NIRMA UNIVERSITY

INSTITUTE OF TECHNOLOGY

**Course : Digital Signal Processing - 2EC502**

**Topic : Compressed Sensing**

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## Compressed Sensing

The Data is the most important asset in todays world. There have been various methods to compress the data to be stored however when talking about data acquisition the number samples the devices captures are way too many. This is because according to the Shannon-Nyquist Theorem one must sample the two times the signal bandwidth. Hence the data acquired contains a lot of samples which then is compressed in the device for storing.

The process discussed above is very inefficient as we have to acquire samples which will be redundant. Hence compressed sensing focus on reducing the number of sample that the device acquire. Compressed sensing defies the Shannon-Nyquist theorem by sampling the signal less than the signal bandwidth.

## Theory

Most natural signals are highly compressible. This compressibility means that when the signals are written In an appropriate basis only few modes are active, thus reducing the number of values that must be stored for accurate representation.

$$x = \Psi s$$

The above equations shows the compressible signal x can be written as sparse vector s in transform basis $\Psi$. The vector s is called K-sparse in $\Psi$, if there are K nonzero elements. If the basis $\Psi$ is generic, such as Fourier or wavelet basis, then only few active terms in s are needed to reconstruct the original signal x.

Instead of measuring x directly and then compressing, one could collect dramatically fewer randomly chosen and then solve for non-zero elements of s in the transformed coordinate system. The measurement y are given by

$$y = Cx$$

The measurement matrix $C$ represent the set of p-linear measurements on set $x$. Typically the measurements consist of random projections of the state, in which case the entries of $C$ are Gaussian or Bernoulli random variable.

$$y = C\Psi s = \Theta s$$

The equation above is underdetermined since there are infinite consistent solution of s. The sparsest solution $\hat{s}$ satisfies the following optimisation problem;

$$\hat{s} = argmin\|s\|_1 \text{ subject to } y = C\Psi s$$

There are very specific condition that must be met for the $l$-1 minimisation to converge with high probability to the sparsest solution. The condition can be summarised as;

1. The measurement matrix $C$ must be incoherent with respect to the sparsifying basis $\Psi$.
2. The number of measurement must be sufficiently large.

### L1 and sparse solution to an underdetermined system

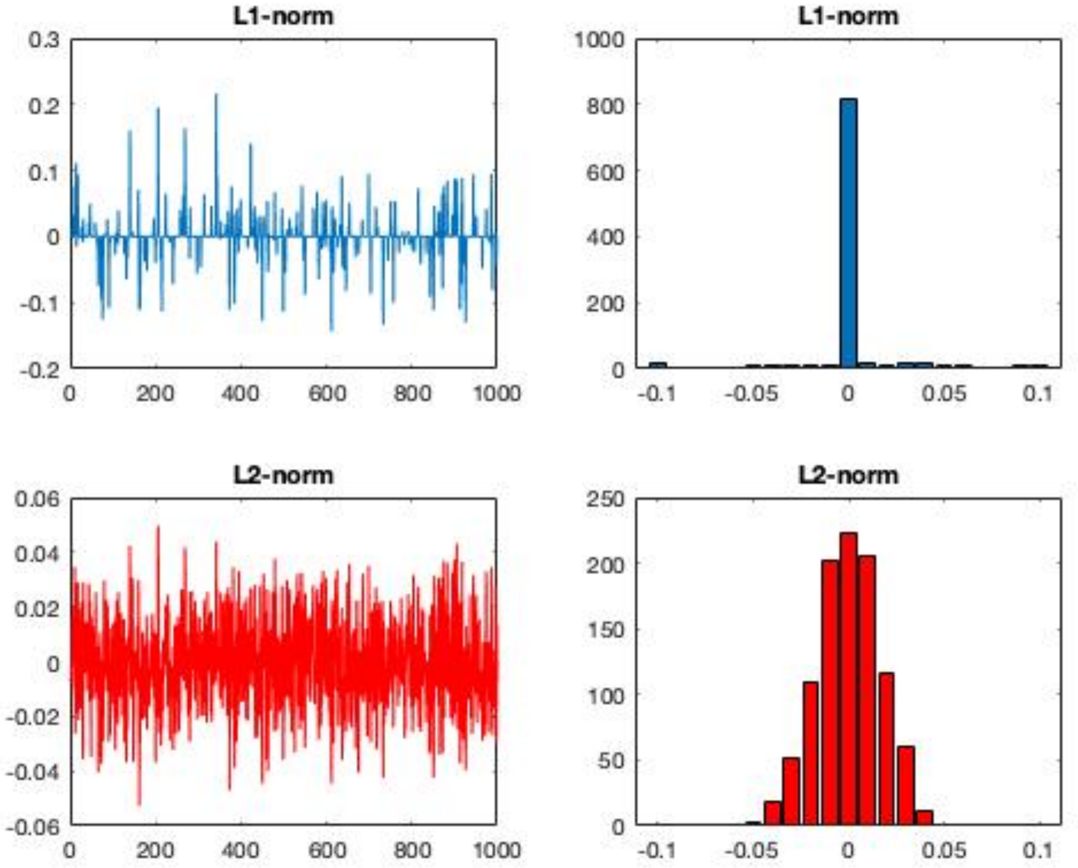
To visualise the sparsity promoting effect of the $l1$ norm the code is as shown.

