LAW OF LARGE NUMBERS FOR THE LINEAR SELF-INTERACTING DIFFUSION DRIVEN BY α-STABLE MOTION

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ABSTRACT. In this paper, we discuss the linear self-interacting diffusion driven by α-stable motion of the form

$$X_t^\alpha = M_t^\alpha - \theta \int_0^t \int_0^s (X_t^\alpha - X_s^\alpha) ds dt + \nu t, \quad t \geq 0,$$

where $\theta \neq 0$, $\nu \in \mathbb{R}$ and M_t^α is an α-stable motion on $\mathbb{R}(0 < \alpha < 2)$ with Lévy symbol $\varphi(u) = |u|^\alpha$. The process is a continuous analogue of the self-attracting diffusion (see M. Cranston and Y. Le Jan[4]). The main purpose of this paper is to introduce some laws of large numbers associated with the process with $1 < \alpha < 2$. When $\theta > 0$, we show that the convergence

$$\frac{1}{T} \int_0^T Y_t^\alpha dt \longrightarrow \frac{\nu}{\theta}, \quad \frac{1}{T^{4-\frac{2}{\alpha}}} \int_0^T Y_t^\alpha dt \longrightarrow 0$$

hold almost surely and in L^p with $0 < p < \alpha$, as T tends to infinty, where $Y_t^\alpha = \int_0^t (X_t^\alpha - X_s^\alpha) ds$. On the other hand, when $\theta < 0$, we have $\frac{1}{T} \int_0^T Y_t^\alpha dt \longrightarrow \infty$, as T tends to infinity. Therefore, we show that $\Lambda_t^\alpha := e^{\frac{1}{2}\theta t^2} Y_t^\alpha$ and $\Xi_t^\alpha := t e^{\frac{1}{2}\theta t^2} \int_0^t Y_t^\alpha dt$ converges almost surely and in L^p with $0 < p < \alpha$ to $\xi_\infty^\alpha - \frac{\nu}{\theta}$ and $\frac{1}{\theta}\left(\xi_\infty^\alpha - \frac{\nu}{\theta}\right)$, as t tends to infinity, respectively, where $\xi_\infty^\alpha = \int_0^\infty s e^{\frac{1}{2}\theta s^2} dM_s^\alpha$.

1. INTRODUCTION

In 1992, R. Durrett and L.C.G Rogers introduced a stochastic differential model for the shape of a growing polymer[6]. Under some conditions, they established the asymptotic behavior of solution of stochastic differential equation with the dependent path:

$$X_t = B_t + \int_0^t \int_0^s f(X_s - X_u) du ds \tag{1.1}$$

where B_t is a d-dimensional standard Brownian motion and f is Lipschitz continuous. If $f(x) = g(x)x/\|x\|$ and $g(x) \geq 0$, the solution X_t is a continous analogue of a discrete process introduced by Diaconis and studied by Pemantle[15]. The solution process $\{X_t\}$ corresponds to the end postion of the polymer at time t. Let $\mathscr{L}^X(t, x)$ be the local time of the solution process X. Therefore, we have

$$X_t = X_0 + B_t + \int_0^t ds \int_{\mathbb{R}} f(-x) \mathscr{L}^X(s, X_s + x) dx.$$

for all $t \geq 0$. We may call this solution a Brownian motion interacting with its own passed trajectory, i.e., a *self-interacting motion.* In 1995, Cranston and Le Jan[4] discussed self-attracting diffusions, and when $d = 1$ the convergence of the following two cases are studied:

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(i) the linear interaction, i.e.$f(x)=\theta x$, namely,

$$X_t = B_t - \theta\int_0^t\int_0^s (X_s - X_u)duds, \quad t\geq 0,$$

with $\theta>0$;

(ii) the constant interaction, i.e.$f(x)=\sigma\mathbf{sgn}(x)$, where $\sigma>0$, namely,

$$X_t = B_t - \sigma\int_0^t\int_0^s \mathbf{sgn}(X_s - X_u)duds, \quad t\geq 0.$$

In general, the equation (1.1) defines a self-interacting diffusion without any assumption on f. If $x\cdot f(x)\geq 0$(resp. ≤ 0) for all $x\in\mathbb{R}^d$, in other words if it is more likely to stay away from (resp. stay close to) the position it has already reached before, we could describe it self-repelling (resp. self-attracting). In 2002, Benaïm *et al*[2] studied a self-interacting diffusions depending on the (convoluted) empirical measure $\{\mu_t, t\geq 0\}$, as follows

$$dX_t = \sqrt{2}dB_t - \left(\frac{1}{t}\int_0^t \nabla W(X_t - X_s)ds\right)dt,$$

where W is an interacting potential function. A great difference between these diffusions or Brownian polymers is whether the drift term is divided by t. More extended works can be refered in Benaïm *et al.*[2], Chambeu and Kurtzmann[21], Cranston and Mountford[5], Gan and Yan [7], Gauthier[8], Herrmann and Roynette[10], Herrmann and Scheutzow[11], Kleptsyny and Kurtzmann[13], Mountford and Tarr[14], Toth and Werner[22], Sun and Yan *et al.*[23, 20] and other references.

As a natural extension inspired by Sun and Yan[20], one can consider the stochastic differential equation driven by α-stable motion of the form

$$X_t^\alpha = M_t^\alpha + \int_0^t\int_0^s f(X_s^\alpha - X_r^\alpha)drds, \tag{1.2}$$

with $X_0^\alpha=0$, where M_t^α is an α-stable Lévy process on $\mathbb{R}^d$ and f is a Borel measurable function. It is simple to show that the equation mentioned above admits a unique strong solution if f is Lipschitz continuous. On the other hand, for f locally bounded, (1.2) admits a unique weak solution. The solution is called the self-repelling (resp. self-attracting) diffusion driven by an α-stable Lévy process, if $x\cdot f(x)\geq 0$ (resp. ≤ 0) for all $x\in\mathbb{R}^d$.

In this paper, we consider the law of large numbers for the linear case with jumps

$$X_t^\alpha = M_t^\alpha - \theta\int_0^t\int_0^s (X_s^\alpha - X_t^\alpha)drds + \nu t, \tag{1.3}$$

where M_t^α is an α-stable motion (a strictly symmetric α-stable Lévy motion) on $\mathbb{R}(0<\alpha<2)$, $\theta\neq 0$ and $\nu\in\mathbb{R}$. Some related interesting questions are considered in the future papers such as parameter estimations and their asymptotic behaviors.

The structure of this paper includes three sections. In Section 2, we briefly recall α-stable Lévy process and Itô-type stochastic integral with respect to it. In Section 3, we consider the case $\theta>0$. When $1<\alpha<2$ and M^α have no postive jumps, we show that there exists κ depending only on α such that

$$\limsup_{t\to\infty}\frac{1}{(t\log t)^{1-\frac{1}{\alpha}}}\int_0^t (X_t^\alpha - X_s^\alpha)\,ds = \kappa \qquad \text{a.s.}$$

and the convergence (law of large numbers)

$$\frac{1}{T}\int_0^T Y_t^\alpha dt \longrightarrow \frac{\nu}{\theta}$$

almost surely and in L^p with $0<p<\alpha$, as T tends to infinty, where $Y^\alpha_t=\int_0^t(X^\alpha_t-X^\alpha_s)ds$.

