# Factor-augmented Regression for High Dimensional Time Series

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# High-dimensional Linear Regression

Linear regression model:

$$y_t = \boldsymbol{x}_t^\top \boldsymbol{\beta} + e_t \text{ or } \boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}$$

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Challenges:

- Dependence
- Correlation
- Non-Gaussianity
- High-dimensionarity

# High-dimensional Linear Regression

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Correlation:

- PCA: Bai (2003), Fan et al.(2013)
- Sparse regression model
  LASSO: Tibshirani (1996), Huang et al. (2008)
  SCAD: Fan and Li, 2001, Xie and Huang (2009)
- Dantzig Selector: Candes and Tao (2007), Bickel et al. (2009)
- Latent factor model: Bai (2003), Bai and Ng (2003), Fan et al. (2011)

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Our new model:

$$y_t = \boldsymbol{f}_t^\top \boldsymbol{\gamma}^* + \boldsymbol{u}_t^\top \boldsymbol{\beta}^* + e_t$$

$$\boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{f}_t + \boldsymbol{u}_t$$

# Factor Model

Model: $x_{it} = \boldsymbol{b}_i^\top \boldsymbol{f}_t + u_{it}$ or $\boldsymbol{X} = \boldsymbol{F}\boldsymbol{B}^\top + \boldsymbol{U}$ and its regular estimation

$$(\hat{\boldsymbol{F}}, \hat{\boldsymbol{B}}) = \arg\min \sum_{i=1}^{p} \sum_{t=1}^{T} (x_{it} - \boldsymbol{b}_i^\top \boldsymbol{f}_t)^2 \text{ subject to } T^{-1}\boldsymbol{F}^\top \boldsymbol{F} = \boldsymbol{I}_K, \boldsymbol{B}^\top \boldsymbol{B} \text{ is diagonal}$$

- Factor: $\hat{\boldsymbol{F}} = \sqrt{T}\hat{\Lambda}$, where $\hat{\Lambda}$ is the matrix of the $K$ eigenvectors corresponding to the first $K$ largest eigenvalues of the $T \times T$ matrix $\boldsymbol{X}\boldsymbol{X}^\top$
- Factor Loading: $\hat{\boldsymbol{B}} = \boldsymbol{X}^\top \hat{\boldsymbol{F}}(\hat{\boldsymbol{F}}^\top \hat{\boldsymbol{F}})^{-1} = T^{-1}\boldsymbol{X}^\top \hat{\boldsymbol{F}}$
- Idiosyncratic Error: $\hat{\boldsymbol{U}} = \boldsymbol{X} - \hat{\boldsymbol{F}}\hat{\boldsymbol{B}}^\top = (\boldsymbol{I}_K - T^{-1}\hat{\boldsymbol{F}}\hat{\boldsymbol{F}}^\top)\boldsymbol{X}$

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# A Framework of High-dimensional Temporal Data

Weak stationary and causal processes of the nonlinear form

$$\boxed{X_t = (X_{1t}, \ldots, X_{pt})^\top = G(\mathcal{F}_t)}$$

- Input: $\mathcal{F}_t = (\varepsilon_t, \varepsilon_{t-1}, \ldots)$, where $\varepsilon_t$, $t \in \mathbb{Z}$, are i.i.d innovations.
- Function: $G(\cdot) = (g_1(\cdot), \ldots, g_p(\cdot))^\top$.

# Functional Dependence Measures

- Functional dependence measure

$q \geq 2,\ t \geq 0,\ 1 \leq j \leq p$

$$X_t = G(\varepsilon_t, \ldots, \varepsilon_1, \varepsilon_0, \varepsilon_{-1}, \ldots)$$
$$\Downarrow$$
$$X_t^* = G(\varepsilon_t, \ldots, \varepsilon_1, \varepsilon_0^*, \varepsilon_{-1}, \ldots)$$

$$\begin{aligned}\delta_{t,q,j} &= \|X_{jt} - X_{jt}^*\|_q \\ &= \|g_j(\varepsilon_t, \ldots, \varepsilon_1, \varepsilon_0, \varepsilon_{-1}, \ldots) \\ &\quad - g_j(\varepsilon_t, \ldots, \varepsilon_1, \varepsilon_0^*, \varepsilon_{-1}, \ldots)\|_q\end{aligned}$$

$$\|X_{\cdot j}\|_q = \sup_{m\geq 0} \rho^{-m} \sum_{t=m}^{\infty} \delta_{t,q,j}, \rho \in (0,1)$$

- Dependence-adjusted Orlicz norm

One dimension

$$\|X_\cdot\|_{\psi_\nu} = \sup_{q\geq 2} q^{-\nu} \sum_{t=0}^{\infty} \|X_t - X_t^*\|_q$$

