Problem 2

Evaluate the sum S for the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}}.$$

Solution

Step 1: Identify the Series Type

The given series can be rewritten to reveal its structure:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{1} \cdot \left(\frac{-1}{2}\right)^{n-1} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

This is a geometric series of the form $\sum_{n=1}^{\infty} ar^{n-1}$, where:

- The first term is $a = \frac{(-1)^{1-1}}{2^{1-1}} = \frac{1}{1} = 1$.
- The common ratio is $r = \frac{-1}{2}$.

Step 2: Check for Convergence

The series converges if and only if the absolute value of the common ratio $|r|$ is less than 1:

$$|r| = \left|\frac{-1}{2}\right| = \frac{1}{2}.$$

Since $|r| = 1/2 < 1$, the series converges.

Step 3: Calculate the Sum S

The sum of a convergent geometric series starting at $n = 1$ is given by the formula:

$$S = \frac{a}{1 - r}.$$

Substituting the values $a = 1$ and $r = -1/2$:

$$S = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}}.$$

$$S = \frac{2}{3}.$$

Final Answer

The sum of the series is:

$$\boxed{S = \frac{2}{3}.}$$