### Code

```
% Solve y = Theta * s for "s"
n = 1000; % dimension of s
p = 200; % number of measurements, dim(y)
Theta = randn(p,n);
y = randn(p,1);
% L1 minimum norm solution s_L1
cvx_begin;
    variable s_L1(n);
    minimize( norm(s_L1,1) );
    subject to
    Theta*s_L1 == y;
cvx_end;

s_L2 = pinv(Theta)*y;%l2norm

figure();
subplot(221)
plot(s_L1);
title('L1-norm');
subplot(222)
[hc,h] = hist(s_L1,[-0.1:0.01:0.1]);
bar(h,hc);
title('L1-norm');
subplot(223)
plot(s_L2,'r');
title('L2-norm');
subplot(224)
[hs,h2] = hist(s_L2,[-0.1:0.01:0.1]);
bar(h2,hs,'r');
title('L2-norm');
```

### Results

The results clearly shows that the $l$-1 norm minimum solution is in fact sparse meaning containing the most zeroes entries whereas $l$-2 minimum solution is dense, with distributed energy in each vector coefficient.




**Compressed sensing Code**

In Matlab, it is straightforward to solve the underdetermined linear systems for both minimum $l1$ and $l2$ norm solution. The minimum $l2$ norm solution can be obtained using pseudo inverse or “pinv” function however has seen in above topic the $l1$ minimum solution is more accurate and so it is obtained via the **cvx** optimisation package. CVX optimisation package is Matlab based modelling system for convex optimisation. So it will helpful to solve the under-determined system.

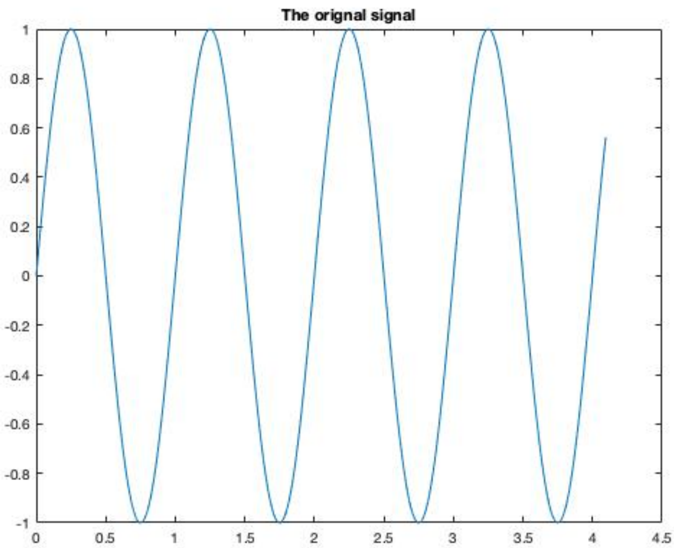
For reconstruction of audio signal

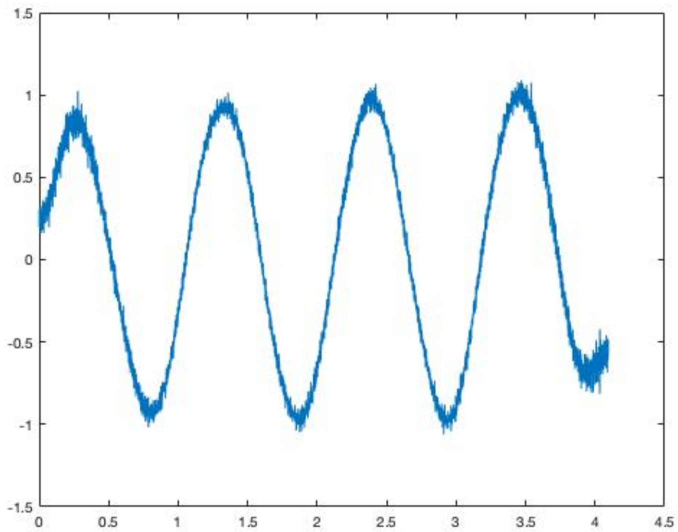
first a signal is generated that is to be measured. The original signal contains 4096 samples and it is a simple cosine signal. Now a random measurement matrix is created which will take the random measurement of the signal. The number of measurement taken are 128 which is much less compared the sample the original signal comprises of. Then solving the compressed sensing system for the given random measurement will give the original signal. The Matlab function dct return the discrete cosine transform to the input vector and the idct returns the inverse discrete cosine transform.

