Problem 4

Evaluate the integral:

$$I = \int \left(1 - \frac{1}{x^2}\right) \sqrt{x\sqrt{x}}\, dx.$$

Solution

To solve the given integral, we will first simplify the integrand and then integrate term by term.

Step 1: Simplify the integrand

The term $\sqrt{x\sqrt{x}}$ can be rewritten as:

$$\sqrt{x\sqrt{x}} = \sqrt{x \cdot x^{1/2}} = \sqrt{x^{3/2}} = x^{3/4}.$$

Substituting this back into the integral, the expression becomes:

$$I = \int \left(1 - \frac{1}{x^2}\right) x^{3/4}\, dx.$$

Now expand the terms:

$$\int \left(1 - \frac{1}{x^2}\right) x^{3/4}\, dx = \int x^{3/4}\, dx - \int \frac{x^{3/4}}{x^2}\, dx.$$

Simplify each term: 1. The first term remains $\int x^{3/4}\, dx$. 2. The second term simplifies as:

$$\frac{x^{3/4}}{x^2} = x^{3/4-2} = x^{-5/4}.$$

Thus, the integral becomes:

$$I = \int x^{3/4}\, dx - \int x^{-5/4}\, dx.$$

Step 2: Integrate each term

We use the power rule for integration:

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for } n \neq -1.$$

1. **First term: $\int x^{3/4}\, dx$:**

$$\int x^{3/4}\, dx = \frac{x^{3/4+1}}{3/4+1} = \frac{x^{7/4}}{7/4} = \frac{4}{7}x^{7/4}.$$

2. **Second term: $\int x^{-5/4}\, dx$:**

$$\int x^{-5/4}\, dx = \frac{x^{-5/4+1}}{-5/4+1} = \frac{x^{-1/4}}{-1/4} = -4x^{-1/4}.$$

Step 3: Combine the results

Combine the results of both integrals:

$$I = \frac{4}{7}x^{7/4} + 4x^{-1/4} + C,$$

where C is the constant of integration.

Final Answer

The solution to the integral is:

$$I = \frac{4}{7}x^{7/4} + 4x^{-1/4} + C.$$