Problem 83

Evaluate the integral:

$$I = \int \frac{dx}{e^{x/2} + e^x}.$$

Solution

We start by making the substitution:

$$t = e^{x/2}, \quad \text{so that} \quad dt = \frac{1}{2}e^{x/2}dx = \frac{1}{2}tdx.$$

Rewriting the integral in terms of t:

$$I = \int \frac{dx}{t + t^2}.$$

Since $dx = \frac{2dt}{t}$, we substitute:

$$I = \int \frac{\frac{2dt}{t}}{t + t^2}.$$

Simplifying:

$$I = 2\int \frac{dt}{t^2 + t}.$$

Factoring the denominator:

$$I = 2\int \frac{dt}{t(t+1)}.$$

Now, we use partial fraction decomposition:

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}.$$

Multiplying both sides by $t(t+1)$, we get:

$$1 = A(t+1) + Bt.$$

Expanding:

$$1 = At + A + Bt.$$

Grouping like terms:

$$1 = (A+B)t + A.$$

Setting up equations:

$$A + B = 0, \quad A = 1.$$

Solving for B:

$$B = -1.$$

Thus:

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}.$$

Substituting back into the integral:

$$I = 2\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt.$$

Splitting into two integrals:

$$I = 2\left(\int \frac{dt}{t} - \int \frac{dt}{t+1}\right).$$

Evaluating each integral:

$$I = 2(\ln|t| - \ln|t+1|) + C.$$

Using logarithm properties:

$$I = 2\ln\left|\frac{t}{t+1}\right| + C.$$

Substituting back $t = e^{x/2}$:

$$I = 2\ln\left|\frac{e^{x/2}}{e^{x/2}+1}\right| + C.$$