# Problem 15

Solve the equation:

$$\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$$

# Solution

## Step 1: Use double angle formulas

Recall:

$$\sin 2x = 2\sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x$$

Substitute into the equation:

$$2\sin x \cos x + (\cos^2 x - \sin^2 x) + \sin x + \cos x + 1 = 0$$

## Step 2: Rearrange and factor

Group the terms:

$$2\sin x \cos x + \cos^2 x - \sin^2 x + \sin x + \cos x + 1 = 0$$

Notice that $\cos^2 x + 1 = \cos^2 x + \sin^2 x + \cos^2 x = 2\cos^2 x + \sin^2 x - \sin^2 x$... Let me try a different approach. Rewrite using $\sin^2 x + \cos^2 x = 1$:

$$2\sin x \cos x + \cos^2 x - \sin^2 x + \sin x + \cos x + \sin^2 x + \cos^2 x = 0$$

$$2\sin x \cos x + 2\cos^2 x + \sin x + \cos x = 0$$

Factor:

$$2\cos x(\sin x + \cos x) + (\sin x + \cos x) = 0$$

$$(\sin x + \cos x)(2\cos x + 1) = 0$$

## Step 3: Solve each factor

**Case 1:** $\sin x + \cos x = 0$

$$\sin x = -\cos x$$

$$\tan x = -1$$

The general solution is:

$$x = -\frac{\pi}{4} + k\pi = \frac{3\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

Or more compactly:

$$\boxed{x = -\frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}}$$

In degrees: $x = -45 + 180k$ or $x = 135 + 180k$

**Case 2:** $2\cos x + 1 = 0$

$$\cos x = -\frac{1}{2}$$

The general solution is:

$$\boxed{x = \pm\frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}}$$

In degrees: $x = \pm 120 + 360k$

## Step 4: Complete solution set

$$\boxed{x \in \left\{-\frac{\pi}{4} + k\pi, \pm\frac{2\pi}{3} + 2k\pi : k \in \mathbb{Z}\right\}}$$

Or listing the fundamental solutions in $[0, 2\pi)$:

$$x \in \left\{\frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}\right\}$$

In degrees: $x \in \{120, 135, 240, 315\}$

## Verification

Let's verify one solution from each case.

**For $x = \frac{3\pi}{4}$ (or 135):**

Calculate each term:

$$\sin 2x = \sin\left(\frac{3\pi}{2}\right) = -1$$

$$\cos 2x = \cos\left(\frac{3\pi}{2}\right) = 0$$

$$\sin x = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\cos x = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Sum:

$$-1 + 0 + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1 = 0 \quad \checkmark$$

**For $x = \frac{2\pi}{3}$ (or 120):**

Calculate each term:

$$\sin 2x = \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos 2x = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

$$\sin x = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$\cos x = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

Sum:

$$-\frac{\sqrt{3}}{2} - \frac{1}{2} + \frac{\sqrt{3}}{2} - \frac{1}{2} + 1 = -1 + 1 = 0 \quad \checkmark$$

## Alternative Solution Method

Here's another way to see the factorization:

Starting from:

$$\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$$

Use $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 2\cos^2 x - 1$:

$$2\sin x\cos x + 2\cos^2 x - 1 + \sin x + \cos x + 1 = 0$$

$$2\sin x\cos x + 2\cos^2 x + \sin x + \cos x = 0$$

Factor by grouping:

$$2\cos x(\sin x + \cos x) + 1(\sin x + \cos x) = 0$$

$$(\sin x + \cos x)(2\cos x + 1) = 0$$

This confirms our factorization.

## Summary

The equation $\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$ factors as:

$$(\sin x + \cos x)(2\cos x + 1) = 0$$

Solutions:

- From $\sin x + \cos x = 0$: $x = -\frac{\pi}{4} + k\pi$
- From $\cos x = -\frac{1}{2}$: $x = \pm\frac{2\pi}{3} + 2k\pi$

## Remark

This problem demonstrates the power of substitution and factoring in solving trigonometric equations. The key steps were:

1. Applying double angle formulas to express everything in terms of $\sin x$ and $\cos x$
2. Recognizing that $\cos^2 x + 1 = 2\cos^2 x$ (using $\sin^2 x + \cos^2 x = 1$)
3. Factoring by grouping to obtain $(\sin x + \cos x)(2\cos x + 1) = 0$

The resulting factored form makes it straightforward to find all solutions by solving two simpler equations.