{Present value} = \frac{\text{Future value}}{(1 + \text{Rate} / t)^{nxt}}$$

$$= \text{Future value} \times D_n$$

Where:
t = compounding frequency (1 = annually, 2 = semi-annually, etc.)
N = number of years
N x t = number of compounding periods, or interest periods.
D_n **Discount factor** = $1 / (1 + \text{Rate}/t)^{nxt}$

The discount factor is the present value of $1 payable in nxt interest periods and its value is between 0 and 1. For example, a sum of money receivable today will have a discount factor of 1, making its present and future values the same. The further out in time the cash flow is received, the smaller will be its discount factor and hence its present value.

The formula for discounting future cash flows in the money markets is different from this one and is developed in Money Market Cash Instruments - Present & Future Value.

1.4. Pricing a Bond

Multiple Cash Flows

The price of any financial instrument is the present value of its expected future cash flows.

This fundamental principle applies to any financial instrument, whether the instrument is a bond, an equity security or even a derivative such as an option. In this section, we illustrate how the discounting technique may be used to price a straight bond.

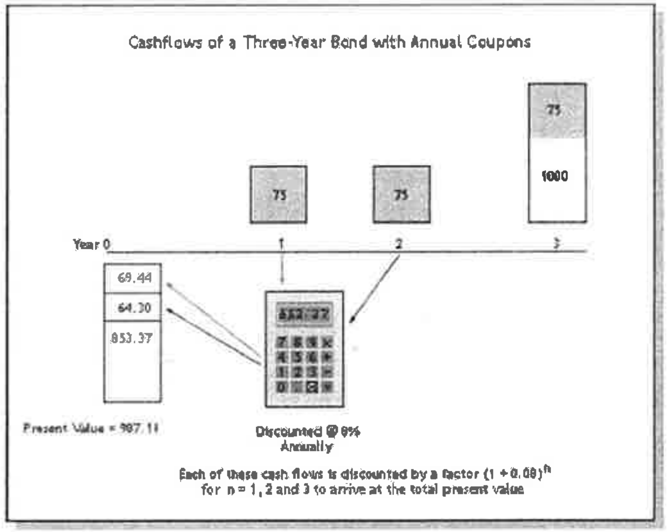
Example

Name: Coca Cola 7½% maturing 19 July 2004
Type: Eurobond, annual coupons
Settlement date: 19 July 2002
Denomination: USD 1,000

? How much should you pay for this bond to achieve a return (or **yield**) of 8%?

Analysis
The figure below displays on a time-line the three cash flows that the bond will generate:

1. Coupon: USD 75 (71/2% of USD 1,000) at the end of year 1, on 19 July 2003
2. Coupon: another USD 75 at the end of year 2
3. Principal repayment plus final coupon: USD 1075, at the end of year 3.



The sum of the present values of the expected future cash flows using an annual discount rate of 8% is $987.1, so this is what the bond should be worth.

The Pricing Formula

Generalising what we did in the example on the previous page to any maturity gives the formula for the price of a bond shown below.

$$\text{Bond Price} = \frac{C/t}{(1 + R/t)} + \frac{C/t}{(1 + R/t)^2} + \frac{C/t}{(1 + R/t)^3} + \ldots + \frac{\text{Principal} + C/t}{(1 + R/t)^{nxt}}$$

Where:
C = Annual coupon rate
R = Return on the investment (the **yield**)
t = Number of coupon payments per year (= compounding period)
n = Number of years

Equivalent formulations:

$$\text{Present value} = C/t \times \Sigma \frac{1}{(1 + R/t)^i} + \frac{\text{Principal}}{(1 + R/t)^{nxt}}$$

for i = 0 ... nxt

$$= C/t \times \frac{(D_1 - D_{nxt+1})}{(1 - D_1)} + \text{Principal} \times D_{nxt}$$

Where D_i = Discount factor = $1/(1 + R/t)^i$

To get to the second equivalent formulation using discount factors, we have used a mathematical technique to reduce the sum of a series.

Properties

In our example we priced the bond on a coupon date, so the next coupon is exactly one coupon period away. In *Bond Pricing* we shall adapt this basic formula to handle situations when a bond may be traded part-way through a coupon period, so the next coupon may not be exactly one coupon period away.

Two important points follow from the pricing formula developed here:

1. The higher the required yield on the bond - i.e. the higher the rate used to discount its future cash flows - the lower will be its market price. This illustrates:

   Bond Market Law No 1: bond price varies inversely with yield

2. The convention in the bond market is to apply the same rate to discount all of the bond's future cash flows.

   In reality we know that the cost of one-year money is rarely the same as the cost of two-year money, or the cost of three-year money. So we could apply a different discount rate to present value each cash flow. This is the technique used in the derivatives markets. The market value of a bond may be different when its cash flows are traded separately - i.e. as a strip of zero-coupon bonds - than it is when it is traded whole, as a family of cash flows.