$p$ dimension

$$\|X_\cdot\|_{\psi_\nu} = \sup_{|v|_2\neq 0} \|v^\top X_\cdot\|_{\psi_\nu}/|v|_2$$

# Functional Dependence Measures

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$p$ dimension

$$\|X_\cdot\|_{\psi_\nu} = \sup_{|v|_2 \neq 0} \|v^\top X_\cdot\|_{\psi_\nu} / |v|_2$$

# Convergence Rate: Regular Conditions

- Factors: $(\boldsymbol{f}_t)_{t\geq 1}$ is weakly stationary with mean zero
- Factor Loadings: $\boldsymbol{B}$ satisfies
  - $p/\tau \leq \lambda_{\min}\left(\boldsymbol{B}^\top \boldsymbol{B}\right) \leq \lambda_{\max}\left(\boldsymbol{B}^\top \boldsymbol{B}\right) \leq p\tau$ with $\tau > 1$
  - $|\boldsymbol{B}|_{\max} \leq c$ with $c > 0$
- Time and Cross-Section Dependence and Heteroskedasticity:
  - $(\boldsymbol{u}_t)_{t\geq 1}$ is weakly stationary with mean zero
  - $c_1 < \lambda_{\min}(\boldsymbol{\Sigma}_u)$, $|\boldsymbol{\Sigma}_u|_1 < c_2$ and $\min_{i,j\leq p} \mathrm{Var}\left(u_{it}u_{jt}\right) > c_1$ with $c_1 < c_2$
  - $\mathbb{E}|\boldsymbol{u}_s^\top \boldsymbol{u}_t - E\boldsymbol{u}_s^\top \boldsymbol{u}_t|^4 < cp^2$ with $c > 0$
- Uncorrelation between factors and idiosyncratic errors: $\mathbb{E}u_{it}f_{jt} = 0$
- $\mathbb{E}|\sum_{i=1}^p \boldsymbol{b}_i u_{it}|_2^4 < cp^2$ with $c > 0$

# Convergence Rate: Temporal Dependence

$(\boldsymbol{u}_t, \boldsymbol{f}_t)_{t\geq 1}$ satisfies

- $(\boldsymbol{u}_t, \boldsymbol{f}_t)_{t\geq 1} \in \mathcal{L}^q$ for all $q > 2$
- Let

$$\|\boldsymbol{u}_{\cdot j}\|_{\psi_\mu} := \sup_{q\geq 2} q^{-\mu}\|\boldsymbol{u}_{\cdot j}\|_q, \text{ and } \|\boldsymbol{f}_{\cdot j}\|_{\psi_\nu} := \sup_{q\geq 2} q^{-\nu}\|\boldsymbol{f}_{\cdot j}\|_q$$

and

$$\|\boldsymbol{u}_{\cdot}\|_{\psi_\mu} = \max_{1\leq j\leq p} \|\boldsymbol{u}_{\cdot j}\|_{\psi_\mu} \text{ and } \|\boldsymbol{f}_{\cdot}\|_{\psi_\nu} = \max_{1\leq j\leq p} \|\boldsymbol{f}_{\cdot j}\|_{\psi_\nu}$$

$\|\boldsymbol{u}_{\cdot}\|_{\psi_\mu}, \|\boldsymbol{f}_{\cdot}\|_{\psi_\nu} < \infty$ for $0 < \rho < 1$ and some $\mu, \nu > 0$

# Convergence Rate

Let

$$\boldsymbol{H} = T^{-1}\boldsymbol{V}^{-1}\hat{\boldsymbol{F}}^\top \boldsymbol{F}\boldsymbol{B}^\top \boldsymbol{B},$$

where $\boldsymbol{V} \in \mathbb{R}^{K\times K}$ is a diagonal matrix consisting of the first $K$ largest eigenvalues of the matrix $T^{-1}\boldsymbol{X}\boldsymbol{X}^\top$. Then, we have

- $|\hat{\boldsymbol{F}} - \boldsymbol{F}\boldsymbol{H}^\top|_{\mathbb{F}} = O_{\mathbb{P}}\left(\sqrt{\frac{T}{p}} + \frac{1}{\sqrt{T}}\right)$.
- $|\boldsymbol{H}^\top\boldsymbol{H} - \boldsymbol{I}_K|_{\mathbb{F}} = O_{\mathbb{P}}\left(\sqrt{\frac{1}{T}} + \sqrt{\frac{1}{p}}\right)$.
- $\max_{j\le p} |\hat{\boldsymbol{b}}_j - \boldsymbol{H}\boldsymbol{b}_j|_2 = O_{\mathbb{P}}\left(\sqrt{1/p} + (\log p)^{1/2+\mu+\nu}/\sqrt{T}\right)$.
- For any $\mathcal{I} \subset \{1, 2, \ldots, p\}$, we have

$$\max_{j\in\mathcal{I}} \sum_{t=1}^{T} \left|\hat{u}_{tj} - u_{tj}\right|^2 = O_{\mathbb{P}}\left((\log|\mathcal{I}|)^{1+2\mu+2\nu} + T/p\right).$$

