Online Appendix to Ippei Fujiwara and Yuichiro Waki,
"The Delphic forward guidance puzzle in New Keynesian models"
(Not to be published with the paper)

B Appendix

B.1 Derivation of the quadratic social welfare in equation (4)

Is the standard quadratic approximation in equation (4) valid in the present setting with communication? A short answer is yes. This is because the standard approximation is realization-by-realization and never uses equations that involve conditional expectations. More specifically, the standard procedure approximates $\mathbb{E}_0[\sum_{t=0}^{\infty}\beta^t u(c_t, h_t)]$, where c is consumption and h is labor, by first obtaining a quadratic approximation of a momentary utility function in period t, $u(c_t, h_t)$, using deterministic equations only, sums them up from time 0 to infinity with discounting, and then take expectation based on the initial information set. When computing ex ante utility, the last step is replaced to that taking unconditional expectation, and there is no step in which conditional expectations based on the private sector's information set are taken. We have an additively separable term that captures unconditional expectation of *terms independent of policy* (t.i.p.) from period 0 on, but the term is unaffected by communication as it is unconditional expectation and, therefore, we can drop it in our analysis. This is a benefit of using ex ante utility. In contrast, if one were to evaluate the representative household's expected utility from period t on based on the period-t information available to it, then there is an additively separable term that captures *conditional* expectation of t.i.p. based on the household's information set in period t, and it is affected by the communication policy. In this case, one should not drop t.i.p. when examine the effect of communication policy.

To illustrate the above point, we follow Woodford (2010) and derive the second-order approximation step by step. When necessary, we refer to equation numbers in his handbook chapter. Throughout, we will assume that the steady state is efficient. Therefore, Section 3.4.1 in Woodford (2010) is the relevant section. He uses a model in which the household supplies a set of differentiated labor to intermediate goods producers, and shows that the representative household's utility in equilibrium, evaluated in period 0, can be expressed as a function of output, Y, and the measure of price dispersion, Δ, as

$$\mathbb{E}_0^P\left[\sum_{t=0}^{\infty}\beta^t\{u(Y_t;\xi_t)-v(Y_t;\xi_t)\Delta_t\}\right],$$

where ξ_t is a preference shock. The price dispersion measure Δ_t is defined as

$$\Delta_t := \int_0^1\left(\frac{p_t(i)}{P_t}\right)^{-\theta(1+\nu)}di,$$

where $\theta > 1$ is the Dixit-Stiglitz elasticity of substitution parameter and $\nu > 0$ is the inverse of Frisch elasticity of labor supply (equation 61).[1]

Let $U(Y,\Delta;\xi) := u(Y;\xi) - v(Y;\xi)\Delta$. Its second order Taylor expansion around the steady state yields (equation 92):

$$\begin{aligned} U(Y_t,\Delta_t;\xi_t) &= \overline{Y}U_Y\hat{Y}_t + U_\Delta\hat{D}_t + \frac{1}{2}(\overline{Y}U_Y + \overline{Y}^2U_{YY})\hat{Y}_t^2 \\ &\quad + \overline{Y}U_{Y\Delta}\hat{Y}_t\hat{\Delta}_t + \overline{Y}U'_{Y\xi}\hat{\xi}_t\hat{Y}_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned}$$

Here t.i.p. refers to terms "that do not involve endogenous variables" and, therefore, consists of linear and quadratic terms in $\hat{\xi}_t$ and a constant.

When the steady state is efficient, we have $U_Y = 0$ and therefore

$$\begin{aligned} U(Y_t,\Delta_t;\xi_t) &= U_\Delta\hat{D}_t + \frac{1}{2}\overline{Y}^2U_{YY}(\hat{Y}_t - \hat{Y}_t^e)^2 \\ &\quad + \overline{Y}U_{Y\Delta}\hat{Y}_t\hat{\Delta}_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned}$$

(equation 94). Assuming that the initial $\hat{\Delta}$ is sufficiently small, i.e. $\hat{\Delta}_{-1} = \mathcal{O}(||\xi||^2)$, we

[1] Here we assume, for simplicity, that the intermediate goods production function is linear in labor. Woodford (2010) allows for diminishing marginal product of labor for the intermediate goods production, and ν is a composite of the Frisch elasticity and the production function curvature parameter.

obtain $\hat{\Delta}_t = \mathcal{O}(||\xi||^2)$ for all t, hence

$$U(Y_t, \Delta_t; \xi_t) = \frac{1}{2}\overline{Y}^2 U_{YY}(\hat{Y}_t - \hat{Y}_t^e)^2 - \overline{v}\hat{\Delta}_t + \\ +\text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

where $\overline{v} := v(\overline{Y}; \overline{\xi}) > 0$ (equation 95).