1.5. Annuities & Perpetuities

Annuities

Annuity: a regular stream of equal cash flows payable over a fixed period of time.

These are found in financial contracts such as rents, leases, amortised loans and pension plans. They also exist as components of derivative instruments such as interest swaps; the so-called fixed side of a swap is an annuity.

Valuing annuities involves the same technique that we discussed in the context of pricing a fixed-maturity bond. The only difference here is that all the cash flows are identical, as there is no principal amount repayable. A slightly simpler version of the formula developed in section *Pricing a Bond* is used.

$$\text{PV of Annuity} = \frac{C}{(1 + R/t)} + \frac{C}{(1 + R/t)^2} + \frac{C}{(1 + R/t)^3} + \ldots + \frac{C}{(1 + R/t)^{nxt}}$$

Where:
C = Regular cash flow
R = Discount rate (the required return on the annuity)
t = number of payments per year (= compounding period)
n = number of years

Equivalent formulations:

Present value = $C \times \Sigma\, 1 / (1 + R/t)^i$ for i = 1... nxt
= $C \times (D_1 - D_{txn+1}) / (1 - D_1)$ (closed form equation)

Where D_i = Discount factor = $1/(1 + R/t)^i$

Example

? How much could you borrow, at 5% per annum, if you intended to repay the loan in 4 equal annual instalments of $5,000?

Analysis

Using the formula for the annuity presented above and making
C = 5,000, R = 0.05, n = 4 and t = 1:

$$\text{Present Value of Annuity} = \frac{5{,}000}{1.05} + \frac{5{,}000}{1.05^2} + \frac{5{,}000}{1.05^3} + \frac{5{,}000}{1.05^4}$$

= 4,761.90 + 4,535.15 + 4,319.19 + 4,113.51

= **$17,729.75**

For long-dated annuities it is certainly easier to use the closed form equation, presented above, which has the same size regardless of the number of cash flows.

Perpetuities

Perpetuity: a regular stream of equal cash flows payable forever.

Perpetuities are rarer than annuities but by no means uncommon: some borrowers have issued fixed-coupon perpetual bonds.

When valuing perpetuities, clearly we have to use some closed form equation rather than an arithmetic series! In fact, the pricing formula is very simple indeed. If we take the closed form solution for the present value of an annuity, on the previous page, and we increase n to infinity we find that the discount factor D_{txn+1} = 0 and the formula reduces to:

Present value of perpetuity = $C \times (D_1 - 0) / (1 - D_1)$

$$= \frac{C}{R}$$

Example

The UK Government 2 1/2% Consols are perpetual bonds that were issued after World War I. What is the price of this bond to yield 6.70%?

Analysis

Present value = 2.50 / 0.067
= **37.31%** - simple as that!

1.6. Exercises

Question 1

What is the future value of USD 100 deposited at 10% interest, annually compounded, for 7 years?

a) Write your answer rounded to 2 decimal places .

USD

Question 2

What is the effective interest rate payable on a 1 year loan for GBP 100 at 10% per annum nominal, when the rate is compounded:

a) Annually?

Enter your answers to 2 decimal places.

b) Semi-annually?

c) Daily?

Question 3

What is the price of a 2-year zero coupon bond to yield 10%, when the compounding period is:

a) Annual?

Express prices as a percentage of face value rounded to 2 decimal places.

b) Semi-annual?

c) Price of bond b), above, is lower than price of bond a) because:

- Bond a) earns a higher effective yield
- Bond b) is in higher demand
- Bond a) earns more coupon income
- Bond b) earns a higher effective yield

Question 4

What is the annually compounded yield on a bond yielding 10%, semi-annual?

a) Enter your answer in percent to 2 decimal places.

Question 5

Face value: $100.00
Maturity: 2 years
Yield: 5.90%
Coupon: 7½%, semi-annual

a) Calculate the price of the US Treasury bond above, rounded to 2 decimal places:

Question 6

What is the present value of a 3 year annuity paying USD 100 annually when discounted at 10%?

a) Enter your answer to 2 decimal places.

USD

Question 7

In this question we compare the present value of a redeemable bond with the present value of an equivalent irredeemable bond, or perpetuity, paying the same annual coupon rate.

a) What is the present value of a 50-year bond with a face value of USD 1,000 and an annual coupon rate of 8%, when discounted at 10%?

Round your answers to 2 decimal places.

USD

b) What is the present value of a straight perpetuity paying USD 80 annually, when discounted at 10%?

USD

Question 8

What is the annual amortisation amount on a USD 100 loan repayable over 3 years in fixed instalments (principal plus interest), at a rate of interest of 10%?

a) Enter your answer to 2 decimal places.

USD