Problem 5

Find the general solution of the differential equation whose solution is given by

$$y^2 + cx = x^3.$$

State the differential equation, solve it (showing steps), and give any domain restrictions.

Solution

1. Determine the differential equation

Assume the solution has the form

$$y^2(x) + cx = x^3,$$

where c is an arbitrary constant. Differentiate both sides with respect to x:

$$\frac{d}{dx}(y^2 + cx) = \frac{d}{dx}(x^3).$$

Using the chain rule on the left-hand side and the power rule on the right-hand side:

$$2y\,\frac{dy}{dx} + c = 3x^2.$$

To eliminate the arbitrary constant c, we solve for it from the original equation:

$$c = x^3 - y^2.$$

Substitute this expression for c into the differentiated equation:

$$2y\,y' + (x^3 - y^2) = 3x^2.$$

Rearranging:

$$2y\,y' = 3x^2 - x^3 + y^2.$$

Thus the differential equation satisfied by y is

$$\boxed{2y\,y' = y^2 + 3x^2 - x^3, \qquad y \neq 0.}$$

Alternatively, this can be written as

$$\boxed{y' = \frac{y^2 + 3x^2 - x^3}{2y}, \qquad y \neq 0.}$$

2. Solve the differential equation

We solve the first-order ODE

$$2y\,y' = y^2 + 3x^2 - x^3.$$

This equation is separable. Rearrange as:

$$2y\,dy = (y^2 + 3x^2 - x^3)\,dx.$$

However, notice that y^2 appears on the right-hand side, making direct separation difficult. Instead, we use the fact that from the differentiated form

$$2y\,y' + c = 3x^2,$$

we can write

$$2y\,y' = 3x^2 - c.$$

Integrate both sides with respect to x:

$$\int 2y\,\frac{dy}{dx}\,dx = \int (3x^2 - c)\,dx.$$

The left-hand side is

$$\int 2y\,dy = y^2 + C_1.$$

The right-hand side is

$$\int (3x^2 - c)\,dx = x^3 - cx + C_2.$$

Equating and combining constants:

$$y^2 = x^3 - cx + C_0,$$

where $C_0 = C_2 - C_1$. Setting $C_0 = 0$ (which can be absorbed into c):

$$\boxed{y^2 + cx = x^3,}$$

which matches the given form. The constant c is arbitrary.

3. Verification

Differentiate $y^2 + cx = x^3$ implicitly:

$$2y\,y' + c = 3x^2.$$

From the original equation, $c = x^3 - y^2$, so:

$$2y\,y' + (x^3 - y^2) = 3x^2,$$

which simplifies to

$$2y\,y' = y^2 + 3x^2 - x^3,$$

confirming that the function indeed satisfies the derived ODE. **Remark.** The solution

is defined implicitly and requires $y \neq 0$ for the differential equation to be well-defined. For different values of the constant c, we obtain a family of curves. The solution can be written explicitly as

$$y = \pm\sqrt{x^3 - cx},$$

which requires $x^3 - cx \geq 0$ for real solutions, giving domain restrictions that depend on the value of c.