Problem 36

Evaluate the limit of the function $Y(x) = \sinh(x) - x$ as $x \to 0$.

Solution

To find $\lim_{x\to 0} Y(x) = \lim_{x\to 0}(\sinh(x) - x)$, we proceed as follows:

1. Recall the definition of the hyperbolic sine function:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

2. Using the Taylor series expansion of $\sinh(x)$ about $x = 0$:

$$\sinh(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \dots$$

Substituting this into $Y(x)$, we get:

$$Y(x) = \sinh(x) - x = \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \dots\right) - x.$$

Simplifying:

$$Y(x) = \frac{x^3}{6} + \frac{x^5}{120} + \dots$$

3. As $x \to 0$, higher-order terms $(x^5, x^7, \dots)$ become negligible. Thus:

$$Y(x) \sim \frac{x^3}{6}.$$

4. Evaluating the limit:

$$\lim_{x\to 0} Y(x) = \lim_{x\to 0} \frac{x^3}{6} = 0.$$

Therefore:

$$\lim_{x\to 0}(\sinh(x) - x) = 0.$$