RINCE preferences: a comment*

Yuichiro Waki[‡]

Aoyama Gakuin University and the University of Queensland

January 28, 2020

Abstract

Farmer (1990) formulated the Risk Neutral Constant Elasticity (RINCE) preferences and obtained a closed-form solution in the presence of idiosyncratic risk for income and investment return. His solution, however, implicitly assumes that the natural borrowing limit never binds, and, when it binds, a closed-form solution is unavailable. A counterexample is provided.

JEL codes: D10, D15

Keywords: Consumption-saving problems; the Risk Neutral Constant Elasticity preferences; Borrowing constraints

Consumption-saving problems in the presence of shocks to income as well as to return on saving do not generally permit closed-form solutions. Farmer (1990) argued that a closed-form solution is available for a particular class of preferences, which he coined the RINCE (RIsk-Neutral Constant Elasticity) preferences.

*I am grateful for financial support from JSPS KAKENHI Grant-in-Aid for Scientific Research Activity start-up Grant Number 19K23199.

†Associate Professor, Department of Economics, Aoyama Gakuin University. Email: ywakiecon@gmail.com. Address: 4-4-25 Shibuya, Shibuya-ku, Tokyo, Japan.

‡Honorary Fellow, the School of Economics, the University of Queensland

1 The stochastic decision problem in Farmer (1990)

The optimization problem is given as follows:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^T} v_0$$

subject to the flow budget constraints:

$$c_t + a_{t+1} \leq R_t a_t + \omega_t, \forall t = 0, \cdots, T,$$

the initial and the terminal conditions:

$$a_0 = \overline{a}_0, \quad a_{T+1} \geq 0,$$

the recursion for the utility:

$$v_t = w(c_t, \mathbb{E}_t v_{t+1}), \forall t = 0, \cdots, T-1, \quad v_T = w_T(c_T),$$

and the non-negativity constraint of consumption: $c_t \geq 0$, for all $t = 0, \cdots, T$.

Here the function w is a homogeneous function and is given by:

$$w(x, y) = (x^\rho + \beta y^\rho)^{1/\rho}$$

with $\beta \in [0, \infty)$ and $\rho \in (-\infty, 0) \cup (0, 1]$.[1] The function w_T is simply $w_T(x) = x$. Therefore, the utility function here is a special case of the Kreps-Porteus preferececes (Kreps and Porteus, 1978, 1979) that is parameterized later in Epstein and Zin (1989). It assumes the risk neutrality but the intertemporal elasticity of substitution is finite and

[1] Farmer also considers a Cobb-Douglous aggregator where ρ is taken to zero, but for notational simplicity I restrict attention to the $\rho \neq 0$ cases. When the planning horizen is infinite, β needs to be further restricted to be less than one.

constant. Hence it is coined a RIsk-Neutral Constant Elasticity (RINCE) preference.

Importantly, income, $\{\omega_t\}_{t=0}^T$, as well as return on saving, $\{R_t\}_{t=0}^T$, are subject to exogenous shocks. When both shocks are present, a closed-form solution does not exist for general preference specifications.

1.1 Characterization of the solution in Farmer (1990)

Farmer (1990) argued that the above problem permits a rather simple, closed-form solution that is linear in a certain measure of wealth. First define two functions, F and G, as

$$F(x) := \left(1 + \beta^{\frac{1}{1-\rho}} x^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} \quad \text{and} \quad G(x) := \left(1 + \beta^{\frac{1}{1-\rho}} x^{\frac{\rho}{1-\rho}}\right)^{-1}.$$

Then define two stochastic processes $\{Q_t\}_{t=0}^T$ and $\{h_t\}_{t=0}^T$ recursively:

$$F(Q_T) = 1; \tag{1}$$

$$Q_t = \mathbb{E}_t[R_{t+1} F(Q_{t+1})]; \tag{2}$$

$$h_T = \omega_T; \tag{3}$$

$$h_t = \omega_t + \mathbb{E}_t\left[h_{t+1} \frac{F(Q_{t+1})}{Q_t}\right] \tag{4}$$

These are defined recursively from the terminal period T. Here $F(Q_{t+1})/Q_t$ is a stochastic discount factor and h_t can be interpreted as human wealth that equals the period-t present discounted value of ω's from period t on. The above recursion does not envolve endogenous variables. Therefore these two stochastic processes are exogenously determined.

