Problem 286

Evaluate the following integral:

$$\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx$$

Solution

We will solve this integral using substitution to eliminate the square root.

Step 1: Choose the substitution

Let $u = \sqrt{x+1}$. Then:

$$\begin{aligned} u &= \sqrt{x+1} \\ u^2 &= x+1 \\ x &= u^2 - 1 \\ dx &= 2u\,du \end{aligned}$$

Also, $\sqrt{x+1} = u$.

Step 2: Substitute into the integral

$$\begin{aligned} \int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx &= \int \frac{u+1}{u-1} \cdot 2u\,du \\ &= 2\int \frac{u(u+1)}{u-1}\,du \\ &= 2\int \frac{u^2+u}{u-1}\,du \end{aligned}$$

Step 3: Perform polynomial long division

We divide $u^2 + u$ by $(u-1)$:

First step: $u^2 \div (u-1)$ gives quotient u:

$$u \cdot (u-1) = u^2 - u$$

Subtracting: $(u^2 + u) - (u^2 - u) = 2u$

Second step: $2u \div (u-1)$ gives quotient 2:

$$2 \cdot (u-1) = 2u - 2$$

Subtracting: $2u - (2u-2) = 2$

Therefore:

$$\frac{u^2+u}{u-1} = u + 2 + \frac{2}{u-1}$$

Step 4: Verify the division

We can verify:

$$(u+2)(u-1) + 2 = u^2 - u + 2u - 2 + 2 = u^2 + u \quad \checkmark$$

Step 5: Integrate term by term

$$\begin{aligned} 2\int \frac{u^2+u}{u-1}\,du &= 2\int \left(u + 2 + \frac{2}{u-1}\right) du \\ &= 2\left[\int u\,du + \int 2\,du + \int \frac{2}{u-1}\,du\right] \end{aligned}$$

$$= 2\left[\frac{u^2}{2} + 2u + 2\ln|u-1|\right] + C$$

$$= u^2 + 4u + 4\ln|u-1| + C$$

Step 6: Substitute back $u = \sqrt{x+1}$

$$= (\sqrt{x+1})^2 + 4\sqrt{x+1} + 4\ln|\sqrt{x+1}-1| + C$$

$$= (x+1) + 4\sqrt{x+1} + 4\ln|\sqrt{x+1}-1| + C$$

$$= x + 1 + 4\sqrt{x+1} + 4\ln|\sqrt{x+1}-1| + C$$

Since the constant can absorb the +1, we can write:

$$= x + 4\sqrt{x+1} + 4\ln|\sqrt{x+1}-1| + C$$

Step 7: Final answer

$$\boxed{\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx = x + 4\sqrt{x+1} + 4\ln|\sqrt{x+1}-1| + C}$$

Verification:
We can verify our answer by differentiation:

$$\frac{d}{dx}\left[x + 4\sqrt{x+1} + 4\ln|\sqrt{x+1}-1|\right] = 1 + 4\cdot\frac{1}{2\sqrt{x+1}} + 4\cdot\frac{1}{\sqrt{x+1}-1}\cdot\frac{1}{2\sqrt{x+1}}$$

$$= 1 + \frac{2}{\sqrt{x+1}} + \frac{2}{\sqrt{x+1}(\sqrt{x+1}-1)}$$

$$= 1 + \frac{2}{\sqrt{x+1}}\left(1 + \frac{1}{\sqrt{x+1}-1}\right)$$

$$= 1 + \frac{2}{\sqrt{x+1}}\cdot\frac{\sqrt{x+1}-1+1}{\sqrt{x+1}-1}$$

$$= 1 + \frac{2}{\sqrt{x+1}}\cdot\frac{\sqrt{x+1}}{\sqrt{x+1}-1}$$

$$= 1 + \frac{2}{\sqrt{x+1}-1}$$

$$= \frac{\sqrt{x+1}-1+2}{\sqrt{x+1}-1}$$

$$= \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1} \quad \checkmark$$