Kinematics of 2⟶ 2 Scattering

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Last Update: January 18, 2021

Introduction

Here is a list of the relevant kinematics parameters for a 2⟶ 2 scattering process in the reference system of the centre of mass. As variables we consider the masses of the particles $\{m_i\}_{i=1}^4$, the centre of mass energy $\sqrt{s}$ and the angle of scattering θ.

Kinematics

We consider a process of the type $1+2 \longrightarrow 3+4$; θ is the angle between $\vec{p}_1$ and $\vec{p}_3$. The centre of mass energy $\sqrt{s}$ and the other Mandelstam variables are defined as

$$\begin{aligned} s &= (p_1^\mu + p_2^\mu)^2 = (p_3^\mu + p_4^\mu)^2 \ , \\ t &= (p_1^\mu - p_3^\mu)^2 = (p_2^\mu - p_4^\mu)^2 \ , \\ u &= (p_1^\mu - p_4^\mu)^2 = (p_2^\mu - p_3^\mu)^2 \ . \end{aligned} \tag{1}$$

Recall they satisfy the property

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \tag{2}$$

Energy and Momentum The momenta of the initial or final pair (i, j) of particles are given by

$$|\vec{p}_i| = |\vec{p}_j| = \frac{1}{2}\sqrt{\frac{(s-(m_i-m_j)^2)(s-(m_i+m_j)^2)}{s}} \ . \tag{3}$$

The energies of the particles are obtained through the Einstein relation $E^2 = m^2 + |\vec{p}|^2$; this amounts to:

$$E_i = \frac{1}{2}\sqrt{\frac{\left(m_i^2 - m_j^2 + s\right)^2}{s}} \quad , \quad E_j = \frac{1}{2}\sqrt{\frac{\left(m_j^2 - m_i^2 + s\right)^2}{s}} \ . \tag{4}$$

Scalar products We use the convention $p \cdot k \equiv p^\mu k_\mu$.

$$
\begin{aligned}
p_1 \cdot p_2 &= \frac{1}{2}\left(s - m_1^2 - m_2^2\right) \\
p_3 \cdot p_4 &= \frac{1}{2}\left(s - m_3^2 - m_4^2\right)
\end{aligned}
\tag{5}
$$

The other cases can be obtained by using Equations 3 and 4 through

$$
\begin{aligned}
p_1 \cdot p_3 &= E_1 E_3 - \cos\theta |\vec{p}_1||\vec{p}_3| \ , \\
p_1 \cdot p_4 &= E_1 E_4 + \cos\theta |\vec{p}_1||\vec{p}_4| \ , \\
p_2 \cdot p_3 &= E_2 E_3 + \cos\theta |\vec{p}_2||\vec{p}_3| \ , \\
p_2 \cdot p_4 &= E_2 E_4 - \cos\theta |\vec{p}_2||\vec{p}_4| \ .
\end{aligned}
\tag{6}
$$

The expressions for the Mandelstam variables can be obtained using these results and expanding the definitions of Equation 1.