Problem 12

Evaluate the following limit:

$$\lim_{x\to+\infty} \frac{(x+1)(x^2+1)\cdots(x^n+1)}{((nx)^n+1)^{\frac{n+1}{2}}}$$

Solution

We begin by simplifying both the numerator and the denominator for large x.

Simplifying the Numerator

The numerator is a product:

$$(x+1)(x^2+1)\cdots(x^n+1)$$

For large x, each term (x^k+1) behaves approximately like x^k, so:

$$(x+1)(x^2+1)\cdots(x^n+1) \approx x^1 \cdot x^2 \cdot \cdots \cdot x^n = x^{1+2+\cdots+n} = x^{\frac{n(n+1)}{2}}$$

Simplifying the Denominator

The denominator is:

$$((nx)^n+1)^{\frac{n+1}{2}}$$

For large x, $(nx)^n$ dominates over 1, so:

$$(nx)^n + 1 \approx (nx)^n$$

Thus, the denominator becomes:

$$((nx)^n)^{\frac{n+1}{2}} = (nx)^{n\cdot\frac{n+1}{2}} = n^{\frac{n(n+1)}{2}} x^{n\cdot\frac{n+1}{2}}$$

Final Expression

Now we can write the limit as:

$$\frac{x^{\frac{n(n+1)}{2}}}{n^{\frac{n(n+1)}{2}} x^{n\cdot\frac{n+1}{2}}}$$

Simplifying:

$$\frac{1}{n^{\frac{n(n+1)}{2}}}$$

Conclusion

Therefore, the limit is:

$$\lim_{x\to+\infty} \frac{(x+1)(x^2+1)\cdots(x^n+1)}{((nx)^n+1)^{\frac{n+1}{2}}} = \frac{1}{n^{\frac{n(n+1)}{2}}}$$