Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

# Constrained Navigation Algorithms for Strapdown Inertial Navigation Systems with Reduced Set of Sensors

Aleksandr Brandt
AlliedSignal Defense & Space Systems
Teterboro NJ 07608-1173

John F. Gardner
Penn State
Mechanical Engineering Dept.
University Par, PA

**ABSTRACT**

This paper develops a family of algorithms for low-cost strapdown inertial navigation system for land vehicles. Constraints on the motion of land vehicles are defined. They include constraints on vehicle's orientation relative to the Earth surface, and relationship between vehicle's attitude and its velocity direction. Navigation equations are derived that assume validity of these constraints on the vehicle's motion. Compared to standard strapdown inertial navigation, these algorithms reduce navigation errors in the presence of relatively high instrument noise, and at the same reduce number of required inertial sensors. Navigation errors are analyzed and used in an error model for the Kalman filter. Derived algorithms are applied to processing of experimental data.

## 1 Introduction

The objective of this research is to develop algorithms for low-cost aided inertial navigation system for land vehicles. The needs of traffic control, safety, fleet management, optimization of mass transit scheduling, require development of Automatic Vehicle Location Systems (AVLS) for land vehicles such as buses, trains, cars, etc. The principal purpose of these systems is to determine continuously, in real-time, the position of moving vehicles with sufficient level of accuracy and reliability.

In recent years, several AVLS that use Global Positioning System (GPS) have been developed. However, application of GPS in urban conditions presents serious problems due to blocking of satellite signals by tall buildings and trees. Experiments done by Tsuji et al. (1991), Vlcek et al. (1993) and others show, that in urban conditions obscuration of GPS signals may lasts for several minutes and that, in general, GPS is available less than 50% of total time. Reflection of the satellite signals from the buildings introduces additional errors. Also, GPS is unavailable in tunnels and for subway trains.

An alternative to the GPS approach is dead reckoning. Dead reckoning is based on continuous measurements of vehicle's heading and speed or traveled distance which are used to compute trajectory. These systems require initialization, i. e. starting position of the vehicle must be provided to the AVLS.

Several existing dead reckoning systems use various sensors to measure direction of vehicle's motion, for example magnetic compass (Lezniak, et al. , 1977), differential odometer (Harris and Krakiwsky, 1989), and gyroscope (Tsui et al. , 1991; Vlcek, et al. , 1993). All these systems use an odometer as the distance measuring device. Distance errors arise in these systems because of uncertainty in odometer scaling factor and skidding.

Another specialized form of dead reckoning is inertial navigation. Inertial navigation is based on measurements of the acceleration and the angular rotation of the vehicle by accelerometers and gyroscopes respectively.

The high cost of inertial sensors and strict maintenance requirements restricted applicability of the inertial navigation systems in the past to high-end military applications. They have been used for applications where their utilization was justified by the unique system requirements.

Recent advances in the development of relatively inexpensive and small inertial instruments lead to development of inertial navigation systems (INS) for civilian land vehicles. Lower accuracy of the affordable sensors and constraints on the motion of the land vehicles suggest non-traditional approaches to interpretation and processing of the sensor signals.

Daum et al. (1994) used 3 inertial instruments - one gyroscope and two accelerometers in their Aided Inertial Land Navigation System (ILANA). The instrument set also included an odometer and GPS receiver. However, in this system accelerometers were used for establishment of a level plane (i. e. to measure pitch and roll angles), rather than for measurement of vehicle acceleration, as in regular inertial navigation systems. Such arrangement is sensitive by principle to accelerations experienced by the vehicle, yielding pitch and roll angle errors, and it does not allow to measure acceleration inertially. An odometer was used as primary instrument for measurement of the transitional motion of the vehicle. GPS was used as an aiding source of the position, velocity, and heading information.

In this paper, constraints on the motion of the land vehicles are used to derive set of navigation equations for the reduced set of four inertial instruments: 1 accelerometer and 3 gyroscopes. Transitional motion of the vehicle is measured by accelerometer. In addition, the odometer is used as a tachometer to provide auxiliary information about vehicle's speed. A second accelerometer may be also used to provide redundant data that helps to reduce overall system errors.

