Problem 9

Solve the differential equation
$$y = 1 + (y')^2.$$

Solution

1. Rewrite the differential equation

The given differential equation is
$$y = 1 + (y')^2,$$
which can be rewritten as:
$$(y')^2 = y - 1.$$
Taking the square root of both sides:
$$y' = \pm\sqrt{y-1}.$$
This is a first-order separable differential equation, requiring $y \geq 1$ for real solutions.

2. Solve the differential equation

We solve the equation
$$\frac{dy}{dx} = \pm\sqrt{y-1}.$$
Separate the variables:
$$\frac{dy}{\sqrt{y-1}} = \pm dx.$$
Integrate both sides:
$$\int \frac{dy}{\sqrt{y-1}} = \pm \int dx.$$
For the left-hand side, use the substitution $u = y - 1$, so $du = dy$:
$$\int \frac{dy}{\sqrt{y-1}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\, du = 2\sqrt{u} + C_1 = 2\sqrt{y-1} + C_1.$$
The right-hand side gives:
$$\pm \int dx = \pm x + C_2.$$
Equating the two integrals:
$$2\sqrt{y-1} = \pm x + C_2 - C_1.$$
Let $C = C_2 - C_1$, and absorb the $\pm$ sign into the constant:
$$2\sqrt{y-1} = x + C.$$
Solving for y:
$$\sqrt{y-1} = \frac{x+C}{2}.$$

Square both sides:

$$y - 1 = \frac{(x + C)^2}{4}.$$

Therefore:

$$y = 1 + \frac{(x + C)^2}{4}.$$

Letting $c = -C$ for convenience:

$$\boxed{y = 1 + \frac{(x - c)^2}{4}, \qquad c \in \mathbb{R}.}$$

3. Singular solution

Note that from the original equation $y = 1 + (y')^2$, if $y' = 0$, then $y = 1$. Check if $y = 1$ is a solution:

$$y = 1 \implies y' = 0 \implies (y')^2 = 0 \implies 1 + (y')^2 = 1.$$

Thus $y = 1$ satisfies the equation. This is the **singular solution**, which can be viewed as the envelope of the family of parabolas (obtained as $c \to \pm\infty$ or by considering the limit where the discriminant vanishes).

The complete solution is:

$$\boxed{y = 1 + \frac{(x - c)^2}{4} \quad \text{(general solution)} \qquad \text{and} \qquad y = 1 \quad \text{(singular solution).}}$$

4. Verification

Differentiate the general solution $y = 1 + \frac{(x-c)^2}{4}$:

$$y' = \frac{2(x - c)}{4} = \frac{x - c}{2}.$$

Compute $(y')^2$:

$$(y')^2 = \left(\frac{x - c}{2}\right)^2 = \frac{(x - c)^2}{4}.$$

Now check the original equation:

$$1 + (y')^2 = 1 + \frac{(x - c)^2}{4} = y.$$

Thus the general solution satisfies the differential equation.

For the singular solution $y = 1$:

$$y' = 0 \implies (y')^2 = 0 \implies 1 + (y')^2 = 1 = y.$$

The singular solution also satisfies the equation.

Remark. The general solution represents a family of parabolas opening upward with vertices at $(c, 1)$ for various values of the constant c. The singular solution $y = 1$ is the horizontal line that forms the envelope of this family of parabolas—each parabola is tangent to the line $y = 1$ at its vertex. Geometrically, all solution curves lie on or above the line $y = 1$, consistent with the requirement $y \geq 1$ from $(y')^2 = y - 1 \geq 0$.