Problem 186

Evaluate the integral:

$$I = \int \frac{x^5}{\sqrt{1-x^4}}\, dx.$$

Solution

We use the substitution:

$$t = x^2 \Rightarrow dt = 2x\, dx.$$

Rewriting the integral in terms of t:

$$I = \int \frac{x^5}{\sqrt{1-x^4}}\, dx.$$

Rewriting x^5 as:

$$x^5 = x^2 \cdot x^3 = t \cdot x^3.$$

Differentiating both sides of $t = x^2$:

$$dt = 2x\, dx \Rightarrow x^3 dx = \frac{dt}{2} x.$$

Rewriting $\sqrt{1-x^4}$:

$$\sqrt{1-x^4} = \sqrt{1-t^2}.$$

Substituting these into the integral:

$$I = \int \frac{t \cdot x^3}{\sqrt{1-t^2}}\, dx.$$

Using the previous substitution for $x^3 dx$:

$$I = \int \frac{t}{\sqrt{1-t^2}} \cdot \frac{dt}{2} x.$$

Rewriting in terms of t:

$$I = \frac{1}{2} \int \frac{t dt}{\sqrt{1-t^2}}.$$

Using the standard integral result:

$$\int \frac{t dt}{\sqrt{1-t^2}} = -\frac{t^2}{2}\sqrt{1-t^2} + \frac{1}{2}\arcsin t.$$

Substituting back $t = x^2$:

$$I = -\frac{x^2}{4}\sqrt{1-x^4} + \frac{1}{4}\arcsin x^2 + C.$$

Thus, the final result is:

$$\int \frac{x^5}{\sqrt{1-x^4}}\, dx = -\frac{x^2}{4}\sqrt{1-x^4} + \frac{1}{4}\arcsin x^2 + C.$$