Problem 73

Evaluate the integral

$$\int \frac{x\,dx}{5+x^4}.$$

Solution

We solve this integral using the substitution method.

Let

$$u = x^2.$$

Then, differentiate both sides with respect to x:

$$\frac{du}{dx} = 2x,$$

so

$$du = 2x\,dx.$$

Therefore,

$$x\,dx = \frac{du}{2}.$$

Substitute this into the original integral:

$$\int \frac{x\,dx}{5+x^4} = \int \frac{\frac{du}{2}}{5+u^2}.$$

Simplify the expression:

$$= \frac{1}{2}\int \frac{du}{5+u^2}.$$

We recognize that the integral

$$\int \frac{du}{a^2+u^2}$$

is a standard integral, whose solution is

$$\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right).$$

In our case, $a^2 = 5$, so $a = \sqrt{5}$. Thus, we have:

$$\frac{1}{2}\int \frac{du}{5+u^2} = \frac{1}{2}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right).$$

Now, substitute $u = x^2$ back into the solution:

$$\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2}{\sqrt{5}}\right).$$

Therefore, the solution to the integral is:

$$\boxed{\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2}{\sqrt{5}}\right)}.$$