Finally, using the second order approximation of the dynamic equation for Δ, i.e. $\Delta_t = h(\Delta_{t-1}, 1 + \pi_t)$, Woodford (2010) shows

$$\sum_{t=0}^{\infty} \beta^t \hat{\Delta}_t = \frac{1}{2}\frac{h_{\pi\pi}}{1 - \alpha\beta}\sum_{t=0}^{\infty} \beta^t \pi_t^2 + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

(equation 97). Because the dynamic equation for Δ is deterministic, the term t.i.p. refers only to terms proportional to $\hat{\Delta}_{-1}$ and does not involve any exogenous variables.

As a result, we have a realization-by-realization approximation:

$$\sum_{t=0}^{\infty} \beta^t U(Y_t, \Delta_t; \xi_t) = \frac{1}{2}\sum_{t=0}^{\infty} \beta^t \{\overline{Y}^2 U_{YY} x_t^2 - (1 - \alpha\beta)^{-1}\overline{v} h_{\pi\pi}\pi_t^2\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

where $x_t = \hat{Y}_t - \hat{Y}_t^e$ is the welfare-relevant measure of the output gap.

Observe that so far we have not used any equations that involve conditional expectations based on either the private sector's information or the central bank's information. All equations that are used are either static or backward-looking. Hence, the term t.i.p. does not include conditional expectations of variables.

By taking the unconditional expectation (given the initial condition $\hat{\Delta}_{-1}$), we obtain

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U(Y_t, \Delta_t; \xi_t)\right] = \mathbb{E}\left[\frac{1}{2}\sum_{t=0}^{\infty} \beta^t \{\overline{Y}^2 U_{YY} x_t^2 - (1 - \alpha\beta)^{-1}\overline{v} h_{\pi\pi}\pi_t^2\}\right] + \mathbb{E}[\text{t.i.p.}] + \mathbb{E}[\mathcal{O}(||\xi||^3)].$$

Recall that the term t.i.p. is the sum of two terms: one that comes from the second order approximation of the utility function and the other that comes from the approximation of the dynamic equation of Δ. The former includes the sum of constants and first and second order terms in $\{\hat{\xi}_t\}_{t=0}^{\infty}$. The latter includes linear terms in $\hat{\Delta}_{-1}$, which is exogenously given. Neither includes endogenous variables. Hence, the unconditional expectation of the sum of these terms, $\mathbb{E}$[t.i.p.], is independent of how much information the private sector obtains along the way. Therefore, we can treat it as constant in our setting with communication, as far as we are concerned with ex ante welfare.

If one instead attempts to evaluate the household's expected utility in equilibrium from period s on, based on the information available to the household, it is expressed as:

$$\mathbb{E}_s^P \left[\frac{1}{2} \sum_{t=s}^{\infty} \beta^t \{ \overline{Y}^2 U_{YY} x_t^2 - (1-\alpha\beta)^{-1} \overline{v} h_{\pi\pi} \pi_t^2 \} \right] + \mathbb{E}_s^P[\text{t.i.p.}] + \mathbb{E}_s^P[\mathcal{O}(||\xi||^3)].$$

Then the term $\mathbb{E}_s^P$[t.i.p.] depends on the information available to the household in period s and, therefore, depends on communication policy as well as what the household has observed up to period s.

B.2 Optimality of secrecy in other purely forward-looking models

Commitment to secrecy remains optimal even if we augment the model with an effective lower bound on the nominal interest rate and with a shock to the social loss function.

B.2.1 A New Keynesian model with the zero lower bound

We use a model along the lines of Eggertsson and Woodford (2003) and Adam and Billi (2006), in which the zero lower bound on nominal interest rates can bind when a large, negative shock to the natural rate of interest hits the economy. Our model is more general than the conventional, simple one. The zero lower bound may bind multiple times and may not be binding at time 0, and the central bank can act differently when it foresees that the zero bound will bind or that it will cease to bind in the near future.