# Lasso Estimation

- Motivation:
  - High dimension setting: sample size $T <$ dimension $p$
  - Assume the true parameter $\boldsymbol{\beta}$ is sparse
- Regularization:

$$(\hat{\boldsymbol{\beta}}_\lambda, \hat{\boldsymbol{\gamma}}) = \underset{\boldsymbol{\beta}\in\mathbb{R}^p, \boldsymbol{\gamma}\in\mathbb{R}^K}{\arg\min} \left\{ \frac{1}{2T} |\boldsymbol{Y} - \hat{\boldsymbol{U}}\boldsymbol{\beta} - \hat{\boldsymbol{F}}\boldsymbol{\gamma}|_2^2 + \lambda |\boldsymbol{\beta}|_1 \right\},$$

  where $\lambda > 0$ controls the strength of regularization.
- Equivalence:

$$\begin{aligned} \tilde{\boldsymbol{Y}} &= (\boldsymbol{I} - \hat{\boldsymbol{P}})\boldsymbol{Y} \\ &= (\boldsymbol{I} - \tfrac{1}{T}\hat{\boldsymbol{F}}\hat{\boldsymbol{F}}^\top)\boldsymbol{Y} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \hat{\boldsymbol{\beta}}_\lambda &= \underset{\boldsymbol{\beta}\in\mathbb{R}^p}{\arg\min} \left\{ \frac{1}{2T} |\tilde{\boldsymbol{Y}} - \hat{\boldsymbol{U}}\boldsymbol{\beta}|_2^2 + \lambda |\boldsymbol{\beta}|_1 \right\} \\ \hat{\boldsymbol{\gamma}} &= (\hat{\boldsymbol{F}}^\top \hat{\boldsymbol{F}})^{-1} \hat{\boldsymbol{F}}^\top \boldsymbol{Y} = \frac{1}{T} \hat{\boldsymbol{F}}^\top \boldsymbol{Y} \end{aligned}$$

# Convergence Rate

$(e_t)_{t\geq 1}$ satisfies

- weakly stationary with mean 0 and uncorrelated with $\boldsymbol{u}_t$ and $\boldsymbol{f}_t$,
- $\|e_.\|_q := \sup_{m\geq 0} \rho^{-m}\Psi_{m,q} < \infty$ with $\Psi_{m,q} = \sum_{i=m}^{\infty} \psi_{i,q}$ for some $0 < \rho < 1$,
- $\|e_.\|_{\psi_\gamma} := \sup_{q\geq 2} q^{-\gamma}\|e_.\|_q < \infty$. for some $\gamma \geq 0$.

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- $\|e_\cdot\|_{\psi_\gamma} := \sup_{q\geq 2} q^{-\gamma} \|e_\cdot\|_q < \infty$. for some $\gamma \geq 0$.

If $\boldsymbol{\varphi}^* = \boldsymbol{\gamma}^* - \boldsymbol{B}^\top \boldsymbol{\beta}^*$, then we have

$$|\hat{\boldsymbol{\gamma}} - \boldsymbol{H}\boldsymbol{\gamma}^*|_2 = O_{\mathbb{P}} \left\{ \frac{1}{\sqrt{T}} + \left( \frac{1}{\sqrt{T}} + \frac{1}{\sqrt{p}} \right) |\boldsymbol{\varphi}^*|_2 + \left( \frac{(\log |\mathcal{S}_\star|)^{\frac{1}{2}+\nu+\mu}}{\sqrt{T}} + \frac{1}{\sqrt{p}} \right) |\boldsymbol{\beta}^*|_1 \right\}$$

where $\mathcal{S}_* = \{j : \beta_j^* \neq 0, 1 \leq j \leq p\}$ and $|\mathcal{S}_*|$ is cardinality of set $\mathcal{S}_*$.

# Convergence Rate

$(e_t)_{t\geq 1}$ satisfies

- weakly stationary with mean 0 and uncorrelated with $\boldsymbol{u}_t$ and $\boldsymbol{f}_t$,
- $\|e_\cdot\|_q := \sup_{m\geq 0} \rho^{-m}\Psi_{m,q} < \infty$ with $\Psi_{m,q} = \sum_{i=m}^{\infty} \psi_{i,q}$ for some $0 < \rho < 1$,
- $\|e_\cdot\|_{\psi_\gamma} := \sup_{q\geq 2} q^{-\gamma}\|e_\cdot\|_q < \infty$. for some $\gamma \geq 0$.

