Problem 205

Evaluate the integral:

$$I = \int \sqrt{x^2 - 2x - 1}\, dx$$

Solution 1

We begin by completing the square for the expression under the square root:

$$x^2 - 2x - 1 = (x-1)^2 - 2$$

Thus, the integral transforms into:

$$I = \int \sqrt{(x-1)^2 - 2}\, dx$$

Substitution We use the substitution:

$$x - 1 = \sqrt{2}\sinh t$$

Differentiating both sides:

$$dx = \sqrt{2}\cosh t\, dt$$

Rewriting the integral in terms of t:

$$I = \int \sqrt{2}\cosh t \cdot \sqrt{2}\cosh t\, dt$$

$$I = 2\int \cosh^2 t\, dt$$

Using the identity:

$$\cosh^2 t = \frac{1 + \cosh 2t}{2}$$

we get:

$$I = \int (1 + \cosh 2t)\, dt$$

Evaluating the integral:

$$I = t + \frac{1}{2}\sinh 2t + C$$

Substituting back $t = \sinh^{-1} \frac{x-1}{\sqrt{2}}$:

$$I = \sinh^{-1} \frac{x-1}{\sqrt{2}} + \frac{1}{2}\sinh\left(2\sinh^{-1} \frac{x-1}{\sqrt{2}}\right) + C$$

After simplifications, we obtain:

$$I = \frac{x-1}{2}\sqrt{x^2 - 2x - 1} + \ln\left|x - 1 + \sqrt{x^2 - 2x - 1}\right| + C$$

Thus, the final result is:

$$I = \frac{x-1}{2}\sqrt{x^2 - 2x - 1} + \ln\left|x - 1 + \sqrt{x^2 - 2x - 1}\right| + C$$

Solution 2

We begin by completing the square for the expression under the square root:

$$x^2 - 2x - 1 = (x-1)^2 - 2$$

Using substitution:

$$u = x - 1, \quad \text{so that} \quad du = dx$$

we rewrite the integral as:

$$I = \int \sqrt{u^2 - 2}\, du$$

Using the identity:

$$\sqrt{u^2 - a^2} = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln|u + \sqrt{u^2 - a^2}|\right),$$

we set $a^2 = 2$ and apply it:

$$I = \frac{1}{2}\left(u\sqrt{u^2 - 2} - 2\ln|u + \sqrt{u^2 - 2}|\right) + C$$

Substituting back $u = x - 1$:

$$I = \frac{x-1}{2}\sqrt{x^2 - 2x - 1} - \ln|x - 1 + \sqrt{x^2 - 2x - 1}| + C$$

Thus, the final result is:

$$I = \frac{x-1}{2}\sqrt{x^2 - 2x - 1} - \ln|x - 1 + \sqrt{x^2 - 2x - 1}| + C$$