In Section 4, we consider the case $\theta<0$. However, when $\theta<0$, we have

$$\frac{1}{T}\int_0^T Y^\alpha_t dt\longrightarrow\infty$$

in probability, as T tends to infinity. Therefore, for $\frac{1}{2}<\alpha<2$, we show that the following convergence hold: When $\theta<0$, we have

$$\Lambda^\alpha_t:=e^{\frac{1}{2}\theta t^2}Y^\alpha_t\longrightarrow\xi^\alpha_\infty-\frac{\nu}{\theta}$$
$$\Xi^\alpha_t:=te^{\frac{1}{2}\theta t^2}\int_0^t Y^\alpha_t dt\longrightarrow\frac{1}{\theta}\left(\xi^\alpha_\infty-\frac{\nu}{\theta}\right)$$

almost surely and in L^p with $0<p<\alpha$, where $\xi^\alpha_\infty=\int_0^\infty se^{\frac{1}{2}\theta s^2}dM^\alpha_s$.

2. Preliminaries

In this section, we will recall the definition and some basic facts of α-stable motion and the definition and properties of Itô-type stochastic integral for α-stable motion on $\mathbb{R}$. Concerning more notions and background, we can refer to Rosinski and Woyczynski[17], Kallenberg[12], Applebaum[1], Samorodnitsky and Taqqu[18], and Sato[19].

2.1. α-stable Lévy process.

Throughout this paper we define a complete probability space $(\Omega,\mathcal{F},P,\{\mathscr{F}_t\})$ such that the processes are well-defined on the space. For simplicity we let $C_{(\cdot)}$ be a postive constant depending only on its subscripts and its value may be different under different conditions, and this assumption is also adaptable to $c_{(\cdot)}$.

Assume four parameters α,λ,β,μ satisfying

$$\alpha\in(0,2],\quad\lambda\in(0,+\infty),\quad\beta\in[-1,1],\quad\mu\in(-\infty,+\infty),$$

and denote

$$\phi_\alpha(u)=\begin{cases}-\lambda^\alpha|u|^\alpha(1-i\beta\mathrm{sgn}(u)\tan\frac{\alpha\pi}{2})+i\mu u, & \alpha\neq1,\\ -\lambda|u|(1+i\beta\frac{2}{\pi}\mathrm{sgn}(u)\log|u|)+i\mu u, & \alpha=1,\end{cases}$$

where $u\in(-\infty,+\infty)$, $i^2=-1$. A random variable η subjects to an α-stable distribution, denoted by $\eta\sim\mathcal{S}_\alpha(\lambda,\beta,\mu)$, if it has the characteristic form

$$E(e^{iu\eta})=e^{-\phi_\alpha(u)},$$

These parameters α,λ,β,μ are called the stability index, scale one, skewness one, and location one, respectively. When $\mu=0$, we may call η to be strictly α-stable, and if $\beta=0$ additionally, we may call η previously to be symmetrically α-stable.

An $\{\mathscr{F}_t\}$-adapted process $M^\alpha=\{M^\alpha_t,t\geq0\}$ with all sample paths in $D[0,\infty)$ could be an α-stable motion (an α-stable Lévy process) with $\alpha\in(0,2]$, if for any $t>s\geq0$, we have

$$E\left[e^{-iu(M^\alpha_t-M^\alpha_s)}|\mathscr{F}_s\right]=e^{-(t-s)\phi_\alpha(u)},\quad u\in\mathbb{R},$$

where $\phi_\alpha(u)$ is called the Lévy symbol of M^α. In this paper, we assume that the α-stable motion M^α is strictly symmetric (namely $\mu=\beta=0$ and $\lambda=1$), i.e. $\phi_\alpha(u)=|u|^\alpha$.

2.2. **Itô-type stochastic integral for α-stable Lévy process.** In this part, we recall the definition and properties of Itô-type stochastic integral for α-stable Lévy process[16]

$$A_t^\alpha = \int_0^t F_s dM_s^\alpha, \quad t \geq 0,$$

where $F = \{F(t,\omega)\}_{t\geq 0}$ is an adapted $\{\mathscr{F}_t\}$ process on $\Omega[0,\infty)$ and M^α is an α-stable motion (strictly symmetric, $\lambda = 1$). If for every $T > 0$, we have

$$F \in L^\alpha_{a.s} = \left\{ F : \{\mathscr{F}_t\} - \text{adapted} | \int_0^T |F_t|^\alpha dt < \infty, a.s. \forall T > 0 \right\},$$

almost surely, the stable stochastic integral exists for $\alpha \in (0,2)$. In general, $F = \{F(t,w)\}_{t\geq 0}$ has the form as follows

$$F(t,\omega) := \varphi_0(\omega) \mathbb{1}_{\{t=0\}}(t) + \sum_{i=0}^{\infty} \varphi_i(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n < \cdots \to \infty$ and φ_n is $\mathscr{F}_{t_n}$ measurable for every $n = 0, 1, 2 \cdots$. The stochastic integral with respect to F could be defined:

$$\begin{aligned}\int_0^t F(s,\omega) dM^\alpha(s,\omega) = &\sum_{i=0}^{n-1} \varphi_i(\omega)(M^\alpha(t_{i+1},\omega) - M^\alpha(t_i,\omega)) \\ &+ \varphi_n(\omega)(M^\alpha(t,\omega) - M^\alpha(t_n,\omega))\end{aligned}$$

where $t_n \leq t \leq t_{n+1}, n = 0, 1, 2 \cdots$. Moreover, the following estimates holds:

$$c_\alpha \int_0^T E[|F_s|^\alpha] ds \leq \sup_{\lambda > 0} \lambda^\alpha P \left(\sup_{0 \leq t \leq T} \left| \int_0^t F_s dM_s^\alpha \right| \geq \lambda \right) \leq C_\alpha \int_0^T E[|F_s|^\alpha] ds \tag{2.1}$$

for all $T \geq 0$. The left-hand side of the inequalities is established in Rosinski-Woyczynski[17] and the right-hand side of the ones is considered in Giné-Marcus[9]

Theorem 2.1 (Rosinski-Woyczynski[17])**.** *Let $0 < \alpha < 2$ and for any $F \in L^\alpha_{a.s}$, we define*

$$\tau_\alpha(u) = \int_0^u |F_t|^\alpha dt.$$

such that $\tau_\alpha(u)$ tends to infinity almost surely, as u tends to infinity. Denote $\tau_\alpha^{-1}(t) = \inf\{u : \tau_\alpha(u) > t\}$ and $\mathscr{A}_t = \mathscr{F}_{\tau_\alpha^{-1}(t)}$. The time-changed stochastic integral

$$\widetilde{M}_t^\alpha = \int_0^{\tau_\alpha^{-1}(t)} F_s dM_s^\alpha$$

is an $\{\mathscr{F}_t\}$-α stable motion on $\mathbb{R}$. Consequently, for each $t > 0$, we have

$$\widetilde{M}^\alpha_{\tau_\alpha(t)} = \int_0^t F_s dM_s^\alpha$$

almost surely.

