Problem 10

Solve the differential equation
$$\frac{x}{y} = \frac{y'}{x+1}.$$

Solution

1. Rewrite the differential equation

The given differential equation is
$$\frac{x}{y} = \frac{y'}{x+1}.$$
Cross-multiply to obtain:
$$x(x+1) = yy',$$
or equivalently:
$$x^2 + x = y\,\frac{dy}{dx}.$$
This is a first-order separable differential equation.

2. Solve the differential equation

Separate the variables:
$$y\,dy = (x^2 + x)\,dx.$$
Integrate both sides:
$$\int y\,dy = \int (x^2 + x)\,dx.$$
The left-hand side gives:
$$\int y\,dy = \frac{y^2}{2} + C_1.$$
The right-hand side gives:
$$\int (x^2 + x)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + C_2.$$
Equating the two integrals:
$$\frac{y^2}{2} + C_1 = \frac{x^3}{3} + \frac{x^2}{2} + C_2.$$
Combining the constants, let $C = C_2 - C_1$:
$$\frac{y^2}{2} = \frac{x^3}{3} + \frac{x^2}{2} + C.$$
Multiply through by 6 to clear denominators:
$$3y^2 = 2x^3 + 3x^2 + 6C.$$

Letting $c = 6C$ (where c is an arbitrary constant):

$$\boxed{3y^2 = 2x^3 + 3x^2 + c, \qquad c \in \mathbb{R},}$$

or solving for y:

$$\boxed{y = \pm\sqrt{\frac{2x^3 + 3x^2 + c}{3}}.}$$

3. Domain restrictions

For the solution $y = \pm\sqrt{\frac{2x^3+3x^2+c}{3}}$ to be real-valued, we require:

$$\frac{2x^3 + 3x^2 + c}{3} \geq 0,$$

which simplifies to:

$$2x^3 + 3x^2 + c \geq 0.$$

The domain depends on the value of the constant c. The polynomial $2x^3 + 3x^2 + c$ determines where the solution is real.

Additionally, from the original equation $\frac{x}{y} = \frac{y'}{x+1}$, we require:

- $y \neq 0$ (to avoid division by zero),
- $x \neq -1$ (to avoid division by zero in the original form).

4. Verification

Differentiate the implicit solution $3y^2 = 2x^3 + 3x^2 + c$ with respect to x:

$$\frac{d}{dx}(3y^2) = \frac{d}{dx}(2x^3 + 3x^2 + c).$$

$$6y\,\frac{dy}{dx} = 6x^2 + 6x.$$

Dividing by 6:

$$y\,\frac{dy}{dx} = x^2 + x.$$

This can be rewritten as:

$$yy' = x(x+1).$$

Dividing both sides by $y(x+1)$ (assuming $y \neq 0$ and $x \neq -1$):

$$\frac{y'}{x+1} = \frac{x(x+1)}{y(x+1)} = \frac{x}{y}.$$

Thus:

$$\frac{x}{y} = \frac{y'}{x+1},$$

which is exactly the original differential equation. The solution is verified.

Remark. The implicit solution $3y^2 = 2x^3 + 3x^2 + c$ represents a family of curves. For different values of c, we obtain different members of this family. The solution curves are symmetric about the x-axis due to the $\pm$ sign in the explicit form. Special care must be taken at points where $x = -1$ or $y = 0$, as the original equation is not defined there.