The Ramsey problem is the same as before except that it must respect the nonnegativity constraint on nominal interest rates:

$$i_t \geq 0. \tag{B.1}$$

An optimal secretive commitment policy is $\{(\pi_t^{SEC}, x_t^{SEC}, i_t^{SEC})\}_{t=0}^{\infty}$ that minimizes the loss function in equation (3) subject to the New Keynesian Phillips curve in equation (1), the dynamic IS equation in equation (2), and the zero lower bound (ZLB) constraint in equation (B.1). Then we have the following result:

Corollary B.1 *Proposition 1 and Corollary 1 hold in the presence of the zero lower bound.*

Our proof of Proposition 1 is valid in the presence of the ZLB because the process of the nominal interest rate that is constructed in the proof, $\{i_t^{ALT}\}$, satisfies the ZLB if $\{i_t\}$ does.

This proposition implies that, from the ex ante point of view, the central bank should be secretive even if the zero lower bound is already binding at time 0 and if it may, for example, receive private news that a negative natural rate shock disappears in near future or that a future cost-push shock is positive. This might appear to contradict with the literature, which has shown that raising inflation expectations can be welfare-improving at the zero lower bound, but it is not. From the ex post point of view, once the central bank observes, *e.g.*, the short duration of a negative natural rate shock or a positive future cost-push shock, ex post welfare improves if the private sector is also informed about the information. However, from the ex ante point of view, the duration of a negative natural rate shock may be much longer and a future cost-push shock may be negative; transparency lowers ex post welfare in that scenario. On average, it is better to leave the private sector uninformed.

B.2.2 A three-equation model with the Taylor rule

It is straightforward to extend our theoretical results in Section 2 to the case where the nominal interest rate follows a simple Taylor rule:

$$i_t = \phi_t^{\pi} \pi_t + \phi_t^{x} x_t + \eta_t, \quad \forall t, \tag{B.2}$$

where its coefficients, (ϕ_t^π, ϕ_t^x), and the monetary policy shock, η_t, are stochastic and their realized values in period t are assumed to be observed by the private sector at the beginning of period t.

In this setting, the only choice of the central bank is the message it sends to the private sector, and we need to alter the definition of a rational expectation equilibrium by including (B.2) as an equilibrium condition. Using an argument similar to the proof of Proposition 1, we can show that the equilibrium ex ante welfare loss is minimized with central bank secrecy.

B.2.3 Forward guidance about the central bank's future policy goals

Delphic forward guidance can be used to communicate information not only about future cost-push shocks but also about the central bank's objective in the future. Let $\{\theta_t\}_{t=0}^{\infty}$ be an exogenous stochastic process. Ex ante welfare loss is now given by

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t L(\pi_t, x_t, \theta_t)\right].$$

Proposition B.1 *Suppose that θ_t is observed by the private sector at the beginning of period t. Then Proposition 1 and Corollary 1 hold in the presence of shocks to the social welfare function.*

If θ_t is publicly observed at the beginning of period t, we can replace equation (13) in the proof of Proposition 1 to

$$\begin{aligned}\mathbb{E}[L(\pi_t, x_t, \theta_t)] &= \mathbb{E}[\mathbb{E}[L(\pi_t, x_t, \theta_t)|\mathcal{G}_t^{SEC}]] \\ &\geq \mathbb{E}[L(\mathbb{E}[\pi_t|\mathcal{G}_t^{SEC}], \mathbb{E}[x_t|\mathcal{G}_t^{SEC}], \theta_t)] = \mathbb{E}[L(\tilde{\pi}_t, \tilde{x}_t, \theta_t)],\end{aligned}$$

and Proposition B.1 follows.

Our focus on the future shock is crucial for this result. When a contemporaneous shock to θ is observed by the central bank but not by the private sector, then the central bank generally faces a trade-off: there are gains from making period-t actions contingent on θ_t, but that can reveal to the private sector some information about θ_t

and possibly about future θ's, which is detrimental to welfare. Therefore, for contemporaneous shocks, secrecy is not in general optimal.

