Problem 4

Find the differential equation whose general solution is the one-parameter family

$$x^2 + c\,y^2 = 2y,$$

and show (by elimination of the parameter and by solving the resulting ODE) that the family above is the general solution. State any domain restrictions or special cases.

Solution

1. Differentiate the family (relation) with respect to x

Treat c as an arbitrary constant and differentiate

$$x^2 + c\,y^2 = 2y$$

implicitly with respect to x. This gives

$$2x + 2cy\,y' = 2y',$$

or, simplifying,

$$x + y'(cy - 1) = 0. \tag{1}$$

2. Eliminate the parameter c

From the original algebraic relation we can solve for c:

$$c = \frac{2y - x^2}{y^2}.$$

Form the combination $cy - 1$ which appears in (1):

$$cy - 1 = \frac{2y - x^2}{y} - 1 = \frac{y - x^2}{y}.$$

Substitute this into (1). Multiplying by y to clear the denominator yields the first-order differential equation

$$(y - x^2)y' + xy = 0. \tag{2}$$

Thus every curve of the family $x^2 + c\,y^2 = 2y$ satisfies the first-order ODE (2). Next we solve (2) and recover the one-parameter family.

3. Solve the ODE

The ODE (2) is

$$(y - x^2)\frac{dy}{dx} + xy = 0.$$

This is a nonlinear first-order equation. Use the substitution

$$y(x) = x^2 u(x) \qquad (x \neq 0)$$

so that $u = \dfrac{y}{x^2}$. Then
$$\frac{dy}{dx} = 2xu + x^2u'.$$
Substitute into (2):
$$(x^2u - x^2)(2xu + x^2u') + x \cdot x^2u = 0.$$
Divide through by x^3 (assuming $x \neq 0$):
$$(u-1)(2u + xu') + u = 0.$$
Expand and collect terms in u':
$$2u^2 + xuu' - 2u - xu' + u = 0 \quad \Longrightarrow \quad x(u-1)u' + u(2u-1) = 0.$$
Solve for u':
$$\frac{du}{dx} = -\frac{u(2u-1)}{x(u-1)}.$$
Separate variables:
$$\frac{u-1}{u(2u-1)}\, du = -\frac{dx}{x}.$$
Perform partial fraction decomposition:
$$\frac{u-1}{u(2u-1)} = \frac{1}{u} - \frac{1}{2u-1},$$
so integrating both sides gives
$$\int \left(\frac{1}{u} - \frac{1}{2u-1}\right) du = -\int \frac{dx}{x}.$$
Thus
$$\ln|u| - \tfrac{1}{2}\ln|2u-1| = -\ln|x| + C,$$
where C is a constant of integration. Exponentiate (and combine constants):
$$\frac{|u|}{|2u-1|^{1/2}} = \frac{K}{|x|}, \qquad K = e^C > 0.$$
Squaring both sides gives
$$\frac{u^2}{2u-1} = \frac{K^2}{x^2}.$$
Rearrange and substitute back $u = \dfrac{y}{x^2}$. After a short algebraic simplification one obtains
$$y^2 = K^2(2y - x^2),$$
or equivalently
$$x^2 + \frac{1}{K^2}\, y^2 = 2y.$$
If we rename the arbitrary constant $c = \dfrac{1}{K^2}$ (note $K \neq 0$, so c is an arbitrary real constant; allowing the sign of the integration constant leads to c taking any real value and the special limiting case $c = 0$ is treated separately), we recover the family
$$\boxed{x^2 + c\,y^2 = 2y}$$
as the general solution of the ODE (2).

4. Special cases and domain remarks

- The derivation assumed $x \neq 0$ when dividing by powers of x. The original algebraic family must be checked at $x = 0$ separately: plugging $x = 0$ into $x^2 + cy^2 = 2y$ gives $cy^2 = 2y$, so either $y = 0$ or $y = 2/c$ (if $c \neq 0$). These are isolated $x = 0$ values consistent with the family and do not alter the conclusion that the one-parameter family is the general solution of the ODE on intervals where the operations above are valid.

- The algebraic manipulations used absolute values and squaring; this can introduce sign issues for particular branches. The constant c is an arbitrary real parameter (including positive, zero and negative values) that indexes the distinct solution curves; the case $c = 0$ gives the parabola $y = \frac{x^2}{2}$.

- The ODE (2) is satisfied by each member of the family $x^2 + cy^2 = 2y$ (for any real c), and solving (2) by the substitution above reproduces that family.

5. Verification

Differentiate $x^2 + cy^2 = 2y$ implicitly to get $2x + 2cyy' = 2y'$. Solving for y' yields

$$y'(cy - 1) = -x,$$

and substituting $c = (2y - x^2)/y^2$ recovers the ODE $(y - x^2)y' + xy = 0$. Conversely, integrating the ODE as done above leads back to $x^2 + cy^2 = 2y$. Hence the two descriptions are equivalent: the family $x^2 + cy^2 = 2y$ is the general solution of the differential equation

$$\boxed{(y - x^2)\,\frac{dy}{dx} + xy = 0\,.}$$