Problem 7

Prove the identity:

$$\frac{1}{\sqrt{x}+\sqrt{x+2}}+\frac{1}{\sqrt{x+2}+\sqrt{x+4}}+\cdots+\frac{1}{\sqrt{x+2n-2}+\sqrt{x+2n}}=\frac{n}{\sqrt{x+2n}+\sqrt{x}}$$

for $x > 0$ and $n \in \mathbb{N}$.

Solution

Key principle: Rationalization

To simplify each term, we rationalize the denominator by multiplying by the conjugate.

Step 1: Rationalize a general term

Consider the k-th term of the sum:

$$\frac{1}{\sqrt{x+2k-2}+\sqrt{x+2k}}$$

Multiply numerator and denominator by the conjugate $\sqrt{x+2k}-\sqrt{x+2k-2}$:

$$\frac{1}{\sqrt{x+2k-2}+\sqrt{x+2k}}\cdot\frac{\sqrt{x+2k}-\sqrt{x+2k-2}}{\sqrt{x+2k}-\sqrt{x+2k-2}}$$

The denominator becomes:

$$(\sqrt{x+2k-2}+\sqrt{x+2k})(\sqrt{x+2k}-\sqrt{x+2k-2})=(x+2k)-(x+2k-2)=2$$

Therefore:

$$\frac{1}{\sqrt{x+2k-2}+\sqrt{x+2k}}=\frac{\sqrt{x+2k}-\sqrt{x+2k-2}}{2}$$

Step 2: Apply to each term

Using the rationalization formula, we can rewrite each term:

First term ($k = 1$):

$$\frac{1}{\sqrt{x}+\sqrt{x+2}}=\frac{\sqrt{x+2}-\sqrt{x}}{2}$$

Second term ($k = 2$):

$$\frac{1}{\sqrt{x+2}+\sqrt{x+4}}=\frac{\sqrt{x+4}-\sqrt{x+2}}{2}$$

Third term ($k = 3$):

$$\frac{1}{\sqrt{x+4}+\sqrt{x+6}} = \frac{\sqrt{x+6}-\sqrt{x+4}}{2}$$

$\vdots$

Last term ($k = n$):

$$\frac{1}{\sqrt{x+2n-2}+\sqrt{x+2n}} = \frac{\sqrt{x+2n}-\sqrt{x+2n-2}}{2}$$

Step 3: Sum the telescoping series

The sum becomes:

$$\frac{1}{\sqrt{x}+\sqrt{x+2}} + \frac{1}{\sqrt{x+2}+\sqrt{x+4}} + \cdots + \frac{1}{\sqrt{x+2n-2}+\sqrt{x+2n}}$$
$$= \frac{\sqrt{x+2}-\sqrt{x}}{2} + \frac{\sqrt{x+4}-\sqrt{x+2}}{2} + \frac{\sqrt{x+6}-\sqrt{x+4}}{2} + \cdots + \frac{\sqrt{x+2n}-\sqrt{x+2n-2}}{2}$$

Factor out $\frac{1}{2}$:

$$= \frac{1}{2}\left[(\sqrt{x+2}-\sqrt{x}) + (\sqrt{x+4}-\sqrt{x+2}) + (\sqrt{x+6}-\sqrt{x+4}) + \cdots + (\sqrt{x+2n}-\sqrt{x+2n-2})\right]$$

Step 4: Observe the telescoping pattern

Notice that this is a telescoping sum where consecutive terms cancel:

$$= \frac{1}{2}\left[\sqrt{x+2}-\sqrt{x}+\sqrt{x+4}-\sqrt{x+2}+\sqrt{x+6}-\sqrt{x+4}+\cdots+\sqrt{x+2n}-\sqrt{x+2n-2}\right]$$

After cancellation, only the first and last terms remain:

$$= \frac{1}{2}\left[\sqrt{x+2n}-\sqrt{x}\right] = \frac{\sqrt{x+2n}-\sqrt{x}}{2}$$

Step 5: Transform to the required form

We have shown that the left side equals $\frac{\sqrt{x+2n}-\sqrt{x}}{2}$. Now we need to show this equals $\frac{n}{\sqrt{x+2n}+\sqrt{x}}$.