B.3 A model with backward indexation and an ELB

B.3.1 Derivation of the quasi-difference inflation and the output gap in period 1

Our assumption is that the endogenous variables are determined by the optimal discretionary policy from period 1 onwards. From period 1 onwards, once the natural rate is normalized (*i.e.*, $r_t^n = r^n$) there is no state variable or shocks in the economy. Hence, both the output gap and the quasi-difference of inflation become zero.

Now consider the period-1 endogenous variables when $r_1^n = r_{elb}^n < 0$. The New Keynesian Phillips curve is given by

$$\hat{\pi}_1 = \kappa x_1$$

because the one-period-ahead quasi-difference in inflation is zero. The Dynamic IS equation is given by

$$x_1 = \sigma^{-1}\{i_1 - (\gamma\hat{\pi}_1 + \gamma^2\pi_0) - r_{elb}^n\}$$

because the output gap in period 2 is zero and because the period-2 inflation equals $\gamma\hat{\pi}_1 + \gamma^2\pi_0$.

As long as the zero lower bound is binding when $r_1^n = r_{elb}^n$, these two equations can be solved, using $i_1 = 0$, and the solution, $(x_{1,elb}(\pi_0), \hat{\pi}_{1,elb}(\pi_0))$, is given by

$$(x_{1,elb}(\pi_0), \hat{\pi}_{1,elb}(\pi_0)) = \left(\frac{\gamma^2/\sigma}{1-\gamma\kappa/\sigma}\pi_0 + \frac{1/\sigma}{1-\gamma\kappa/\sigma}r_{elb}^n, \frac{\kappa\gamma^2/\sigma}{1-\gamma\kappa/\sigma}\pi_0 + \frac{\kappa/\sigma}{1-\gamma\kappa/\sigma}r_{elb}^n\right).$$

B.3.2 Messages are virtually irrelevant

This section provides the proof that the central bank can *virtually* achieve C^* even without messages, *i.e.*, for any prior probability p and any $\epsilon > 0$, the central bank can achieve an ex ante loss that is lower than $C^*(p) + \epsilon$ without sending messages.

To understand the reason, it is instructive to consider two cases: (i) $C^*(p) = C(p)$ at p and (ii) $C^*(p) < C(p)$ at p. In Case (i), secrecy is optimal at p and, therefore, prohibiting the central bank from sending messages is irrelevant for welfare. What about Case (ii)? Because C is a continuous function on a compact interval $[0,1]$ in $\mathbb{R}$ and because C^* is a convexification of C, for each such $p \in (0,1)$, we can find p_1, p_2, and α between 0 and 1 such that $p = \alpha p_1 + (1-\alpha)p_2$ and $C^*(p) = \alpha C(p_1) + (1-\alpha)C(p_2)$.

Let $i^*(\rho)$ denote the optimal belief-dependent nominal interest rate given belief ρ. If $i^*(p_1)$ and $i^*(p_2)$ are different, the central bank can exactly achieve $C^*(p)$ by setting the nominal interest rate as follows: if $r_1^n = r_{elb}^n$, then

$$i_0 = \begin{cases} i^*(p_1) & \text{with probability } \frac{p_1(p_2-p)}{p(p_2-p_1)} \\ i^*(p_2) & \text{with the remaining probability} \end{cases}$$

and if $r_1^n = r^n$, then

$$i_0 = \begin{cases} i^*(p_1) & \text{with probability } \frac{(1-p_1)(p_2-p)}{(1-p)(p_2-p_1)} \\ i^*(p_2) & \text{with the remaining probability.} \end{cases}$$

Then, the probability of $i_0 = i^*(p_1)$ equals

$$p \times \frac{p_1(p_2-p)}{p(p_2-p_1)} + (1-p) \times \frac{(1-p_1)(p_2-p)}{(1-p)(p_2-p_1)} = \alpha,$$

and the probability of $i_0 = i^*(p_2)$ equals $1-\alpha$. Note also that, after observing $i_0 = i^*(p_1)$, the private agents' posterior probability of the period-1 natural rate shock being negative becomes p_1, and that after observing $i_0 = i^*(p_2)$ the posterior becomes p_2.

