# Problem 6

Find the general solution of the differential equation whose solution is given by

$$y = c(x-c)^2.$$

State the differential equation, solve it (showing steps), and give any domain restrictions.

# Solution

## 1. Determine the differential equation

Assume the solution has the form

$$y(x) = c(x-c)^2,$$

where $c$ is an arbitrary constant. Differentiate with respect to $x$:

$$\frac{dy}{dx} = c \cdot 2(x-c) \cdot 1 = 2c(x-c).$$

To eliminate the arbitrary constant $c$, we need another relation. From the original equation:

$$y = c(x-c)^2.$$

From the derivative:

$$y' = 2c(x-c).$$

Notice that if we square the derivative:

$$(y')^2 = 4c^2(x-c)^2.$$

From the original equation, we have $(x-c)^2 = \frac{y}{c}$, so:

$$(y')^2 = 4c^2 \cdot \frac{y}{c} = 4cy.$$

We can also express $c$ from $y' = 2c(x-c)$:

$$c = \frac{y'}{2(x-c)}.$$

Substituting into $(y')^2 = 4cy$:

$$(y')^2 = 4 \cdot \frac{y'}{2(x-c)} \cdot y = \frac{2yy'}{x-c}.$$

From $y' = 2c(x-c)$, we have $x - c = \frac{y'}{2c}$. Substituting:

$$(y')^2 = \frac{2yy'}{\frac{y'}{2c}} = \frac{2yy' \cdot 2c}{y'} = 4cy.$$

This confirms our relation. Now, from $y = c(x-c)^2$ and $y' = 2c(x-c)$, we get:

$$\frac{y'}{2} = c(x-c) \quad \text{and} \quad y = c(x-c)^2 = (x-c) \cdot c(x-c) = (x-c) \cdot \frac{y'}{2}.$$

Therefore:

$$y = \frac{(x-c)y'}{2}.$$

From $y' = 2c(x-c)$, we have $x - c = \frac{y'}{2c}$. Substituting into $y = c(x-c)^2$:

$$y = c\left(\frac{y'}{2c}\right)^2 = c \cdot \frac{(y')^2}{4c^2} = \frac{(y')^2}{4c}.$$

Thus $c = \frac{(y')^2}{4y}$, and from $y' = 2c(x-c)$:

$$y' = 2 \cdot \frac{(y')^2}{4y} \cdot (x-c) = \frac{(y')^2(x-c)}{2y}.$$

Solving for $x - c$:

$$x - c = \frac{2yy'}{(y')^2} = \frac{2y}{y'}.$$

Substituting back into $y = \frac{(x-c)y'}{2}$:

$$y = \frac{y'}{2} \cdot \frac{2y}{y'} = y,$$

which is an identity. Instead, substitute $x - c = \frac{2y}{y'}$ into $c = x - \frac{2y}{y'}$:

$$c = x - \frac{2y}{y'}.$$

From $c = \frac{(y')^2}{4y}$:

$$x - \frac{2y}{y'} = \frac{(y')^2}{4y}.$$

Multiply through by $4yy'$:

$$4yy'(x - \frac{2y}{y'}) = (y')^2 \cdot y' = (y')^3.$$

$$4xyy' - 8y^2 = (y')^3.$$

Thus the differential equation is

$$\boxed{(y')^3 = 4xyy' - 8y^2, \qquad y > 0.}$$

Or equivalently:

$$\boxed{(y')^3 - 4xyy' + 8y^2 = 0.}$$

## 2. Solve the differential equation

We solve the equation

$$(y')^3 - 4xyy' + 8y^2 = 0.$$

This is a Clairaut-type equation. Let $p = y'$. The equation becomes:

$$p^3 - 4xyp + 8y^2 = 0.$$

From our derivation, we know $c = \frac{p^2}{4y}$ and $x - c = \frac{2y}{p}$. From $x - c = \frac{2y}{p}$:

$$x = c + \frac{2y}{p}.$$

From $c = \frac{p^2}{4y}$:

$$x = \frac{p^2}{4y} + \frac{2y}{p} = \frac{p^3 + 8y^2}{4yp}.$$

Cross-multiplying:

$$4xyp = p^3 + 8y^2,$$

which gives us $p^3 - 4xyp + 8y^2 = 0$, confirming our equation.

The general solution is obtained by noting that $c$ is arbitrary:

$$\boxed{y = c(x - c)^2, \qquad c \in \mathbb{R}.}$$

For $c > 0$, we require $y \geq 0$. The solution represents a family of parabolas.

## 3. Verification

Differentiate $y = c(x - c)^2$:

$$y' = 2c(x - c).$$

We have $x - c = \frac{2y}{y'}$, so:

$$(y')^3 = [2c(x - c)]^3 = 8c^3(x - c)^3.$$

Also, $4xyy' - 8y^2 = 4xy \cdot 2c(x - c) - 8c^2(x - c)^4$.

From $y' = 2c(x - c)$, we get $c = \frac{y'}{2(x-c)}$. Substituting $c = \frac{(y')^2}{4y}$:

$$(y')^3 = 4xy \cdot y' - 8y^2,$$

confirming the differential equation is satisfied. **Remark.** The solution $y = c(x - c)^2$

represents a one-parameter family of parabolas with vertices at $(c, 0)$. The differential equation is of third order in $y'$, which is characteristic of certain envelope problems.