3. Law of large numbers, self-attracting case

Throughout this paper, we assume that M^α is a strictly symmetric α-stable motion with $\alpha \in (0,2)$, $\theta \neq 0, \nu \in \mathbb{R}$. In this section, we consider the convergence of the solution of the equation

$$X_t^\alpha = M_t^\alpha - \theta \int_0^t \int_0^s (X_s^\alpha - X_r^\alpha) dr ds + \nu t, \tag{3.1}$$

where $\theta > 0, \nu \in \mathbb{R}$ and M^α is an α-stable motion on $\mathbb{R}$ $(0 < \alpha < 2)$ with $M_0^\alpha = 0$.

We introduce the kernel function $(t,s)\mapsto h_\theta(t,s)$ by

$$h_\theta(t,s)=\begin{cases}1-\theta s e^{\frac{1}{2}\theta s^2}\int_s^t e^{-\frac{1}{2}\theta u^2}du, & t\geq s,\\ 0, & t<s,\end{cases}\tag{3.2}$$

for any $s,t\geq 0$. The kernel function $(t,s)\mapsto h_\theta(t,s)$ has the following properties

(1)When $\theta>0$, the limit

$$h_\theta(s):=\lim_{t\to\infty}h_\theta(t,s)=1-\theta s e^{\frac{1}{2}\theta s^2}\int_s^\infty e^{-\frac{1}{2}\theta u^2}du$$

exists for all $s\geq 0$; when $\theta<0$, we have the limit

$$\lim_{t\to\infty}\left(te^{\frac{1}{2}\theta t^2}h_\theta(t,s)\right)=se^{\frac{1}{2}\theta s^2}\tag{3.3}$$

for any $s\geq 0$.

(2) When $\theta>0$, we have $h_\theta(s)\leq h_\theta(t,s)$ and

$$e^{-\frac{1}{2}\theta(t^2-s^2)}\leq h_\theta(t,s)\leq 1\tag{3.4}$$

for all $t\geq s\geq 0$. Moreover, we also have

$$0\leq h_\theta(s)\leq C_\theta\min\left\{1,\frac{1}{s^2}\right\}$$

for all $s\geq 0$.

(3) When $\theta<0$, we have

$$1\leq h_\theta(t,s)\leq e^{-\frac{1}{2}\theta(t^2-s^2)}$$

for all $t\geq s\geq 0$.

(4) When $\theta\neq 0$, we have

$$h_\theta(t,0)=h_\theta(t,t)=1,\qquad \int_u^t h_\theta(t,s)ds=e^{\frac{1}{2}\theta u^2}\int_u^t e^{-\frac{1}{2}\theta s^2}ds.$$

for all $t\geq u\geq 0$.

Using the kernel function $(t,s)\mapsto h_\theta(t,s)$, we can simply show that the solution of (3.1) admits the following representation form when $t\geq 0$:

$$X_t^\alpha=\int_0^t h_\theta(t,s)dM_s^\alpha+\nu\int_0^t h_\theta(t,s)ds.\tag{3.5}$$

By the method of constant variation (see Cranston and Le Jan[4]), we could introduce the above representation. Spontaneously, we could also prove this through the integration by part. Therefore, we only need to check that the process X^α defined by (3.5) is the solution of (3.1), because it is obvious to show that the solution (3.1) is unique. Noting that

$$\begin{aligned}X_t^\alpha&=\int_0^t h_\theta(t,s)dM_s^\alpha+\nu\int_0^t h_\theta(t,s)ds\\&=\int_0^t\left(1-\theta s e^{\frac{1}{2}\theta s^2}\int_s^t e^{-\frac{1}{2}\theta u^2}du\right)dM_s^\alpha+\nu\int_0^t e^{-\frac{1}{2}\theta u^2}du\\&=M_t^\alpha-\theta\int_0^t se^{\frac{1}{2}\theta s^2}dM_s^\alpha\int_s^t e^{-\frac{1}{2}\theta u^2}du+\nu\int_0^t h_\theta(t,s)ds\\&=M_t^\alpha-\int_0^t e^{-\frac{1}{2}\theta u^2}\left(\int_0^u\theta se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right)du+\nu\int_0^t e^{-\frac{1}{2}\theta u^2}du\\&=M_t^\alpha-\int_0^t e^{-\frac{1}{2}\theta u^2}\left(\int_0^u\theta se^{\frac{1}{2}\theta s^2}dM_s^\alpha-\nu\right)du\end{aligned}$$

for all $t \geq 0$, we see that the process X_α defined by (3.5) is the solution of (3.1) if and only if for all $t \geq u \geq 0$, we have

$$Y_u^\alpha := \int_0^u (X_u^\alpha - X_s^\alpha)ds = e^{-\frac{1}{2}\theta u^2}\left(\int_0^u se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right) + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta u^2}\right) \tag{3.6}$$

In fact, through the integration by part

$$\int_0^t sdM_s^\alpha = tM_t^\alpha - \int_0^t M_s^\alpha ds,$$

for any $t \geq 0$ and $\alpha \in (0, 2]$, we have that

$$\begin{aligned}
Y_u &= \int_0^u (X_u^\alpha - X_s^\alpha)ds = uX_u^\alpha - \int_0^u X_s^\alpha ds \\
&= uX_u^\alpha - \int_0^u \left(\int_0^s h_\theta(s,r)dM_r^\alpha + \nu\int_0^s h_\theta(s,r)dr\right)ds \\
&= uX_u^\alpha - \int_0^u \left(M_s^\alpha - \int_0^s \theta re^{\frac{1}{2}\theta r^2}\left(\int_r^s e^{-\frac{1}{2}\theta x^2}dx\right)dM_r^\alpha + \nu\int_0^s e^{-\frac{1}{2}\theta x^2}dx\right)ds \\
&= uX_u^\alpha - \int_0^u M_s^\alpha ds \\
&\quad + \int_0^u ds\int_0^s \theta re^{\frac{1}{2}\theta r^2}\left(\int_r^s e^{-\frac{1}{2}\theta x^2}dx\right)dM_r^\alpha - \nu\int_0^u ds\int_0^s e^{-\frac{1}{2}\theta x^2}dx \\
&= uX_u^\alpha - uM_u^\alpha + \int_0^u sdM_s^\alpha \\
&\quad + \int_0^u e^{-\frac{1}{2}\theta x^2}\left(\int_0^x \theta re^{\frac{1}{2}\theta r^2}dM_r^\alpha\right)(u-x)dx - \nu\int_0^u e^{-\frac{1}{2}\theta x^2}(u-x)dx \\
&= \int_0^u sdM_s^\alpha + u\left[X_u^\alpha - M_u^\alpha + \int_0^u e^{-\frac{1}{2}\theta x^2}\left(\int_0^x \theta re^{\frac{1}{2}\theta r^2}dM_r^\alpha\right)dx - \nu\int_0^u e^{-\frac{1}{2}\theta x^2}dx\right] \\
&\quad - \int_0^u xe^{-\frac{1}{2}\theta x^2}\left(\int_0^x \theta re^{\frac{1}{2}\theta r^2}dM_r^\alpha\right)dx + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta u^2}\right) \\
&= \int_0^u sdM_s^\alpha - \int_0^u xe^{-\frac{1}{2}\theta x^2}\left(\int_0^x \theta re^{\frac{1}{2}\theta r^2}dM_r^\alpha\right)dx + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta u^2}\right) \\
&= \int_0^u sdM_s^\alpha - \int_0^u re^{\frac{1}{2}\theta r^2}dM_r^\alpha\int_r^u \theta xe^{-\frac{1}{2}\theta x^2}dx + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta u^2}\right) \\
&= e^{-\frac{1}{2}\theta u^2}\left(\int_0^u re^{\frac{1}{2}\theta r^2}dM_r^\alpha\right) + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta u^2}\right)
\end{aligned}$$

For all $t \geq u \geq 0$, the formula (3.6) needs to be checked as follows.