Even if $i^*(p_1)$ and $i^*(p_2)$ happen to be identical, the central bank can achieve the ex ante loss that is arbitrarily close to $C^*(p)$. Fix an arbitrary $\delta \neq 0$ such that $i^*(p_2+\delta) \neq i^*(p_2) = i^*(p_1)$. Imagine that the central bank sets the nominal rate as follows: if $r_1^n = r_{elb}^n$, then

$$i_0 = \begin{cases} i^*(p_1) & \text{with probability } \frac{p_1(p_2-p)}{p(p_2-p_1)} \\ i^*(p_2+\delta) & \text{with the remaining probability} \end{cases}$$

and if $r_1^n = r^n$, then

$$i_0 = \begin{cases} i^*(p_1) & \text{with probability } \frac{(1-p_1)(p_2-p)}{(1-p)(p_2-p_1)} \\ i^*(p_2+\delta) & \text{with the remaining probability.} \end{cases}$$

Then, as before, after observing $i_0 = i^*(p_1)$, the private agents' posterior probability of the period-1 natural rate shock being negative becomes p_1, and that after observing $i_0 = i^*(p_2+\delta)$ the posterior becomes p_2.

As shown in lemmas below, i^* is continuous and there is no interval on which i^* is constant. It then follows that we can let δ arbitrarily close to zero while maintaining $i^*(p_2+\delta) \neq i^*(p_2)$. Continuity of i^* implies that, by letting $\delta \to 0$, the achieved ex ante loss converges to $C^*(p)$. This completes the proof.

Lemma B.1 *i^* is differentiable.*

Proof. Recall that the optimal belief-dependent policy solves the following problem:

$$C(\rho) := \min_{(\pi_0, x_0)} L(\pi_0, x_0) + \beta\rho\mathcal{L}_{1,elb}(\pi_0)$$

subject to the Phillips curve,

$$\pi_0 = \kappa x_0 + \beta\rho\hat{\pi}_{1,elb}(\pi_0).$$

Because this is a linear-quadratic problem, its solution, which we denote by $\pi^*(\rho)$ and $x^*(\rho)$, can be obtained analytically as follows. Let

$$\begin{aligned} A_\pi &= \frac{\gamma^2/\sigma}{1-\gamma\kappa/\sigma} \\ A_r &= +\frac{1/\sigma}{1-\gamma\kappa/\sigma}, \end{aligned}$$

and the solution is given by

$$\pi^*(\rho) = -\frac{\beta\rho \times F_1 - 2b\left(\frac{1}{\kappa} - \beta A_\pi \rho\right)\rho\beta A_r r_{elb}^n}{2\left[1 + b\left(\frac{1}{\kappa} - \beta A_\pi\rho\right)^2 + \beta\rho F_2\right]} \tag{B.3}$$

and

$$x^*(\rho) = \left(\frac{1}{\kappa} - \beta A_\pi \rho\right) \pi^*(\rho) - \rho\beta A_r r^n_{elb}.$$

From the Dynamic IS equation, the optimal belief-dependent interest rate is given by

$$i^*(\rho) = \gamma\pi^*(\rho) + \rho\hat{\pi}_{1,elb}(\pi^*(\rho)) + r^n - \sigma\left(x^*(\rho) - \rho x_{1,elb}(\pi^*(\rho))\right).$$

Clearly, π^*, x^*, and i^* are all differentiable. ■

Lemma B.2 *There is no interval on which i^* is constant.*

Proof. Suppose to the contrary that there is such a subinterval. Because i^* is differentiable, its derivative must be zero on that subinterval. Differentiating i^*, we obtain

$$(i^*)'(\rho) = (\pi^*)'(\rho) \times \left\{\gamma - \frac{\sigma}{\kappa} + (\kappa + \sigma(1+\beta))A_\pi \rho\right\}.$$

Hence, the aforementioned subinterval must be contained in the following union of two sets:

$$\{\rho \in (0,1) | (\pi^*)'(\rho) = 0\} \cup \left\{-\frac{\gamma - \frac{\sigma}{\kappa}}{(\kappa + \sigma(1+\beta))A_\pi}\right\}.$$

Because the latter set is a singleton, the former must contain an interval. Consider such an interval. Then π^* is constant on it. Let π^{const} denote the value of π^* on the interval. Then, from equation (B.3), the following equality must hold for any ρ on the interval:

$$2\left[1 + b\left(\frac{1}{\kappa} - \beta A_\pi \rho\right)^2 + \beta\rho F_2\right]\pi^{const} \quad = \quad -\beta\rho \times F_1 - 2b\left(\frac{1}{\kappa} - \beta A_\pi \rho\right)\rho\beta A_r r^n_{zlb}.$$

