Problem 1

Evaluate the partial sum S_n and the sum $S = \lim_{n\to\infty} S_n$ for the series:

$$\sum_{n=1}^{\infty} q^n.$$

Solution

The given series is a geometric series starting from $n = 1$.

Step 1: Determine the Partial Sum S_n

The n-th partial sum S_n is the sum of the first n terms of the series:

$$S_n = q^1 + q^2 + q^3 + \cdots + q^n. \quad \text{(Equation 1)}$$

To find a closed-form expression for S_n, we multiply Equation 1 by q:

$$qS_n = q^2 + q^3 + q^4 + \cdots + q^{n+1}. \quad \text{(Equation 2)}$$

Subtracting Equation 2 from Equation 1:

$$S_n - qS_n = (q + q^2 + \cdots + q^n) - (q^2 + q^3 + \cdots + q^{n+1}).$$

Most terms cancel out, leaving:

$$S_n(1 - q) = q - q^{n+1}.$$

Assuming $q \neq 1$, we solve for S_n:

$$S_n = \frac{q - q^{n+1}}{1 - q} = \frac{q(1 - q^n)}{1 - q}.$$

Step 2: Determine the Sum S

The sum of the series S is the limit of the partial sum S_n as $n \to \infty$:

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{q - q^{n+1}}{1 - q}.$$

The convergence depends on the term q^{n+1}:

1. **Case $|q| < 1$ (Convergence):** If $|q| < 1$, then $\lim_{n\to\infty} q^{n+1} = 0$.

$$S = \frac{q - 0}{1 - q} = \frac{q}{1 - q}.$$

2. **Case $|q| \geq 1$ or $q = 1$ (Divergence):** If $|q| > 1$, $\lim_{n\to\infty} |q^{n+1}| = \infty$, so the series diverges. If $q = 1$, $S_n = 1 + 1 + \cdots + 1 = n$, so $\lim_{n\to\infty} S_n = \infty$, and the series diverges. If $q = -1$, $S_n = -1 + 1 - 1 + \cdots + (-1)^n$, which oscillates, so the series diverges.

The sum S only exists for $|q| < 1$.

Final Answer

Partial Sum S_n

The n-th partial sum is:

$$\boxed{S_n = \frac{q(1 - q^n)}{1 - q}, \quad \text{for } q \neq 1.}$$

Sum S

The sum of the series is:

$$\boxed{S = \lim_{n\to\infty} S_n = \begin{cases} \frac{q}{1-q}, & \text{if } |q| < 1 \\ \text{Diverges}, & \text{if } |q| \geq 1 \end{cases}.}$$