## 2 Strapdown Inertial Navigation Algorithm with Reduced Set of Inertial Instruments

Inertial navigation is based on Newton's Second Law:

$$\ddot{\mathbf{r}}|_{\mathrm{i}} = \frac{1}{m}\mathbf{F} \tag{2.1}$$

where $\ddot{\mathbf{r}}|_{\mathrm{i}}$ is second derivative of the position vector as seen from the inertial coordinate frame (or i-frame, see Appendix A for definition of the various coordinate frames) , $\mathbf{F}$ is a resultant of the forces acting upon the body; and $m$ is the mass of the body. In alternate form Newton's Second Law may be expressed as

$$\ddot{\mathbf{r}}|_{\mathrm{i}} = \mathbf{f} + \mathbf{g}_m \tag{2.2}$$

where $\mathbf{f}$ is specific force and $\mathbf{g}_m$ is gravitation vector at the body's location.

Specific force $\mathbf{f}$ is vector equal in magnitude and opposite in direction to resultant of inertia reaction force and gravitation force per unit mass. Specific force is the physical quantity which is measured by accelerometers, and therefore the latter form of the Newton's Second Law is used for derivation of navigation equations.

Equation (2.2) governs body motion in an inertial coordinate frame. However, for practical navigation we are interested in vehicle's motion as it seen from Earth-fixed coordinate frame. In this research we use an Earth-fixed tangential s-frame with origin on the Earth surface as Earth-fixed coordinate frame. (See Appendix A)

Earth-fixed coordinate frames are not inertial, because of Earth rotation around its axis. Therefore, equation (2.2) should be expressed in terms of vehicle's Earth-referenced velocity $\mathbf{v} = \dot{\mathbf{r}}|_{\mathrm{s}}$.

Using relationship for acceleration in rotating coordinate frame we get

$$\begin{aligned}\ddot{\mathbf{r}}\big|_{\mathrm{i}} &= \ddot{\mathbf{r}}\big|_{\mathrm{s}} + 2\omega_{\mathrm{is}} \times \mathbf{v} + \omega_{\mathrm{is}} \times (\omega_{\mathrm{is}} \times \mathbf{r}) = \\ &= \dot{\mathbf{v}}\big|_{\mathrm{s}} + 2\omega_{\mathrm{is}} \times \mathbf{v} + \omega_{\mathrm{is}} \times (\omega_{\mathrm{is}} \times \mathbf{r})\end{aligned} \tag{2.3}$$

where $\omega_{\mathrm{is}}$ is vector angular rotation of the s-frame relative to i-frame. Since s-frame is Earth-fixed, $\omega_{\mathrm{is}}$ is entirely due to the earth's rotation.

Combining equations (2. 2) and (2. 3) we get

$$\dot{\mathbf{v}}|_{\mathbf{s}} = \mathbf{f} - 2\omega_{\mathrm{is}} \times \mathbf{v}_{\mathbf{s}} + [\mathbf{g}_{\mathrm{m}} - \omega_{\mathrm{is}} \times (\omega_{\mathrm{is}} \times \mathbf{r}_{\mathrm{i}})] \tag{2.4}$$

The quantity in the brackets is the sum of the effects of gravity and the earth's rotation and denoted as $\mathbf{g}$, where

$$\mathbf{g} = \mathbf{g}_{\mathbf{m}} - \omega_{\mathrm{is}} \times (\omega_{\mathrm{is}} \times \mathbf{r}_{\mathrm{i}}) \tag{2.5}$$

Substituting (2. 5) in (2. 4) we get vector equation

$$\dot{\mathbf{v}}\big|_{\mathrm{s}} = \mathbf{f} - 2\omega_{\mathrm{is}} \times \mathbf{v}_{\mathrm{s}} + \mathbf{g} \tag{2 6}$$

Equation (2.6) is vector strapdown navigation equation for 6-DOF motion.

