

Problem 17

Solve the equation:

$$\tan^2 x + \cot^2 x - 14 = 0, \quad (0 < x < \pi)$$

tocsectionProblem 17: Tangent and cotangent equation

Solution

Step 1: Use the relationship between tangent and cotangent

Recall that $\cot x = \frac{1}{\tan x}$. Let $t = \tan x$, so $\cot x = \frac{1}{t}$.

The equation becomes:

$$t^2 + \frac{1}{t^2} - 14 = 0$$

Step 2: Multiply through by t^2

Multiply both sides by t^2 (assuming $t \neq 0$):

$$t^4 + 1 - 14t^2 = 0$$

$$t^4 - 14t^2 + 1 = 0$$

Step 3: Solve as a quadratic in t^2

Let $u = t^2 = \tan^2 x$. The equation becomes:

$$u^2 - 14u + 1 = 0$$

Using the quadratic formula:

$$u = \frac{14 \pm \sqrt{196 - 4}}{2} = \frac{14 \pm \sqrt{192}}{2} = \frac{14 \pm 8\sqrt{3}}{2} = 7 \pm 4\sqrt{3}$$

Step 4: Find $\tan^2 x$

We have two cases:

$$\tan^2 x = 7 + 4\sqrt{3} \quad \text{or} \quad \tan^2 x = 7 - 4\sqrt{3}$$

Let's simplify these values:

For $7 + 4\sqrt{3}$:

$$7 + 4\sqrt{3} \approx 7 + 6.928 = 13.928$$

For $7 - 4\sqrt{3}$:

$$7 - 4\sqrt{3} \approx 7 - 6.928 = 0.072$$

Step 5: Recognize special values

Notice that:

$$(2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}$$

$$(2 - \sqrt{3})^2 = 4 - 4\sqrt{3} + 3 = 7 - 4\sqrt{3}$$

Therefore:

$$\tan^2 x = (2 + \sqrt{3})^2 \quad \text{or} \quad \tan^2 x = (2 - \sqrt{3})^2$$

Taking square roots:

$$|\tan x| = 2 + \sqrt{3} \quad \text{or} \quad |\tan x| = 2 - \sqrt{3}$$

Step 6: Solve for x in the interval $(0, \pi)$

Since $0 < x < \pi$, we need to consider the sign of $\tan x$ in each quadrant:

- For $0 < x < \frac{\pi}{2}$: $\tan x > 0$
- For $\frac{\pi}{2} < x < \pi$: $\tan x < 0$

Case 1: $\tan x = 2 + \sqrt{3}$

Note that $2 + \sqrt{3} = \tan 75$ (since $\tan 75 = \tan(45 + 30) = \frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}$).

Therefore:

$$x = \frac{5\pi}{12} \quad (\text{or } 75)$$

Case 2: $\tan x = -(2 + \sqrt{3})$

This occurs in the second quadrant:

$$x = \pi - \frac{5\pi}{12} = \frac{7\pi}{12}$$

$$x = \frac{7\pi}{12} \quad (\text{or } 105)$$

Case 3: $\tan x = 2 - \sqrt{3}$

Note that $2 - \sqrt{3} = \tan 15$ (since $\tan 15 = \tan(45 - 30) = \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}$).

Therefore:

$$x = \frac{\pi}{12} \quad (\text{or } 15)$$

Case 4: $\tan x = -(2 - \sqrt{3})$

This occurs in the second quadrant:

$$x = \pi - \frac{\pi}{12} = \frac{11\pi}{12}$$

$$\boxed{x = \frac{11\pi}{12}} \quad (\text{or } 165)$$

Step 7: Complete solution set

In the interval $(0, \pi)$:

$$\boxed{x \in \left\{ \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12} \right\}}$$

Or in degrees:

$$\boxed{x \in \{15, 75, 105, 165\}}$$

Verification

Let's verify with $x = \frac{\pi}{12}$ (or 15), where $\tan x = 2 - \sqrt{3}$:

$$\begin{aligned} \tan^2 x &= (2 - \sqrt{3})^2 = 7 - 4\sqrt{3} \\ \cot^2 x &= \frac{1}{\tan^2 x} = \frac{1}{7 - 4\sqrt{3}} = \frac{7 + 4\sqrt{3}}{(7 - 4\sqrt{3})(7 + 4\sqrt{3})} = \frac{7 + 4\sqrt{3}}{49 - 48} = 7 + 4\sqrt{3} \end{aligned}$$

Therefore:

$$\tan^2 x + \cot^2 x = (7 - 4\sqrt{3}) + (7 + 4\sqrt{3}) = 14 \quad \checkmark$$

Summary

The equation $\tan^2 x + \cot^2 x - 14 = 0$ was transformed to:

$$t^4 - 14t^2 + 1 = 0 \quad \text{where } t = \tan x$$

This gives $\tan^2 x = 7 \pm 4\sqrt{3} = (2 \pm \sqrt{3})^2$.

The four solutions in $(0, \pi)$ are:

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

These correspond to $|\tan x| = 2 \pm \sqrt{3}$ with appropriate signs for each quadrant.

Remark

This problem demonstrates several key techniques:

1. Using the substitution $\cot x = \frac{1}{\tan x}$ to convert to a polynomial equation
2. Recognizing that the resulting quartic factors as a quadratic in $\tan^2 x$
3. Identifying that $7 \pm 4\sqrt{3} = (2 \pm \sqrt{3})^2$ are perfect squares
4. Recognizing that $2 + \sqrt{3} = \tan 75$ and $2 - \sqrt{3} = \tan 15$

These special tangent values arise from the tangent addition formulas for 45 ± 30 and are worth memorizing for solving trigonometric equations efficiently.