Problem 44

Evaluate the following limit using the substitution:

$$\lim_{x \to 0} \frac{\cos(xe^x) - \cos(xe^{-x})}{x^3},$$

where

$$\cos(xe^x) - \cos(xe^{-x}) = -2\sin\left(\frac{xe^x + xe^{-x}}{2}\right)\sin\left(\frac{xe^x - xe^{-x}}{2}\right).$$

Solution

We begin by applying the given substitution:

$$\cos(xe^x) - \cos(xe^{-x}) = -2\sin\left(\frac{xe^x + xe^{-x}}{2}\right)\sin\left(\frac{xe^x - xe^{-x}}{2}\right).$$

Now, we rewrite the limit expression:

$$\lim_{x \to 0} \frac{-2\sin\left(\frac{xe^x + xe^{-x}}{2}\right)\sin\left(\frac{xe^x - xe^{-x}}{2}\right)}{x^3}.$$

For small x, we expand the exponential terms using the Taylor series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4), \quad e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \mathcal{O}(x^4).$$

We simplify $xe^x + xe^{-x}$ and $xe^x - xe^{-x}$:

$$xe^x + xe^{-x} = 2x + \mathcal{O}(x^3), \quad xe^x - xe^{-x} = 2x^2 + \mathcal{O}(x^4).$$

Thus, we have:

$$\frac{xe^x + xe^{-x}}{2} \approx x, \quad \frac{xe^x - xe^{-x}}{2} \approx x^2.$$

Using the small-angle approximation $\sin\theta \approx \theta$ for small θ, we approximate:

$$\sin\left(\frac{xe^x + xe^{-x}}{2}\right) \approx x, \quad \sin\left(\frac{xe^x - xe^{-x}}{2}\right) \approx x^2.$$

Substitute these into the limit expression:

$$\lim_{x \to 0} \frac{-2 \cdot x \cdot x^2}{x^3} = \lim_{x \to 0} \frac{-2x^3}{x^3} = -2.$$

Final Answer

$$\boxed{-2}.$$