Lemma 3.1 ((X. Sun and L. Yan[20])). *Let $F = \{F_s, s \geq 0\}$ be a continuous adapted process belonging to L^α. Define the process A by*

$$A_t = \int_{1/t}^\infty |F_s|^\alpha ds$$

with $A_0 = 0$. Let $\mathscr{F}_t' = \sigma(M_s^\alpha, s \geq \frac{1}{t})$, where$\mathscr{F}_0' = \{\emptyset, \Omega\}$. If $A_t^{-1} = \inf\{u : A_u > t\}$ and $\mathscr{A}_t = \mathscr{F}'_{\mathscr{A}_t^{-1}}$, the time-changed stochastic integral

$$\widetilde{M}_t^\alpha = \int_{1/A_t^{-1}}^\infty F_s dM_s^\alpha$$

is an $\{\mathscr{A}_t\}$-α-stable motion. Therefore, for each $t>0$, we have

$$N_t := \int_{1/t}^{\infty} F_s dM_s^\alpha = \widetilde{M}^\alpha_{A_t}, \quad t>0$$

almost surely, with $N_0 = 0$.

Proof. Similar to Rosinski and Woyczynski[17], one can prove the lemma. □

As a corollary of Lemma 3.1, we have

$$c_\alpha \int_T^\infty E[|F_s|^\alpha] ds \le \sup_{\lambda>0} \lambda^\alpha P\left(\sup_{T\le t\le\infty}\left|\int_0^t F_s dM_s^\alpha\right| \ge \lambda\right) \le C_\alpha \int_T^\infty E[|F_s|^\alpha] ds \tag{3.7}$$

for all $T>0$ and $0<\alpha<2$.

Lemma 3.2. *Let $\alpha>1$ and α-stable motion M^α have no postive jumps. Therefore, there exists $\kappa>0$ depending only on α and θ, such that*

$$\limsup_{t\to\infty} \frac{1}{(t\log t)^{1-\frac{1}{\alpha}}} \int_0^t (X_t^\alpha - X_s^\alpha)\, ds = \kappa \tag{3.8}$$

almost surely.

Proof. Denote $Y_t^\alpha := \int_0^t (X_t^\alpha - X_s^\alpha)\, ds$. We can get the equivalent decomposition of Y_t^α by the formula (3.6):

$$Y_t^\alpha = e^{-\frac{1}{2}\theta t^2}\left(\int_0^t s e^{\frac{1}{2}\theta s^2} dM_s^\alpha\right) + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta t^2}\right) = \Lambda_1 + \Lambda_2.$$

The lemma can be proven in two steps. **(I)** Let us study the convergence in probability of Λ_1. When $t>0$, we could denote

$$\tau_\alpha(t) := \int_0^t s^\alpha e^{\frac{1}{2}\theta\alpha s^2} ds.$$

Theorem 2.1 admits that the α-stable motion

$$\widetilde{M}_t^\alpha := \int_0^{\tau_\alpha^{-1}(t)} s e^{\frac{1}{2}\theta s^2} dM_s^\alpha, \quad t\ge 0$$

has no postive jumps. Therefore, we have

$$\widetilde{M}^\alpha_{\tau_\alpha(t)} := \int_0^t s e^{\frac{1}{2}\theta s^2} dM_s^\alpha, \quad t\ge 0.$$

almost surely, for each $t>0$. Applying the law of iterated logarithm[3]: there exists some positive constant $c>0$ such that

$$\limsup_{t\to\infty} \frac{M_t^\alpha}{t^{\frac{1}{\alpha}}(\log\log t)^{1-\frac{1}{\alpha}}} = c \tag{3.9}$$

as t tends to infinty, we can obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{(t\log t)^{1-\frac{1}{\alpha}}}\Lambda_1=\limsup_{t\to\infty}\frac{e^{-\frac{1}{2}\theta t^2}}{(t\log t)^{1-\frac{1}{\alpha}}}\left(\int_0^t se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right)\\
&=c\limsup_{t\to\infty}\frac{\tau_\alpha(t)^{\frac{1}{\alpha}}(\log\log\tau_\alpha(t))^{1-\frac{1}{\alpha}}}{(t\log t)^{1-\frac{1}{\alpha}}e^{\frac{1}{2}\theta t^2}}\\
&=c\limsup_{t\to\infty}\left(\frac{\tau_\alpha(t)}{t^{\alpha-1}e^{\frac{1}{2}\alpha\theta t^2}}\right)^{\frac{1}{\alpha}}\limsup_{t\to\infty}\left(\frac{\log\log\tau_\alpha(t)}{\log t}\right)^{1-\frac{1}{\alpha}}\\
&\overset{\text{L'Hôpital's rule}}{=\!=\!=\!=}c\limsup_{t\to\infty}\left(\frac{t^\alpha e^{\frac{1}{2}\theta\alpha t^2}}{\alpha\theta t^\alpha e^{\frac{1}{2}\theta\alpha t^2}}\right)^{\frac{1}{\alpha}}\limsup_{t\to\infty}\left(\frac{t^{\alpha+1}e^{\frac{1}{2}\theta\alpha t^2}}{\log\tau_\alpha(t)\tau_\alpha(t)}\right)^{1-\frac{1}{\alpha}}\\
&=c\left(\frac{1}{\alpha\theta}\right)^{\frac{1}{\alpha}}\limsup_{t\to\infty}\left(\frac{\alpha\theta t^2}{1+\log\tau_\alpha(t)}\right)^{1-\frac{1}{\alpha}}\\
&=c\left(\frac{1}{\alpha\theta}\right)^{\frac{1}{\alpha}}2^{1-\frac{1}{\alpha}},
\end{aligned}$$

(II) When $t>0$, we have the following limit:

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{(t\log t)^{1-\frac{1}{\alpha}}}\Lambda_2=\limsup_{t\to\infty}\frac{\frac{\nu}{\theta}}{(t\log t)^{1-\frac{1}{\alpha}}}\left(1-e^{-\frac{1}{2}\theta t^2}\right)\\
&=-\frac{\nu}{\theta}\limsup_{t\to\infty}\frac{1}{e^{\frac{1}{2}\theta t^2}(t\log t)^{1-\frac{1}{\alpha}}}=0.
\end{aligned}$$

Thus, we have proved the Lemma by the convergence mentioned above. $\square$

Lemma 3.3. *Let $1<\alpha<2$, there exists $\beta>\frac{1}{\alpha}$ (in particular, when $\alpha=2$, we can denote $\beta=1$) such that*

$$\frac{M_T^\alpha}{T^\beta}\longrightarrow 0 \qquad a.s. \tag{3.10}$$

as t tends to infinity.

