On the (non-)linearity of the value and the policy functions under RINCE preferences*

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Abstract

Farmer (1990, Quarterly Journal of Economics) formulated the Risk Neutral Constant Elasticity (RINCE) preferences and obtained a closed-form solution to the consumption-saving problem in the presence of idiosyncratic shocks to flow income and to investment return. Both the value and the policy functions are linear in appropriately defined wealth, and their linearity has been exploited to facilitate aggregation in macroeconomic models. His solution, however, implicitly assumes that the natural borrowing limit never binds, and, when it binds, a closed-form solution is unavailable and the linearity no longer holds. To preserve linearity one needs to specify the shock process carefully so that the natural borrowing limit never binds.

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1 Introduction

Consumption-saving problems in the presence of shocks to income as well as to return on saving do not generally permit closed-form solutions. Farmer (1990) argued that a closed-form solution is available for a particular class of preferences, which he coined the RINCE (RIsk-Neutral Constant Elasticity) preferences. It is a special case of the Kreps-Porteus preferences (Kreps and Porteus, 1978, 1979) and assumes the risk neutrality but the intertemporal elasticity of substitution is finite and constant.

An important property of this closed-form solution is that both the value and the policy functions of a decision-maker are linear in her wealth. The decision-maker's past actions affect the current value and policy functions only through her current wealth. All coefficients on wealth in these functions are determined exogenously by the stochastic process of real interest rates and by the preference parameters.

Since Gertler (1999), the RINCE preferences have been widely used in macroeconomic models because the linearity of the value and the policy functions can be exploited to facilitate aggregation in the presence of uninsurable, idiosyncratic shocks.[1] Given that all households face the same real interest rate process, aggregate consumption and saving are linear in aggregate wealth if all households have the same RINCE preferences. This property makes the model tractable because we do not need to keep

[1]Examples include Fujiwara and Teranishi (2008), Carvalho, Ferrero, and Nechio (2016), Basso and Rachedi (2018) and Fujiwara, Hori, and Waki (2019). These papers and Gertler (1999) assume probabilistic aging to incorporate an overlapping generations structure in general equilibrium models, while maintaining tractability using age-group-specific aggregation.

track of the wealth distribution when computing aggregate variables. Even if the economy consists of many but finite groups of households, where groups differ in their RINCE preference specifications, we can still conduct within-group aggregation and the model remains tractable as long as the number of groups is small.

This paper shows that there is an implicit assumption made in Farmer (1990) to obtain the linear closed-form solution, and provides a counterexample in which this assumption is invalid. The assumption is that the "natural" borrowing limit never binds, i.e. the decision-maker never finds it optimal to borrow more than she can repay with certainty. When formulating the recursive version of the decision problem, Farmer (1990) only requires that the decision-maker does not leave any debt upon dying (i.e. in the terminal period), but imposes no borrowing constraints before the terminal period. His linear closed-form solution is the solution to this recursive problem, but it is not a solution to the original, non-recursive problem when the aforementioned implicit assumption is violated.[2]

Indeed, there are actually cases in which the decision-maker who solves the recursive problem without borrowing constraints accumulates so much debt that she cannot repay when her flow income turns out to be low. The counterexample in the present paper is a two-period one in which the second period flow income is random and may become zero with strictly positive probability. Given that the second period value function is linear in wealth as described in Farmer (1990) and that no borrowing constraint is imposed in the first period, the decision-maker has an incentive to borrow in the first period. This is because her future income is, on average, higher than the current cash-on-hand, and because she has a consumption-smoothing motive. However, if her flow income in the second period turns out to be zero, she cannot repay debt while entertaining non-negative consumption.

[2]Another way to put it is that the recursive formulation in Farmer (1990) is incorrect when the implicit assumption is not satisfied.

