Problem 9: Comparing Numbers

Determine which number is larger in each pair:

1. $\frac{5}{7}$ or $\frac{7}{9}$
2. $\sqrt[3]{5}$ or $\sqrt{3}$
3. $\log_{10} 5$ or $\log_9 7$
4. $\frac{3\sqrt{7}+5\sqrt{2}}{\sqrt{5}}$ or 6
5. $\sqrt{8\sqrt{15}}$ or $\frac{1}{2}(\sqrt{30}-\sqrt{2})$

Solutions

1° Compare $\frac{5}{7}$ and $\frac{7}{9}$

Method: Cross multiplication

Compare $\frac{5}{7}$ and $\frac{7}{9}$ by cross multiplying:

$$5 \cdot 9 = 45 \quad \text{vs} \quad 7 \cdot 7 = 49$$

Since $45 < 49$, we have $\frac{5}{7} < \frac{7}{9}$.

Alternative: Common denominator

$$\frac{5}{7} = \frac{45}{63}, \quad \frac{7}{9} = \frac{49}{63}$$

Since $45 < 49$, we have $\frac{45}{63} < \frac{49}{63}$.

$$\boxed{\frac{7}{9} \text{ is larger}}$$

2° Compare $\sqrt[3]{5}$ and $\sqrt{3}$

Method: Raise to the 6th power

To compare these, raise both to the 6th power (LCM of 2 and 3):

$$(\sqrt[3]{5})^6 = (5^{1/3})^6 = 5^2 = 25$$

$$(\sqrt{3})^6 = (3^{1/2})^6 = 3^3 = 27$$

Since $25 < 27$ and the function x^6 is increasing for positive x:

$$\sqrt[3]{5} < \sqrt{3}$$

Numerical verification:

$$\sqrt[3]{5} \approx 1.710, \quad \sqrt{3} \approx 1.732$$

$$\boxed{\sqrt{3} \text{ is larger}}$$

3° Compare $\log_{10} 5$ and $\log_9 7$

Method: Convert to common form or estimate

First, estimate each value:

$$\log_{10} 5 = \log_{10} \frac{10}{2} = \log_{10} 10 - \log_{10} 2 = 1 - \log_{10} 2 \approx 1 - 0.301 = 0.699$$

For $\log_9 7$, use the change of base formula:

$$\log_9 7 = \frac{\log_{10} 7}{\log_{10} 9} = \frac{\log_{10} 7}{2 \log_{10} 3}$$

We know:

$$\log_{10} 7 \approx 0.845, \quad \log_{10} 3 \approx 0.477$$

Therefore:

$$\log_9 7 \approx \frac{0.845}{2 \times 0.477} = \frac{0.845}{0.954} \approx 0.886$$

Since $0.699 < 0.886$:

$$\boxed{\log_9 7 \text{ is larger}}$$

4° Compare $\frac{3\sqrt{7}+5\sqrt{2}}{\sqrt{5}}$ and 6

Method: Square both sides after manipulation

First, rationalize and simplify the left side:

$$\frac{3\sqrt{7} + 5\sqrt{2}}{\sqrt{5}} = \frac{(3\sqrt{7} + 5\sqrt{2})\sqrt{5}}{5} = \frac{3\sqrt{35} + 5\sqrt{10}}{5}$$

Compare with 6:

$$\frac{3\sqrt{35} + 5\sqrt{10}}{5} \overset{?}{<} 6$$

Multiply both sides by 5:

$$3\sqrt{35} + 5\sqrt{10} \overset{?}{<} 30$$

Estimate:

$$\sqrt{35} \approx 5.916, \quad \sqrt{10} \approx 3.162$$

$$3\sqrt{35} + 5\sqrt{10} \approx 3(5.916) + 5(3.162) = 17.748 + 15.810 = 33.558$$

Since $33.558 > 30$:

$$\boxed{\frac{3\sqrt{7} + 5\sqrt{2}}{\sqrt{5}} \text{ is larger}}$$

5° Compare $\sqrt{8\sqrt{15}}$ and $\frac{1}{2}(\sqrt{30}-\sqrt{2})$

Method: Square both sides

Let $A = \sqrt{8\sqrt{15}}$ and $B = \frac{1}{2}(\sqrt{30}-\sqrt{2})$.

Square both:

$$A^2 = 8\sqrt{15}$$

$$B^2 = \frac{1}{4}(\sqrt{30}-\sqrt{2})^2 = \frac{1}{4}(30 - 2\sqrt{30}\sqrt{2} + 2)$$
$$= \frac{1}{4}(32 - 2\sqrt{60}) = \frac{1}{4}(32 - 2\cdot 2\sqrt{15}) = \frac{1}{4}(32 - 4\sqrt{15})$$
$$= 8 - \sqrt{15}$$

Now compare A^2 and B^2:

$$A^2 = 8\sqrt{15}, \quad B^2 = 8 - \sqrt{15}$$

Compare these by checking if $8\sqrt{15} > 8 - \sqrt{15}$:

$$8\sqrt{15} + \sqrt{15} \overset{?}{>} 8$$

$$9\sqrt{15} \overset{?}{>} 8$$
$$\sqrt{15} \overset{?}{>} \frac{8}{9}$$

Square both sides:

$$15 \overset{?}{>} \frac{64}{81}$$
$$15 \approx 15 > 0.790$$

This is clearly true. Therefore $A^2 > B^2$, which means:

$$\boxed{\sqrt{8\sqrt{15}} \text{ is larger}}$$

Summary of Results

1. $\frac{7}{9} > \frac{5}{7}$
2. $\sqrt{3} > \sqrt[3]{5}$
3. $\log_9 7 > \log_{10} 5$
4. $\frac{3\sqrt{7}+5\sqrt{2}}{\sqrt{5}} > 6$
5. $\sqrt{8\sqrt{15}} > \frac{1}{2}(\sqrt{30}-\sqrt{2})$

Remark

These comparison problems require different techniques:

- **Fractions:** Cross multiplication or common denominators
- **Different roots:** Raise to a common power (LCM of indices)
- **Logarithms:** Change of base formula and estimation
- **Radicals:** Squaring, rationalization, and algebraic manipulation

The key principle is to transform both numbers into comparable forms while preserving the inequality relationship (being careful with operations like squaring that require both sides to be positive).