Impermanent Loss: Math Derivation

Impermanent loss is a popular concept when it comes to Automated Market Makers (AMMs) like Uniswap.

A liquidity provider puts an initial amount into two tokens and makes them available for traders to trade with each other. Impermanent loss is the loss incurred when market prices change, increasing the number of lower relative value tokens It is defined as the percentage difference between the value of the new token composition versus the value if the initial composition (hold) was maintained.

There are great articles that explain the concept well and provide examples, but they all show a formula for Impermanent Loss (IL) without offering a derivation:

$$IL = \frac{2\sqrt{d}}{1+d} - 1$$

Let's see how to get it step by step:

Considerations
Automated Market Maker protocols like Uniswap and SushiSwap are based on a very simple equation:

$$x * y = K \qquad (1)$$

Where, x is the number of tokens for asset X, y is the number of tokens for asset Y and K is the constant product of the pool.

The value of the initial position is:

$$V_0 = x * p_x + y * p_y \qquad (2)$$

An equal value of both tokens is supplied to the pool, therefore:

$$x * p_x = y * p_y$$

$$p_x = \frac{y}{x} * p_y \qquad p_y = \frac{x}{y} * p_x \qquad (3)$$

From (1) we know that $x = K/y$, likewise $y = K/x$,
Substitute it in (3):

$$p_x = \frac{\frac{K}{x}}{x} * p_y \qquad p_y = \frac{\frac{K}{y}}{y} * p_x$$

$$p_x = \frac{K}{x^2} * p_y \qquad p_y = \frac{K}{y^2} * p_x$$

We solve for x and y in terms of $K,\ p_x,\ p_y$.

$$x^2 = \frac{K}{p_x} * p_y \qquad y^2 = \frac{K}{p_y} * p_x$$

$$x = \sqrt{\frac{K}{p_x} * p_y} \qquad y = \sqrt{\frac{K}{p_y} * p_x} \qquad (4)$$

Then, we rewrite V_0 according to (4):

$$V_0 = \sqrt{\frac{K}{p_x} * p_y} * p_x + \sqrt{\frac{K}{p_y} * p_x} * p_y$$

$$V_0 = \sqrt{K * p_y * p_x} + \sqrt{K * p_x * p_y}$$

$$V_0 = 2\sqrt{K * p_y * p_x}$$

What happens if prices change from p_x to p_{x1} and from p_y to p_{y1}?
The new quantities would be:

$$x_1 = \sqrt{\frac{K}{p_{x1}} * p_{y1}} \qquad y_1 = \sqrt{\frac{K}{p_{y1}} * p_{x1}} \qquad (5)$$

We calculate the value with the new composition x_1 and y_1.

$$V_1 = x_1 * p_{x1} + y_1 * p_{y1}$$

$$V_1 = \sqrt{\frac{K}{p_{x1}} * p_{y1}} * p_{x1} + \sqrt{\frac{K}{p_{y1}} * p_{x1}} * p_{y1}$$

$$V_1 = 2\sqrt{K * p_{y1} * p_{x1}} \qquad (6)$$

While holding does not change the quantities, x and y remain the same.
Therefore, the value would be:

$$Hold = x * p_{x1} + y * p_{y1}$$

$$Hold = \sqrt{\frac{K}{p_x} * p_y} * p_{x1} + \sqrt{\frac{K}{p_y} * p_x} * p_{y1} \qquad (7)$$

To get the impermanent loss, we calculate the difference between V_1 and $Hold$.

$$V_1 - Hold = 2\sqrt{K * p_{y1} * p_{x1}} - (\sqrt{\frac{K}{p_x} * p_y} * p_{x1} + \sqrt{\frac{K}{p_y} * p_x} * p_{y1})$$