This implies that the quadratic equations for ρ on both sides must have the same set of coefficients. However, this cannot be true. If $\pi^{const} \neq 0$, the left-hand side has a constant while the right-hand side does not. If $\pi^{const} = 0$, then the coefficients on ρ and ρ^2 on the right-hand side must be zero, but they are not. This is a contradiction and, therefore, there is no subinterval of $(0,1)$ on which i^* is constant. ■

B.4 A canonical DSGE model

B.4.1 The representative household

The representative household's expected utility at the beginning of period 0 is:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(C_t - X_t\right) - v\left(L_t\right) \right].$$

where C_t is consumption, L_t is labor, and X_t is external habit that equals bC_{t-1} $(b \geq 0)$ in equilibrium.

The household accumulates capital according to the capital accumulation equation:

$$K_{t+1} = (1-\delta)K_t + \left[1 - S\left(I_t, I_{t-1}\right)\right] I_t,$$

where I_t is investment and K_t is capital. The function S represents investment adjustment costs. The flow budget constraint is given by:

$$A_{t+1} + P_t\left(C_t + I_t\right) \quad \leq \quad \tilde{W}_t L_t + R_{t-1} A_t + P_t r_t^K K_t + \text{profits and transfers},$$

where A_t is the nominal risk-free bond, P_t the nominal price index, $\tilde{W}_t$ the nominal wage, R_t the gross nominal interest rate, and r_t^K the rental rate of capital, respectively. Although it is not included in the equation, the household also trades a complete set of state-contingent claims.

We assume the following functional forms:

$$\begin{aligned}
u\left(C_t - X_t\right) &= \frac{\left(C_t - X_t\right)^{1-\sigma}}{1-\sigma}, \\
v\left(L_t\right) &= \frac{L_t^{1+\eta}}{1+\eta}, \\
S\left(I_t, I_{t-1}\right) &= \frac{s}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2.
\end{aligned}$$

B.4.2 Labor union

We model sticky nominal wages following Smets and Wouters (2007). There is a unit measure of labor unions, each of which is indexed by $l \in [0, 1]$. A union l collects the homogeneous labor supplied from households and transforms it into a differentiated labor good indexed by l using a linear technology. It faces a downward-sloping demand curve for its labor good and chooses the nominal wage subject to the Calvo-style probability. Firms combine differentiated labor goods into a single, composite labor using a CES aggregator $h_t = [\int_0^1 h_t(l)^{\frac{\epsilon_h - 1}{\epsilon_h}} dl]^{\frac{\epsilon_h}{\epsilon_h - 1}}$. The nominal wage of this composite labor good is denoted by W_t.

When the union l is given an opportunity to change its wage in period t, it solves

$$\max_{W_t^*} \mathbb{E}_t \sum_{s=t}^{\infty} \theta_h^{s-t} m_{t,s} \left[W_s(l) - \frac{\tilde{W}_s}{1 + \tau_s^h} \right] h_s(l),$$

subject to

$$\begin{aligned} h_s(l) &= \left[\frac{W_s(l)}{W_s} \right]^{-\varepsilon_h} h_s, \forall s \geq t, \\ W_{t+i}(l) &= W_t(l) \prod_{n=1}^{i} \Pi_{t+n-1}^{\gamma_h}, \forall i \geq 1, \\ W_t(l) &= W_t^*. \end{aligned}$$

Here, $m_{t,s}$ denotes the stochastic discount factor for nominal payoffs, τ_t^h the subsidy to the union, and $\Pi_t = P_t / P_{t-1}$ the gross consumer price inflation rates, respectively. θ_h, ε_h andγ_h are the Calvo parameter for staggered nominal wages, elasticity of substitution among differentiated labor, and the degree of nominal wage indexation on past inflation rates.