Farmer argues that the solution to the decision problem is written as

$$c_t = G(Q_t)W_t, \tag{5}$$

$$v_t = F(Q_t)W_t, \tag{6}$$

for all $t = 0, 1, ..., T$, where W_t denotes the beginning-of-period-t total wealth, defined as the sum of financial and human wealth, $W_t := Ra_t + h_t$.

The solution characterized above has a nice property: the optimal consumption and saving are linear in wealth. This property is used to obtain aggregation results in some macroeconomic studies discussed later in Section 3.

1.2 Sketch of the proof in Farmer (1990)

Farmer (1990) provides a sketch of the proof that is based on a recursive formulation.

For any t, any a, and any history of exogenous shocks up to period t, let

$$V_t(a) \quad := \quad F(Q_t)(R_t a + h_t), \tag{7}$$

$$C_t(a) \quad := \quad F(Q_t)(R_t a + h_t), \tag{8}$$

and

$$A'_t(a) \quad := \quad R_t a + \omega_t - C_t(a). \tag{9}$$

Here the dependence of these functions on a exogenous shock history is kept implicit and represented by the subscript t.

Then the sequence of value functions, $\{V_t\}_{t=0}^T$, and policy functions, $\{C_t, A'_t\}_{t=0}^T$, solves the following functional equations: in the terminal period T, for any a, and any history of exogenous shocks up to period T,

$$V_T(a) = \max_{c,a'} c$$

subject to the budget constraint, $c + a' \leq R_T a + \omega_T$, and the non-negativity constraint: $c, a' \geq 0$; and, for any $t = 0, 1, \cdots, T-1$, any a, and any history of exogenous shocks up to period t,

$$V_t(a) = \max_{c,a'} \left\{ c^\rho + \beta \left(\mathbb{E}_t V_{t+1}(a') \right)^\rho \right\}^{1/\rho}$$

subject to the budget constraint, $c + a' \leq R_t a + \omega_t$ and the non-negativity constraint for consumption: $c_t \geq 0$.

The proof is based on the backward induction and the first-order condition of the above recursive problem.

2 What is the problem?

Then what is a problem? The problem is that the above recursive formulation is incorrect. It is correct only when the natural borrowing limit is slack in all periods and in all states. In this model, however, the natural borrowing limit can bind and the solution proposed in Farmer (1990) is incorrect when it binds. The following example illustrates this point.

2.1 An example where Farmer's solution is incorrect

Consider the following two-period model, which is a special case of one in Farmer (1990) with $T = 1$. Set parameters as follows: $R_0 a_0 + \omega_0 < 0.5$, $R_1 = 1$ (with certainty), and $\omega_1 = 0$ with probability $\epsilon < 0.5$ and $\omega_1 = 1$ with probability $1 - \epsilon$.

Then,

$$\begin{aligned} F(Q_1) &= 1; &(10)\\ Q_0 &= \mathbb{E}_0[F(Q_1)] = 1; &(11)\\ h_1 &= \omega_1; &(12)\\ h_0 &= \omega_0 + \mathbb{E}_0 h_1 = \omega_0 + \mathbb{E}_0\omega_1 &(13) \end{aligned}$$

and

$$\begin{aligned} c_0 &= G(1)W_0 = \frac{W_0}{2}, &(14)\\ c_1 &= a_1 + \omega_1, &(15)\\ v_0 &= F(1)W_0 = 2^{\frac{1-\rho}{\rho}} W_0, &(16)\\ v_1 &= a_1 + \omega_1. &(17) \end{aligned}$$

The optimal saving a_1 is therefore,

$$\begin{aligned} a_1 &= R_0 a_0 + \omega_0 - c_0 \\ &= R_0 a_0 + \omega_0 - \frac{W_0}{2} \\ &= R_0 a_0 + \omega_0 - \frac{R_0 a_0 + \omega_0 + \mathbb{E}_0\omega_1}{2} \\ &= \frac{1}{2}\{R_0 a_0 + \omega_0 - \mathbb{E}_0\omega_1\}. \end{aligned}$$

Because $R_0 a_0 + \omega_0 < 0.5$ and $\mathbb{E}_0\omega_1 = 1 - \epsilon > 0.5$, the last expression is negative. Hence, the optimal saving according to Farmer's solution is negative and the decision maker chooses to borrow in period 0.