We consider that $\Delta_x = p_{x1}/p_x$ and $\Delta_y = p_{y1}/p_y$.
So, $p_{x1} = p_x * \Delta_x$ and $p_{y1} = p_y * \Delta_y$

$$V_1 - Hold = 2\sqrt{K * p_y * \Delta_y * p_x * \Delta_x} - (\sqrt{\frac{K}{p_x} * p_y} * p_x * \Delta_x + \sqrt{\frac{K}{p_y} * p_x} * p_y * \Delta_y)$$

$$V_1 - Hold = 2\sqrt{K * p_y * p_x * \Delta_y * \Delta_x} - (\sqrt{K * p_y * p_x} * \Delta_x + \sqrt{K * p_x * p_y} * \Delta_y)$$

We calculate it as a percentage:

$$\frac{V_1 - Hold}{Hold} = \frac{2\sqrt{K*p_y*p_x*\Delta_y*\Delta_x} - (\sqrt{K*p_y*p_x}*\Delta_x + \sqrt{K*p_x*p_y}*\Delta_y)}{(\sqrt{K*p_y*p_x}*\Delta_x + \sqrt{K*p_x*p_y}*\Delta_y)}$$

We factor $\sqrt{K * p_y * p_x}$:

$$\frac{V_1 - Hold}{Hold} = \frac{\sqrt{K*p_y*p_x}*(2\sqrt{\Delta_y*\Delta_x} - (\Delta_x + \Delta_y))}{\sqrt{K*p_y*p_x}*(\Delta_x + \Delta_y)}$$

$$\frac{V_1 - Hold}{Hold} = \frac{2\sqrt{\Delta_y*\Delta_x} - (\Delta_x + \Delta_y)}{(\Delta_x + \Delta_y)}$$

$$\frac{V_1 - Hold}{Hold} = \frac{2\sqrt{\Delta_y*\Delta_x}}{(\Delta_x + \Delta_y)} - 1 \qquad (8)$$

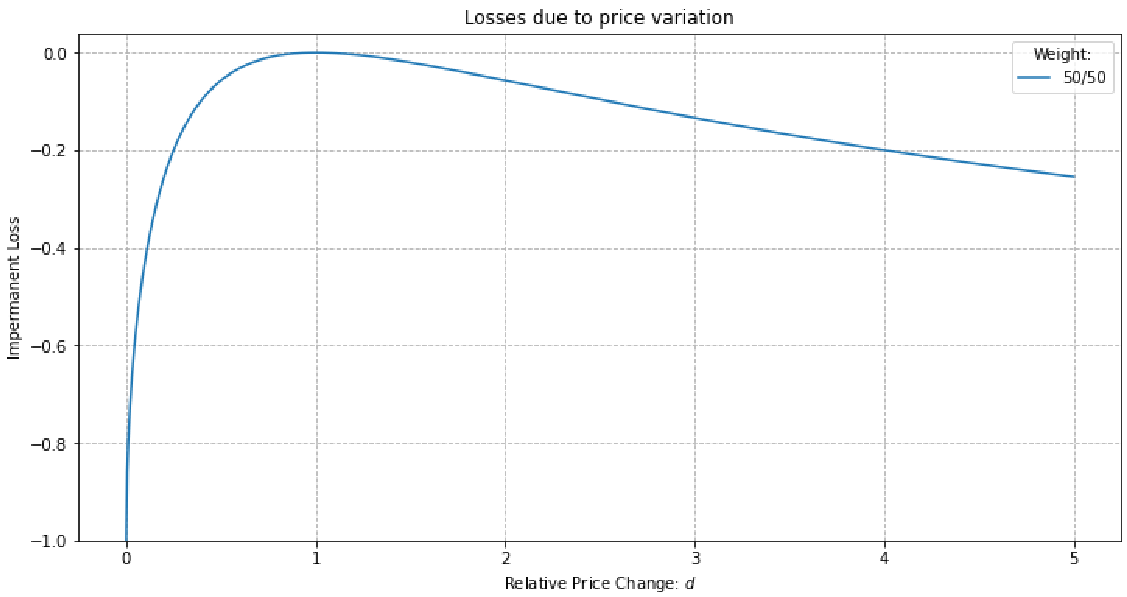
Consider that $d = \Delta_x/\Delta_y$, then $\Delta_x = d * \Delta_y$:

$$\frac{V_1 - Hold}{Hold} = \frac{2\sqrt{{\Delta_y}^2*d}}{(d*\Delta_y + \Delta_y)} - 1$$