Resolving equation (2.6) in tangential s-frame yields Earth-frame strapdown navigation matrix equations

$$\begin{aligned}\dot{\mathbf{r}}^{\mathrm{s}} &= \mathbf{v}^{\mathrm{s}} \\ \dot{\mathbf{v}}^{\mathrm{s}} &= \mathbf{C}_{\mathrm{b}}^{\mathrm{s}}\mathbf{f}^{\mathrm{b}} - 2\Omega_{\mathrm{is}}^{\mathrm{s}}\mathbf{v}^{\mathrm{s}} + \mathbf{g}^{\mathrm{s}} \\ \dot{\mathbf{C}}_{\mathrm{b}}^{\mathrm{s}} &= \mathbf{C}_{\mathrm{b}}^{\mathrm{s}}(\Omega_{\mathrm{ib}}^{\mathrm{b}} - \Omega_{\mathrm{is}}^{\mathrm{b}})\end{aligned} \tag{2.7}$$

where $\mathbf{r}^{\mathrm{s}}$ is vehicle's position vector in the s-frame, $\mathbf{C}_{\mathrm{b}}^{\mathrm{s}}$ is the coordinate transformation matrix from body-fixed b-frame to s-frame, $\Omega_{\mathrm{ib}}^{\mathrm{b}}$ - skew-symmetric matrix of the angular velocities of the vehicle relative to the inertial i-frame, resolved in b-frame; $\Omega_{\mathrm{is}}^{\mathrm{s}}$ - skew-symmetric matrix of the Earth rotation angular velocity components resolved in s-frame.

We can see that implementation of navigation equations (2.7) would require 3 accelerometers to measure specific force $\mathbf{f}$ and 3 gyroscopes to compute changing direction of specific force, i. e. components of $\Omega_{\mathrm{sb}}^{\mathrm{b}}$.

Let's now define constraints on the motion of the land vehicles:

1) Direction of the vehicle's velocity coincides with direction of the vehicle's longitudinal axis;
2) Pitch and roll angles of the vehicle's body relative to the Earth surface are small;
3) Vehicle always remains on the Earth surface.

In order to derive the relationship between vehicle speed and rotation on the one side and sensor readings, we will use another frame, the velocity v-frame. This frame is defined so that its x-axis direction coincides with vehicle's earth-referenced velocity direction. Therefore, in the v-frame

$$\mathbf{v}^{\mathrm{v}} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \tag{2.8}$$

where $v$ is vehicle speed.

Using relationship for vector derivatives in rotating frames, we have

$$\dot{\mathbf{v}}\big|_{\mathrm{s}} = \dot{\mathbf{v}}\big|_{\mathrm{v}} + \omega_{\mathrm{sv}} \times \mathbf{v} \tag{2.9}$$

Combining (2. 6) and (2. 9) we can get

$$\dot{\mathbf{v}}\,|_{\mathrm{v}} = \mathbf{f} - (\omega_{\mathrm{ib}} - \omega_{\mathrm{vb}} + \omega_{\mathrm{is}}) \times \mathbf{v} + \mathbf{g} \tag{2.10}$$

Earth angular rotation $\omega_{\mathrm{is}}$ is small and its magnitude is at the sensor noise level for low-cost gyros, therefore we shall ignore it.

Resolving (2.10) in velocity frame, we get equation in matrix-vector form

$$(\dot{\mathbf{v}}\,|_{\mathrm{v}})^{\mathrm{v}} = \mathbf{C}_{\mathrm{b}}^{\mathrm{v}}\mathbf{f}^{\mathrm{b}} - \mathbf{C}_{\mathrm{b}}^{\mathrm{v}}(\Omega_{\mathrm{ib}}^{\mathrm{b}} - \Omega_{\mathrm{vb}}^{\mathrm{b}})\mathbf{C}_{\mathrm{v}}^{\mathrm{b}}\mathbf{v}^{\mathrm{v}} + \mathbf{C}_{\mathrm{s}}^{\mathrm{v}}\mathbf{g}^{\mathrm{s}} \tag{2. 11}$$

where $\mathbf{f}^{\mathrm{b}}$ is measured by the accelerometers.

$$\mathbf{f}^{\mathrm{b}} = \begin{bmatrix} \mathrm{f}^{\mathrm{x}} \\ \mathrm{f}^{\mathrm{y}} \\ \mathrm{f}^{\mathrm{z}} \end{bmatrix} \tag{2.12}$$

where $f^x$, $f^y$, and $f^z$ are respective accelerometer measurements.