However, this cannot be a solution to the original problem. In period 1, the decision maker's flow income ω_1 may be zero with positive probability ϵ. In that state with zero income, the decision maker cannot repay her debt while entertaining non-negative consumption. Hence, the terminal condition $a_2 \geq 0$ has to be violated in the zero-income state.

2.2 Diagnosis

What is wrong? In Farmer's recursive formulation, the household's choices are restricted only by the flow budget constraints, except for the terminal period in which saving has to be non-negative. However, in order to ensure that the household does not leave any debt upon dying in all states, one has to impose some borrowing constraints, such as the natural borrowing limit or an ad-hoc borrowing limit (Aiyagari, 1994), that restricts the household's ability to borrow *before the terminal period*. When the borrowing constraint binds, the value function is no longer linear in a proposed measure of wealth and has a kink above which the borrowing constraint becomes slack.

The natural borrowing limit is the maximal amount of debt a decision maker can repay with certainty, without violating the non-negativity constraint for consumption. The natural borrowing limit in the two-period example is

$$a_1 \geq -\min \frac{\omega_1}{R_1},$$

where the minimization on the right-hand side is over the set of all possible states in period 1. Because $R_1 = 1$ with certainty and $\min \omega_1 = 0$, the natural borrowing limit in the example is equivalent to $a_1 \geq 0$, i.e. no borrowing is allowed. This constraint binds as far as $R_0 a_0 + \omega_0 < 0.5$, and it is slack for $R_0 a_0 + \omega_0 \geq 0.5$. Failing to take this constraint into account can leads us to an incorrect solution. The period-0 value and policy functions take the form as in equations (7)-(9) for $a_0 \geq (0.5 - \omega_0)/R_0$. For

$a_0 < (0.5 - \omega_0)/R_0$, however, they are given by

$$
\begin{aligned}
C_0(a) &= R_0 a_0 + \omega_0, \qquad (18) \\
A_0'(a) &= 0, \qquad (19)
\end{aligned}
$$

and

$$
V_0(a) = ((R_0 a_0 + \omega_0)^\rho + (\mathbb{E}_0 \omega_1)^\rho)^{1/\rho} = ((R_0 a_0 + \omega_0)^\rho + (1-\epsilon)^\rho)^{1/\rho} \qquad (20)
$$

In incomplete market models with income risk, if utility function is a time-separable, expected utility with geometric discounting, Inada condition ensures that the natural borrowing limit never binds. This is because, if it binds, the household's consumption becomes zero after some history of shocks, and it is suboptimal under Inada condition.

Does the assumption of CES preference aggregator imply Inada condition? Althouh the preference aggregator is nonlinear in current consumption, the answer is no. This is most clearly seen in the above two-period example. Substitute the the identity $v_1 = c_1$ into the preference aggregator in period 0 to obtain:

$$
v_0 = \max_{c_0, a_0} \{c_0^\rho + \beta(\mathbb{E}_0 c_1)^\rho\}^{1/\rho}.
$$

Hence, the agent has an incentive to smooth average consumptions over time, but as far as the expected future consumption is sufficiently high, the agent does not really mind consumption (or the continuation value) becoming zero in some future states. This is because of risk neutrality. When the expected future income is sufficiently high relative to the current cash on hands, the decision maker wants to borrow as much as possible so that the natural borrowing limit binds.

2.3 A correct recursive formulation with the natural borrowing limit

It is possible to incorporate occasionally binding natural borrowing constraints.

The maximal amount that can be repaid is also defined recursively. Define $\{a_{t+1}^{LB}\}_{t=0}^{T}$ as follows:

$$a_{T+1}^{LB} = 0$$

and

$$a_t^{LB} = \max \frac{a_{t+1}^{LB} - \omega_t}{R_t}, \quad \forall t = 0, 1, \cdots, T.$$

Note that for all t, though implicit, the left-hand side object a_t^{LB} is contingent on the history of shocks up to period $t-1$ (inclusive). Each variable in the maximand on the right-hand side shares the same history up to period $t-1$ with the left-hand side variable, but also depends on the shock realization in period t. The maximization is over the set of all the possible shock realizations in period t. This variable is forward-looking but determined exogenously. Hence, it can be computed separately from endogenous variables.

