Problem 16

Solve the equation:

$$32\cos^6 x - \cos 6x = 1$$

Solution

Step 1: Use Chebyshev polynomial or multiple angle formula

We need to express $\cos 6x$ in terms of powers of $\cos x$. Using the multiple angle formula:

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$$

Step 2: Substitute into the equation

$$32\cos^6 x - (32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1) = 1$$

$$32\cos^6 x - 32\cos^6 x + 48\cos^4 x - 18\cos^2 x + 1 = 1$$

$$48\cos^4 x - 18\cos^2 x = 0$$

Step 3: Factor

Factor out $\cos^2 x$:

$$\cos^2 x(48\cos^2 x - 18) = 0$$

$$6\cos^2 x(8\cos^2 x - 3) = 0$$

This gives us two cases:

Case 1: $\cos^2 x = 0$

$$\cos x = 0$$

The general solution is:

$$\boxed{x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}}$$

In degrees: $x = 90 + 180k$

Case 2: $8\cos^2 x - 3 = 0$

$$\cos^2 x = \frac{3}{8}$$

$$\cos x = \pm\sqrt{\frac{3}{8}} = \pm\frac{\sqrt{3}}{2\sqrt{2}} = \pm\frac{\sqrt{6}}{4}$$

The general solutions are:

$$\boxed{x = \pm \arccos\left(\frac{\sqrt{6}}{4}\right) + 2k\pi, \quad k \in \mathbb{Z}}$$

Numerically: $\arccos\left(\frac{\sqrt{6}}{4}\right) \approx 52.24$
So: $x \approx \pm 52.24 + 360k$

Step 4: Complete solution set

$$\boxed{x \in \left\{\frac{\pi}{2} + k\pi, \pm \arccos\left(\frac{\sqrt{6}}{4}\right) + 2k\pi : k \in \mathbb{Z}\right\}}$$

Or in $[0, 2\pi)$:

$$x \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \arccos\left(\frac{\sqrt{6}}{4}\right), 2\pi - \arccos\left(\frac{\sqrt{6}}{4}\right), \pi - \arccos\left(\frac{\sqrt{6}}{4}\right), \pi + \arccos\left(\frac{\sqrt{6}}{4}\right)\right\}$$

Verification

For $x = \frac{\pi}{2}$ (or 90):

$$\cos x = 0, \quad \cos 6x = \cos(3\pi) = -1$$
$$32(0)^6 - (-1) = 0 + 1 = 1 \quad \checkmark$$

For $\cos x = \frac{\sqrt{6}}{4}$:
We have $\cos^2 x = \frac{3}{8}$, so:

$$\cos^4 x = \frac{9}{64}, \quad \cos^6 x = \frac{27}{512}$$

Left side:

$$32 \cdot \frac{27}{512} = \frac{27}{16}$$

For $\cos 6x$, we use:

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$$

$$= 32 \cdot \frac{27}{512} - 48 \cdot \frac{9}{64} + 18 \cdot \frac{3}{8} - 1$$
$$= \frac{27}{16} - \frac{27}{4} + \frac{27}{4} - 1 = \frac{27}{16} - 1 = \frac{11}{16}$$

Therefore:

$$32\cos^6 x - \cos 6x = \frac{27}{16} - \frac{11}{16} = \frac{16}{16} = 1 \quad \checkmark$$

Alternative: Using the identity directly

There's a known identity:

$$32\cos^6 x = \cos 6x + 6\cos 4x + 15\cos 2x + 10$$

Using this:

$$\cos 6x + 6\cos 4x + 15\cos 2x + 10 - \cos 6x = 1$$

$$6\cos 4x + 15\cos 2x + 9 = 0$$

$$2\cos 4x + 5\cos 2x + 3 = 0$$

Let $u = \cos 2x$. Using $\cos 4x = 2\cos^2 2x - 1 = 2u^2 - 1$:

$$2(2u^2 - 1) + 5u + 3 = 0$$

$$4u^2 + 5u + 1 = 0$$

Using the quadratic formula:

$$u = \frac{-5 \pm \sqrt{25 - 16}}{8} = \frac{-5 \pm 3}{8}$$

So $u = -\frac{1}{4}$ or $u = -1$.
Case 1: $\cos 2x = -1$

$$2x = \pi + 2k\pi \implies x = \frac{\pi}{2} + k\pi$$

Case 2: $\cos 2x = -\frac{1}{4}$

$$2x = \pm \arccos\left(-\frac{1}{4}\right) + 2k\pi$$

$$x = \pm \frac{1}{2} \arccos\left(-\frac{1}{4}\right) + k\pi$$

Note: $\arccos(-\frac{1}{4}) = \pi - \arccos(\frac{1}{4})$
Using $\cos 2x = 2\cos^2 x - 1 = -\frac{1}{4}$:

$$2\cos^2 x = \frac{3}{4} \implies \cos^2 x = \frac{3}{8} \implies \cos x = \pm\frac{\sqrt{6}}{4}$$

This confirms our earlier result!

Summary

The equation $32\cos^6 x - \cos 6x = 1$ simplifies to:

$$48\cos^4 x - 18\cos^2 x = 0$$

$$6\cos^2 x(8\cos^2 x - 3) = 0$$

Solutions:

- From $\cos x = 0$: $x = \frac{\pi}{2} + k\pi$
- From $\cos x = \pm\frac{\sqrt{6}}{4}$: $x = \pm \arccos\left(\frac{\sqrt{6}}{4}\right) + 2k\pi$

Remark

This problem demonstrates the power of multiple angle formulas, particularly the Chebyshev polynomial representation of $\cos 6x$ in terms of powers of $\cos x$. The key insight is that when we substitute the expansion of $\cos 6x$, the sixth-degree terms cancel, leaving a simpler fourth-degree equation in $\cos x$ that factors nicely.

The identity $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$ is part of a general pattern where $\cos(nx)$ can be expressed as a polynomial in $\cos x$, known as Chebyshev polynomials of the first kind.