Elements of skew-symmetric matrix $\Omega_{ib}^{b}$ are measured by gyros

$$\Omega_{ib}^{b} = \begin{bmatrix} 0 & -\omega^z & \omega^y \\ \omega^z & 0 & -\omega^x \\ -\omega^y & \omega^x & 0 \end{bmatrix} \qquad (2.13)$$

where $\omega^x$, $\omega^y$, and $\omega^z$ are respective gyroscope measurements.

Coordinate transform matrices $\mathbf{C}_b^v$ and $\mathbf{C}_s^v$ can be expressed through yaw($\psi$), pitch ($\theta$), and roll($\varphi$) angles of the body b-frame relative to s-frame

$$\mathbf{C}_b^v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \qquad (2.14)$$

$$\mathbf{C}_b^e = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \qquad (2.15)$$

Also, gravity vector resolved in s-frame is

$$\mathbf{g}^s = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \quad (2.16)$$

where g=9. 8027 m/s$^2$ (this value is for State College, Pa).

Substituting (2.8) and (2.12)-(2.16) into (2.11) we can get following relationships

$$\dot{v} = f^x + g\sin\theta \qquad (2.17)$$

$$f^y = v\omega^z + g\sin\varphi\cos\theta \qquad (2.18)$$

From (2.17) follows that just one accelerometer is enough to determine speed $v$ of the vehicle. In addition, second accelerometer may be used in conjunction with equation (2.18) to define constraint on navigation quantities and to reduce system error.

Now, let's get equations for propagation of yaw, pitch, and roll angles. Using last equation in (2.7), ignoring Earth rotation, and considering that

$$\mathbf{C}_b^s = \begin{bmatrix} \cos\theta\cos\psi & -\cos\varphi\sin\psi + \sin\varphi\sin\theta\cos\psi & \sin\varphi\sin\psi + \cos\varphi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\varphi\cos\psi + \sin\varphi\sin\theta\sin\psi & -\sin\varphi\cos\psi + \cos\varphi\sin\theta\sin\psi \\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta \end{bmatrix}$$

we can derive the following relationships

$$\begin{aligned} \dot{\varphi} &= \omega^x + (\sin\varphi\tan\theta)\omega^y + (\cos\varphi\tan\theta)\omega^z \\ \dot{\theta} &= \cos\varphi\,\omega^y - \sin\varphi\,\omega^z \\ \dot{\psi} &= \frac{\sin\varphi}{\cos\theta}\omega^y + \frac{\cos\varphi}{\cos\theta}\omega^z \end{aligned} \qquad (2.9)$$

Finally, speed and attitude angles may be used to compute vehicle's trajectory in the XY plane of the Earth-fixed tangent frame (s-frame)

$$\begin{aligned} \dot{x}_s &= v\cos\theta\cos\psi \\ \dot{y}_s &= v\cos\theta\sin\psi \end{aligned} \qquad (2.20)$$

Equations (2.17), (2.19), and (2.20) represent navigator which utilizes reduced set of inertial instruments. It takes readings from the longitudinal accelerometer $f^x$ and from 3 gyros $\omega^x, \omega^y, \omega^z$ as input and computes speed, roll, pitch angles and position in Earth-fixed tangent frame.

