Problem 14

Evaluate the following limit:

$$\lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{\sqrt{x} - 4}$$

Solution

We start by substituting $x = 16$ into both the numerator and the denominator:

Substitute $x = 16$

- Numerator:

$$\sqrt[4]{x} - 2 = \sqrt[4]{16} - 2 = 2 - 2 = 0$$

- Denominator:

$$\sqrt{x} - 4 = \sqrt{16} - 4 = 4 - 4 = 0$$

Since both the numerator and denominator are 0, we have an indeterminate form $\frac{0}{0}$. Therefore, we will apply L'Hôpital's Rule.

Apply L'Hôpital's Rule

Differentiate the numerator and denominator:

- Numerator:

$$\frac{d}{dx}\left(\sqrt[4]{x}\right) = \frac{d}{dx}\left(x^{1/4}\right) = \frac{1}{4}x^{-3/4}$$

- Denominator:

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}$$

Evaluate the New Limit

Now, we evaluate the limit of the new expression:

$$\lim_{x \to 16} \frac{\frac{1}{4}x^{-3/4}}{\frac{1}{2}x^{-1/2}}$$

Simplifying:

$$\frac{\frac{1}{4}x^{-3/4}}{\frac{1}{2}x^{-1/2}} = \frac{1}{4} \times \frac{2}{1} \times \frac{x^{-3/4}}{x^{-1/2}} = \frac{1}{2} \times x^{-3/4+1/2} = \frac{1}{2} \times x^{-1/4}$$

Substitute $x = 16$

Substitute $x = 16$ into the simplified expression:

$$\frac{1}{2} \times 16^{-1/4}$$

Since $16 = 2^4$, we have:

$$16^{-1/4} = (2^4)^{-1/4} = 2^{-1} = \frac{1}{2}$$

Thus, the limit is:

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Conclusion

Therefore, the limit is:

$$\lim_{x \to 16} \frac{\sqrt[4]{x} - 2}{\sqrt{x} - 4} = \frac{1}{4}$$