Problem 41

Evaluate the limit:

$$\lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2}.$$

Solution

To solve this, we proceed step by step.

Step 1: Behavior of $\tan \frac{\pi x}{2}$ as $x \to 1$

As $x \to 1$, the argument of the tangent, $\frac{\pi x}{2}$, approaches $\frac{\pi}{2}$, where $\tan \frac{\pi x}{2}$ becomes unbounded. This suggests the product $(1 - x) \tan \frac{\pi x}{2}$ requires further analysis.

Step 2: Change variables

Let:

$$t = \frac{\pi}{2} - \frac{\pi x}{2}.$$

As $x \to 1$, $t \to 0^+$, and:

$$\frac{\pi x}{2} = \frac{\pi}{2} - t.$$

From the definition of t, we can rewrite:

$$1 - x = \frac{2t}{\pi}.$$

The limit becomes:

$$\lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2} = \lim_{t \to 0^+} \frac{2t}{\pi} \cdot \tan \left(\frac{\pi}{2} - t \right).$$

Step 3: Simplify $\tan \left(\frac{\pi}{2} - t \right)$

Using the trigonometric identity:

$$\tan \left(\frac{\pi}{2} - t \right) = \cot t = \frac{1}{\tan t}.$$

Thus, the limit becomes:

$$\lim_{t \to 0^+} \frac{2t}{\pi} \cdot \frac{1}{\tan t}.$$

Step 4: Approximate $\tan t$ for small t

For small t, $\tan t \approx t$. Substituting this approximation:

$$\frac{1}{\tan t} \approx \frac{1}{t}.$$

Substituting back into the limit:

$$\lim_{t \to 0^+} \frac{2t}{\pi} \cdot \frac{1}{t} = \lim_{t \to 0^+} \frac{2}{\pi}.$$

Final Answer

The limit evaluates to:

$$\boxed{\frac{2}{\pi}}.$$