B.4.3 Intermediate goods producer

The sticky price friction is modeled in a standard fashion. When the intermediate-goods producer f is given an opportunity to change its price in period t, it solves:

$$\max_{P_t^*} \mathbb{E}_t \sum_{s=t}^{\infty} \theta^{s-t} m_{t,s} \left[(1+\tau_s) P_s(f) - P_s MC_s \right] Y_t(f),$$

subject to

$$\begin{aligned} Y_s(f) &= \left[\frac{P_s(f)}{P_s} \right]^{-\varepsilon} Y_s, \\ P_{t+i}(f) &= P_t(f) \prod_{n=1}^{i} \Pi_{t+n-1}^{\gamma}, \forall i \geq 1, \\ P_t(f) &= P_t^*. \end{aligned}$$

The sales subsidy is denoted by τ_t. γ is the degree of price indexation on past inflation rates. Real marginal cost MC_t is the Lagrange multiplier in the cost minimization problem:

$$\min_{h_t, K_t} \frac{W_t}{P_t} h_t + r_t^K K_t,$$

subject to the production technology:

$$Y_t = K_t^{\alpha} \left[\exp(z_t) h_t \right]^{1-\alpha},$$

where z_t denotes the aggregate technology shock.

B.4.4 Final good producer

A final good producer minimizes the total cost $\int_0^1 P_t(f) Y_t(f) \mathrm{d}f$ subject to the aggregating technology:

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}f \right]^{\frac{\epsilon}{\epsilon-1}}.$$

B.4.5 Monetary policy

The central bank set nominal interest rates following the Taylor type rule:

$$R_t - 1 = \rho(R_{t-1} - 1) + (1-\rho)\left[\phi^\pi\left(\Pi_t - 1\right) + \phi^y\left(\frac{Y_t}{Y_{t-1}} - 1\right)\right] + \eta_t,$$

where ρ denotes the degree of policy inertia.

B.4.6 System of equations

We have 19 equations for 19 endogenous variables: Y_t, λ_t, Π_t, w_t, q_t, r_t^K, I_t, MC_t, K_t, $\bar{F}_t$, $\bar{K}_t$, C_t, Δ_t, $\Pi_{W,t}$, $\bar{F}_t^h$, $\bar{K}_t^h$, Δ_t^h, and R_t.

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + \left\{1 - \frac{s}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right\} I_t, \\
Y_t &= C_t + I_t, \\
\lambda_t &= \left(C_t - bC_{t-1}\right)^{-\sigma}, \\
\lambda_t &= \beta\mathbb{E}_t \frac{R_t}{\Pi_{t+1}}\lambda_{t+1}, \\
q_t &= \beta\mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t}\left[r_{t+1}^K + q_{t+1}(1-\delta)\right], \\
1 &= q_t\left\{1 - \frac{s}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - s\left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right\} + \beta\mathbb{E}_t\frac{\lambda_{t+1}}{\lambda_t}q_{t+1}s\left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2, \\
w_t &= (1-\alpha)\exp\left(z_t\right)^{1-\alpha} MC_t K_t^\alpha h_t^{-\alpha}, \\
r_t^K &= \alpha\exp\left(z_t\right)^{1-\alpha} MC_t K_t^{\alpha-1} h_t^{1-\alpha}, \\
\bar{F}_t &= 1 + \theta\beta\mathbb{E}_t\frac{\lambda_{t+1}Y_{t+1}}{\lambda_t Y_t}\pi_{t+1}^\varepsilon \pi_t^\gamma \bar{F}_{t+1}, \\
\bar{K}_t &= \exp\left(u_t\right) MC_t + \theta\beta\mathbb{E}_t\frac{\lambda_{t+1}Y_{t+1}}{\lambda_t Y_t}\pi_{t+1}^{1+\varepsilon}\bar{K}_{t+1}, \\
\bar{K}_t &= \left[\frac{1-\theta\left(\frac{\pi_{t-1}^\gamma}{\pi_t}\right)^{1-\varepsilon}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} \bar{F}_t,
\end{aligned}$$