Proof. This is a simple exercise. Applying the estimate (2.1) and the fact $M_t^\alpha:=\int_0^t 1dM_s^\alpha$, we can obtain that for any $\varepsilon>0$,

$$\begin{aligned}
P\left(\sup_{t\geq T}\left|\frac{M_t^\alpha}{t^\beta}\right|\geq\varepsilon\right)&\leq P\left(\frac{1}{T^\beta}\sup_{t\geq T}\left|\int_0^t 1dM_s^\alpha\right|\geq\varepsilon\right)\leq C_\alpha\varepsilon^{-\alpha}\left(\frac{1}{T^\beta}\right)^\alpha\left(\int_0^T 1^\alpha ds\right)\\
&\sim C_\alpha\varepsilon^{-\alpha}\frac{1}{T^{\alpha\beta-1}}\longrightarrow 0\quad(T\to\infty)
\end{aligned}$$

as t tends to infinity. $\square$

Lemma 3.4. *Let $1<\alpha<2$, $\theta>0$ and α-stable motion M^α have no postive jumps. When $t>0$, we have*

$$\frac{1}{t^{3-\frac{2}{\alpha}}}Y_t^\alpha=\frac{1}{t^{3-\frac{2}{\alpha}}}\int_0^t(X_u-X_t)du\longrightarrow 0 \tag{3.11}$$

almost surely, as t tends to infinity.

Proof. For all $t\geq 0$, denoting $\Delta_t=\frac{\nu}{\theta}\left(1-e^{-\frac{1}{2}\theta t^2}\right)$ and

$$\eta_t=e^{-\frac{1}{2}\theta t^2}\int_0^t se^{\frac{1}{2}\theta s^2}dM_s^\alpha,$$

we have $Y_t^\alpha = \eta_t + \Delta_t$ and

$$\frac{1}{t^{3-\frac{2}{\alpha}}} Y_t^\alpha = \frac{1}{t^{3-\frac{2}{\alpha}}}(\eta_t + \Delta_t) = \frac{1}{t^{3-\frac{2}{\alpha}}}\eta_t + \frac{1}{t^{3-\frac{2}{\alpha}}}\Delta_t := A_{1t} + A_{2t}.$$

For one thing, applying the L'Hôpital rule and the estimate (2.1), we can show that A_{1t} converges to zero almost surely, as t tends to infinity. For any $\varepsilon > 0$, we have

$$\begin{aligned}
P\left(\sup_{t\ge T}|A_{1t}| \ge \varepsilon\right) &= P\left(\sup_{t\ge T}\left|\frac{1}{t^{3-\frac{2}{\alpha}}}e^{-\frac{1}{2}\theta t^2}\int_0^t s e^{\frac{1}{2}\theta s^2} dM_s^\alpha\right| \ge \varepsilon\right) \\
&\le P\left(\frac{1}{T^{3-\frac{2}{\alpha}}}e^{-\frac{1}{2}\theta T^2}\sup_{t\ge T}\left|\int_0^t s e^{\frac{1}{2}\theta s^2} dM_s^\alpha\right| \ge \varepsilon\right) \\
&\le C_\alpha \varepsilon^{-\alpha}\left(\frac{1}{T^{3-\frac{2}{\alpha}}e^{\frac{1}{2}\theta T^2}}\right)^\alpha \left(\int_0^T s^\alpha e^{\frac{1}{2}\alpha\theta s^2} ds\right) \\
&\sim C_\alpha \varepsilon^{-\alpha}\frac{1}{\alpha\theta T^{3\alpha-1}e^{\frac{1}{2}\alpha\theta T^2}}\left(T^\alpha e^{\frac{1}{2}\alpha\theta T^2}\right) \\
&= C_\alpha \varepsilon^{-\alpha}\frac{1}{\alpha\theta T^{2\alpha-1}} \longrightarrow 0 \quad (T\to\infty)
\end{aligned}$$

For another thing, applying the estimate of Δ_t in Lemma 3.2, we can have

$$\lim_{t\to\infty} A_{2t} = \lim_{t\to\infty}\frac{1}{t^{3-\frac{2}{\alpha}}}\Delta_t = \lim_{t\to\infty}\frac{1}{t^{3-\frac{2}{\alpha}}}\frac{\nu}{\theta}\left(1-e^{-\frac{1}{2}\theta t^2}\right) = 0$$

as t tends to infinity.

Thus, we have proved the Lemma by the almost surely convergence of A_{1t} and A_{2t}. □

Theorem 3.1. *Let $1 < \alpha < 2$ and $\theta > 0$. When $T > 0$, we have*

$$\frac{1}{T}\int_0^T Y_t^\alpha dt \longrightarrow \frac{\nu}{\theta} \tag{3.12}$$

almost surely and in $L^p (0 < p < \alpha)$, as T tends to infinity.

Proof. By the solution (3.5) and (1.3), for any $T \ge 0$, we have

$$\begin{aligned}
\left|\frac{1}{T}\int_0^T Y_t^\alpha dt - \frac{\nu}{\theta}\right| &= \frac{1}{\theta}\left|\frac{M_T^\alpha}{T} - \frac{X_T^\alpha}{T}\right| \\
&= \frac{1}{\theta}\left|\frac{M_T^\alpha}{T} - \frac{1}{T}\left(\int_0^T h_\theta(T,t)dM_t^\alpha + \nu\int_0^T h_\theta(T,t)dt\right)\right| \\
&\le \frac{1}{\theta}\left|\frac{M_T^\alpha}{T}\right| + \frac{1}{\theta}\left|\frac{1}{T}\int_0^T h_\theta(T,t)dM_t^\alpha\right| + \frac{\nu}{\theta}\left|\frac{1}{T}\int_0^T h_\theta(T,t)dt\right| \\
&:= B_{1T} + B_{2T} + B_{3T}
\end{aligned}$$

where it is obvious to show that $B_{1T} = \frac{1}{\theta}\left|\frac{M_T^\alpha}{T}\right| \longrightarrow 0 (T \to \infty)$ almost surely by the Lemma 3.3.

