Problem 55

Evaluate the following limit:

$$\lim_{x\to 0} \frac{\ln\cos(ax)}{\ln\cos(bx)}.$$

Solution

We start by expanding $\cos(ax)$ and $\cos(bx)$ using the Taylor series expansion around $x = 0$. For small x, we have:

$$\cos(y) \approx 1 - \frac{y^2}{2}.$$

Thus:

$$\cos(ax) \approx 1 - \frac{(ax)^2}{2}, \quad \cos(bx) \approx 1 - \frac{(bx)^2}{2}.$$

Next, we use the approximation for $\ln(1-u)$ when $u \to 0^+$:

$$\ln(1-u) \approx -u.$$

Applying this to $\ln(\cos(ax))$ and $\ln(\cos(bx))$:

$$\ln(\cos(ax)) \approx -\frac{(ax)^2}{2}, \quad \ln(\cos(bx)) \approx -\frac{(bx)^2}{2}.$$

Substituting these approximations into the limit:

$$\lim_{x\to 0} \frac{\ln\cos(ax)}{\ln\cos(bx)} \approx \lim_{x\to 0} \frac{-\frac{(ax)^2}{2}}{-\frac{(bx)^2}{2}}.$$

Simplify the fraction:

$$\frac{-\frac{(ax)^2}{2}}{-\frac{(bx)^2}{2}} = \frac{(ax)^2}{(bx)^2} = \frac{a^2}{b^2}.$$

Since this expression is independent of x, the limit is:

$$\lim_{x\to 0} \frac{\ln\cos(ax)}{\ln\cos(bx)} = \frac{a^2}{b^2}.$$

Final Answer

The value of the limit is:

$$\lim_{x\to 0} \frac{\ln\cos(ax)}{\ln\cos(bx)} = \frac{a^2}{b^2}.$$