

## Problem 225

Evaluate the integral

$$\int \frac{x^5 dx}{x^6 - x^3 - 2}.$$

## Solution

Let us first observe that the derivative of the denominator is similar to the numerator:

$$\frac{d}{dx}(x^6 - x^3 - 2) = 6x^5 - 3x^2.$$

Since this derivative is not directly proportional to the numerator, we look for a substitution.

Let us set:

$$u = x^3,$$

thus:

$$du = 3x^2 dx, \quad \text{so} \quad x^2 dx = \frac{du}{3}.$$

Rewrite the integral:

$$\int \frac{x^5 dx}{x^6 - x^3 - 2} = \int \frac{x^3 \cdot x^2 dx}{(x^3)^2 - x^3 - 2} = \int \frac{u \cdot \frac{du}{3}}{u^2 - u - 2}.$$

Simplify:

$$= \frac{1}{3} \int \frac{u du}{u^2 - u - 2}.$$

Now use substitution: Let us set

$$v = u^2 - u - 2,$$

then

$$\frac{dv}{du} = 2u - 1, \quad \text{thus} \quad dv = (2u - 1)du.$$

We express  $u du$  in terms of  $dv$ :

$$u du = \frac{1}{2}(dv + du).$$

But since the appearance is complicated, instead, use another method: partial fractions.

Factor the denominator:

$$u^2 - u - 2 = (u - 2)(u + 1).$$

Thus, we can decompose:

$$\frac{u}{(u - 2)(u + 1)} = \frac{A}{u - 2} + \frac{B}{u + 1}.$$

Multiplying through:

$$u = A(u + 1) + B(u - 2),$$

$$u = Au + A + Bu - 2B,$$

$$u = (A + B)u + (A - 2B).$$

Comparing coefficients:

$$A + B = 1, \quad A - 2B = 0.$$

From the second equation:

$$A = 2B,$$

substituting into the first:

$$2B + B = 1, \quad 3B = 1, \quad B = \frac{1}{3}.$$

Thus:

$$A = \frac{2}{3}.$$

Therefore:

$$\frac{u}{(u-2)(u+1)} = \frac{2/3}{u-2} + \frac{1/3}{u+1}.$$

Thus:

$$\frac{1}{3} \int \frac{u}{(u-2)(u+1)} du = \frac{1}{3} \left( \frac{2}{3} \int \frac{du}{u-2} + \frac{1}{3} \int \frac{du}{u+1} \right).$$

Simplify:

$$= \frac{2}{9} \int \frac{du}{u-2} + \frac{1}{9} \int \frac{du}{u+1}.$$

Now integrate:

$$= \frac{2}{9} \ln |u-2| + \frac{1}{9} \ln |u+1| + C.$$

Recall  $u = x^3$ , thus:

$$= \frac{2}{9} \ln |x^3-2| + \frac{1}{9} \ln |x^3+1| + C.$$