Rationalize $\frac{n}{\sqrt{x+2n}+\sqrt{x}}$ by multiplying by the conjugate:

$$\frac{n}{\sqrt{x+2n}+\sqrt{x}} \cdot \frac{\sqrt{x+2n}-\sqrt{x}}{\sqrt{x+2n}-\sqrt{x}} = \frac{n(\sqrt{x+2n}-\sqrt{x})}{(x+2n)-x}$$
$$= \frac{n(\sqrt{x+2n}-\sqrt{x})}{2n} = \frac{\sqrt{x+2n}-\sqrt{x}}{2}$$

This matches our result from Step 4! Therefore:

$$\frac{\sqrt{x+2n}-\sqrt{x}}{2} = \frac{n}{\sqrt{x+2n}+\sqrt{x}}$$

Final result

$$\boxed{\frac{1}{\sqrt{x}+\sqrt{x+2}}+\frac{1}{\sqrt{x+2}+\sqrt{x+4}}+\cdots+\frac{1}{\sqrt{x+2n-2}+\sqrt{x+2n}}=\frac{n}{\sqrt{x+2n}+\sqrt{x}}}$$

Verification

Let's verify with specific values.

For $x = 1$, $n = 1$:

$$\text{Left side: } \frac{1}{\sqrt{1}+\sqrt{3}} = \frac{1}{1+\sqrt{3}} = \frac{\sqrt{3}-1}{(\sqrt{3})^2-1^2} = \frac{\sqrt{3}-1}{2}.$$

$$\text{Right side: } \frac{1}{\sqrt{3}+\sqrt{1}} = \frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{2}. \quad \checkmark$$

For $x = 4$, $n = 2$:

$$\text{Left side: } \frac{1}{\sqrt{4}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{8}} = \frac{1}{2+\sqrt{6}} + \frac{1}{\sqrt{6}+2\sqrt{2}}.$$

$$\text{Using Step 3: } \frac{\sqrt{8}-\sqrt{4}}{2} = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1.$$

$$\text{Right side: } \frac{2}{\sqrt{8}+\sqrt{4}} = \frac{2}{2\sqrt{2}+2} = \frac{2}{2(\sqrt{2}+1)} = \frac{1}{\sqrt{2}+1}$$

$$= \frac{\sqrt{2}-1}{(\sqrt{2})^2-1^2} = \frac{\sqrt{2}-1}{1} = \sqrt{2}-1. \quad \checkmark$$

For $x = 9$, $n = 3$:

$$\text{Using Step 3: } \frac{\sqrt{15}-\sqrt{9}}{2} = \frac{\sqrt{15}-3}{2}.$$

$$\text{Right side: } \frac{3}{\sqrt{15}+\sqrt{9}} = \frac{3}{\sqrt{15}+3} = \frac{3(\sqrt{15}-3)}{(\sqrt{15})^2-3^2} = \frac{3(\sqrt{15}-3)}{15-9}$$

$$= \frac{3(\sqrt{15}-3)}{6} = \frac{\sqrt{15}-3}{2}. \quad \checkmark$$

Remark

This is a classic example of a **telescoping sum**. The key technique is to rationalize each denominator, which transforms the sum into a form where consecutive terms cancel out. Only the first and last terms survive, yielding $\frac{\sqrt{x+2n}-\sqrt{x}}{2}$.

The identity $\frac{\sqrt{x+2n}-\sqrt{x}}{2} = \frac{n}{\sqrt{x+2n}+\sqrt{x}}$ follows from rationalizing the right side. These two forms are equivalent, and both represent the simplified result of the telescoping sum. Such problems demonstrate the power of algebraic manipulation in revealing hidden patterns.