CONVEX OPTIMIZATION

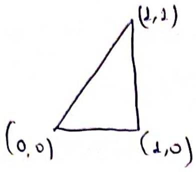
ASSIGNMENT No. 1

EE21MTECH14002

Q.3

Sol:- (a) $S = \{(0,0), (1,1), (1,0)\}$

The Convex hull conv(S) of S is



(b) $S = \{\underline{x}_1, \underline{x}_2, \ldots \ldots \underline{x}_m\}$ (Set of all vertices of conv(S))

To Prove

$$f(x) \le \max\{f(\underline{x}_1), f(\underline{x}_2), \ldots f(\underline{x}_m)\},$$
for $\underline{x} \in \text{Conv}(S)$

Let us take λ such that $\lambda \in R_+^m$ with

$$\lambda_1 + \lambda_2 + \cdots \lambda_m = 1$$

We can say that $\boxed{S\lambda = x}$ $\longrightarrow$ $\because$ The convex combination of the vertices can make all the pts inside the convex hull.

$$f(x) = f(S\lambda) \le \sum_{i=1}^{M} \lambda_i f(x_i)$$

(As f is convex func. it follows this property)

We also know that $\sum_{i=1}^{M} \lambda_i f(x_i) \le \max\{f(x_1) + f(x_2) + \cdots f(x_m)\}$

$\because \sum_{i=1}^{m} \lambda_i = 1$, the total sum is always less than the max of $f(\underline{x})$ where $\underline{x} \in S$

Or the weighted avg. is always less than max.

$\therefore$ we can say that

$$f(x) = f(S\lambda) \le \sum_{i=1}^{M} \lambda_i f(x_i) \le \max_{i=1,2,\ldots m} f(x_i)$$

$$\Rightarrow f(x) \le \max\{f(x_1), f(x_2) \cdots f(\underline{x}_m)\}$$

$\because$ All the $x_i \in S$ (set of vertices of conv(S))
we can say that maximum of f over Conv(S) occurs at one of the vertices of Conv(S).

Q.2

Sol:- Given, $C = \{x : x^T y \geq 0$ for every $y \in S\}$

(a) Let's say $x_1 \in C$ and $x_2 \in C$, then

i.e $x_1^T y \geq 0$ and $x_2^T y \geq 0 \ \forall \ y \in S$

$\Rightarrow \alpha_1 (x_1^T y) + \alpha_2 (x_2^T y) \geq 0 \ \forall y \in S$

$\Rightarrow (\alpha_1 x_1^T + \alpha_2 x_2^T) y \geq 0 \ \forall \ y \in S$

iff $\alpha_1 x_1^T + \alpha_2 x_2^T \geq 0$

This is not possible for $\alpha_1, \alpha_2 \in \mathbb{R}$

$\therefore$ C is not a subspace

(b) Lets say $x_1 \in C$ and $x_2 \in C$ then,

$x_1^T y \geq 0$ and $x_2^T y \geq 0 \ \forall \ y \in S$

for C to be affine,

$(\theta x_1^T + 1-\theta x_2^T) y \geq 0 \ \forall \ y \in S$

$\Rightarrow \theta x_1^T + (1-\theta) x_2^T \geq 0 \ \forall \ \theta \in \mathbb{R}$

which is not possible

(simply solve for $\theta x_1^T + (1-\theta) x_2^T < 0$)

(c) Let say $x_1 \in C$ and $x_2 \in C$, then

$x_1^T y \geq 0$ and $x_2^T y \geq 0 \ \forall \ y \in S$

$\Rightarrow \theta x_1^T y \geq 0$ and $(1-\theta) x_2^T y \geq 0$

$\forall \ \theta \geq 0 \qquad \forall \ \theta \leq 1$

i.e. $\theta x_1^T y + (1-\theta) x_2^T y \geq 0 \ \forall \ y \in S$

$\forall \ 0 \leq \theta \leq 1$

i.e C is Convex set

(d) Let say $x_1 \in C$

$\Rightarrow x_1^T y \geq 0 \quad \forall \ y \in S$

$\Rightarrow \theta x_1^T y \geq 0 \ \forall \ \theta \geq 0$

$\forall \ y \in S$

$\therefore$ C is a Cone

Ans

Q5(a)

