Problem 20

Consider the function:

$$f(x) = \frac{x(x^2 - 1)}{(x^2 + 1)(x - 1)}.$$

Find the simplification of the function and identify its behavior.

Solution

Factor the numerator and denominator

The numerator $x(x^2 - 1)$ can be factored using the difference of squares:

$$x(x^2 - 1) = x(x - 1)(x + 1).$$

The denominator is already factored:

$$(x^2 + 1)(x - 1).$$

Thus, the function becomes:

$$f(x) = \frac{x(x - 1)(x + 1)}{(x^2 + 1)(x - 1)}.$$

Cancel out common factors

We can cancel the factor $(x - 1)$ from the numerator and denominator (assuming $x \neq 1$):

$$f(x) = \frac{x(x + 1)}{x^2 + 1}, \quad \text{for} \quad x \neq 1.$$

Final simplified form

The simplified form of the function is:

$$f(x) = \frac{x(x + 1)}{x^2 + 1}, \quad \text{for} \quad x \neq 1.$$

Behavior at $x = 1$

The original function has a discontinuity at $x = 1$ due to the factor $(x - 1)$ in the denominator. As $x \to 1$, the function approaches:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x(x + 1)}{x^2 + 1} = \frac{1(1 + 1)}{1^2 + 1} = \frac{2}{2} = 1.$$

Conclusion

The function simplifies to:

$$f(x) = \frac{x(x + 1)}{x^2 + 1}, \quad \text{for} \quad x \neq 1.$$

The limit of the function as $x \to 1$ is:

$$\lim_{x \to 1} f(x) = 1.$$