Problem 3

Find the differential equation satisfied by

$$y = \sin(x + c),$$

and solve that differential equation. Present the solution and any domain remarks.

Solution

1. Determine the differential equation

Assume

$$y(x) = \sin(x + c),$$

where c is an arbitrary constant. Differentiate with respect to x:

$$y'(x) = \cos(x + c), \qquad y''(x) = -\sin(x + c) = -y(x).$$

Therefore y satisfies the second-order linear homogeneous ordinary differential equation

$$\boxed{y'' + y = 0.}$$

2. Solve the differential equation

We solve the ODE

$$y'' + y = 0$$

using the characteristic equation. Assume $y = e^{rx}$. Substituting gives

$$r^2 + 1 = 0 \quad \Longrightarrow \quad r = \pm i.$$

Hence the general real solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

where C_1 and C_2 are arbitrary real constants.

3. Equivalence to the phase-shift form

The linear combination above can be written in amplitude–phase form. Let

$$R = \sqrt{C_1^2 + C_2^2} \quad (\geq 0).$$

If $R > 0$, define an angle φ such that

$$\cos\varphi = \frac{C_2}{R}, \qquad \sin\varphi = \frac{C_1}{R},$$

so that

$$C_1 \cos x + C_2 \sin x = R \sin(x + \varphi).$$

Thus an equivalent representation of the general solution is

$$\boxed{y(x) = R\sin(x + \varphi), \quad R \geq 0,\ \varphi \in \mathbb{R}.}$$

In particular, the special case $R = 1$ and $\varphi = c$ gives the form

$$y(x) = \sin(x + c),$$

which is the expression in the problem.

4. Remarks on the domain

All functions above are defined for every real x; the natural domain of real solutions is $x \in \mathbb{R}$. The constants C_1, C_2 (or R, φ) are determined by initial conditions if provided.

5. Verification

Differentiate $y = C_1 \cos x + C_2 \sin x$ twice:

$$y' = -C_1 \sin x + C_2 \cos x, \qquad y'' = -C_1 \cos x - C_2 \sin x = -y,$$

so $y'' + y = 0$ is satisfied.

Final answers:

$\boxed{\text{Differential equation: } y'' + y = 0}$ and $\boxed{\text{General solution: } y = C_1 \cos x + C_2 \sin x = R \sin(x +}$