If

$$|\mathcal{S}|\left(\frac{1}{p} + \frac{(\log p)^{1+2\mu+2\nu}}{T}\right) \to 0,$$

then choosing appropriate $\lambda = \frac{c}{T}|\hat{\boldsymbol{U}}^\top(\tilde{\boldsymbol{Y}} - \hat{\boldsymbol{U}}\boldsymbol{\beta}^*)|_\infty$ with some constant $c \geq 2$, we have $\hat{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^* \in \mathcal{C}(\mathcal{S}_*, 3)$ and

$$|\hat{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*|_2 = O_{\mathbb{P}}\left(\sqrt{\frac{|\mathcal{S}_*|(\log p)^{1+2\gamma+2\mu}}{T}} + \frac{\mathcal{V}_{T,p}|\boldsymbol{\varphi}^*|_2\sqrt{|\mathcal{S}_*|}}{T}\right),$$

$$|\hat{\boldsymbol{U}}(\hat{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*)|_2^2 = O_{\mathbb{P}}\left(|\mathcal{S}_*|(\log p)^{1+2\gamma+2\mu} + \frac{\mathcal{V}_{T,p}^2|\boldsymbol{\varphi}^*|_2^2|\mathcal{S}_*|}{T}\right)$$

with

$$\mathcal{V}_{T,p} = \frac{T}{p} + \sqrt{\frac{(\log p)^{1+2\mu+2\nu}}{T}} + \sqrt{\frac{T(\log p)^{1+2\mu+2\max\{\nu,\mu\}}}{p}}.$$

# Debiased Lasso Estimator

Let $\hat{\mathbf{\Theta}}$ be an approximation for the inverse of $\tilde{\mathbf{\Sigma}}_{\boldsymbol{u}} = \frac{1}{T}\hat{\boldsymbol{U}}^\top \hat{\boldsymbol{U}}$, we get the error between de-biased estimator $\tilde{\boldsymbol{\beta}}$ and $\boldsymbol{\beta}^*$ by

$$
\begin{aligned}
\tilde{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^* =& \hat{\boldsymbol{\beta}}_\lambda + \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^\top \left(\boldsymbol{Y} - \hat{\boldsymbol{U}}\hat{\boldsymbol{\beta}}_\lambda\right) - \boldsymbol{\beta}^* \\
=& \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^\top \boldsymbol{e} + \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^\top \boldsymbol{F}\boldsymbol{\varphi}^* + (\boldsymbol{I}_p - \hat{\mathbf{\Theta}}\tilde{\mathbf{\Sigma}}_{\boldsymbol{u}})(\hat{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*).
\end{aligned}
$$

# Debiased Lasso Estimator

Let $\hat{\mathbf{\Theta}}$ be an approximation for the inverse of $\tilde{\mathbf{\Sigma}}_{\boldsymbol{u}} = \frac{1}{T}\hat{\boldsymbol{U}}^\top \hat{\boldsymbol{U}}$, we get the error between de-biased estimator $\tilde{\boldsymbol{\beta}}$ and $\boldsymbol{\beta}^*$ by

$$
\begin{aligned}
\tilde{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^* &= \hat{\boldsymbol{\beta}}_\lambda + \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^\top \left(\boldsymbol{Y} - \hat{\boldsymbol{U}}\hat{\boldsymbol{\beta}}_\lambda\right) - \boldsymbol{\beta}^* \\
&= \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^\top \boldsymbol{e} + \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^\top \boldsymbol{F}\boldsymbol{\varphi}^* + (\boldsymbol{I}_p - \hat{\mathbf{\Theta}}\tilde{\mathbf{\Sigma}}_{\boldsymbol{u}})(\hat{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*).
\end{aligned}
$$

## Gaussian Approximation

Assume that $e_t$ satisfies the regular conditions and temporal dependence, and $\min_{1\le j\le p} \sigma_{jj} \ge b_0$ for some universal constant $b_0 > 0$. Then, we have

$$
d_n := \sup_{x\ge 0}\left|\mathbb{P}\left(\frac{1}{\sqrt{T}}|\mathbf{\Theta}\boldsymbol{U}^\top \boldsymbol{e}|_\infty \le x\right) - \mathbb{P}(|\boldsymbol{Z}|_\infty \le x)\right| \lesssim |\mathbf{\Theta}|_\infty \|\boldsymbol{u}_\cdot\|_{\psi_\mu} \|e_\cdot\|_{\psi_\gamma}\left(\frac{\log^c(p\vee T)}{T}\right)^{\frac{1}{9}},
$$

with $c = \max\{4\mu + 4\gamma + 7, 2\mu + 2\gamma + 10\}$, and the constant in $\lesssim$ depends on $\mu$, $\gamma$, $\rho$ and $b_0$.