$$\begin{aligned} \dot{v} &= f^x + g\sin\theta \\ \dot{\varphi} &= \omega^x + (\sin\varphi\tan\theta)\omega^y + (\cos\varphi\tan\theta)\omega^z \\ \dot{\theta} &= \cos\varphi\,\omega^y - \sin\varphi\,\omega^z \\ \dot{\psi} &= \frac{\sin\varphi}{\cos\theta}\omega^y + \frac{\cos\varphi}{\cos\theta}\omega^z \\ \dot{x}_s &= v\cos\theta\cos\psi \\ \dot{y}_s &= v\cos\theta\sin\psi \end{aligned} \qquad (2.21)$$

## 3 Error Model for the Reduced Instrument Set Inertial Algorithm

In order to reduce overall system errors, we need to estimate system errors using Kalman filter. Kalman filtering was applied only to first 3 equations in (2.21). It was possible to do because first 3 equations in (2.21) are decoupled form the last 3. Errors arise in the system primarily due to the sensor errors and initialization errors. Additional error source is possible violation of assumption that direction of the vehicle's velocity coincides with direction of the vehicle's longitudinal axis. Using perturbation technique, we can derive the following error model for the navigator with reduced set of instruments

$$\begin{aligned} \delta\dot{v} &= \gamma_1 f^x + g[\cos\theta\delta\theta + \gamma_1\sin\theta] + b_x \\ \delta\dot{\varphi} &= (\sin\varphi\delta\theta + \cos\varphi\tan\theta\delta\varphi)\omega_y \\ &\quad + (\cos\varphi\delta\theta - \sin\varphi\tan\theta\delta\varphi)\omega_z + \\ &\quad d_x + \sin\varphi\tan\theta d_y + \cos\varphi\tan\theta d_z \\ \delta\dot{\theta} &= (-\sin\varphi\delta\varphi)\omega_y - (\cos\varphi\delta\varphi)\omega_z + (\cos\varphi + \sin\varphi\delta\varphi)d_y \\ &\quad + (-\sin\varphi + \cos\varphi\delta\varphi)d_z \end{aligned} \qquad (3.1)$$

where $b_x$ is accelerometer bias, $d_x$, $d_y$, and $d_z$ are gyro drifts, and $\gamma_1$ is parameter that defines noncollinearity between velocity direction and body frame x axis.

Equations (3.1) were used in Kalman filter for error estimation. Velocity from odometer was used as and external measurement.

One issue to be decided in Kalman filter design is the choice between feedback and feedforward mechanization of the filter. In feedback configuration, after each update KF sends estimated errors back to the navigator to correct its equations, and all future navigator outputs are computed with regard to these corrections. In feedforward configuration, navigator works all time independently of KF error estimations, and navigator's outputs are corrected by KF a posteriori.

For the navigator with reduced instrument set a mixed mechanization was chosen, were some KF outputs (particularly, gyros' drifts and accelerometer bias) were fed back to navigator and others were fed forward. It was necessary to feed back gyros' drifts estimates to correct navigators outputs, because, otherwise, roll and pitch angles errors $\delta\varphi$ and $\delta\theta$ grew large, thus violating assumption made in derivation of the error model. Mixed configuration of the KF is shown in the Figure 3.1

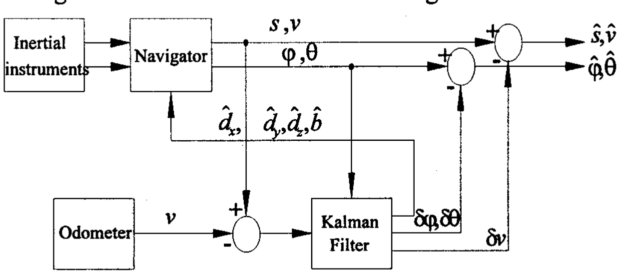


Figure 3. 1 Kalman Filter Configuration.

# 4 Experimental Results

The algorithms for a strapdown inertial navigator with reduced set of inertial instruments developed above were tested in experimental bus navigation system developed in Pennsylvania Transportation Institute. The system was based on portable PC Dolch-488DX2. The instrument set included a MotionPak Inertial Measurement Unit from Systron Donner, 2 inclinometers, and a tachometer. The system was used to evaluate different navigation technologies for bus navigation in urban environment.