$$
\begin{aligned}
\Delta_t &= (1-\theta)\left[\frac{1-\theta\left(\frac{\pi_{t-1}^{\gamma}}{\pi_t}\right)^{1-\varepsilon}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta\left(\frac{\pi_t}{\pi_{t-1}^{\gamma}}\right)^{\varepsilon}\Delta_{t-1}, \\
\Delta_t Y_t &= K_t^{\alpha}\left[\exp(z_t)\,h_t\right]^{1-\alpha}, \\
\Pi_t^w &= \Pi_t w_t / w_{t-1}, \\
\bar{F}_t^h &= \lambda_t w_t h_t + \beta\theta_h \mathbb{E}_t\left[\bar{F}_{t+1}^h (\Pi_{t+1}^W)^{\epsilon_h-1}\right]\Pi_t^{(1-\epsilon_h)\gamma_h}, \\
\bar{K}_t^h &= \exp(\mu_t)\,h_t^{1+\eta}(\Delta_t^h)^{\eta} + \beta\theta_h \mathbb{E}_t\left[\bar{K}_{t+1}^h (\Pi_{t+1}^W)^{\epsilon_h}\right]\Pi_t^{-\epsilon_h\gamma_h}, \\
\bar{K}_t^h &= \left[\frac{1-\theta_h\left(\frac{\pi_{t-1}^{\gamma_h}}{\pi_{W,t}}\right)^{1-\varepsilon_h}}{1-\theta_h}\right]^{\frac{1}{1-\varepsilon_h}} \bar{F}_t^h,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_t^h &= (1-\theta_h)\left[\frac{1-\theta_h\left(\frac{\pi_{t-1}^{\gamma_h}}{\pi_{W,t}}\right)^{1-\varepsilon_h}}{1-\theta_h}\right]^{\frac{\varepsilon_h}{\varepsilon_h-1}} + \theta_h\left(\frac{\pi_{W,t}}{\pi_{t-1}^{\gamma_h}}\right)^{\varepsilon_h}\Delta_{t-1}^h, \\
R_t - 1 &= \rho(R_{t-1}-1) + (1-\rho)\left[\phi^{\pi}(\pi_t - 1) + \phi^{y}\left(\frac{Y_t}{Y_{t-1}} - 1\right)\right] + \eta_t.
\end{aligned}
$$

Here Δ_t and Δ_t^h denote relative price dispersion terms for prices and wages, defined as

$$
\begin{aligned}
\Delta_t &:= \int_0^1 \left[\frac{P_t(f)}{P_t}\right]^{-\varepsilon} \mathrm{d}f, \\
\Delta_t^h &:= \int_0^1 \left[\frac{W_t(l)}{W_t}\right]^{-\varepsilon_h} \mathrm{d}l.
\end{aligned}
$$

$\bar{F}_t$, $\bar{K}_t$, $\bar{F}_t^h$, and $\bar{K}_t^h$ are auxiliary variables. Price and wage markup shocks are defined as

$$
\begin{aligned}
\exp(u_t) &:= \frac{\varepsilon}{(1+\tau_t)(\varepsilon-1)}, \\
\exp(\mu_t) &:= \frac{\varepsilon_h}{\left(1+\tau_t^h\right)(\varepsilon_h-1)}.
\end{aligned}
$$

All shocks are assumed to follow AR(1) processes:

$$\begin{aligned} z_t &= \rho_z z_{t-1} + \omega_{z,t}, \\ u_t &= \rho_u u_{t-1} + \omega_{u,t}, \\ \mu_t &= \rho_\mu \mu_{t-1} + \omega_{\mu,t}, \\ \eta_t &= \rho_\eta \eta_{t-1} + \omega_{\eta,t}, \end{aligned}$$

where

$$\begin{aligned} \omega_{z,t} &\sim N\left(0, \sigma_z^2\right), \\ \omega_{u,t} &\sim N\left(0, \sigma_u^2\right), \\ \omega_{\mu,t} &\sim N\left(0, \sigma_\mu^2\right), \\ \omega_{\eta,t} &\sim N\left(0, \sigma_\eta^2\right). \end{aligned}$$

B.4.7 Welfare cost

Let $\{(C_t^n, L_t^n)\}$ be the equilibrium consumption and labor in an economy where the agents observe n-period ahead shocks. Ex ante welfare of the household is

$$V^n = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \{u(C_t^n - bC_{t-1}^n) - v(L_t^n)\}\right].$$

For each n, we measure welfare gain/loss in consumption unit defined as

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \{u((1+\lambda_n)C_t^0 - b(1+\lambda_n)C_{t-1}^0) - v(L_t^0)\}\right] = V^n.$$

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