Problem 1

Find the general solution of the differential equation whose solution is given by

$$y = Ce^{\sqrt{x^2-1}}.$$

State the differential equation, solve it (showing steps), and give any domain restrictions.

Solution

1. Determine the differential equation

Assume the solution has the form

$$y(x) = Ce^{\sqrt{x^2-1}},$$

where C is an arbitrary constant. Differentiate y with respect to x. Using the chain rule and the derivative

$$\frac{d}{dx}\sqrt{x^2-1} = \frac{x}{\sqrt{x^2-1}} \qquad (\text{for } |x| > 1),$$

we obtain

$$\frac{dy}{dx} = Ce^{\sqrt{x^2-1}} \cdot \frac{x}{\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}\,y.$$

Thus the differential equation satisfied by y is

$$\boxed{y' = \frac{x}{\sqrt{x^2-1}}\,y, \qquad |x| > 1.}$$

2. Solve the differential equation

Solve the first-order linear ODE by separation of variables. Rewrite the equation as

$$\frac{1}{y}\,dy = \frac{x}{\sqrt{x^2-1}}\,dx.$$

Integrate both sides:

$$\int \frac{1}{y}\,dy = \int \frac{x}{\sqrt{x^2-1}}\,dx.$$

The left-hand side gives

$$\int \frac{1}{y}\,dy = \ln|y| + C_1.$$

For the right-hand side use the substitution $u = x^2 - 1$, so $du = 2x\,dx$ and

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \int \frac{1}{2}\frac{1}{\sqrt{u}}\,du = \sqrt{u} + C_2 = \sqrt{x^2-1} + C_2.$$

Equating the integrals and combining the integration constants,

$$\ln|y| = \sqrt{x^2-1} + C,$$

where $C = C_2 - C_1$. Exponentiate both sides:

$$|y| = e^C e^{\sqrt{x^2-1}}.$$

Introduce the arbitrary real constant $C_0 = \pm e^C$ (which can be zero as well) to get the general solution

$$\boxed{y = C_0 e^{\sqrt{x^2-1}}, \qquad |x| > 1,}$$

which matches the given form. The domain restriction $|x| > 1$ is required for the expression $\sqrt{x^2 - 1}$ to be real.

3. Verification

Differentiate $y = C_0 e^{\sqrt{x^2-1}}$ to verify:

$$y' = C_0 e^{\sqrt{x^2-1}} \cdot \frac{x}{\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}\, y,$$

so the function indeed satisfies the derived ODE on $|x| > 1$.

Remark. The zero solution $y \equiv 0$ corresponds to $C_0 = 0$. If one allows complex-valued functions, the domain restriction can be relaxed, but for real solutions the natural domain is $|x| > 1$.