Problem 10

Evaluate the sum S for the series:

$$S = \sum_{n=1}^{\infty} \operatorname{arctg} \frac{1}{2n^2}.$$

Solution

Step 1: Decomposing the General Term

The general term of the series is $a_n = \operatorname{arctg} \frac{1}{2n^2}$. We use the arctangent subtraction formula:

$$\operatorname{arctg}(x) - \operatorname{arctg}(y) = \operatorname{arctg}\left(\frac{x-y}{1+xy}\right).$$

We will show that a_n can be expressed as a difference of two terms from a known sequence:

$$\operatorname{arctg}\left(\frac{1}{2n-1}\right) - \operatorname{arctg}\left(\frac{1}{2n+1}\right).$$

Applying the formula with $x = \frac{1}{2n-1}$ and $y = \frac{1}{2n+1}$:

$$x - y = \frac{1}{2n-1} - \frac{1}{2n+1} = \frac{(2n+1)-(2n-1)}{(2n-1)(2n+1)} = \frac{2}{4n^2-1}.$$

$$1 + xy = 1 + \frac{1}{(2n-1)(2n+1)} = 1 + \frac{1}{4n^2-1} = \frac{(4n^2-1)+1}{4n^2-1} = \frac{4n^2}{4n^2-1}.$$

The argument of the resulting arctangent is:

$$\frac{x-y}{1+xy} = \frac{\frac{2}{4n^2-1}}{\frac{4n^2}{4n^2-1}} = \frac{2}{4n^2} = \frac{1}{2n^2}.$$

Thus, the general term is:

$$a_n = \operatorname{arctg}\left(\frac{1}{2n-1}\right) - \operatorname{arctg}\left(\frac{1}{2n+1}\right).$$

This expression is in the form $b_n - b_{n+1}$, where $b_n = \operatorname{arctg}\left(\frac{1}{2n-1}\right)$ and $b_{n+1} = \operatorname{arctg}\left(\frac{1}{2(n+1)-1}\right) = \operatorname{arctg}\left(\frac{1}{2n+1}\right)$.

Step 2: Determine the Partial Sum S_N

The series is a **telescoping series**. The N-th partial sum S_N is the sum of the first N terms:

$$S_N = \sum_{n=1}^{N} \left[\operatorname{arctg}\left(\frac{1}{2n-1}\right) - \operatorname{arctg}\left(\frac{1}{2n+1}\right)\right].$$

Writing out the terms:

$$\begin{aligned} S_N = &\left[\operatorname{arctg}\left(\frac{1}{1}\right) - \operatorname{arctg}\left(\frac{1}{3}\right)\right] + \left[\operatorname{arctg}\left(\frac{1}{3}\right) - \operatorname{arctg}\left(\frac{1}{5}\right)\right] + \cdots \\ &+ \left[\operatorname{arctg}\left(\frac{1}{2N-1}\right) - \operatorname{arctg}\left(\frac{1}{2N+1}\right)\right]. \end{aligned}$$

After cancellation, the simplified partial sum is left with the first term of the first parenthesis and the last term of the last parenthesis:

$$S_N = \operatorname{arctg}(1) - \operatorname{arctg}\left(\frac{1}{2N+1}\right).$$

Step 3: Calculate the Sum S

The sum of the series S is the limit of the partial sum S_N as $N \to \infty$:

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left[\operatorname{arctg}(1) - \operatorname{arctg}\left(\frac{1}{2N+1}\right)\right].$$

We evaluate the limit of the second term:

$$\lim_{N\to\infty} \frac{1}{2N+1} = 0.$$

Since the arctangent function is continuous, $\lim_{x\to 0} \operatorname{arctg}(x) = 0$:

$$\lim_{N\to\infty} \operatorname{arctg}\left(\frac{1}{2N+1}\right) = \operatorname{arctg}(0) = 0.$$

The final sum is:

$$S = \operatorname{arctg}(1) - 0 = \frac{\pi}{4}.$$

Final Answer

Partial Sum S_N

The N-th partial sum is:

$$\boxed{S_N = \operatorname{arctg}(1) - \operatorname{arctg}\left(\frac{1}{2N+1}\right) = \frac{\pi}{4} - \operatorname{arctg}\left(\frac{1}{2N+1}\right).}$$

Sum S

The sum of the series is:

$$\boxed{S = \frac{\pi}{4}.}$$