Code is as shown on the next page.

```
%Compressed Sensing for Audio signal
clc;clear all;close all;

%Generating a signal that is to be measured
n = 4096;
f = 1;
fs = 1000;
t = 0:1/fs:4095/fs;

x = sin(2*pi*f*t);
figure();
plot(t,x);
title('The orignal signal');

%Measuring the signal
p = n/32;
c = round(rand(p,1)*n);
y = x(c); %Random measurement of x stored in y


%Compressed Sensing
Psi = dct(eye(n));
Theta = Psi(c,:);
cvx_begin;
    variable s(n);
    minimize(norm(s,1));
    subject to
        Theta*s == y';
cvx_end;

xrecon = idct(s);
figure();
plot(t,xrecon);
```

## Results



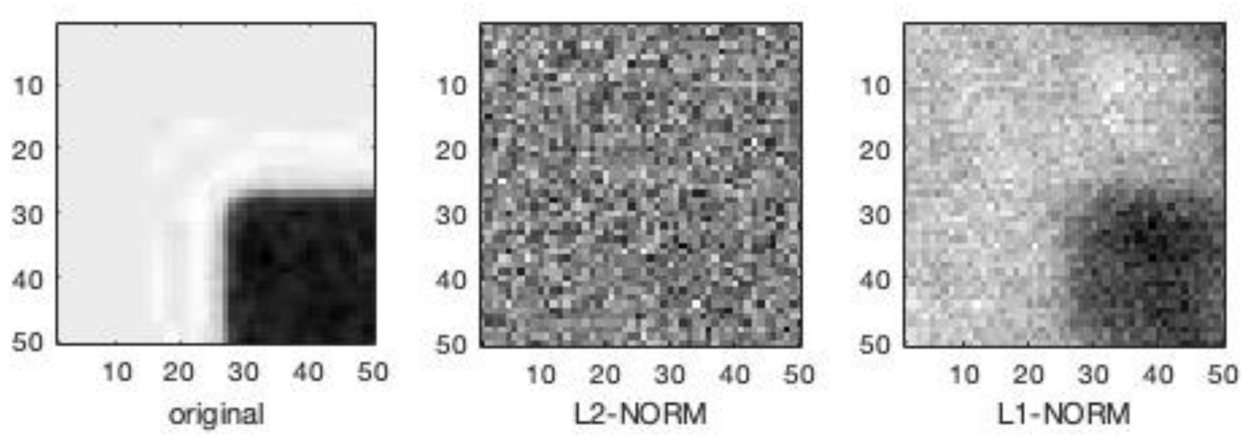
From the result it can be concluded that using compressed sensing it is possible to reconstruct a high-dimensional signal from a sparse set of random measurement.

For Image reconstruction

```
%Compressed sensing for images
clc;clear all; close all;

Img = imread('stairs.jpeg');
Img = Img([50:99],[50:99]);
x = double(Img(:));
n = length(x);

m = 250;
Phi = randn(m,n);

%___COMPRESSION___
y = Phi*x;

%___THETA___
Theta = zeros(m,n);
for ii = 1:n
    ii;
    ek = zeros(1,n);
    ek(ii) = 1;
    psi = idct(ek)';
    Theta(:,ii) = Phi*psi;
end

%___l2 NORM SOLUTION___
s2 = pinv(Theta)*y;

%___l1 NORM SOLUTION__
cvx_begin;
    variable s1(n);
    minimize(norm(s1,1));
    subject to
        Theta*s1 == y;
cvx_end;


%___IMAGE RECONSTRUCTIONS___
x2 = zeros(n,1);
for ii = 1:n
    ii;
    ek = zeros(1,n);
    ek(ii) = 1;
    psi = idct(ek)';
    x2 = x2+psi*s2(ii);
end

x1 = zeros(n,1);
for ii = 1:n
    ii;
```

```
    ek = zeros(1,n);
    ek(ii) = 1;
    psi = idct(ek)';
    x1 = x1+psi*s1(ii);
end

figure('name','Compressive sensing image reconstructions')
subplot(1,3,1), imagesc(reshape(x,50,50)), xlabel('original'), axis image
subplot(1,3,2), imagesc(reshape(x2,50,50)), xlabel('L2-NORM'), axis image
subplot(1,3,3), imagesc(reshape(x1,50,50)), xlabel('L1-NORM'), axis image
colormap gray
```

