Problem 1

Evaluate the integral:

$$I = \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}}\, dx.$$

Solution

To solve the given integral, we first simplify the terms in the integrand and express all powers of x in terms of rational exponents. Let us proceed step by step.

Step 1: Rewrite the terms with rational exponents

The numerator of the integrand is:

$$\sqrt{x} - 2\sqrt[3]{x^2} + 1,$$

which can be rewritten as:

$$x^{1/2} - 2x^{2/3} + 1.$$

The denominator is:

$$\sqrt[4]{x} = x^{1/4}.$$

Thus, the integrand becomes:

$$\frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} = \frac{x^{1/2}}{x^{1/4}} - \frac{2x^{2/3}}{x^{1/4}} + \frac{1}{x^{1/4}}.$$

Simplify each term:

$$\frac{x^{1/2}}{x^{1/4}} = x^{1/2-1/4} = x^{1/4}, \quad \frac{2x^{2/3}}{x^{1/4}} = 2x^{2/3-1/4}, \quad \frac{1}{x^{1/4}} = x^{-1/4}.$$

The integrand becomes:

$$I = \int \left(x^{1/4} - 2x^{5/12} + x^{-1/4}\right) dx.$$

Step 2: Integrate each term

The integral is now split into three simpler terms:

$$I = \int x^{1/4}\, dx - 2\int x^{5/12}\, dx + \int x^{-1/4}\, dx.$$

1. **First term: $\int x^{1/4}\, dx$:** Using the power rule $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$:

$$\int x^{1/4}\, dx = \frac{x^{1/4+1}}{1/4+1} = \frac{x^{5/4}}{5/4} = \frac{4}{5}x^{5/4}.$$

2. **Second term: $-2\int x^{5/12}\, dx$:** Again, using the power rule:

$$\int x^{5/12}\, dx = \frac{x^{5/12+1}}{5/12+1} = \frac{x^{17/12}}{17/12} = \frac{12}{17}x^{17/12}.$$

Multiply by -2:

$$-2\int x^{5/12}\, dx = -2 \cdot \frac{12}{17}x^{17/12} = -\frac{24}{17}x^{17/12}.$$

3. **Third term: $\int x^{-1/4}\, dx$:** Using the power rule:

$$\int x^{-1/4}\, dx = \frac{x^{-1/4+1}}{-1/4+1} = \frac{x^{3/4}}{3/4} = \frac{4}{3}x^{3/4}.$$

Step 3: Combine the results

Combine all three terms:

$$I = \frac{4}{5}x^{5/4} - \frac{24}{17}x^{17/12} + \frac{4}{3}x^{3/4} + C,$$

where C is the constant of integration.

Final Answer

The solution to the integral is:

$$I = \frac{4}{5}x^{5/4} - \frac{24}{17}x^{17/12} + \frac{4}{3}x^{3/4} + C.$$