# Debiased Lasso Estimator

Let $\hat{\mathbf{\Theta}}$ be an approximation for the inverse of $\tilde{\mathbf{\Sigma}}_{\boldsymbol{u}} = \frac{1}{T}\hat{\boldsymbol{U}}^{\top}\hat{\boldsymbol{U}}$, we get the error between de-biased estimator $\tilde{\boldsymbol{\beta}}$ and $\boldsymbol{\beta}^*$ by

$$
\begin{aligned}
\tilde{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^* &= \hat{\boldsymbol{\beta}}_\lambda + \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^{\top}\left(\boldsymbol{Y} - \hat{\boldsymbol{U}}\hat{\boldsymbol{\beta}}_\lambda\right) - \boldsymbol{\beta}^* \\
&= \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^{\top}\boldsymbol{e} + \frac{1}{T}\hat{\mathbf{\Theta}}\hat{\boldsymbol{U}}^{\top}\boldsymbol{F}\boldsymbol{\varphi}^* + (\boldsymbol{I}_p - \hat{\mathbf{\Theta}}\tilde{\mathbf{\Sigma}}_{\boldsymbol{u}})(\hat{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*).
\end{aligned}
$$

## Gaussian Multiplier

Assume that

$$
(\lambda_{\max}|\mathcal{S}_*| + \Delta_\infty)(\log p)^{1+\mu+\gamma} \to 0
$$

$$
\left(\mathcal{V}_{T,p}|\boldsymbol{\varphi}^*|_2 + \sqrt{\frac{T}{p} + (\log p)^{1+\mu+\gamma}}\right)|\mathbf{\Theta}|_\infty \sqrt{\frac{(\log p)^{1+2\mu+2\gamma}}{T}} \to 0
$$

and $\min_{1\le j\le p}\sigma_{jj} \ge b_0$ for some universal constant $b_0 > 0$. Then, we have

$$
\sup_{x\ge 0}\left|\mathbb{P}\left(\sqrt{T}|\tilde{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*|_\infty \le x\right) - \mathbb{P}(|\boldsymbol{Z}|_\infty \le x)\right| \to 0
$$

# Estimation of Unknown Long-run Covariance Matrices

Covariance matrix of $\boldsymbol{Z}$: $\boldsymbol{\Sigma}_{\boldsymbol{Z}} = \mathrm{Cov}(\boldsymbol{Z}) = \boldsymbol{\Theta}\boldsymbol{\Sigma}_{\boldsymbol{u}e}\boldsymbol{\Theta}^\top$

- Estimate $\boldsymbol{\Theta}$ by node-wise regression in Van de Geer et al. (2014)

$$\hat{\boldsymbol{w}}_j = \underset{\boldsymbol{w}\in\mathbb{R}^{p-1}}{\arg\min}\left\{\frac{1}{2T}\sum_{t=1}^{T}|\hat{u}_{tj} - \boldsymbol{w}^\top\hat{\boldsymbol{u}}_{t,-j}|^2 + \lambda_j|\boldsymbol{w}|_1\right\}$$

  and $\hat{\boldsymbol{\Theta}}$ with $\hat{\Theta}_{jl} = -\hat{w}_{jl}/\hat{v}_j^2$ for all $j \neq l$ and $\hat{\Theta}_{jj} = -1/\hat{v}_j^2$ with

$$\hat{v}_j^2 = \frac{1}{T}\sum_{t=1}^{T}|\hat{u}_{tj} - \hat{\boldsymbol{w}}_j^\top\hat{\boldsymbol{u}}_{t,-j}|^2 + \lambda_j|\hat{\boldsymbol{w}}_j|_1.$$

- Estimate $\boldsymbol{\Sigma}_{\boldsymbol{u}e} := (\sigma_{jk})_{j,k=1}^{p} := \sum_{\ell=-\infty}^{\infty}\Gamma_{\boldsymbol{u}}(\ell)\gamma_e(\ell)$ by

$$\hat{\boldsymbol{\Sigma}}_{\boldsymbol{u}e} = \frac{1}{Mw}\sum_{b=1}^{w}\left(\sum_{t\in L_b}\hat{\boldsymbol{u}}_t\hat{e}_t - M\bar{\boldsymbol{Q}}\right)\left(\sum_{t\in L_b}\hat{\boldsymbol{u}}_t\hat{e}_t - M\bar{\boldsymbol{Q}}\right)^\top$$

  where $\bar{\boldsymbol{Q}} = (Mw)^{-1}\sum_{t=1}^{Mw}\hat{\boldsymbol{u}}_t\hat{e}_t$, the $b$-th window $L_b = \{1+(b-1)M,\ldots,bM\}$, $b = 1,2,\ldots,w$, the window size $|L_b| = M \to \infty$ and the number of blocks $w = \lfloor T/M \rfloor$.

