Problem 19

Evaluate the following limit:

$$\lim_{x \to 0} \sqrt[x]{\cos\left(\sqrt{x}\right)}.$$

Solution

We start by rewriting the expression $\sqrt[x]{f(x)}$ as:

$$\sqrt[x]{f(x)} = e^{\frac{\ln(f(x))}{x}}.$$

Thus, the original limit becomes:

$$\lim_{x \to 0} \sqrt[x]{\cos\left(\sqrt{x}\right)} = \lim_{x \to 0} e^{\frac{\ln\left(\cos\left(\sqrt{x}\right)\right)}{x}}.$$

Behavior of $\cos\left(\sqrt{x}\right)$ as $x \to 0$

Since $\cos(0) = 1$, we have:

$$\cos\left(\sqrt{x}\right) \to 1 \quad \text{as} \quad x \to 0.$$

Using the Taylor expansion for $\cos(z)$ around 0:

$$\cos(z) = 1 - \frac{z^2}{2} + O(z^4),$$

we substitute $z = \sqrt{x}$ to get:

$$\cos\left(\sqrt{x}\right) = 1 - \frac{x}{2} + O(x^2).$$

Taking the Natural Logarithm

We now take the natural logarithm of $\cos\left(\sqrt{x}\right)$:

$$\ln\left(\cos\left(\sqrt{x}\right)\right) = \ln\left(1 - \frac{x}{2} + O(x^2)\right).$$

Using the approximation $\ln(1+z) \approx z$ for small z, we get:

$$\ln\left(1 - \frac{x}{2} + O(x^2)\right) \approx -\frac{x}{2}.$$

Substituting into the Limit Expression

Substituting this approximation into the limit expression:

$$\lim_{x \to 0} e^{\frac{\ln\left(\cos\left(\sqrt{x}\right)\right)}{x}} = \lim_{x \to 0} e^{\frac{-\frac{x}{2}}{x}} = \lim_{x \to 0} e^{-\frac{1}{2}}.$$

Final Value of the Limit

Since the exponent is constant, we obtain:

$$e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

Conclusion

Therefore, the value of the limit is:

$$\lim_{x \to 0} \sqrt[x]{\cos\left(\sqrt{x}\right)} = \frac{1}{\sqrt{e}}.$$