The correct recursive formulation is thus given as follows. First, in the terminal period T, the problem is identical to that in Farmer (1990). For any $t \leq T-1$, any a, and any history of exogenous shocks up to period t,

$$V_t(a) = \max_{c,a'} \left\{ c^\rho + \beta \left(\mathbb{E}_t V_{t+1}(a') \right)^\rho \right\}^{1/\rho}$$

subject to the budget constraint, $c + a' \leq R_t a + \omega_t$, and the natural borrowing limit:

$$a' \geq a_t^{LB}.$$

A solution to this problem is not as simple as in Farmer (1990). The natural borrowing limit is an occasionally binding constraint, and the decision rule is necessarily nonlinear. Once the natural borrowing limit binds, consumption can fall to zero permanently, which is not desirable if the model is used to match actual data.

3 Discussion

In order to exploit the linearity, one has to set up a model carefully so that the natural borrowing limit never binds. For example, one has to assume away the possibility of a temporary, negative shock to income, which incentivizes households to borrow. This limits the set of labor income processes that permit linear decision rules under the RINCE preferences. When studying e.g. temporary unemployment shocks, the RINCE preferences won't result in tractable, linear decision rules.

Nonetheless, they may still be useful in studying important macroeconomic issues such as aging. Gertler (1999) formulated a tractable overlapping generations model using the RINCE preferences and probabilistic aging. The young ages (or does not age) with a constant probability, while the old dies with a constant probability. As far as the old's labor income is sufficiently lower than the young's, and the old's labor income does not increase so much over time, nobody in the economy has incentives to borrow, and consumption and savings are linear in wealth, as in Farmer (1990). Because of the linearity of the decision rules, aggregate equilibrium behavior of the model depends on the wealth distribution only through the total amount of wealth (the sum of caiptal stock and human wealth) in the economy and the fraction (or the amount) of wealth held by the young. This property facilitates equilibrium computation. The same modeling strategy is used in Fujiwara and Teranishi (2008), Carvalho, Ferrero, and Nechio (2016), Basso and Rachedi (2018) and Fujiwara, Hori, and Waki (2019), to introduce an OLG structure into New Keynesian models.[2]

[2] In Basso and Rachedi (2018), there are three states (young, mature, and old) instead of two (young and old). The young workers receive lower labor income than do the mature workers. Therefore, the young workers have an incentive to borrow. However, borrowing is prevented by an ad-hoc borrowing

References

AIYAGARI, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109(3), 659–684.

BASSO, H. S., AND O. RACHEDI (2018): "The young, the old, and the government: demographics and fiscal multipliers," Working Papers 1837, Banco de España Working Papers.

CARVALHO, C., A. FERRERO, AND F. NECHIO (2016): "Demographics and real interest rates: Inspecting the mechanism," *European Economic Review*, 88(C), 208–226.

EPSTEIN, L. G., AND S. E. ZIN (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4), 937–969.

FARMER, R. E. A. (1990): "RINCE Preferences," *The Quarterly Journal of Economics*, 105(1), 43–60.

FUJIWARA, I., S. HORI, AND Y. WAKI (2019): "Generational War on Inflation: Optimal Inflation Rates for the Young and the Old," Globalization Institute Working Papers 372, Federal Reserve Bank of Dallas.

FUJIWARA, I., AND Y. TERANISHI (2008): "A dynamic new Keynesian life-cycle model: Societal aging, demographics, and monetary policy," *Journal of Economic Dynamics and Control*, 32(8), 2398–2427.

GERTLER, M. (1999): "Government debt and social security in a life-cycle economy," *Carnegie-Rochester Conference Series on Public Policy*, 50(1), 61–110.

constraint that sets the borrowing limit to zero. Hence, before solving the model, it is a priori known that the borrowing constraint binds for the young households. Other households does not have an incentive to borrow. The ad-hoc borrowing limit also prevents individual consumption from falling to zero.

KREPS, D. M., AND E. L. PORTEUS (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46(1), 185–200.

——— (1979): "Dynamic Choice Theory and Dynamic Programming," *Econometrica*, 47(1), 91–100.