# Convergence of Covariance Estimate

Assume that $e_t$ satisfies the regular conditions and temporal dependence, and $\min_{1\le j\le p}\sigma_{jj} \ge b_0$ for some universal constant $b_0 > 0$. Then, we have

$$\mathbb{P}(T|\hat{\boldsymbol{\Sigma}}_{\boldsymbol{u}e} - \boldsymbol{\Sigma}_{\boldsymbol{u}e}|_\infty \ge x) \lesssim p^2 \exp\left(-\frac{x^c}{4ec(\sqrt{w}\|\boldsymbol{u}_\cdot\|_{\psi_\mu}^2\|e_\cdot\|_{\psi_\gamma}^2)^c}\right),$$

with $c = 1/(1 + 2\mu + 2\gamma)$ and the constants in $\lesssim$ only depend on $\mu$ and $\gamma$.

# Hypothesis test

$$H_0 : \boldsymbol{\beta}^* = 0 \text{ versus } H_1 : \boldsymbol{\beta}^* \neq 0.$$

- Given the level $\alpha \in (0, 1)$, we can reject $H_0$ if $\sqrt{T}|\tilde{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}_0|_\infty \geq \hat{\chi}_{1-\alpha}$
- $(1-\alpha)$th confidence intervals for $\boldsymbol{\beta}^*$ can be constructed as $\tilde{\boldsymbol{\beta}}_\lambda \pm \hat{\chi}_{1-\alpha}/\sqrt{T}$.

## Theorem

Assume $|\boldsymbol{I}_p - \hat{\boldsymbol{\Theta}}\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{u}}|_{\max} = O_{\mathbb{P}}(\Lambda_{\max}), |\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}|_\infty = O_{\mathbb{P}}(\Delta_\infty)$, and $|\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}|_{\max} = O_{\mathbb{P}}(\Delta_{\max})$. Further assume that

$$\Delta_{\max}^2 \frac{(\log p)^{1+2\mu+2\gamma}}{T} + \Delta_\infty^2 + \Delta_\infty|\boldsymbol{\Theta}|_\infty = o\left(\frac{1}{\log p}\right)$$

Then

$$\sup_{x \geq 0} \left|\mathbb{P}(\sqrt{T}|\tilde{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*|_\infty \leq x) - \mathbb{P}^*(|\hat{\boldsymbol{\Sigma}}_{\boldsymbol{Z}}^{1/2}\boldsymbol{\xi}|_\infty \leq x)\right| \xrightarrow{\mathbb{P}} 0$$

# Simulation Study: Setting A

- $\boldsymbol{\gamma}^* = (0.5, 0.5)$, $K = 2$
- $\boldsymbol{\beta}^* = (0.5, 0.5, 0.5, 0, \ldots, 0)$, dimension $p = 100$ and sparsity $s = 3$
- $\boldsymbol{f}_t \sim \mathcal{N}(0, \boldsymbol{I}_K)$
- $\boldsymbol{u}_t \sim \mathcal{N}(0, \boldsymbol{I}_p)$
- $\boldsymbol{B}$ ~Unif$(-1, 1)$
- $e_t = 0.1e_{t-1} + \eta_t$ with innovation $\eta_t$ i.i.d from
  - Gaussian distribution $\mathcal{N}(0, 0.5^2)$, $\gamma = 1/2$
  - double exponential distribution (Laplace$(0, \frac{1}{2\sqrt{2}})$), $\gamma = 1$
- Sample size $T$
  - sub-Gaussian tail: $s\sqrt{(\log p)^{1+2\gamma}/T} \in [0.30, 0.60]$
  - sub-Exponential tail: $s\sqrt{(\log p)^{1+2\gamma}/T} \in [0.50, 0.80]$

# Simulation Study: Result

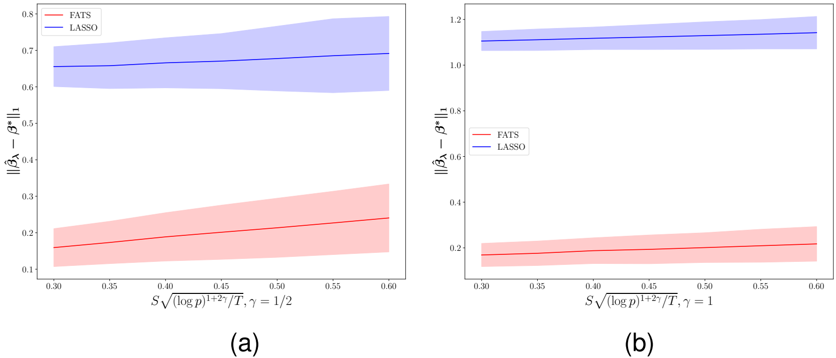



Figure: Accuracy for with $|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*|_1$ based on 500 replications with different innovations in noise $\boldsymbol{e}$

