Problem 29

Solve and discuss the roots of the equation:

$$x^3 - 1 + kx(x-1) = 0$$

Solution

Step 1: Factor and simplify

Notice that $x^3 - 1 = (x-1)(x^2+x+1)$. Rewrite the equation:

$$(x-1)(x^2+x+1) + kx(x-1) = 0$$

$$(x-1)[x^2+x+1+kx] = 0$$

$$(x-1)(x^2+(1+k)x+1) = 0$$

Step 2: Find the roots

First root: $x = 1$

From the factor $(x-1) = 0$:

$$\boxed{x_1 = 1}$$

This root exists for all values of k.

Other roots: from $x^2 + (1+k)x + 1 = 0$

Using the quadratic formula:

$$x = \frac{-(1+k) \pm \sqrt{(1+k)^2 - 4}}{2}$$

$$= \frac{-(1+k) \pm \sqrt{1 + 2k + k^2 - 4}}{2}$$

$$= \frac{-(1+k) \pm \sqrt{k^2 + 2k - 3}}{2}$$

Factor the discriminant:

$$k^2 + 2k - 3 = (k+3)(k-1)$$

Therefore:

$$\boxed{x_{2,3} = \frac{-(1+k) \pm \sqrt{(k+3)(k-1)}}{2}}$$

Step 3: Discuss the nature of roots based on k

The discriminant is $\Delta = (k+3)(k-1)$.

Case 1: $k < -3$

$\Delta = (k+3)(k-1) > 0$ (both factors negative)
Two additional distinct real roots plus $x = 1$:

$$\boxed{\text{Three distinct real roots}}$$

Case 2: $k = -3$

$\Delta = 0$
The quadratic has a double root:

$$x = \frac{-(1-3)}{2} = \frac{2}{2} = 1$$

So $x = 1$ is a triple root!

$$\boxed{\text{One triple root: } x = 1}$$

Verification: $(x-1)^3 = x^3 - 3x^2 + 3x - 1 = x^3 - 1 - 3x(x-1)$

Case 3: $-3 < k < 1$

$\Delta = (k+3)(k-1) < 0$ (one positive, one negative factor)
The quadratic has complex conjugate roots:

$$x_{2,3} = \frac{-(1+k) \pm i\sqrt{|k^2 + 2k - 3|}}{2}$$

$$\boxed{\text{One real root } (x = 1) \text{ and two complex conjugate roots}}$$

Case 4: $k = 1$

$\Delta = 0$
The quadratic has a double root:

$$x = \frac{-(1+1)}{2} = -1$$

So we have $x = 1$ (simple) and $x = -1$ (double):

$$\boxed{\text{Two distinct real roots: } x = 1 \text{ (simple), } x = -1 \text{ (double)}}$$

Case 5: $k > 1$

$\Delta = (k+3)(k-1) > 0$ (both factors positive)
Two additional distinct real roots plus $x = 1$:

$$\boxed{\text{Three distinct real roots}}$$

Step 4: Special values and verification

For $k = 0$:

$$x^3 - 1 = 0 \implies x = 1, \omega, \omega^2$$

where $\omega = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$ are the complex cube roots of unity.

From our formula with $k = 0$:

$$x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

This matches!

For $k = 2$:

$$x^2 + 3x + 1 = 0$$

$$x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

Three distinct real roots: $x = 1$, $x = \frac{-3+\sqrt{5}}{2} \approx -0.382$, $x = \frac{-3-\sqrt{5}}{2} \approx -2.618$

Step 5: Summary table

Range of k	Δ	Nature of roots
$k < -3$	> 0	Three distinct real roots
$k = -3$	$= 0$	One triple root: $x = 1$
$-3 < k < 1$	< 0	One real root and two complex conjugates
$k = 1$	$= 0$	$x = 1$ (simple), $x = -1$ (double)
$k > 1$	> 0	Three distinct real roots

Step 6: Additional observations

Product of all roots:

From Vieta's formulas for $x^3 + 0x^2 + (k-1)x + (-1-k) = 0$:

$$x_1 x_2 x_3 = -\frac{-1-k}{1} = 1 + k$$

Sum of all roots:

$$x_1 + x_2 + x_3 = 0$$

Since $x_1 = 1$:

$$x_2 + x_3 = -1$$

From the quadratic: $x_2 + x_3 = -(1+k)$... wait, that gives $-(1+k) = -1$?
Let me recalculate. From $x^2 + (1+k)x + 1 = 0$:

$$x_2 + x_3 = -(1+k)$$

So the sum of all three roots is:

$$1 + (-(1+k)) = 1 - 1 - k = -k$$

Actually, expanding $(x-1)(x^2+(1+k)x+1)$:

$$x^3+(1+k)x^2+x-x^2-(1+k)x-1=x^3+kx^2-kx-1$$

So the original equation is $x^3+kx^2-kx-1=0$... Let me verify:

$$x^3-1+kx(x-1)=x^3-1+kx^2-kx=x^3+kx^2-kx-1$$

From Vieta: sum of roots $=-k$.

Conclusion

The equation $x^3-1+kx(x-1)=0$ factors as $(x-1)(x^2+(1+k)x+1)=0$.

It always has $x=1$ as a root, and the other roots depend on the discriminant $(k+3)(k-1)$: - For $k<-3$ or $k>1$: three distinct real roots - For $k=-3$: triple root at $x=1$ - For $-3<k<1$: one real root and two complex conjugates - For $k=1$: double root at $x=-1$ and simple root at $x=1$