Therefore, a valid recursive formulation requires some borrowing constraints not only in the terminal period but also in periods before that. By adding some borrowing constraints, can we keep entertaining the linearity? Unfortunately, we cannot. Once we impose some borrowing constraints, whether natural or ad-hoc (Aiyagari, 1994), we have occasionally binding constraints in the decision problem, and this creates kinks in the value and the policy functions. I demonstrate this point by adding a borrowing constraint to the above counterexample.

One source of this problem is that Farmer (1990) allows the flow income and the real interest rates to follow *any* non-negative joint stochastic process. It may be possible to preserve linearity by imposing restrictions on the stochastic process of the real interest rates and the flow income, so that the borrowing constraints never bind. I will discuss this point in Section 4.

2 The stochastic decision problem in Farmer (1990)

A decision-maker lives from $t = 0$ to T and maximizes her lifetime utility evaluated in period 0, v_0, by choosing a consumption-saving plan, $\{c_t, a_{t+1}\}_{t=0}^{T}$, subject to the flow budget constraints.

The flow budget constraints are given by:

$$c_t + a_{t+1} \leq R_t a_t + \omega_t, \quad \forall t = 0, \cdots, T, \tag{1}$$

where R_t denotes the gross real interest rate earned on the period $t-1$ saving a_t and ω_t denotes the flow income in period t. Both R_t and ω_t are allowed to follow any non-negative, joint stochastic process.

The initial and the terminal conditions for her saving $\{a_t\}$ are:

$$a_0 = \overline{a}_0 \tag{2}$$

and

$$a_{T+1} \geq 0. \tag{3}$$

The decision-maker's lifetime utility from $\{c_t\}$ follows a recursion:

$$v_t = w(c_t, \mathbb{E}_t v_{t+1}), \forall t = 0, \cdots, T-1, \quad v_T = w_T(c_T). \tag{4}$$

The decision-maker's problem is to maximize v_0 by choosing $\{c_t, a_{t+1}\}_{t=0}^{T}$ subject to the flow budget constraints (equation 1), the initial and the terminal condition for saving (equations 2 and 3), and the recursion for utility (equation 4), and the non-negativity constraint for consumption: $c_t \geq 0$, for all $t = 0, \cdots, T$.

Here the function w in equation (4) is a homogeneous function and is given by:

$$w(x, y) = (x^\rho + \beta y^\rho)^{1/\rho}$$

with $\beta \in [0, \infty)$ and $\rho \in (-\infty, 0) \cup (0, 1]$.[3] The function w_T is simply $w_T(x) = x$. Therefore, the utility function here is a special case of the Kreps-Porteus preferences (Kreps and Porteus, 1978, 1979) that is parameterized later in Epstein and Zin (1989). It assumes the risk neutrality but the intertemporal elasticity of substitution is finite and constant. Hence it is coined a RIsk-Neutral Constant Elasticity (RINCE) preference.

Importantly, income, $\{\omega_t\}_{t=0}^{T}$, as well as return on saving, $\{R_t\}_{t=0}^{T}$, are subject to exogenous shocks. When both shocks are present, a closed-form solution does not

[3] Farmer also considers a Cobb-Douglous aggregator where ρ is taken to zero, but for notational simplicity I restrict attention to the $\rho \neq 0$ cases. When the planning horizon is infinite, β needs to be further restricted to be less than one.

exist for general preference specifications.

2.1 Characterization of the solution in Farmer (1990)

Farmer (1990) argued that the above problem permits a rather simple, closed-form solution that is linear in a certain measure of wealth. First define two functions, F and G, as

$$F(x) := \left(1 + \beta^{\frac{1}{1-\rho}} x^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} \quad \text{and} \quad G(x) := \left(1 + \beta^{\frac{1}{1-\rho}} x^{\frac{\rho}{1-\rho}}\right)^{-1}.$$

Then define two stochastic processes $\{Q_t\}_{t=0}^T$ and $\{h_t\}_{t=0}^T$ recursively:

$$F(Q_T) = 1, \text{ and } Q_t = \mathbb{E}_t[R_{t+1} F(Q_{t+1})], \text{ for all } t = 0, 1, \cdots, T-1,$$

and

$$h_T = \omega_T, \text{ and } h_t = \omega_t + \mathbb{E}_t\left[h_{t+1} \frac{F(Q_{t+1})}{Q_t}\right], \text{ for all } t = 0, 1, \cdots, T-1.$$

Here $F(Q_{t+1})/Q_t$ is a stochastic discount factor and h_t can be interpreted as human wealth that equals the period-t present discounted value of income from period t on. The above recursion does not envolve endogenous variables. Therefore these two stochastic processes are exogenously determined.