# Simulation Study: Setting B

- $\boldsymbol{\gamma}^* = (0.5, 0.5)$, $K = 2$
- $\boldsymbol{\beta}^* = (0.5, 0.5, 0.5, 0, \ldots, 0)$, dimension $p = 100$ and sparsity $s = 3$
- $t$-th row of $\boldsymbol{F}$ and $\boldsymbol{U}$ satisfy
  - $\boldsymbol{f}_t = \boldsymbol{\Phi} \boldsymbol{f}_{t-1} + \boldsymbol{\epsilon}_t$ with $\boldsymbol{\Phi}_{i,j} = 0.1^{|i-j|+1}$, $1 \le i, j \le K$ and $\boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \boldsymbol{I}_K)$; $\boldsymbol{u}_t \sim \mathcal{N}(0, \boldsymbol{I}_p)$
  - $\boldsymbol{u}_t = \boldsymbol{\Phi} \boldsymbol{u}_{t-1} + \boldsymbol{\varepsilon}_t$ with $\boldsymbol{\Phi}_{i,j} = 0.1^{|i-j|+1}$, $1 \le i, j \le p$ and $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, \boldsymbol{I}_p)$; $\boldsymbol{f}_t \sim \mathcal{N}(0, \boldsymbol{I}_K)$
- $\boldsymbol{e} \sim \mathcal{N}(0, 0.5^2)$
- $\boldsymbol{B}$ ~Unif$(-1, 1)$
- Sample size $T$:
  - $\boldsymbol{F}$ with VAR(1): $s\sqrt{\log p/T} \in [0.30, 0.60]$
  - $\boldsymbol{U}$ with VAR(1): $s\sqrt{(\log p)^{1+2\mu}/T} \in [0.30, 0.60]$

# Simulation Study: Result

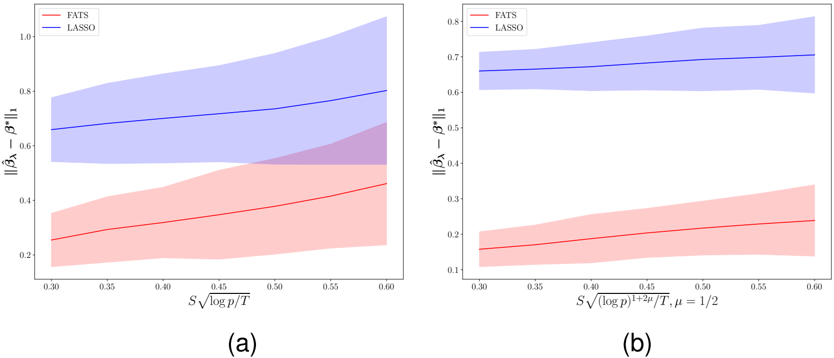



Figure: Accuracy for with $|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*|_1$ based on 500 replications with VAR(1) $\boldsymbol{F}$ and $\boldsymbol{U}$.

# Simulation Study: Gaussian Approximation

Settings:

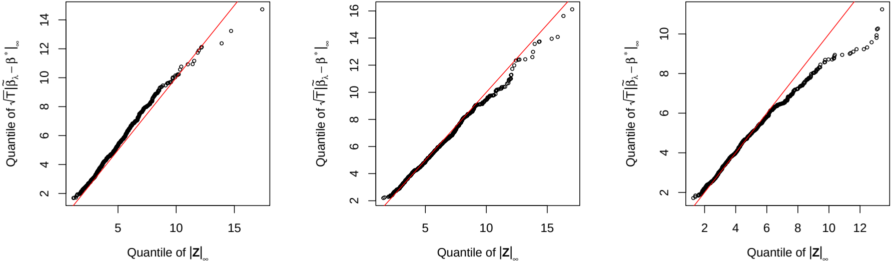
- $\boldsymbol{\gamma}^* = (0.5, 0.5)^\top$, $K = 2$
- $\boldsymbol{\beta}^* = (0.5, 0.5, 0.5, 0, \ldots, 0)^\top$ with dimension $p = 100$ and sparsity $s = 3$
- $\boldsymbol{f}_t \sim \mathcal{N}(0, \boldsymbol{I}_K)$
- $\boldsymbol{u}_t \sim \mathcal{N}(0, \boldsymbol{I}_p)$
- $\boldsymbol{B} \sim$ Unif $(-1, 1)$
- $e_t = \phi e_{t-1} + \eta_t = \sum_{i=0}^{\infty} \phi^i \eta_{t-i}$ with (1) $\phi = 0.1$; (2) $\phi = 0.5$; (3) $\phi = 0.9$
- Sample size $T = 300$

