Problem 3

Given: $y = \left(\frac{5}{2}x^2 - x + 5\right)^2 - \left(\frac{3}{2}x^2 + 5x - 4\right)^2$.
1° Show that $y > 0$.
2° Solve the equation $y = 0$.
3° If $z = \frac{y}{(4x^2-1)^2}$, find x for which $z = 1$.

Solution

Part 1°: Show that $y > 0$

Step 1. Apply the difference of squares formula $A^2 - B^2 = (A+B)(A-B)$ where:

$$\begin{aligned} A &= \frac{5}{2}x^2 - x + 5, \\ B &= \frac{3}{2}x^2 + 5x - 4. \end{aligned}$$

Step 2. Calculate $A + B$:

$$\begin{aligned} A + B &= \left(\frac{5}{2}x^2 - x + 5\right) + \left(\frac{3}{2}x^2 + 5x - 4\right) \\ &= \frac{5}{2}x^2 + \frac{3}{2}x^2 - x + 5x + 5 - 4 \\ &= 4x^2 + 4x + 1 \\ &= (2x+1)^2. \end{aligned}$$

Step 3. Calculate $A - B$:

$$\begin{aligned} A - B &= \left(\frac{5}{2}x^2 - x + 5\right) - \left(\frac{3}{2}x^2 + 5x - 4\right) \\ &= \frac{5}{2}x^2 - \frac{3}{2}x^2 - x - 5x + 5 + 4 \\ &= x^2 - 6x + 9 \\ &= (x-3)^2. \end{aligned}$$

Step 4. Therefore:

$$y = (A+B)(A-B) = (2x+1)^2(x-3)^2 = [(2x+1)(x-3)]^2.$$

Step 5. Since y is a perfect square:

$$y = [(2x+1)(x-3)]^2 \geq 0 \text{ for all real } x.$$

Step 6. To show $y > 0$, we need to show that $(2x+1)(x-3) \neq 0$ for all x, or find when it equals zero.

$(2x+1)(x-3) = 0$ when $x = -\frac{1}{2}$ or $x = 3$.

Therefore: $y = 0$ when $x = -\frac{1}{2}$ or $x = 3$, and $y > 0$ for all other values of x.

$$\boxed{y \geq 0 \text{ for all } x, \text{ with } y = 0 \text{ only at } x = -\frac{1}{2} \text{ or } x = 3}.$$

Note: The problem statement "$y > 0$" is not strictly correct as stated. It should be "$y \geq 0$".

Part 2°: Solve $y = 0$

Step 1. From Part 1, we have:

$$y = [(2x+1)(x-3)]^2 = 0.$$

Step 2. This occurs when:

$$(2x+1)(x-3) = 0.$$

Step 3. Solve each factor:

$$\begin{aligned} 2x+1 = 0 &\implies x = -\frac{1}{2}, \\ x-3 = 0 &\implies x = 3. \end{aligned}$$

$$\boxed{x = -\frac{1}{2} \text{ or } x = 3}.$$

Part 3°: Find x for which $z = 1$

Step 1. Given:

$$z = \frac{y}{(4x^2-1)^2} = 1.$$

Step 2. This means:

$$y = (4x^2-1)^2.$$

Step 3. From Part 1, we know:

$$y = [(2x+1)(x-3)]^2.$$

Step 4. Therefore:

$$[(2x+1)(x-3)]^2 = (4x^2-1)^2.$$

Step 5. Taking square roots (considering both signs):

$$(2x+1)(x-3) = \pm(4x^2-1).$$

Step 6. Note that $4x^2 - 1 = (2x-1)(2x+1)$. So:

$$(2x+1)(x-3) = \pm(2x-1)(2x+1).$$

Step 7. Case 1: $(2x+1)(x-3) = (2x-1)(2x+1)$.
If $2x+1 \neq 0$ (i.e., $x \neq -\frac{1}{2}$), divide both sides by $(2x+1)$:

$$x - 3 = 2x - 1.$$

$$-3 + 1 = 2x - x.$$

$$x = -2.$$

Step 8. Case 2: $(2x+1)(x-3) = -(2x-1)(2x+1)$.
If $2x+1 \neq 0$, divide by $(2x+1)$:

$$x - 3 = -(2x-1) = -2x + 1.$$

$$x + 2x = 1 + 3.$$

$$3x = 4.$$

$$x = \frac{4}{3}.$$

Step 9. Verify both solutions:
For $x = -2$:

$$y = [(2(-2)+1)(-2-3)]^2 = [(-3)(-5)]^2 = 15^2 = 225,$$
$$(4x^2-1)^2 = (4(4)-1)^2 = 15^2 = 225. \quad \checkmark$$

For $x = \frac{4}{3}$:

$$y = \left[\left(2 \cdot \frac{4}{3} + 1\right)\left(\frac{4}{3} - 3\right)\right]^2 = \left[\frac{11}{3} \cdot \left(-\frac{5}{3}\right)\right]^2 = \left(-\frac{55}{9}\right)^2 = \frac{3025}{81},$$
$$(4x^2-1)^2 = \left(4 \cdot \frac{16}{9} - 1\right)^2 = \left(\frac{64}{9} - 1\right)^2 = \left(\frac{55}{9}\right)^2 = \frac{3025}{81}. \quad \checkmark$$

$$\boxed{x = -2 \text{ or } x = \frac{4}{3}}.$$