Problem 20

Determine the parameter k so that the given trinomial is a perfect square:

1. $y = (k+2)x^2 - 20x + 10k + 5$
2. $y = kx^2 - x\sqrt{k} + 3k + 1$

Solution

Part 1°: $y = (k+2)x^2 - 20x + 10k + 5$

Condition for perfect square

A quadratic $ax^2 + bx + c$ is a perfect square if and only if its discriminant equals zero:

$$\Delta = b^2 - 4ac = 0$$

For our trinomial:

- $a = k + 2$
- $b = -20$
- $c = 10k + 5$

Apply the discriminant condition

$$(-20)^2 - 4(k+2)(10k+5) = 0$$
$$400 - 4(k+2)(10k+5) = 0$$
$$400 - 4(10k^2 + 5k + 20k + 10) = 0$$
$$400 - 4(10k^2 + 25k + 10) = 0$$
$$400 - 40k^2 - 100k - 40 = 0$$
$$360 - 40k^2 - 100k = 0$$

Divide by 20:

$$18 - 2k^2 - 5k = 0$$
$$2k^2 + 5k - 18 = 0$$

Solve the quadratic equation

Using the quadratic formula:

$$k = \frac{-5 \pm \sqrt{25 + 144}}{4} = \frac{-5 \pm \sqrt{169}}{4} = \frac{-5 \pm 13}{4}$$

Therefore:

$$k = \frac{-5 + 13}{4} = \frac{8}{4} = 2 \quad \text{or} \quad k = \frac{-5 - 13}{4} = \frac{-18}{4} = -\frac{9}{2}$$

Verify the solutions

For $k = 2$:

$$y = (2+2)x^2 - 20x + 10(2) + 5 = 4x^2 - 20x + 25$$

Check if this is a perfect square:

$$4x^2 - 20x + 25 = (2x)^2 - 2 \cdot 2x \cdot 5 + 5^2 = (2x-5)^2 \quad \checkmark$$

For $k = -\frac{9}{2}$:

$$y = \left(-\frac{9}{2} + 2\right) x^2 - 20x + 10\left(-\frac{9}{2}\right) + 5 = -\frac{5}{2}x^2 - 20x - 45 + 5$$

$$= -\frac{5}{2}x^2 - 20x - 40 = -\frac{5}{2}(x^2 + 8x + 16) = -\frac{5}{2}(x+4)^2 \quad \checkmark$$

Answer for Part 1

$$\boxed{k = 2 \quad \text{or} \quad k = -\frac{9}{2}}$$

Part $2°$: $y = kx^2 - x\sqrt{k} + 3k + 1$

Condition for perfect square

For the trinomial to be a perfect square, we need $\Delta = 0$:

Here:

- $a = k$
- $b = -\sqrt{k}$
- $c = 3k + 1$

Note: For $\sqrt{k}$ to be real, we need $k \geq 0$.

Apply the discriminant condition

$$(-\sqrt{k})^2 - 4k(3k+1) = 0$$

$$k - 4k(3k+1) = 0$$

$$k - 12k^2 - 4k = 0$$

$$-3k - 12k^2 = 0$$

$$-3k(1 + 4k) = 0$$

This gives us:

$$k = 0 \quad \text{or} \quad 1 + 4k = 0$$

$$k = 0 \quad \text{or} \quad k = -\frac{1}{4}$$

Check validity

Since we need $k \geq 0$ for $\sqrt{k}$ to be real, we must reject $k = -\frac{1}{4}$.

However, let's check $k = 0$:

For $k = 0$:

$$y = 0 \cdot x^2 - x\sqrt{0} + 3(0) + 1 = 1$$

This is a constant, not a quadratic. While technically a "perfect square" $(1 = 1^2)$, it's degenerate.

Alternative interpretation

If we allow complex values or interpret the problem differently, let's reconsider $k = -\frac{1}{4}$ algebraically (even though $\sqrt{k}$ would be imaginary):

For completeness in the algebraic sense, if we had $k = -\frac{1}{4}$, the coefficient of x would be $-x\sqrt{-1/4} = -x \cdot \frac{i}{2} = -\frac{ix}{2}$.

Reconsider with $k > 0$

Let's verify there might be another solution. Going back to:

$$k - 12k^2 - 4k = 0$$

$$12k^2 + 3k = 0$$

$$3k(4k + 1) = 0$$

So $k = 0$ or $k = -\frac{1}{4}$.

Since $k = -\frac{1}{4} < 0$ makes $\sqrt{k}$ undefined in real numbers, and $k = 0$ gives a degenerate case:

Answer for Part 2

If we require a non-degenerate quadratic with real coefficients:

$$\boxed{\text{No valid solution for } k > 0}$$

If we accept the degenerate case:

$$\boxed{k = 0}$$

Summary

Part 1: $k = 2$ or $k = -\frac{9}{2}$

Part 2: $k = 0$ (degenerate case), or no real solution for non-degenerate quadratics.

Remark

A quadratic trinomial $ax^2 + bx + c$ is a perfect square if and only if $\Delta = b^2 - 4ac = 0$. This is equivalent to the trinomial having a repeated root, which means it can be written as $a(x - r)^2$ for some value r.

For Part 1, both values of k produce valid perfect squares:

- $k = 2$ gives $(2x - 5)^2$
- $k = -\frac{9}{2}$ gives $-\frac{5}{2}(x + 4)^2$

For Part 2, the constraint that $k \geq 0$ (for $\sqrt{k}$ to be real) combined with the discriminant condition leads to only the degenerate solution $k = 0$.