Farmer argues that the solution to the decision problem is written as

$$c_t = G(Q_t) W_t, \tag{5}$$

$$v_t = F(Q_t) W_t, \tag{6}$$

for all $t = 0, 1, ..., T$, where W_t denotes the beginning-of-period-t total wealth, defined as the sum of financial and human wealth, $W_t := R_t a_t + h_t$.

The solution characterized above has a nice property: the optimal consumption and saving are linear in wealth. This property is used to obtain aggregation results in some macroeconomic studies discussed later in Section 4.

2.2 Sketch of the proof in Farmer (1990)

Farmer (1990) provides a sketch of the proof that is based on a recursive formulation.

For any t, any a, and any history of exogenous shocks up to period t, let

$$V_t(a) := F(Q_t)(R_t a + h_t), \tag{7}$$

$$C_t(a) := F(Q_t)(R_t a + h_t), \tag{8}$$

and

$$A'_t(a) := R_t a + \omega_t - C_t(a). \tag{9}$$

Here the dependence of these functions on a exogenous shock history is kept implicit and represented by the subscript t.

Then the sequence of value functions, $\{V_t\}_{t=0}^T$, and policy functions, $\{C_t, A'_t\}_{t=0}^T$, solves the following functional equations: in the terminal period T, for any a, and any history of exogenous shocks up to period T,

$$V_T(a) = \max_{c,a'} c$$

subject to the budget constraint, $c + a' \leq R_T a + \omega_T$, and the non-negativity constraint: $c, a' \geq 0$; and, for any $t = 0, 1, \cdots, T-1$, any a, and any history of exogenous shocks up to period t,

$$V_t(a) = \max_{c,a'} \left\{ c^\rho + \beta \left(\mathbb{E}_t V_{t+1}(a') \right)^\rho \right\}^{1/\rho}$$

subject to the budget constraint, $c + a' \leq R_t a + \omega_t$ and the non-negativity constraint for consumption: $c_t \geq 0$.

The proof is based on the backward induction and the first-order condition of the above recursive problem.

3 What is the problem?

Then what is a problem? The problem is that the above recursive formulation is incorrect. It is correct only when the natural borrowing limit is slack in all periods and in all states. In this model, however, the natural borrowing limit can bind and the solution proposed in Farmer (1990) is incorrect when it binds. The following example illustrates this point.

3.1 An example where Farmer's solution is incorrect

Consider the following two-period model, which is a special case of one in Farmer (1990) with $T = 1$. Set parameters as follows: $R_0 a_0 + \omega_0 < 0.5$, $R_1 = 1$ (with certainty), and $\omega_1 = 0$ with probability $\epsilon < 0.5$ and $\omega_1 = 1$ with probability $1 - \epsilon$.

Let us calculate Farmer's solution. First, using the recursion for Q and h, we obtain

$$Q_0 = \mathbb{E}_0[F(Q_1)] = 1,$$

and

$$h_0 = \omega_0 + \mathbb{E}_0 h_1 = \omega_0 + \mathbb{E}_0 \omega_1.$$

Consumption and value are then obtained using the policy and the value functions as follows:

$$\begin{aligned}
c_0 &= G(1)W_0 = \frac{W_0}{2}, \\
c_1 &= a_1 + \omega_1, \\
v_0 &= F(1)W_0 = 2^{\frac{1-\rho}{\rho}} W_0, \\
v_1 &= a_1 + \omega_1,
\end{aligned}$$

where $W_0 = R_0 a_0 + h_0$.