Sol: (a) given $f: \mathbb{R}^2 \to \mathbb{R}$

and f is convex, $f(x_1, x_2) = f(x_2, x_1) \; \forall \; x_1, x_2$

let us assume that we get $\min_{x \in C} f(x)$ for 2 diff. coordinates

i.e. let min of $f(x)$ occur at $X(x_1, x_2)$ and $Y(y_1, y_2)$

$\because f$ is convex symmetric

$\Rightarrow f(x_1, x_2) = f(x_2, x_1) = m_1$ (say the minimum) — ①

and $\forall \; X(x_1, x_2) \in C \; ; \; \exists \; X'(x_2, x_1)$

s.t. $f(x_1, x_2) = f(x_2, x_1)$

Now $f(y_1, y_2) = f(y_2, y_1) = m_2$ (Say the minimum) — ②

and $\forall \; Y(y_1, y_2) \in C \; ; \; \exists \; Y'(y_2, y_1)$

s.t. $f(y_1, y_2) = f(y_2, y_1)$

① and ②

Since same minima $\Leftrightarrow m_1 = m_2$

and then we get 4 points (different) where the function is maximum

Contradiction.

Since a convex func. has only 1 minima

But, Since $f(x)$ is convex symmetric func. ; it has 2 points at which minima occures i.e.

$X(x_1, x_2)$ and $X'(x_2, x_1)$

$\Rightarrow x_1 = x_2$

$\therefore$ equal coordinates.

(b) Optimal Value of

$$\max_{x} \; x_1 x_2 x_3 \cdots x_n \quad \text{s.t.}$$

$$\sum x_i = 1, \; x_i \geq 0 \;\; i = 1, 2, \ldots, n$$

let $G = \max_{x} x_1 x_2 \cdots x_n$ & $A = \sum x_i$

we know that A.M $\geq$ G.M apply this for $\{x_1, x_2, \cdots x_n\}$

$$\Rightarrow \frac{\sum x_i}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

$$\Rightarrow \frac{\sum x_i}{n} \geq \sqrt[n]{\prod x_i} \quad \Rightarrow \frac{1}{n} \geq \sqrt[n]{\prod x_i}$$

$\Rightarrow$ Since $x_i \geq 0 \ \forall i \Rightarrow \left(\frac{1}{n}\right)^n \geq \prod x_i$

$\Rightarrow$ G: $\max_{x} x_1 x_2 \cdots x_n \Rightarrow \left(\frac{1}{n}\right)^n = \frac{1}{n^n}$ Ans

Q.6
(a)

Convex function $f(x) = \frac{x^T x}{t}$

$\operatorname{dom} f = \{(x, t) : t > 0\}$

To prove: $f(x)$ is convex

Proof :-

Let $g(x) = x^T x$

$\because x^T x = \|x\|_2$

$\Rightarrow g(x) = \|x\|_2$

$\Rightarrow g(x)$ is a convex function

as norms are always convex

$f(x, t)$ can be written in terms of g — (a)

$\Rightarrow f(x, t) = t g\left(\frac{x}{t}\right)$;

by dom f : $t > 0$

$g(x/t)$ Indicates perspective of g

Now, taking epigraph of "f"