Results

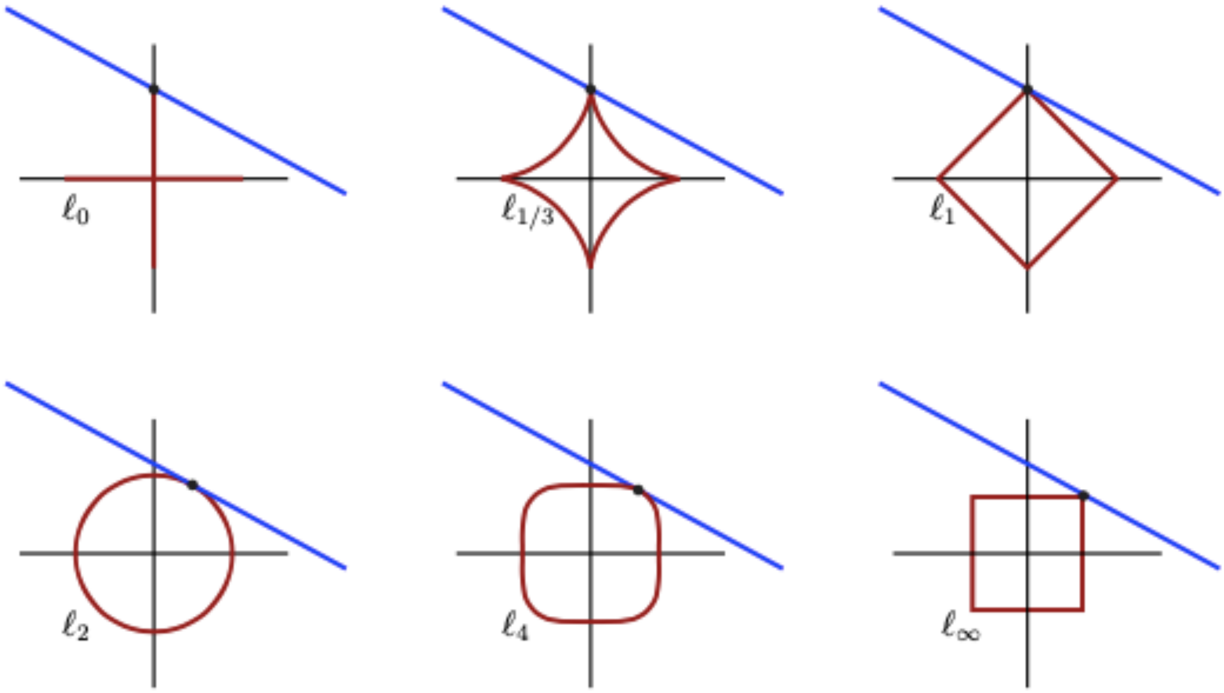



Observing the above result the it clearly shows that using the L1 norm the reconstructed image is very much alike the original image. The L2 norm solution is not able to give the correct output has it is distributing the energy in every vector coefficient.

**Geometry of compression**

Compressed sensing can be summarised in a relatively simple statement: A given signal, if it is sufficiently sparse in a known basis, may be recovered using significantly fewer measurement and these measurement are sufficiently random. Specially, enough good measurements will result in a matrix that preserve the distance and inner product structure of sparse vector s.

Determining how many measurements to take is relatively simple. If the signal is K-sparse in a basis $\Psi$, meaning that all but K coefficients are zero, then number of measurement scales as $p = klog(n/K)$. It is considered that is measurement are incoherent it is good and will give consistent and accurate result.




The minimum norm point on aline in different $lp$ norms. The blue line represents the solution set of an under-determined system of equations, and the red curve represents the the minimum-norm level sets that intersect this blue line for different norms. The minimum norm solution also corresponds to the sparest solution, with only one coordinate active.

**Conclusion**

The compressed sensing makes the signal acquisition much more efficient. Using proper optimisation and norm is very important. From the simulation it can be seen that rather than collecting high-dimensional data, just collecting few random data can will also be able to recreate the original information.