Experiments were run on the PTI Bus Test Track. Tags on track pavement located approximately 61 m (200 ft) intervals were used to evaluate the performance of the system. These tags were surveyed, and their precise positions in the Pennsylvania North Plane Coordinate System were known. The bus moved once around the track at average speed of 64 km/hr with 2 brief stops, covering a total distance of 1.6 km.

Figures 4.1 and 4.2 show vehicle trajectory computed by reduced instrument set inertial algorithm and by by Earth-frame full instrument set strapdown algorithm respectively, each complemented with corresponding Kalman Filter. Figures 4.3 and 4.4 show position errors at the tags for both algorithms. We can see that while position error grows fast for the full instrument set strapdown algorithm due to high uncompensated instrument errors and initialization errors, reduced instrument set inertial algorithm yields much better performance, with position error below 10 m.

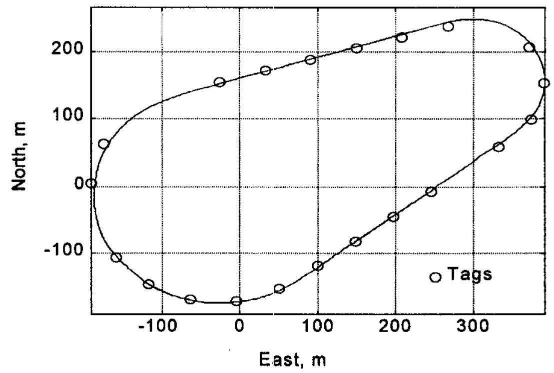


Figure 4.1. Vehicle Trajectory Computed by Reduced Instrument Set Inertial Algorithm.

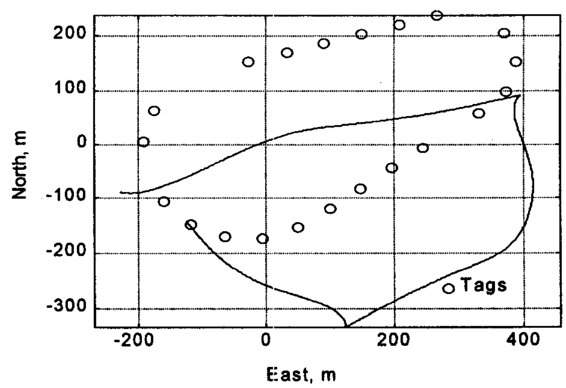


Figure 4. 2. Vehicle Trajectory Computed by Earth-frame Full Instrument Set Strapdown Algorithm.

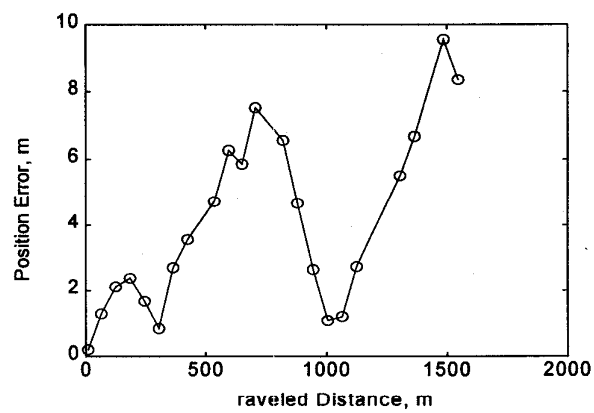


Figure 4. 3. Position Errors at the Tags for Reduced Instrument Set Inertial Algorithm.

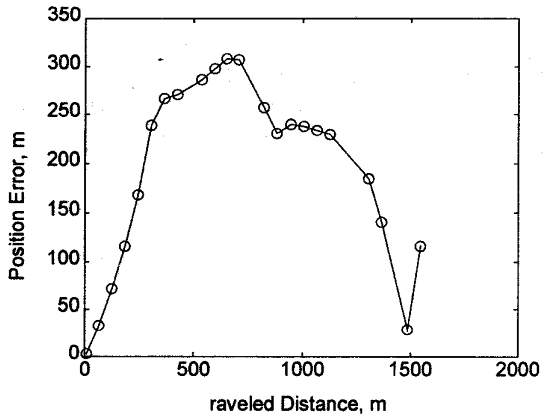


Figure 4. 4. Position Errors at the Tags for Earth-Frame Full Instrument Set Strapdown Algorithm.

