

### Problem 3

Evaluate the partial sum  $S_n$  and the sum  $S = \lim_{n \rightarrow \infty} S_n$  for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

### Solution

#### Step 1: Partial Fraction Decomposition

The general term of the series,  $a_n = \frac{1}{n(n+1)}$ , can be decomposed using partial fractions:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}.$$

Multiplying by  $n(n+1)$ :

$$1 = A(n+1) + Bn.$$

- Set  $n = 0$ :  $1 = A(1) + 0 \implies A = 1$ .
- Set  $n = -1$ :  $1 = 0 + B(-1) \implies B = -1$ .

Thus, the general term is:

$$a_n = \frac{1}{n} - \frac{1}{n+1}.$$

#### Step 2: Determine the Partial Sum $S_n$

The series is a \*\*telescoping series\*\*. The  $n$ -th partial sum  $S_n$  is the sum of the first  $n$  terms:

$$S_n = \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right).$$

Writing out the terms:

$$S_n = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n-1} - \frac{1}{n} \right) + \left( \frac{1}{n} - \frac{1}{n+1} \right).$$

The intermediate terms cancel out in pairs (the negative part of one term cancels the positive part of the next term). This is the characteristic of a telescoping series. The simplified partial sum is left with only the first positive term and the last negative term:

$$S_n = \frac{1}{1} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

#### Step 3: Calculate the Sum $S$

The sum of the series  $S$  is the limit of the partial sum  $S_n$  as  $n \rightarrow \infty$ :

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right).$$

Since  $\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$ :

$$S = 1 - 0 = 1.$$

## Final Answer

### Partial Sum $S_n$

The  $n$ -th partial sum is:

$$S_n = 1 - \frac{1}{n+1}.$$

### Sum $S$

The sum of the series is:

$$S = 1.$$