$\operatorname{epi}(f) = \{(x, t, k) : f(x, t) \leq k\}$

$\operatorname{epi}(f) = \{(x, t, k) : t g\left(\frac{x}{t}\right) \leq k\}$

$\because t > 0$ we can take it on either side of inequality

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$$epi(f) = \left\{ x, t, k) = g\left(\frac{x}{t}\right) \leq \frac{k}{t} \right\} \quad \text{--- (1)}$$

from eqn (1)

$$(\underline{x}, k, t) \in epi(f) \iff \left(\frac{x}{t}, \frac{k}{t}\right) \in epi(g)$$

epi(f) is inverse perspective of epi(g)

Now,

$\Rightarrow$ [g is convex (by a) $\Rightarrow$ epi(g) is convex]

$\because$ epi(f) is inverse perspective of epi(g)

$\Rightarrow$ [epi(g) is convex $\Rightarrow$ epi(f) is convex]

By using property

a function is convex if and only if epi(f) is convex

$\Rightarrow$ epi(f) is convex $\Rightarrow$ $f(\underline{x}, t)$ is convex

hence, $f(\underline{x}, t) = \frac{x^T x}{t}$ is convex

Ans

Q.6
(b)

given

$$f(x) = \frac{x^T x}{t^2}$$

$\text{dom} f = \{(x,t) : t > 0\}$

$$f(x) = \frac{x^T x}{t^2} = \frac{x^T}{t} \cdot \frac{x}{t}$$

Consider $g(x) = x^T x$

$$\Rightarrow f(x,t) = g\left(\frac{x}{t}\right)$$

$g(x)$ is a convex function as it is a composition of 2 convex functions (x^2, $\|x\|$) and x^2 is increasing for $(x) > 0$ (as here $\|x\| > 0$)

Let $x \in \mathbb{R}_+$ s.t

$$S_\alpha = \{(x,t) : f(x,t) \leq \alpha, t > 0\}$$
$$= \{(x,t) : g\left(\frac{x}{t}\right) \leq \alpha, t > 0\}$$

α - sublevel set of $g(x) = c_\alpha = \{x : g(x) \leq \alpha\}$

If $(x,t) \in S_\alpha$ then $\left(\frac{x}{t}\right) \in C_\alpha$

$\Rightarrow$ S_α is inverse perspective of C_α

$\because$ $g(x)$ is convex, its sublevel sets are convex

$\Rightarrow$ C_α is convex

S_α is also convex (inverse perspective preserves convexity)

$S_\alpha = \alpha$ - sublevel set of f

Since sublevel set of f is convex, it is a quasi convexity

$$f(x,t) = \frac{x^T x}{t^2}$$ is a quasi-convex function

where,

$\text{dom} f = \{(x,t)\ t > 0\}$

Q.6 (c) function f is defined as

$$f(x) = \begin{cases} \left\| x - \frac{x}{\|x\|_2} \right\|_2 & ; \text{ if } \|x\|_2 \geq 1 \\ 0 & ; \text{ otherwise} \end{cases}$$

Let us consider $\left\| x - \frac{x}{\|x\|_2} \right\|_2 = \left\| x \left(1 - \frac{1}{\|x\|_2} \right) \right\|_2$

$$= \|x\|_2 \left[\left| \left(1 - \frac{1}{\|x\|_2} \right) \right| \right]$$

$\because \left[\|x k\| = |k| \|x\| \right] ; k \in \mathbb{R}$

$$= \|x\|_2 \left| \frac{[\|x\|_2 - 1]}{\|x\|_2} \right| = \|x\|_2 - 1 \quad \left(\because \|x\|_2 \geq 1 \right)$$

$$f(x) = \max(\|x\|_2 - 1, 0)$$

maximum of convex function is convex, f is also convex

→ Constant func are affine

∴ constant func. are convex

sum of convex func. $(\|x\|_2, -1)$ is also convex

∴ $\max(\|x\|_2 - 1, 0)$ is also convex

$$f(x) = \begin{cases} \left\| x - \frac{x}{\|x\|_2} \right\|_2 & , \|x\|_2 \geq 1 \\ 0 & , \text{o.w.} \end{cases} \text{ is convex}$$

Ans