The optimal saving a_1 is therefore,

$$\begin{aligned}
a_1 &= R_0 a_0 + \omega_0 - c_0 \\
&= R_0 a_0 + \omega_0 - \frac{W_0}{2} \\
&= R_0 a_0 + \omega_0 - \frac{R_0 a_0 + \omega_0 + \mathbb{E}_0 \omega_1}{2} \\
&= \frac{1}{2}\{R_0 a_0 + \omega_0 - \mathbb{E}_0 \omega_1\}.
\end{aligned}$$

Because $R_0 a_0 + \omega_0 < 0.5$ and $\mathbb{E}_0 \omega_1 = 1 - \epsilon > 0.5$, the last expression is negative. Hence, the optimal saving according to Farmer's solution is negative.

However, this cannot be a solution to the original, non-recursive problem. In period 1, the decision-maker's flow income ω_1 may be zero with positive probability ϵ. In that state with zero income, the decision-maker cannot repay her debt while entertaining non-negative consumption. Hence, the terminal condition $a_2 \geq 0$ has to be violated in the zero-income state.

3.2 Diagnosis

What is wrong? In Farmer's recursive formulation, the household's choices are restricted only by the flow budget constraints, except for the terminal period in which saving has to be non-negative. However, in order to ensure that the household does not leave any debt upon dying in all states, one has to impose some borrowing constraints, such as the natural borrowing limit or an ad-hoc borrowing limit (Aiyagari, 1994), that restricts the household's ability to borrow *before the terminal period*. When the borrowing constraint binds, the value function is no longer linear in a proposed measure of wealth and has a kink above which the borrowing constraint becomes slack.

The natural borrowing limit is the maximal amount of debt a decision maker can repay with certainty, without violating the non-negativity constraint for consumption. The natural borrowing limit in the two-period example is

$$a_1 \geq -\min \frac{\omega_1}{R_1},$$

where the minimization on the right-hand side is over the set of all possible states in period 1. Because $R_1 = 1$ with certainty and $\min \omega_1 = 0$, the natural borrowing limit in the example is equivalent to $a_1 \geq 0$, i.e. no borrowing is allowed. This constraint binds as far as $R_0 a_0 + \omega_0 < 0.5$, and it is slack for $R_0 a_0 + \omega_0 \geq 0.5$. Failing to take this constraint into account can leads us to an incorrect solution. The period-0 value and policy functions take the form as in equations (7)-(9) for $a_0 \geq (0.5 - \omega_0)/R_0$. For $a_0 < (0.5 - \omega_0)/R_0$, however, they are given by

$$C_0(a) = R_0 a_0 + \omega_0, \tag{10}$$

$$A_0'(a) = 0, \tag{11}$$

and

$$V_0(a) = \left((R_0 a_0 + \omega_0)^\rho + (\mathbb{E}_0 \omega_1)^\rho\right)^{1/\rho} = \left((R_0 a_0 + \omega_0)^\rho + (1-\epsilon)^\rho\right)^{1/\rho}, \tag{12}$$

which is obviously nonlinear in a.

In incomplete market models with income risk, if utility function is a time-separable, expected utility with geometric discounting, Inada condition ensures that the natural borrowing limit never binds. This is because, if it binds, the household's consumption becomes zero after some history of shocks, and it is suboptimal under Inada condition.

Does the assumption of CES preference aggregator imply Inada condition? Althouh the preference aggregator is nonlinear in current consumption, the answer is no. This is most clearly seen in the above two-period example. Substitute the the identity $v_1 = c_1$ into the preference aggregator in period 0 to obtain:

$$v_0 = \max_{c_0, a_0} \{c_0^\rho + \beta(\mathbb{E}_0 c_1)^\rho\}^{1/\rho}.$$

Hence, the agent has an incentive to smooth average consumptions over time, but as far as the expected future consumption is sufficiently high, the agent does not really mind consumption (or the continuation value) becoming zero in some future states. This is because of risk neutrality. When the expected future income is sufficiently high relative to the current cash on hands, the decision maker wants to borrow as much as possible so that the natural borrowing limit binds.