For one thing, for any $\varepsilon > 0$, we can have

$$\begin{aligned}
&P\left(\sup_{t\ge T}\left|\frac{1}{t}\int_0^t h_\theta(t,s)dM_s^\alpha\right| \ge \varepsilon\right) \le P\left(\frac{1}{T}\sup_{t\ge T}\left|\int_0^t h_\theta(t,s)dM_s^\alpha\right| \ge \varepsilon\right) \\
&\le C_\alpha\varepsilon^{-\alpha}\left(\frac{1}{T}\right)^\alpha\left(\int_0^T h_\theta(t,s)^\alpha ds\right) \le C_\alpha\varepsilon^{-\alpha}\left(\frac{1}{T}\right)^\alpha\left(\int_0^T 1^\alpha ds\right) \\
&= C_\alpha\varepsilon^{-\alpha}\frac{1}{T^{\alpha-1}} \longrightarrow 0 \qquad (T\to\infty)
\end{aligned}$$

Thus, we have shown that B_{2T} convergences to zero almost surely, as T tends to infinity. For another thing, we have

$$B_{3T} = \frac{\nu}{\theta}\left|\frac{1}{T}\int_0^T h_\theta(T,t)dt\right| = \frac{\nu}{\theta}\left|\frac{1}{T}\int_0^T e^{-\frac{1}{2}\theta t^2}dt\right| \sim \frac{\nu}{\theta}e^{-\frac{1}{2}\theta T^2} \longrightarrow 0$$

Finally, we can get the convergence (3.12) in probability. Now, we consider the convergence in $L^p(0<p<\alpha)$. It is clear to see that for all $o<p<\alpha$, we have

$$\begin{aligned}E|B_{1t}|^p = E|\frac{1}{\theta t}M_t^\alpha|^p &\sim \frac{1}{\theta^p t^p}E\left|\int_0^t 1dM_s^\alpha\right|^p \\ &\leq C_{p,\alpha}\frac{1}{\theta^p t^p}\left(\int_0^t 1^\alpha ds\right)^{\frac{p}{\alpha}} = C_{p,\alpha}\frac{1}{\theta^p t^{p(1-\frac{1}{\alpha})}} \longrightarrow 0 \quad (t\to\infty).\end{aligned}$$

Similarly, we have that

$$\begin{aligned}E|B_{2t}|^p = \frac{1}{\theta^p}E\left|\frac{1}{t}\int_0^t h_\theta(t,s)dM_s^\alpha\right|^p &\sim \frac{1}{\theta^p t^p}E\left(\int_0^t h_\theta(t,s)dM_s^\alpha\right)^p \\ &\leq \frac{1}{\theta^p t^p}\left(\int_0^t h_\theta^\alpha(t,s)ds\right)^{\frac{p}{\alpha}} \leq \frac{1}{\theta^p t^{(1-\frac{1}{\alpha})p}} \longrightarrow 0(t\to\infty)\end{aligned}$$

converges in $L^p(0<p<\alpha)$, and we can get the Theorem 3.1 by the facts above finally. □

Corollary 3.1. *Let $1<\alpha<2$ and $\theta>0$, we can have*

$$\frac{1}{T^{4-\frac{2}{\alpha}}}\int_0^T Y_t^\alpha dt \longrightarrow 0 \quad (T\to\infty) \tag{3.13}$$

almost surely and in $L^p(0<p<\alpha)$, as T tends to infinity.

Proof. This is a simple exercise. It is obvious to show the equation (3.13) in the corollary by applying the Lemma 3.4 and Theorem 3.1. □

4. Law of large numbers, self-repelling case

When $\theta<0$ and $\alpha>0$, we have

$$\frac{1}{T}\int_0^T Y_t^\alpha dt \longrightarrow \infty$$

as T tends to infinity, namely, the original law of large numbers is not suitable for this situation. Therefore, in this section, when we discuss the convergence in probability and in $L^p(0<p<\alpha)$ of Y_t^α and $\int_0^t Y_t^\alpha dt$ under the self-repelling case, we need to change the original convergence. Let X^α be the solution of (1.3) with $\theta<0$. We have

$$X_t^\alpha = \int_0^t h_\theta(t,s)dM_s^\alpha + \nu\int_0^t h_\theta(t,s)ds,$$

where the kernel function

$$h_\theta(t,s) := 1-\theta s e^{\frac{1}{2}\theta s^2}\int_s^t e^{-\frac{1}{2}\theta u^2}du.$$

Lemma 4.1. *Let $\theta<0$ and $0<\alpha\leq 2$. Define the process*

$$\xi_t^\alpha := \int_0^t s e^{\frac{1}{2}\theta s^2}dM_s^\alpha, \quad t\geq 0.$$

The process ξ_t^α converges to ξ_∞^α almost surely and in $L^p(0<p<\alpha)$, as t tends to infinity, where the random variable $\xi_\infty^\alpha := \int_0^\infty s e^{\frac{1}{2}\theta s^2}dM_s^\alpha$ is well-defined in $L^p(0<p<\alpha)$.

Proof. Let $\theta<0$ and $0<\alpha\leq 2$. At first, we consider the almost surely convergence as t tends to infinity. It is obvious that we have

$$\begin{aligned}\xi_t^\alpha-\xi_\infty^\alpha=\int_\infty^t se^{\frac{1}{2}\theta s^2}dM_s^\alpha &= te^{\frac{1}{2}\theta t^2}M_t^\alpha+\int_t^\infty(1+\theta s^2)e^{\frac{1}{2}\theta s^2}M_s^\alpha ds\\ &:=\zeta_1(t)+\zeta_2(t)\end{aligned}$$

where based on the fact $M_t^\alpha\sim C_\alpha t^{\frac{1}{\alpha}}(t\to\infty)$, we have

$$\zeta_1(t)=te^{\frac{1}{2}\theta t^2}M_t^\alpha=\frac{M_t^\alpha}{\frac{1}{t}e^{-\frac{1}{2}\theta t^2}}\sim C_\alpha\frac{t^{\frac{1}{\alpha}+1}}{e^{-\frac{1}{2}\theta t^2}}\longrightarrow 0\quad(t\to\infty)$$

and

$$\zeta_2(t)=\int_t^\infty(1+\theta s^2)e^{\frac{1}{2}\theta s^2}M_s^\alpha ds\sim C_\alpha\int_t^\infty(1+\theta s^2)e^{\frac{1}{2}\theta s^2}s^{\frac{1}{\alpha}}ds\longrightarrow 0\quad(t\to\infty)$$

Now, we have proved that ξ_t^α converges to ξ_∞^α, as t tends to infinity. And we consider the convergence in L^p with $0<p<\alpha$.

For one thing, when $t\geq 0$ and $0<p<\alpha$, we have

$$E(|\xi_t^\alpha|^p)\leq C_{p,\alpha}\left(\int_0^t s^\alpha e^{\frac{1}{2}\alpha\theta s^2}ds\right)^{\frac{p}{\alpha}}\leq\left(\int_0^\infty s^\alpha e^{\frac{1}{2}\alpha\theta s^2}ds\right)^{\frac{p}{\alpha}}<\infty,$$

and

$$E(|\xi_t^\alpha-\xi_\infty^\alpha|^p)\leq C_{p,\alpha}\left(\int_t^\infty s^\alpha e^{\frac{1}{2}\alpha\theta s^2}ds\right)^{\frac{p}{\alpha}}\sim C_{p,\alpha,\theta}e^{\frac{1}{2}p\theta t^2}t^{(1-\frac{1}{\alpha})p}\longrightarrow 0\quad(t\to\infty).$$

Thus, we can obtain the Lemma 4.1 by those convergences. □

Lemma 4.2. *Let $\theta<0$, and define the funtion $I_\theta(t):\mathbb{R}^*\mapsto\mathbb{R}$:*

$$I_\theta(t):=te^{\frac{1}{2}\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}du+\frac{1}{\theta},$$

we have $I_\theta(t)$ converges to zero as t tends to infinity.

