Problem 6

Evaluate the sum S for the series:

$$S = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{1}{(n+1)^k} \right].$$

Solution

Step 1: Condition for Convergence

We split the series into two components:

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^k} = S_1 - S_2.$$

A series $\sum a_n$ can be split into $\sum b_n - \sum c_n$ if and only if both $\sum b_n$ and $\sum c_n$ converge.

1. The first series, $S_1 = \sum_{n=1}^{\infty} \frac{1}{n^2}$, is a convergent p-series since $p = 2 > 1$. Its sum is known as the Basel problem solution:

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}.$$

2. The second series, $S_2 = \sum_{n=1}^{\infty} \frac{1}{(n+1)^k}$, is a p-series (shifted index). It converges if and only if $k > 1$.

Therefore, the total series S converges if and only if $\mathbf{k > 1}$.

Step 2: Calculating the Sum S_2

For $k > 1$, the sum S_2 is related to the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$:

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

This is the sum $\zeta(k)$ excluding the first term ($n = 1$):

$$S_2 = \left(\sum_{n=1}^{\infty} \frac{1}{n^k} \right) - \frac{1}{1^k} = \zeta(k) - 1.$$

Step 3: Calculating the Total Sum S

For $k > 1$:

$$S = S_1 - S_2 = \zeta(2) - (\zeta(k) - 1).$$

Substituting $\zeta(2) = \frac{\pi^2}{6}$:

$$S = \frac{\pi^2}{6} - \zeta(k) + 1.$$

Special Case: $k = 2$

If $k = 2$, the series is telescoping:

$$\sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right].$$

The partial sum is $S_N = 1 - \frac{1}{(N+1)^2}$.

$$S = \lim_{N \to \infty} S_N = 1.$$

This result is consistent with the general formula for $k = 2$:

$$S = \frac{\pi^2}{6} - \zeta(2) + 1 = \frac{\pi^2}{6} - \frac{\pi^2}{6} + 1 = 1.$$

Final Answer

The series converges if and only if $\mathbf{k > 1}$. If $k > 1$, the sum S is:

$$\boxed{S = 1 + \frac{\pi^2}{6} - \zeta(k).}$$

Where $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$ is the Riemann zeta function.