# Simulation Study: Gaussian Approximation

Implementation:

- Estimate factor $\hat{\boldsymbol{F}}$, loading matrix $\hat{\boldsymbol{B}}$, and noise $\hat{\boldsymbol{U}}$.
- For the given $\hat{\boldsymbol{U}}$, we estimate $\hat{\boldsymbol{\Theta}}$ by using node-wise regression.
- Use the Lasso method to estimate $\boldsymbol{\beta}_\lambda$ based on data $\hat{\boldsymbol{U}}$ and $\tilde{\boldsymbol{Y}}$ with $\lambda$ chosen by cross-validation.
- Construct a de-biased estimator $\tilde{\boldsymbol{\beta}}_\lambda = \hat{\boldsymbol{\beta}}_\lambda + \frac{1}{T}\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{U}}^\top\left(\boldsymbol{Y} - \hat{\boldsymbol{U}}\hat{\boldsymbol{\beta}}_\lambda\right)$.
- Construct the batched mean estimate $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{Z}}$ from blocks $\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{U}}^\top\hat{e}$ and window size $w = T/M$ with block size $M = 10$.
- Obtain $\boldsymbol{Z} = \hat{\boldsymbol{\Sigma}}_{\boldsymbol{Z}}^{1/2}\boldsymbol{\xi}$ with every entry of $\boldsymbol{\xi}$ simulated from $\mathcal{N}(0,1)$ and repeat this computation 1000 times to obtain the $|\boldsymbol{Z}|_\infty$.

# Simulation Study: Gaussian Approximation




(a) $\phi = 0.1$ (b) $\phi = 0.5$ (c) $\phi = 0.9$

Figure: The QQ-plot of $\sqrt{T}|\tilde{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}^*|_\infty$ versus $|Z|_\infty$ under different dependence $\phi$

# Real Data Study

- Dataset:
  - **IPNCONGD**: industrial production (IP) index with non-durable consumer goods, measuring the real output of all products with few use times located in the United States
  - **HOUSTW**: new privately-owned housing units in the West Census region including California
- Quarter Data from March 1967 to December 2022
- Transformation code: 1 = no transformation, 2 = first difference $\Delta x_t$, 3 = second difference $\Delta^2 x_t$, 4 = $\log(x_t)$, 5 = first difference of logged variables $\Delta \log(x_t)$, 6 = second difference of logged variables $\Delta^2 \log(x_t)$

# Real Data Study

- Evaluation:
  - Given a window size $w$, we use $w$ observation pairs $\{(\boldsymbol{x}_{t-w}, Y_{t-w}), \ldots, (\boldsymbol{x}_{t-1}, Y_{t-1})\}$
  - Output: a prediction $\widehat{Y}_t$ and in-sample mean $\bar{Y}_t = \frac{1}{w}\sum_{i=t-w}^{t-1} Y_i$
  - Out-of-sample $R^2$ is given by

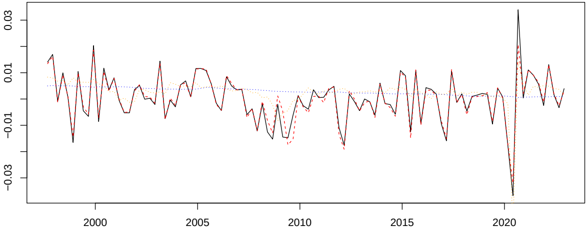
$$R^2 := 1 - \frac{\sum_{t=w+1}^{T}(Y_t - \widehat{Y}_t)^2}{\sum_{t=w+1}^{T}(Y_t - \bar{Y}_t)^2},$$

- Time window $w = 120$ quarterly from March 1967 to December

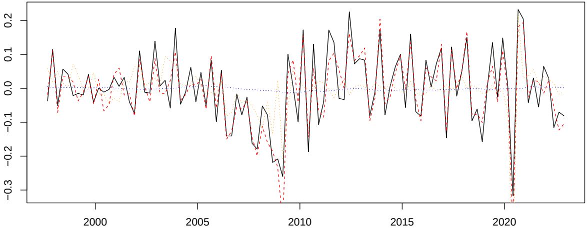
| Data | FATS | PCR |
|---|---|---|
| IPNCONGD | 0.948 | 0.313 |
| HOUSTW | 0.842 | 0.180 |

# Real Data Study: September 1997 to December 2022

- IPNCONG




- HOUSTW



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Thank you for your attention today!