Problem 10

If $\frac{x}{a} = \frac{y}{b}$, prove that:

$$\frac{x^2}{x^2 + y^2} + \frac{b^2}{a^2 + b^2} = 1$$

Proof

Step 1: Use the given condition

Given that $\frac{x}{a} = \frac{y}{b}$, let this common ratio equal k:

$$\frac{x}{a} = \frac{y}{b} = k$$

This gives us:

$$x = ka \quad \text{and} \quad y = kb$$

Step 2: Substitute into the left side

Substitute $x = ka$ and $y = kb$ into the first term:

$$\frac{x^2}{x^2 + y^2} = \frac{(ka)^2}{(ka)^2 + (kb)^2} = \frac{k^2a^2}{k^2a^2 + k^2b^2}$$

Factor out k^2 from the denominator:

$$= \frac{k^2a^2}{k^2(a^2 + b^2)} = \frac{a^2}{a^2 + b^2}$$

Step 3: Add the two terms

Now add both terms:

$$\frac{x^2}{x^2 + y^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}$$

Since both fractions have the same denominator:

$$= \frac{a^2 + b^2}{a^2 + b^2} = 1$$

Conclusion

We have proven that:

$$\boxed{\frac{x^2}{x^2 + y^2} + \frac{b^2}{a^2 + b^2} = 1}$$

Verification with specific values

Let's verify with $a = 3$, $b = 4$, and $k = 2$ (so $x = 6$, $y = 8$):

Check the condition:

$$\frac{x}{a} = \frac{6}{3} = 2, \quad \frac{y}{b} = \frac{8}{4} = 2 \quad \checkmark$$

Left side:

$$\frac{x^2}{x^2+y^2} + \frac{b^2}{a^2+b^2} = \frac{36}{36+64} + \frac{16}{9+16} = \frac{36}{100} + \frac{16}{25}$$

$$= \frac{36}{100} + \frac{64}{100} = \frac{100}{100} = 1 \quad \checkmark$$

Alternative Proof (Direct Manipulation)

Starting with the condition $\frac{x}{a} = \frac{y}{b}$, we can write:

$$bx = ay$$

Square both sides:

$$b^2x^2 = a^2y^2$$

This gives us:

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

Let this common ratio be m:

$$x^2 = ma^2 \quad \text{and} \quad y^2 = mb^2$$

Now evaluate the left side:

$$\frac{x^2}{x^2+y^2} + \frac{b^2}{a^2+b^2} = \frac{ma^2}{ma^2+mb^2} + \frac{b^2}{a^2+b^2}$$

$$= \frac{ma^2}{m(a^2+b^2)} + \frac{b^2}{a^2+b^2} = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}$$

$$= \frac{a^2+b^2}{a^2+b^2} = 1$$

Geometric Interpretation

This identity has a beautiful geometric interpretation. If we consider:

- $\frac{x^2}{x^2+y^2}$ represents the "weight" of x^2 in the sum $x^2 + y^2$
- $\frac{b^2}{a^2+b^2}$ represents the "weight" of b^2 in the sum $a^2 + b^2$

When $\frac{x}{a} = \frac{y}{b}$, the ratio of x to y is the same as the ratio of a to b. The identity states that the "weight" of x^2 in $(x^2 + y^2)$ plus the "weight" of b^2 in $(a^2 + b^2)$ equals 1.

This is related to the fact that in similar triangles or proportional quantities, certain normalized ratios satisfy complementary relationships.

Remark

This identity demonstrates how proportional relationships ($\frac{x}{a} = \frac{y}{b}$) lead to elegant sum properties. The key insight is that the proportionality allows us to factor out the common ratio, revealing that $\frac{x^2}{x^2+y^2} = \frac{a^2}{a^2+b^2}$, which is independent of the specific value of the common ratio k. Such identities are fundamental in the study of similar figures, normalized vectors, and probability theory where quantities are expressed as fractions of totals.