Problem 18

Solve the equation:

$$\sin x - \cos x - |\sin x + \cos x| = 1$$

Solution

Step 1: Consider cases based on the sign of $\sin x + \cos x$

The absolute value requires us to consider two cases:

Case 1: $\sin x + \cos x \geq 0$

Then $|\sin x + \cos x| = \sin x + \cos x$, and the equation becomes:

$$\sin x - \cos x - (\sin x + \cos x) = 1$$

$$\sin x - \cos x - \sin x - \cos x = 1$$

$$-2\cos x = 1$$

$$\cos x = -\frac{1}{2}$$

Case 2: $\sin x + \cos x < 0$

Then $|\sin x + \cos x| = -(\sin x + \cos x)$, and the equation becomes:

$$\sin x - \cos x - (-(\sin x + \cos x)) = 1$$

$$\sin x - \cos x + \sin x + \cos x = 1$$

$$2\sin x = 1$$

$$\sin x = \frac{1}{2}$$

Step 2: Solve Case 1 with constraint

From Case 1: $\cos x = -\frac{1}{2}$

The general solutions are:

$$x = \pm\frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

Or: $x = \frac{2\pi}{3} + 2k\pi$ or $x = \frac{4\pi}{3} + 2k\pi$

Now we must verify the constraint $\sin x + \cos x \geq 0$:

For $x = \frac{2\pi}{3}$ (or 120):

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{2\pi}{3} = -\frac{1}{2}$$

$$\sin x + \cos x = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2} \approx 0.366 > 0 \quad \checkmark$$

For $x = \frac{4\pi}{3}$ (or 240):

$$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}, \quad \cos\frac{4\pi}{3} = -\frac{1}{2}$$

$$\sin x + \cos x = -\frac{\sqrt{3}}{2} - \frac{1}{2} = -\frac{\sqrt{3}+1}{2} \approx -1.366 < 0 \quad \times$$

Therefore, from Case 1, only:

$$\boxed{x = \frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}}$$

Step 3: Solve Case 2 with constraint

From Case 2: $\sin x = \frac{1}{2}$

The general solutions are:

$$x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$$

Now we must verify the constraint $\sin x + \cos x < 0$:

For $x = \frac{\pi}{6}$ (or 30):

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sin x + \cos x = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} \approx 1.366 > 0 \quad \times$$

For $x = \frac{5\pi}{6}$ (or 150):

$$\sin\frac{5\pi}{6} = \frac{1}{2}, \quad \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\sin x + \cos x = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} \approx -0.366 < 0 \quad \checkmark$$

Therefore, from Case 2, only:

$$\boxed{x = \frac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}}$$

Step 4: Complete solution set

$$\boxed{x \in \left\{\frac{2\pi}{3} + 2k\pi, \frac{5\pi}{6} + 2k\pi : k \in \mathbb{Z}\right\}}$$

Or in degrees:

$$\boxed{x \in \{120 + 360k, 150 + 360k : k \in \mathbb{Z}\}}$$

In the interval $[0, 2\pi)$:

$$x \in \left\{\frac{2\pi}{3}, \frac{5\pi}{6}\right\} = \{120, 150\}$$

Verification

For $x = \frac{2\pi}{3}$ (or 120):

$$\sin x = \frac{\sqrt{3}}{2}, \quad \cos x = -\frac{1}{2}$$

$$\sin x + \cos x = \frac{\sqrt{3}-1}{2} > 0$$

$$|\sin x + \cos x| = \frac{\sqrt{3}-1}{2}$$

Left side:

$$\frac{\sqrt{3}}{2} - (-\frac{1}{2}) - \frac{\sqrt{3}-1}{2} = \frac{\sqrt{3}+1-\sqrt{3}+1}{2} = \frac{2}{2} = 1 \quad \checkmark$$

For $x = \frac{5\pi}{6}$ (or 150):

$$\sin x = \frac{1}{2}, \quad \cos x = -\frac{\sqrt{3}}{2}$$

$$\sin x + \cos x = \frac{1-\sqrt{3}}{2} < 0$$

$$|\sin x + \cos x| = \frac{\sqrt{3}-1}{2}$$

Left side:

$$\frac{1}{2} - (-\frac{\sqrt{3}}{2}) - \frac{\sqrt{3}-1}{2} = \frac{1+\sqrt{3}-\sqrt{3}+1}{2} = \frac{2}{2} = 1 \quad \checkmark$$

Summary

The equation $\sin x - \cos x - |\sin x + \cos x| = 1$ was solved by considering two cases based on the sign of $\sin x + \cos x$:

- **Case 1 ($\sin x + \cos x \geq 0$):** Led to $\cos x = -\frac{1}{2}$, with the valid solution $x = \frac{2\pi}{3} + 2k\pi$
- **Case 2 ($\sin x + \cos x < 0$):** Led to $\sin x = \frac{1}{2}$, with the valid solution $x = \frac{5\pi}{6} + 2k\pi$

The key step was verifying that each potential solution satisfied the constraint imposed by the case condition.

Remark

This problem demonstrates the importance of handling absolute values correctly in trigonometric equations. The strategy involves:

1. Identifying the two cases based on the sign of the expression inside the absolute value
2. Solving the simplified equation in each case
3. Verifying that the solutions satisfy the constraint of their respective cases

4. Rejecting solutions that violate their case constraints

This technique is essential for solving equations involving absolute values, where the solution set is often restricted to only those values that are consistent with the assumptions made when removing the absolute value signs.