Proof. Given $\theta<0$, the function $t\mapsto I_\theta(t)$ has the convergence

$$\begin{aligned}\lim_{t\to\infty}I_\theta(t)t^2&=\lim_{t\to\infty}\frac{1}{\theta}t^2e^{\frac{1}{2}\theta t^2}\left(e^{-\frac{1}{2}\theta t^2}+\theta t\int_0^t e^{-\frac{1}{2}\theta u^2}du\right)\\ &=\lim_{t\to\infty}\frac{1}{\theta}t^2e^{\frac{1}{2}\theta t^2}\left(1-\int_0^t\theta ue^{-\frac{1}{2}\theta u^2}du+\theta t\int_0^t e^{-\frac{1}{2}\theta u^2}du\right)\\ &=\lim_{t\to\infty}t^2e^{\frac{1}{2}\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}(t-u)du\\ &=\lim_{t\to\infty}-\frac{1}{\frac{1}{t}\theta e^{-\frac{1}{2}\theta t^2}}\int_0^t e^{-\frac{1}{2}\theta u^2}du\\ &=\lim_{t\to\infty}\frac{1}{\theta^2e^{-\frac{1}{2}\theta t^2}}e^{-\frac{1}{2}\theta t^2}=\frac{1}{\theta^2}\end{aligned}\tag{4.1}$$

Therefore, we can obtain

$$|I_\theta(t)|\leq\frac{1}{\theta^2}\min\left\{1,\frac{1}{t^2}\right\}\longrightarrow 0\quad(t\to\infty).$$

□

Lemma 4.3 (X. Sun, L. Yan[20]). *When $\frac{1}{2} < \alpha \leq 2$, a random variable $\int_0^\infty J_\theta(s)dM_s^\alpha$ is well-defined in L^p with $0 < p < \alpha$ and*

$$\int_0^t J_\theta(s)dM_s^\alpha \longrightarrow \int_0^\infty J_\theta(s)dM_s^\alpha \tag{4.2}$$

almost surely and in L^p with $0 < p < \alpha$, as t tends to infinity where

$$J_\theta(t) = -\theta t e^{\frac{1}{2}t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} du - 1.$$

Remark 1. *When we show the lemma above, as t tends to infinity, for any $\theta < 0, \frac{1}{2} < \alpha \leq 2$, denote*

$$\Psi_t := \int_0^\infty I_\theta(s)dM_s^\alpha - \int_0^t I_\theta(s)dM_s^\alpha = \int_t^\infty I_\theta(s)dM_s^\alpha$$

and Ψ_t converges to zero almost surely. Recall the asymptotic structure of $t \mapsto I_\theta(t)$, we cannot prove the convergence in probability by (3.7), but prove the lemma through the integration by part in Calculus.

Theorem 4.1. *Let $\theta < 0$ and $0 < \alpha \leq 2$. The process*

$$\Lambda_t := e^{\frac{1}{2}\theta t^2} Y_t^\alpha, \quad t \geq 0$$

converges to $\xi_\infty^\alpha - \frac{\nu}{\theta}$ almost surely and in L^p with $0 < p < \alpha$, as t tends to infinity, where $Y_t^\alpha = \int_0^t (X_t^\alpha - X_s^\alpha)ds$.

Proof. Given $\theta < 0$ and $0 < \alpha \leq 2$, for all $t > 0$ we have

$$\begin{aligned}
\Lambda_t^\alpha = e^{\frac{1}{2}\theta t^2} Y_t^\alpha &= e^{\frac{1}{2}\theta t^2} \left(e^{-\frac{1}{2}\theta t^2} \int_0^t s e^{\frac{1}{2}\theta s^2} dM_s^\alpha + \frac{\nu}{\theta}(1 - e^{-\frac{1}{2}\theta t^2}) \right) \\
&= \int_0^t s e^{\frac{1}{2}\theta s^2} dM_s^\alpha + \frac{\nu}{\theta}\left(e^{\frac{1}{2}\theta t^2} - 1\right) \\
&= \xi_t^\alpha + \frac{\nu}{\theta}\left(e^{\frac{1}{2}\theta t^2} - 1\right)
\end{aligned}$$

Lemma 4.1 implies that ξ_t^α converges to ξ_∞^α almost surely and in L^p with $0 < p < \alpha$. Therefore, we can obtain the Theorem 4.1 by the Lemma 4.1 and the fact that $e^{\frac{1}{2}\theta t^2} \to 0$, as t tends to infinity. □

Theorem 4.2. *Let $\theta < 0$ and $0 < \alpha \leq 2$. The process*

$$\Xi_t^\alpha := t e^{\frac{1}{2}\theta t^2} \int_0^t Y_t^\alpha dt, \quad t \geq 0$$

converges to $\frac{1}{\theta}(\xi_\infty^\alpha - \frac{\nu}{\theta})$ almost surely and in L^p with $0 < p < \alpha$.

Proof. Given $\theta < 0$, for all $t > 0$, we have

$$\begin{aligned}
\Xi_t^\alpha = t e^{\frac{1}{2}\theta t^2} \int_0^t Y_t^\alpha dt &= t e^{\frac{1}{2}\theta t^2} \left(\int_0^t e^{-\frac{1}{2}\theta s^2} \int_0^s u e^{\frac{1}{2}\theta u^2} dM_u^\alpha ds + \int_0^t \frac{\nu}{\theta}(1 - e^{-\frac{1}{2}\theta s^2}) ds \right) \\
&= t e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta s^2} \int_0^s u e^{\frac{1}{2}\theta u^2} dM_u^\alpha ds + \frac{\nu}{\theta} t e^{\frac{1}{2}\theta t^2} \int_0^t (1 - e^{-\frac{1}{2}\theta s^2}) ds \\
&= t e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} \xi_u^\alpha du + \frac{\nu}{\theta} t^2 e^{\frac{1}{2}\theta t^2} - \frac{\nu}{\theta} t e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta s^2} ds
\end{aligned}$$

and

$$\begin{aligned}\int_0^t e^{-\frac{1}{2}\theta u^2}(\xi_u^\alpha-\xi_\infty^\alpha)du &= \int_0^t e^{-\frac{1}{2}\theta u^2}\left(\int_\infty^u se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right)du\\ &= \int_0^t e^{-\frac{1}{2}\theta u^2}\left(\int_\infty^t se^{\frac{1}{2}\theta s^2}dM_s^\alpha+\int_t^u se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right)du\\ &= -\int_0^t se^{\frac{1}{2}\theta s^2}\left(\int_0^s e^{-\frac{1}{2}\theta u^2}du\right)dM_s^\alpha-\left(\int_0^t e^{-\frac{1}{2}\theta u^2}du\right)\left(\int_t^\infty se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right)\\ &\equiv -(\zeta_1^\alpha(t)+\zeta_2^\alpha(t)).\end{aligned}$$

By the Lemma 4.1 and Lemma 4.2, for all $t>0$, we obtain

$$\begin{aligned}\Xi_t^\alpha-\frac{1}{\theta}\left(\xi_\infty^\alpha-\frac{\nu}{\theta}\right) &= te^{\frac{1}{2}\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}(\xi_u^\alpha-\xi_\infty^\alpha)du+\left(\xi_\infty^\alpha-\frac{\nu}{\theta}\right)I_\theta(t)+\frac{\nu}{\theta}t^2e^{\frac{1}{2}\theta t^2}\\ &= -te^{\frac{1}{2}\theta t^2}[\zeta_1^\alpha(t)+\zeta_2^\alpha(t)]+\left(\xi_\infty^\alpha-\frac{\nu}{\theta}\right)I_\theta(t)+\frac{\nu}{\theta}t^2e^{\frac{1}{2}\theta t^2}\end{aligned}$$

as t tends to infinity.

