

Problem 26

Find k such that the roots of the trinomial

$$y = (x + k)^2 - 5(x + 3k) + 57$$

are reciprocals.

Solution

Step 1: Expand the trinomial

Expand $(x + k)^2$:

$$y = x^2 + 2kx + k^2 - 5x - 15k + 57$$

$$y = x^2 + (2k - 5)x + (k^2 - 15k + 57)$$

Step 2: Use condition for reciprocal roots

For a quadratic $ax^2 + bx + c = 0$, the roots are reciprocals if and only if $a = c$.

In our case:

- $a = 1$
- $b = 2k - 5$
- $c = k^2 - 15k + 57$

For reciprocal roots, we need:

$$a = c$$

$$1 = k^2 - 15k + 57$$

Step 3: Solve for k

$$k^2 - 15k + 57 = 1$$

$$k^2 - 15k + 56 = 0$$

Using the quadratic formula:

$$k = \frac{15 \pm \sqrt{225 - 224}}{2} = \frac{15 \pm \sqrt{1}}{2} = \frac{15 \pm 1}{2}$$

Therefore:

$$k = \frac{15 + 1}{2} = 8 \quad \text{or} \quad k = \frac{15 - 1}{2} = 7$$

Step 4: Verify the solutions

For $k = 8$:

$$y = x^2 + (2 \cdot 8 - 5)x + (64 - 120 + 57)$$

$$y = x^2 + 11x + 1$$

The roots satisfy:

$$x_1 \cdot x_2 = \frac{c}{a} = \frac{1}{1} = 1$$

So $x_2 = \frac{1}{x_1}$, confirming the roots are reciprocals.

Let's find the actual roots:

$$x = \frac{-11 \pm \sqrt{121 - 4}}{2} = \frac{-11 \pm \sqrt{117}}{2} = \frac{-11 \pm 3\sqrt{13}}{2}$$

Check that they're reciprocals:

$$x_1 x_2 = \frac{(-11 + 3\sqrt{13})(-11 - 3\sqrt{13})}{4} = \frac{121 - 9 \cdot 13}{4} = \frac{121 - 117}{4} = \frac{4}{4} = 1 \quad \checkmark$$

For $k = 7$:

$$y = x^2 + (2 \cdot 7 - 5)x + (49 - 105 + 57)$$

$$y = x^2 + 9x + 1$$

Similarly, $x_1 \cdot x_2 = 1$, so the roots are reciprocals.

Let's find the actual roots:

$$x = \frac{-9 \pm \sqrt{81 - 4}}{2} = \frac{-9 \pm \sqrt{77}}{2}$$

Check:

$$x_1 x_2 = \frac{(-9 + \sqrt{77})(-9 - \sqrt{77})}{4} = \frac{81 - 77}{4} = \frac{4}{4} = 1 \quad \checkmark$$

Step 5: Complete solution

$$\boxed{k = 7 \quad \text{or} \quad k = 8}$$

Alternative approach: Using Vieta's formulas directly

For roots x_1 and x_2 of $x^2 + bx + c = 0$:

$$x_1 + x_2 = -b, \quad x_1 x_2 = c$$

If the roots are reciprocals: $x_2 = \frac{1}{x_1}$, then:

$$x_1 \cdot \frac{1}{x_1} = 1$$

So the product of roots must equal 1:

$$x_1x_2 = c = 1$$

For a general quadratic $ax^2 + bx + c = 0$:

$$x_1x_2 = \frac{c}{a}$$

For reciprocal roots:

$$\frac{c}{a} = 1 \implies c = a$$

This confirms our approach.

Summary

For the trinomial $y = (x + k)^2 - 5(x + 3k) + 57$, which expands to $y = x^2 + (2k - 5)x + (k^2 - 15k + 57)$:

The condition for reciprocal roots is that the constant term equals the leading coefficient:

$$k^2 - 15k + 57 = 1$$

$$k^2 - 15k + 56 = 0$$

$$(k - 7)(k - 8) = 0$$

Therefore: $k = 7$ or $k = 8$.

Remark

The condition for reciprocal roots ($x_2 = \frac{1}{x_1}$) is elegantly captured by requiring $x_1x_2 = 1$. For a monic quadratic ($a = 1$), this means the constant term must equal 1. For a general quadratic $ax^2 + bx + c = 0$, the condition is $c = a$.

This problem demonstrates how algebraic conditions on roots translate to simple relationships between coefficients, making problems involving reciprocal roots particularly tractable.