3.3 A correct recursive formulation with the natural borrowing limit

It is possible to incorporate occasionally binding natural borrowing constraints.

The maximal amount that can be repaid is also defined recursively. Define $\{a_{t+1}^{LB}\}_{t=0}^{T}$ as follows:

$$a_{T+1}^{LB} = 0$$

and

$$a_t^{LB} = \max \frac{a_{t+1}^{LB} - \omega_t}{R_t}, \quad \forall t = 0, 1, \cdots, T.$$

Note that for all t, though implicit, the left-hand side object a_t^{LB} is contingent on the history of shocks up to period $t-1$ (inclusive). Each variable in the maximand on the right-hand side shares the same history up to period $t-1$ with the left-hand side variable, but also depends on the shock realization in period t. The maximization is over the set of all the possible shock realizations in period t. This variable a_t^{LB} is forward-looking but determined exogenously. Hence, it can be computed separately from endogenous variables.

The correct recursive formulation is thus given as follows. First, in the terminal period T, the problem is identical to that in Farmer (1990). For any $t \leq T-1$, any a, and any history of exogenous shocks up to period t,

$$V_t(a) = \max_{c,a'} \left\{ c^\rho + \beta \left(\mathbb{E}_t V_{t+1}(a') \right)^\rho \right\}^{1/\rho}$$

subject to the budget constraint, $c + a' \leq R_t a + \omega_t$, and the natural borrowing limit:

$$a' \geq a_t^{LB}.$$

A solution to this problem is not as simple as in Farmer (1990). The natural borrowing limit is an occasionally binding constraint, and the decision rule is necessarily nonlinear. In addition to non-tractability, the solution may be unrealistic, because once the natural borrowing limit binds, consumption falls to zero permanently. This is not a desirable property if the model is used to match actual data.

4 Discussion

In order to entertain the linearity, one has to set up a model carefully so that the natural borrowing limit never binds. For example, one has to assume away the possibility of a temporary, negative shock to income, which incentivizes households to borrow. This limits the set of flow income processes that permit linear decision rules under the RINCE preferences. When studying e.g. temporary unemployment shocks, the RINCE preferences won't result in tractable, linear decision rules.

Nonetheless, they may still be useful in studying important macroeconomic issues such as aging. Gertler (1999) formulated a tractable overlapping generations model using the RINCE preferences and probabilistic aging. The young ages (or does not age) with a constant probability, while the old dies with a constant probability. As far as the old's labor income is sufficiently lower than the young's, and the old's labor income does not increase so much over time, nobody in the economy has incentives to borrow. In this case, the natural borrowing constraint never binds, and consumption and savings are linear in wealth. Because of the linearity of the decision rules, aggregate equilibrium behavior of the model depends on the wealth distribution only through the total amount of wealth (the sum of capital stock and human wealth) in the economy and the fraction (or the amount) of wealth held by the young. This property facilitates equilibrium computation. The same modeling strategy is used in Fujiwara and Teranishi (2008), Carvalho, Ferrero, and Nechio (2016), Basso and Rachedi (2018) and Fujiwara, Hori, and Waki (2019), to introduce an OLG structure into New Keynesian models.[4]

[4]In Basso and Rachedi (2018), there are three states (young, mature, and old) instead of two (young and old). The young workers receive lower labor income than do the mature workers. Therefore, the young workers have an incentive to borrow. However, borrowing is prevented by an ad-hoc borrowing constraint that sets the borrowing limit to zero. Hence, before solving the model, it is a priori known that the borrowing constraint binds for the young households. Other households does not have an incentive to borrow. The ad-hoc borrowing limit also prevents individual consumption from falling to zero.

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