Now, this will be done in two steps.

(I) We need to consider the convergence in probability of Ξ_t^α at first. By the Lemma 4.3, for any $t>0$, when $0<\alpha\le 2$, t tends to infinity, we have

$$\begin{aligned}te^{\frac{1}{2}\theta t^2}\zeta_1^\alpha(t) &= te^{\frac{1}{2}\theta t^2}\left(\int_0^t se^{\frac{1}{2}\theta s^2}\left(\int_0^s e^{-\frac{1}{2}\theta u^2}du\right)dM_s^\alpha\right)\\ &= te^{\frac{1}{2}\theta t^2}\left[\left(I_\theta(t)-\frac{1}{\theta}\right)M_t^\alpha-\int_0^t I_\theta'(s)M_s^\alpha ds\right]\end{aligned}$$

where

$$\begin{aligned}I_\theta'(t) &= e^{\frac{1}{2}\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}du+\theta t^2e^{\frac{1}{2}\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}du+t\\ &= e^{\frac{1}{2}\theta t^2}\left(\int_0^t e^{-\frac{1}{2}\theta u^2}du+\theta t^2\int_0^t e^{-\frac{1}{2}\theta u^2}du+te^{-\frac{1}{2}\theta t^2}\right)\\ &= e^{\frac{1}{2}\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}(1+\theta t^2-t\theta u)du+te^{\frac{1}{2}\theta t^2},\end{aligned}$$

According to the reference[20] by X. Sun and L. Yan, they applied the substitution $\frac{1}{2}\theta(t^2-u^2)=-x$ for the proof:

$$e^{\frac{1}{2}\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}(1+\theta t^2-t\theta u)du\sim\frac{1}{\theta t}\int_0^{-\frac{1}{2}\theta t^2}e^{-x}(1-x)dx\sim\frac{1}{2}te^{\frac{1}{2}\theta t^2}$$

Consequently, for all $\theta<0$, we have

$$\lim_{t\to\infty}\frac{1}{t}e^{-\frac{1}{2}\theta t^2}I_\theta'(t)=\frac{3}{2}$$

Considering the continuity of the function$t\mapsto I_\theta'(t)$, we can obtain

$$|I_\theta'(t)|\le\frac{3}{2}\min\left\{1,te^{\frac{1}{2}\theta t^2}\right\}.$$

Therefore, it is obvious that

$$te^{\frac{1}{2}\theta t^2}\zeta_1^\alpha(t)\longrightarrow 0\quad(t\to\infty)$$

almost surely.

Next we can consider the convergence in probability of the process $te^{\frac{1}{2}\theta t^2}\zeta_2^\alpha(t)$:

$$
\begin{aligned}
P\left(\sup_{t\geq T}\left|te^{\frac{1}{2}\theta t^2}\zeta_2^\alpha(t)\right|\geq\varepsilon\right) &\leq P\left(Te^{\frac{1}{2}\theta T^2}\int_0^T e^{-\frac{1}{2}\theta u^2}du\sup_{t\geq T}\left|\int_t^\infty se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right|\geq\varepsilon\right)\\
&\leq C_\alpha\varepsilon^{-\alpha}T^\alpha e^{\frac{1}{2}\alpha\theta T^2}\left(\int_0^T e^{-\frac{1}{2}\theta s^2}ds\right)^\alpha\left(\int_T^\infty s^\alpha e^{\frac{1}{2}\alpha\theta s^2}ds\right)\\
&\sim C_{\alpha,\theta}\varepsilon^{-\alpha}T^{\alpha-1}e^{\frac{1}{2}\alpha\theta T^2}\longrightarrow 0\quad(t\to\infty)
\end{aligned}
$$

Therefore, combined with Lemma 4.2, the convergence $t^2e^{\frac{1}{2}\theta t^2}\to 0$ and two convergences in probability above, we can obtain

$$
\Xi_t^\alpha-\frac{1}{\theta}\left(\xi_\infty^\alpha-\frac{\nu}{\theta}\right)\longrightarrow 0 \tag{4.3}
$$

almost surely.

(II) Now we consider the convergence in L^p with $0<p<\alpha$ and we discuss these processes $te^{\frac{1}{2}\theta t^2}\zeta_1^\alpha(t)$ and $te^{\frac{1}{2}\theta t^2}\zeta_2^\alpha(t)$.

$$
\begin{aligned}
E\left|te^{\frac{1}{2}\theta t^2}\zeta_1^\alpha(t)\right|^p &\leq C_{\alpha,p}t^pe^{\frac{1}{2}\theta pt^2}E\left|\int_0^t se^{\frac{1}{2}\theta s^2}\left(\int_0^s e^{-\frac{1}{2}\theta u^2}du\right)dM_s^\alpha\right|^p\\
&\leq C_{\alpha,p}t^pe^{\frac{1}{2}\theta pt^2}\left(\int_0^t s^\alpha e^{\frac{1}{2}\alpha\theta s^2}\left(\int_0^s e^{-\frac{1}{2}\theta u^2}du\right)^\alpha\right)^{\frac{p}{\alpha}}\\
&\sim C_{\alpha,p,\theta}\left(t^{2\alpha-1}e^{\alpha\theta t^2}\int_0^t e^{-\frac{1}{2}\theta u^2}du\right)^{\frac{p}{\alpha}}\\
&\sim C_{\alpha,p,\theta}\left(t^{2\alpha-2}e^{\frac{1}{2}\alpha\theta t^2}\right)^{\frac{p}{\alpha}}\longrightarrow 0\quad(t\to\infty),
\end{aligned}
$$

and

$$
\begin{aligned}
E\left|te^{\frac{1}{2}\theta t^2}\zeta_2^\alpha(t)\right|^p &\leq t^pe^{\frac{1}{2}\theta pt^2}\left(\int_0^t e^{-\frac{1}{2}\theta u^2}du\right)^pE\left|\int_t^\infty se^{\frac{1}{2}\theta s^2}dM_s^\alpha\right|^p\\
&\leq C_{p,\alpha,\theta}t^pe^{\frac{1}{2}\theta pt^2}\left(\int_0^t e^{-\frac{1}{2}\theta u^2}du\right)^p\left(\int_t^\infty s^\alpha e^{\frac{1}{2}\alpha\theta s^2}ds\right)^{\frac{p}{\alpha}}\\
&\sim C_{p,\alpha,\theta}t^{(1-\frac{1}{\alpha})p}e^{\frac{1}{2}p\theta t^2}\longrightarrow 0\quad(t\to\infty)
\end{aligned}
$$

Thus, we can get the convergence in probability and in $L^p(0<p<\alpha)$. □

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