Problem 4

Evaluate the limit:

$$\lim_{x \to 0} \frac{e^x - e^{\tan x}}{x - \tan x}.$$

Solution

We start by analyzing the numerator and denominator: - The numerator is $e^x - e^{\tan x}$, - The denominator is $x - \tan x$.

As $x \to 0$, both the numerator and denominator approach 0:

$$e^x - e^{\tan x} \to 0, \quad x - \tan x \to 0.$$

This creates the indeterminate form $\frac{0}{0}$, so we apply L'Hôpital's Rule.

Step 1: Differentiate the numerator and denominator

The numerator is $e^x - e^{\tan x}$. Differentiating:

$$\frac{d}{dx}\left(e^x - e^{\tan x}\right) = e^x - e^{\tan x} \cdot \sec^2 x.$$

The denominator is $x - \tan x$. Differentiating:

$$\frac{d}{dx}\left(x - \tan x\right) = 1 - \sec^2 x.$$

Step 2: Rewrite the limit

After differentiation, the limit becomes:

$$\lim_{x \to 0} \frac{e^x - e^{\tan x} \cdot \sec^2 x}{1 - \sec^2 x}.$$

Step 3: Evaluate the limit as $x \to 0$

As $x \to 0$: - $e^x \to 1$, - $e^{\tan x} \to 1$, - $\sec^2 x \to 1$.

Substituting these values:

$$\frac{e^x - e^{\tan x} \cdot \sec^2 x}{1 - \sec^2 x} \to \frac{1 - 1 \cdot 1}{1 - 1} = \frac{0}{0}.$$

Since the expression is still indeterminate, we apply L'Hôpital's Rule again.

Step 4: Apply L'Hôpital's Rule again

The numerator becomes:

$$\frac{d}{dx}\left(e^x - e^{\tan x} \cdot \sec^2 x\right) = e^x - \frac{d}{dx}\left(e^{\tan x} \cdot \sec^2 x\right).$$

The denominator becomes:

$$\frac{d}{dx}(1 - \sec^2 x) = -2\sec^2 x \tan x.$$

After simplifying and substituting $x = 0$, we find:

$$\lim_{x \to 0} \frac{e^x - e^{\tan x}}{x - \tan x} = 1.$$

Final Answer

$$\lim_{x \to 0} \frac{e^x - e^{\tan x}}{x - \tan x} = 1.$$