We factor Δ_y:

$$\frac{V_1 - Hold}{Hold} = \frac{2*\Delta_y*\sqrt{d}}{\Delta_y(d+1)} - 1$$

$$\frac{V_1 - Hold}{Hold} = \frac{2\sqrt{d}}{(d+1)} - 1 \qquad (9)$$

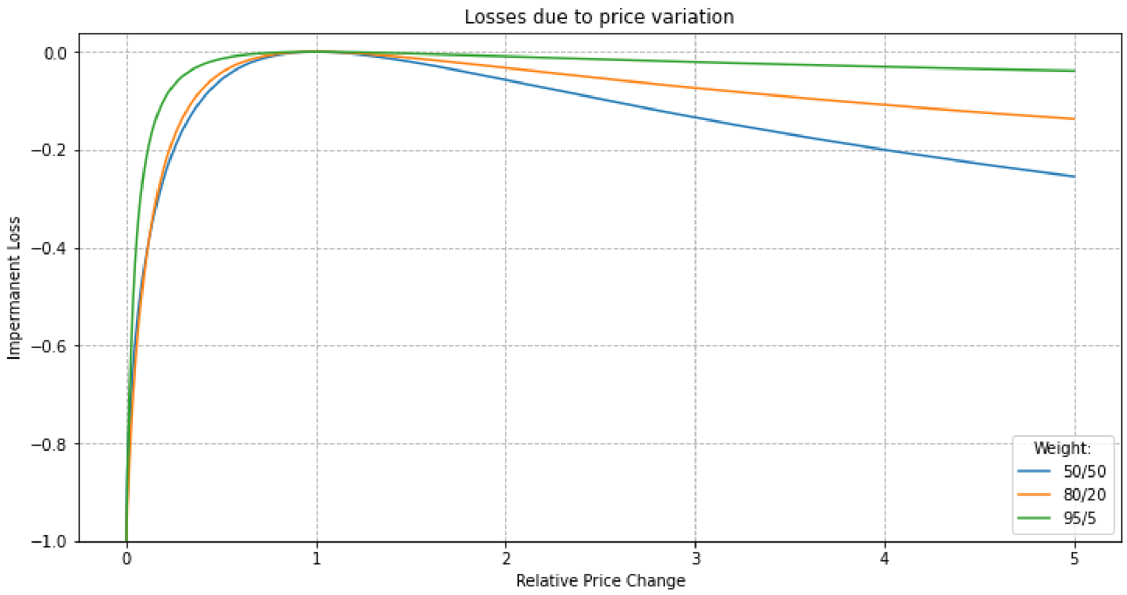


Note 01:
In case the distribution of tokens is not ***50/50***, we will call w_x for the weight of token X and w_y for the weight of token Y.
We rewrite equation 8:

$$\frac{V_1 - Hold}{Hold} = \frac{2\sqrt{\Delta_y * \Delta_x}}{(\Delta_x + \Delta_y)} - 1$$

$$\frac{V_1 - Hold}{Hold} = \frac{{\Delta_x}^{0.5} * {\Delta_y}^{0.5}}{(0.5 * \Delta_x + 0.5 * \Delta_y)} - 1$$

$$\frac{V_1 - Hold}{Hold} = \frac{{\Delta_x}^{w_x} * {\Delta_y}^{w_y}}{(w_x * \Delta_x + w_y * \Delta_y)} - 1 \qquad (10)$$



Note 02:
Generalizing for more than two tokens:

$$\frac{V_1 - Hold}{Hold} = \frac{\prod_i (\Delta_i)^{w_i}}{\sum_i (\Delta_i * w_i)} - 1 \qquad (11)$$

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