Problem 16

Evaluate the following limit:

$$\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$$

Solution

We begin by substituting $x = 1$ into both the numerator and denominator:

Substitute $x = 1$

- First factor:

$$1 - x = 1 - 1 = 0$$

- Second factor:

$$\tan\left(\frac{\pi x}{2}\right) = \tan\left(\frac{\pi}{2}\right)$$

Since $\tan\left(\frac{\pi}{2}\right)$ tends to infinity, we encounter the indeterminate form $0 \times \infty$.

Rewriting the Expression

We rewrite the expression by substituting $y = 1 - x$, so that as $x \to 1$, we have $y \to 0$. The expression becomes:

$$\lim_{y \to 0} y \cdot \tan\left(\frac{\pi(1-y)}{2}\right)$$

Simplifying:

$$\lim_{y \to 0} y \cdot \tan\left(\frac{\pi}{2} - \frac{\pi y}{2}\right)$$

Using the Identity for $\tan\left(\frac{\pi}{2} - z\right)$

We use the identity:

$$\tan\left(\frac{\pi}{2} - z\right) = \cot(z)$$

Thus, the expression becomes:

$$\lim_{y \to 0} y \cdot \cot\left(\frac{\pi y}{2}\right)$$

Asymptotic Behavior of $\cot\left(\frac{\pi y}{2}\right)$

For small y, we use the approximation $\cot(z) \approx \frac{1}{z}$ as $z \to 0$, so:

$$\cot\left(\frac{\pi y}{2}\right) \approx \frac{2}{\pi y} \text{ as } y \to 0$$

Final Expression

Substituting this approximation:

$$\lim_{y \to 0} y \cdot \frac{2}{\pi y} = \lim_{y \to 0} \frac{2}{\pi} = \frac{2}{\pi}$$

Conclusion

Thus, the value of the limit is:

$$\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \frac{2}{\pi}$$