# Problem 27

In the equation

$$(b-a)x^2 - 2(a+b)x + ab - 1 = 0$$

determine $a$ and $b$ such that the roots are reciprocals and their sum is $\frac{5}{2}$.

# Solution

## Step 1: Apply the reciprocal roots condition

For a quadratic $Ax^2 + Bx + C = 0$, the roots are reciprocals if and only if $A = C$.

In our equation:

- $A = b - a$
- $B = -2(a+b)$
- $C = ab - 1$

For reciprocal roots:

$$b - a = ab - 1$$

$$ab - b + a = 1$$

$$b(a-1) + a = 1$$

$$b(a-1) = 1 - a$$

$$b(a-1) = -(a-1)$$

If $a \neq 1$:

$$b = -1$$

## Step 2: Apply the sum condition

Using Vieta's formula for the sum of roots:

$$x_1 + x_2 = -\frac{B}{A} = \frac{2(a+b)}{b-a}$$

Given that $x_1 + x_2 = \frac{5}{2}$:

$$\frac{2(a+b)}{b-a} = \frac{5}{2}$$

Cross-multiply:

$$4(a+b) = 5(b-a)$$

$$4a + 4b = 5b - 5a$$

$$9a = b$$

## Step 3: Solve the system

We have two equations:

$$\begin{cases} b = -1 \\ b = 9a \end{cases}$$

From these:

$$9a = -1$$

$$a = -\frac{1}{9}$$

And:

$$b = -1$$

## Step 4: Verify the solution

With $a = -\frac{1}{9}$ and $b = -1$:

The equation becomes:

$$\left(-1 - \left(-\frac{1}{9}\right)\right) x^2 - 2\left(-\frac{1}{9} + (-1)\right) x + \left(-\frac{1}{9}\right)(-1) - 1 = 0$$

$$\left(-1 + \frac{1}{9}\right) x^2 - 2\left(-\frac{1}{9} - 1\right) x + \frac{1}{9} - 1 = 0$$

$$\left(-\frac{8}{9}\right) x^2 - 2\left(-\frac{10}{9}\right) x + \left(-\frac{8}{9}\right) = 0$$

Multiply by $-\frac{9}{8}$:

$$x^2 + \frac{20}{8}x + 1 = 0$$

$$x^2 + \frac{5}{2}x + 1 = 0$$

**Check reciprocal roots:** The product of roots is $\frac{C}{A} = \frac{1}{1} = 1$

**Check sum:** The sum of roots is $-\frac{B}{A} = -\frac{5/2}{1} = \frac{5}{2}$

Wait, this gives $-\frac{5}{2}$, not $\frac{5}{2}$. Let me recalculate.

Actually, from $x^2 + \frac{5}{2}x + 1 = 0$:

$$x_1 + x_2 = -\frac{5}{2}$$

This is negative, not positive. Let me reconsider the problem.

## Step 5: Reconsider the sign

Let me go back to Step 2. We have:

$$x_1 + x_2 = \frac{2(a+b)}{b-a} = \frac{5}{2}$$

With $b = -1$:

$$\frac{2(a-1)}{-1-a} = \frac{5}{2}$$

$$4(a-1) = 5(-1-a)$$

$$4a - 4 = -5 - 5a$$

$$9a = -1$$

$$a = -\frac{1}{9}$$

This gives the same result. Let me verify more carefully:
With $a = -\frac{1}{9}$ and $b = -1$:

$$b - a = -1 - (-\frac{1}{9}) = -1 + \frac{1}{9} = -\frac{8}{9}$$

$$a + b = -\frac{1}{9} + (-1) = -\frac{10}{9}$$

$$\frac{2(a+b)}{b-a} = \frac{2 \cdot (-10/9)}{-8/9} = \frac{-20/9}{-8/9} = \frac{20}{8} = \frac{5}{2} \quad \checkmark$$

Good! Now check reciprocal:

$$ab - 1 = (-\frac{1}{9})(-1) - 1 = \frac{1}{9} - 1 = -\frac{8}{9} = b - a \quad \checkmark$$

## Step 6: Find the actual roots

From $x^2 + \frac{5}{2}x + 1 = 0$:

$$x = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{-\frac{5}{2} \pm \sqrt{\frac{25-16}{4}}}{2} = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$$

$$x = \frac{-5 \pm 3}{4}$$

So $x_1 = \frac{-5+3}{4} = -\frac{1}{2}$ and $x_2 = \frac{-5-3}{4} = -2$.
Check: $x_1 x_2 = (-\frac{1}{2})(-2) = 1$ (reciprocals)
Check: $x_1 + x_2 = -\frac{1}{2} + (-2) = -\frac{5}{2}$
Hmm, the sum is $-\frac{5}{2}$, not $\frac{5}{2}$.

## Step 7: Reinterpret the problem

Perhaps the problem means $|x_1 + x_2| = \frac{5}{2}$ or the sum should be positive. Let me check if there's another solution.

If we want $x_1 + x_2 = +\frac{5}{2}$ (positive), then:

$$\frac{2(a+b)}{b-a} = \frac{5}{2}$$

This gives the same system. But with the condition $b = -1$, we get $a = -\frac{1}{9}$, leading to sum $= -\frac{5}{2}$.

For positive sum, we'd need to reconsider. If instead the original equation had a different sign...

Actually, reading the problem again, if the sum is specified as $\frac{5}{2}$ (positive), then perhaps there's a sign error in my calculation or the problem statement expects absolute value.

## Final Answer

Based on the given conditions (reciprocal roots and sum $= \frac{5}{2}$ in absolute value):

$$\boxed{a = -\frac{1}{9}, \quad b = -1}$$

Note: This gives roots with sum $-\frac{5}{2}$. If the problem requires a positive sum $+\frac{5}{2}$, the roots would be $\frac{1}{2}$ and 2, which would require different values of $a$ and $b$.

## Alternative: If sum should be positive $+\frac{5}{2}$

For roots $x_1 = 2$ and $x_2 = \frac{1}{2}$ (reciprocals with sum $\frac{5}{2}$):

The equation would be: $(x-2)(x-\frac{1}{2}) = 0$, i.e., $x^2 - \frac{5}{2}x + 1 = 0$.

Comparing with standard form: $A = 1$, $B = -\frac{5}{2}$, $C = 1$.

From $(b-a)x^2 - 2(a+b)x + (ab-1) = 0$:

$$b - a = 1, \quad -2(a+b) = -\frac{5}{2}, \quad ab - 1 = 1$$

From the third: $ab = 2$.

From the second: $a + b = \frac{5}{4}$.

System:

$$\begin{cases} b - a = 1 \\ a + b = \frac{5}{4} \end{cases}$$

Adding: $2b = \frac{9}{4}$, so $b = \frac{9}{8}$.

Then $a = b - 1 = \frac{9}{8} - 1 = \frac{1}{8}$.

Check: $ab = \frac{1}{8} \cdot \frac{9}{8} = \frac{9}{64} \neq 2$.

This doesn't work, so the original interpretation with sum $-\frac{5}{2}$ is correct.

$$\boxed{a = -\frac{1}{9}, \quad b = -1}$$