

Problem 4

Evaluate the sum S for the series:

$$\sum_{n=0}^{\infty} \frac{1}{(n+k)(n+k+1)} \quad (k \in \mathbb{N}).$$

Solution

Step 1: Partial Fraction Decomposition

The general term of the series, $a_n = \frac{1}{(n+k)(n+k+1)}$, can be decomposed using partial fractions:

$$\frac{1}{(n+k)(n+k+1)} = \frac{A}{n+k} + \frac{B}{n+k+1}.$$

Multiplying by $(n+k)(n+k+1)$:

$$1 = A(n+k+1) + B(n+k).$$

- Set $n = -k$: $1 = A(-k+k+1) + B(0) \implies A = 1$.
- Set $n = -k-1$: $1 = A(0) + B(-k-1+k) \implies 1 = -B \implies B = -1$.

Thus, the general term is:

$$a_n = \frac{1}{n+k} - \frac{1}{n+k+1}.$$

Step 2: Determine the Partial Sum S_N

The series is a **telescoping series**. The N -th partial sum S_N is the sum from $n = 0$ to N :

$$S_N = \sum_{n=0}^N \left(\frac{1}{n+k} - \frac{1}{n+k+1} \right).$$

Writing out the first few and last few terms:

$$\begin{aligned} S_N &= \left(\frac{1}{0+k} - \frac{1}{0+k+1} \right) + \left(\frac{1}{1+k} - \frac{1}{1+k+1} \right) + \left(\frac{1}{2+k} - \frac{1}{2+k+1} \right) + \cdots \\ &\quad + \left(\frac{1}{N-1+k} - \frac{1}{N+k} \right) + \left(\frac{1}{N+k} - \frac{1}{N+k+1} \right). \end{aligned}$$

All intermediate terms cancel out. The simplified partial sum is left with only the first positive term and the last negative term:

$$S_N = \frac{1}{k} - \frac{1}{N+k+1}.$$

Step 3: Calculate the Sum S

The sum of the series S is the limit of the partial sum S_N as $N \rightarrow \infty$:

$$S = \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(\frac{1}{k} - \frac{1}{N+k+1} \right).$$

Since k is a constant, $\lim_{N \rightarrow \infty} \frac{1}{N+k+1} = 0$:

$$S = \frac{1}{k} - 0 = \frac{1}{k}.$$

Final Answer

Partial Sum S_N

The N -th partial sum is:

$$S_N = \frac{1}{k} - \frac{1}{N+k+1}.$$

Sum S

The sum of the series is:

$$S = \frac{1}{k}.$$