## 5 Conclusions

A family of algorithms for low-cost land inertial navigation systems have been developed and experimentally tested. The algorithms utilizes constraints inherent in the motion of land vehicles to reduce number of required inertial instruments and to formulate navigation equations such that they yield lower navigation errors. An error model for this algorithm has been developed and used in a Kalman filter for the system. While the Kalman filter uses velocity updates from an odometer as an external measurement, continuous odometer measurement is not required by the navigation system. This avoids problems typical of dead reckoning navigation systems related to skidding and scaling factors. The system maintains position to within 10 m in 1.6 km. (0.63%). This should be sufficient for navigation of buses in urban conditions, where frequent position updates form bus stops and map matching are readily be available.

## 6 References

Daum P., Beyer J., Kohler T. F. W., Aided Inertial Land Navigation System (ILANA) with a Minimum Set of Inertial Sensors, IEEE-IEE Vehicle Navigation & Information Systems Conference, Yokohama-VNIS '94, pp.284-291.

Harris C.B., Krakiwsky E.J., AVLN 2000 Automatic Vehicle Location and Navigation System, IEEE-IEE Vehicle Navigation & Information Systems Conference, Toronto-VNIS'89, 1989, pp.126-130.

Lezniak T.W., Lewis R.W., McMillen R.A., A Dead Reckoning/Map Correlation System for Automatic Vehicle Tracking, IEEE Transactions on Vehicular Technology, vol.VT-26, no.1, February 1977.

Tsuji H., Maeda H., Shibata A., Morisue F., Evaluation of Location System Combining a GPS Receiver with Inertial Sensor, Proceedings -Society of Automotive Engineers, no. P-253, pt. 2 1991, pp. 645-649.

Vlcek C., McLain P., Murphy M., GPS/Dead Reckoning for Vehicle Tracking in the "Urban Canyon" Environment, IEEE-IEE Vehicle Navigation & Information Systems Conference, Ottawa-VNIS '93, pp.A34-A41.

## Appendix A. Coordinate Frames Used in Inertial Navigation

Several different coordinate frames are defined here.

**Inertial Frame (i-frame)**

Inertial frame is coordinate frame that does not rotate and does not accelerate with respect to inertial space. We will define inertial frame as frame with origin at the mass center of the Earth and axes non-rotating with respect to the distant stars and such that z-axis points towards North celestial pole, x-axis in equatorial plane points toward the intersection point of the local meridian and the equatorial plane at time $t_0$, and y-axis in equatorial plane so that it completes a right-handed coordinate system.

**Earth-fixed Tangent Frame (s-frame)**

Earth-fixed Tangent frame has origin at fixed point on the Earth surface , z-axis pointing up, x-axis pointing to the ellipsoidal east, and y-axis pointing to ellipsoidal north.

**Body frame (b-frame)**

Body frame is rigidly attached to the vehicle. We assume that output axes of strapdown IMU are orthogonal (ignoring possible non-orthogonality of the sensor axes), and align body coordinate frame with IMU axes. Body frame has origin at the center of strapdown IMU, x-axis toward front side of the IMU, z-axis upwards, and y-axis completing right-handed coordinate frame. Body frame is a frame in which the measurements of a strapdown IMU are made.

**Velocity frame (v-frame)**

Velocity coordinate frame has origin at the center of strapdown IMU, x-axis in direction of vehicle's velocity with respect to the tangent frame, y-axis parallel to x-y plane of tangent frame, and z-axis completing right-handed coordinate frame. Assuming that x-axis of velocity frame is almost parallel to the x-y plane of the tangent frame, direction of the y-axis of the velocity frame is chosen so that its z-axis points up.