Problem 5

Taking the arithmetic value of the roots, perform the indicated operations:

$$\sqrt{(x-2)^2} - \sqrt{(x+3)^2} + \sqrt{(5-x)^2}.$$

Solution

Key principle: For any real number a, we have $\sqrt{a^2} = |a|$ (the absolute value of a).

Step 1: Apply the absolute value property

Using $\sqrt{a^2} = |a|$:

$$\begin{aligned}
\sqrt{(x-2)^2} &= |x-2|, \\
\sqrt{(x+3)^2} &= |x+3|, \\
\sqrt{(5-x)^2} &= |5-x|.
\end{aligned}$$

Therefore:

$$\sqrt{(x-2)^2} - \sqrt{(x+3)^2} + \sqrt{(5-x)^2} = |x-2| - |x+3| + |5-x|.$$

Step 2: Identify critical points

The expression changes form at the points where each term inside the absolute value equals zero:

- $x - 2 = 0 \implies x = 2$
- $x + 3 = 0 \implies x = -3$
- $5 - x = 0 \implies x = 5$

These critical points divide the real line into four intervals: $(-\infty, -3)$, $[-3, 2)$, $[2, 5)$, and $[5, \infty)$.

Step 3: Simplify in each interval

Interval 1: $x < -3$

For $x < -3$:

- $x - 2 < 0 \implies |x-2| = -(x-2) = 2 - x$
- $x + 3 < 0 \implies |x+3| = -(x+3) = -x - 3$
- $5 - x > 0 \implies |5-x| = 5 - x$

Therefore:

$$\begin{aligned}|x-2|-|x+3|+|5-x| &= (2-x)-(-x-3)+(5-x)\\ &= 2-x+x+3+5-x\\ &= 10-x.\end{aligned}$$

Interval 2: $-3 \leq x < 2$
For $-3 \leq x < 2$:

- $x-2<0 \implies |x-2| = 2-x$
- $x+3 \geq 0 \implies |x+3| = x+3$
- $5-x>0 \implies |5-x| = 5-x$

Therefore:

$$\begin{aligned}|x-2|-|x+3|+|5-x| &= (2-x)-(x+3)+(5-x)\\ &= 2-x-x-3+5-x\\ &= 4-3x.\end{aligned}$$

Interval 3: $2 \leq x < 5$
For $2 \leq x < 5$:

- $x-2 \geq 0 \implies |x-2| = x-2$
- $x+3>0 \implies |x+3| = x+3$
- $5-x>0 \implies |5-x| = 5-x$

Therefore:

$$\begin{aligned}|x-2|-|x+3|+|5-x| &= (x-2)-(x+3)+(5-x)\\ &= x-2-x-3+5-x\\ &= -x.\end{aligned}$$

Interval 4: $x \geq 5$
For $x \geq 5$:

- $x-2>0 \implies |x-2| = x-2$
- $x+3>0 \implies |x+3| = x+3$
- $5-x \leq 0 \implies |5-x| = -(5-x) = x-5$

Therefore:

$$\begin{aligned}|x-2|-|x+3|+|5-x| &= (x-2)-(x+3)+(x-5)\\ &= x-2-x-3+x-5\\ &= x-10.\end{aligned}$$

Step 4: Write the complete piecewise function

$$\boxed{\sqrt{(x-2)^2}-\sqrt{(x+3)^2}+\sqrt{(5-x)^2}=\begin{cases}10-x & \text{if } x<-3\\ 4-3x & \text{if } -3\le x<2\\ -x & \text{if } 2\le x<5\\ x-10 & \text{if } x\ge 5\end{cases}}$$

Verification

Let's verify with specific values:

For $x = -4$ (in interval 1):

$$\begin{aligned}\text{Direct: } & \sqrt{(-4-2)^2}-\sqrt{(-4+3)^2}+\sqrt{(5-(-4))^2}\\ & =\sqrt{36}-\sqrt{1}+\sqrt{81}=6-1+9=14.\\ \text{Formula: } & 10-(-4)=14. \quad \checkmark\end{aligned}$$

For $x = 0$ (in interval 2):

$$\begin{aligned}\text{Direct: } & \sqrt{(0-2)^2}-\sqrt{(0+3)^2}+\sqrt{(5-0)^2}\\ & =\sqrt{4}-\sqrt{9}+\sqrt{25}=2-3+5=4.\\ \text{Formula: } & 4-3(0)=4. \quad \checkmark\end{aligned}$$

For $x = 3$ (in interval 3):

$$\begin{aligned}\text{Direct: } & \sqrt{(3-2)^2}-\sqrt{(3+3)^2}+\sqrt{(5-3)^2}\\ & =\sqrt{1}-\sqrt{36}+\sqrt{4}=1-6+2=-3.\\ \text{Formula: } & -3. \quad \checkmark\end{aligned}$$

For $x = 6$ (in interval 4):

$$\begin{aligned}\text{Direct: } & \sqrt{(6-2)^2}-\sqrt{(6+3)^2}+\sqrt{(5-6)^2}\\ & =\sqrt{16}-\sqrt{81}+\sqrt{1}=4-9+1=-4.\\ \text{Formula: } & 6-10=-4. \quad \checkmark\end{aligned}$$