

Problem 2

We are given a piecewise function $f(x)$ defined as:

$$f(x) = \begin{cases} x^2 + 1 & \text{for } x < 1, \\ \frac{x^2 - 1}{x - 1} & \text{for } x > 1. \end{cases}$$

The goal is to analyze the function $f(x)$ and determine its behavior at the point where $x = 1$.

Solution

We need to check if the function $f(x)$ is continuous at $x = 1$, i.e., we want to check if:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1).$$

Step 1: Check $f(1)$

The function is not explicitly defined at $x = 1$, but we can check the left-hand limit and the right-hand limit to determine the value at $x = 1$.

Step 2: Calculate the left-hand limit

For $x < 1$, the function is given by $f(x) = x^2 + 1$. Therefore, the left-hand limit is:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + 1) = 1^2 + 1 = 2.$$

Step 3: Calculate the right-hand limit

For $x > 1$, the function is given by $f(x) = \frac{x^2 - 1}{x - 1}$. We simplify this expression:

$$x^2 - 1 = (x - 1)(x + 1),$$

so:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \quad \text{for } x > 1.$$

Thus, the right-hand limit is:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + 1) = 1 + 1 = 2.$$

Step 4: Check the continuity at $x = 1$

Since:

$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 2,$$

the function $f(x)$ is continuous at $x = 1$.

Step 5: Conclusion

Thus, we conclude:

$$\lim_{x \rightarrow 1} f(x) = 2.$$