Problem 8

Find the general solution of the differential equation

$$yy' - x = 0.$$

Solve it (showing steps), and give any domain restrictions.

Solution

1. Rewrite the differential equation

The given differential equation is

$$yy' - x = 0,$$

which can be rewritten as:

$$yy' = x,$$

or equivalently:

$$y\,\frac{dy}{dx} = x.$$

This is a first-order separable differential equation.

2. Solve the differential equation

Separate the variables by writing:

$$y\,dy = x\,dx.$$

Integrate both sides:

$$\int y\,dy = \int x\,dx.$$

The left-hand side gives:

$$\int y\,dy = \frac{y^2}{2} + C_1.$$

The right-hand side gives:

$$\int x\,dx = \frac{x^2}{2} + C_2.$$

Equating the two integrals:

$$\frac{y^2}{2} + C_1 = \frac{x^2}{2} + C_2.$$

Combining the constants, let $C = C_2 - C_1$:

$$\frac{y^2}{2} = \frac{x^2}{2} + C.$$

Multiply through by 2:

$$y^2 = x^2 + 2C.$$

Letting $c = 2C$ (where c is an arbitrary constant), we obtain:

$$y^2 = x^2 + c.$$

Solving for y:

$$\boxed{y = \pm\sqrt{x^2 + c},}$$

or in implicit form:

$$\boxed{y^2 - x^2 = c, \qquad c \in \mathbb{R}.}$$

3. Domain restrictions

For the solution $y = \pm\sqrt{x^2 + c}$ to be real-valued, we require:

$$x^2 + c \geq 0.$$

This condition depends on the value of the constant c:

- If $c \geq 0$, then $x^2 + c \geq 0$ for all $x \in \mathbb{R}$, so the domain is all real numbers.
- If $c < 0$, then we need $x^2 \geq -c$, which gives $|x| \geq \sqrt{-c}$. The domain is $x \in (-\infty, -\sqrt{-c}] \cup [\sqrt{-c}, \infty)$.

The implicit form $y^2 - x^2 = c$ represents:

- A family of hyperbolas when $c \neq 0$.
- The pair of lines $y = \pm x$ when $c = 0$.

4. Verification

Differentiate the implicit solution $y^2 - x^2 = c$ with respect to x:

$$\frac{d}{dx}(y^2 - x^2) = \frac{d}{dx}(c).$$

$$2y\,\frac{dy}{dx} - 2x = 0.$$

Dividing by 2:

$$y\,\frac{dy}{dx} - x = 0,$$

or equivalently:

$$yy' - x = 0,$$

which is exactly the original differential equation. Thus the solution is verified.

Remark. The general solution represents a family of hyperbolas (or degenerate lines when $c = 0$) centered at the origin. Each value of the constant c gives a different member of this family. The solution curves are orthogonal trajectories to the family of hyperbolas $xy = k$ for various constants k.