Problem 30

In the equation $x^3 - kx^2 + 2kx - (k+1) = 0$, determine the parameter k such that:

1. the sum of roots is 7;
2. the product of roots is 8.

Find these roots.

Solution

Preliminary: Vieta's formulas

For the cubic equation $x^3 + ax^2 + bx + c = 0$ with roots x_1, x_2, x_3:

$$x_1 + x_2 + x_3 = -a, \quad x_1x_2 + x_1x_3 + x_2x_3 = b, \quad x_1x_2x_3 = -c$$

Our equation: $x^3 - kx^2 + 2kx - (k+1) = 0$
Therefore:

$$x_1 + x_2 + x_3 = k$$
$$x_1x_2 + x_1x_3 + x_2x_3 = 2k$$
$$x_1x_2x_3 = k + 1$$

Part 1°: Sum of roots equals 7

From Vieta's formula:

$$x_1 + x_2 + x_3 = k = 7$$

Therefore: $\boxed{k = 7}$
The equation becomes:

$$x^3 - 7x^2 + 14x - 8 = 0$$

Find the roots for $k = 7$

Let's try to find a rational root. By the Rational Root Theorem, possible rational roots are divisors of 8: $\pm 1, \pm 2, \pm 4, \pm 8$.

Try $x = 1$:

$$1 - 7 + 14 - 8 = 0 \quad \checkmark$$

So $x = 1$ is a root! Factor out $(x - 1)$:

$$x^3 - 7x^2 + 14x - 8 = (x-1)(x^2 - 6x + 8)$$

Factor the quadratic:

$$x^2 - 6x + 8 = (x-2)(x-4)$$

Therefore, the roots are:

$$\boxed{x_1 = 1, \quad x_2 = 2, \quad x_3 = 4}$$

Verify:

- Sum: $1 + 2 + 4 = 7$
- Product: $1 \cdot 2 \cdot 4 = 8$
- Sum of products of pairs: $1 \cdot 2 + 1 \cdot 4 + 2 \cdot 4 = 2 + 4 + 8 = 14 = 2k$

Part 2°: Product of roots equals 8

From Vieta's formula:

$$x_1 x_2 x_3 = k + 1 = 8$$

Therefore: $k = 7$, so $\boxed{k = 7}$
This gives the same value as Part 1!
The roots are the same:

$$\boxed{x_1 = 1, \quad x_2 = 2, \quad x_3 = 4}$$

Observation

Both conditions lead to the same value $k = 7$. This is not a coincidence! Let's verify consistency:

For $k = 7$:

- Sum of roots: $x_1 + x_2 + x_3 = k = 7$
- Product of roots: $x_1 x_2 x_3 = k + 1 = 8$

Both conditions are satisfied simultaneously.

General factorization

The equation always has $x = 1$ as a root. To see this, substitute $x = 1$:

$$1 - k + 2k - (k + 1) = 1 - k + 2k - k - 1 = 0$$

The general factorization is:

$$x^3 - kx^2 + 2kx - (k + 1) = (x - 1)(x^2 - (k - 1)x + (k + 1))$$

Summary

For the equation $x^3 - kx^2 + 2kx - (k + 1) = 0$:
Part 1 and Part 2 both give: $\boxed{k = 7}$
The roots are: $\boxed{x = 1,\ x = 2,\ x = 4}$

Remark

This problem demonstrates the power of Vieta's formulas in relating the coefficients of a polynomial to properties of its roots. The fact that both conditions (sum = 7 and product = 8) lead to the same value of k is special to this particular equation structure and the chosen target values.

The factorization $(x - 1)(x^2 - (k - 1)x + (k + 1)) = 0$ reveals that $x = 1$ is always a root, making the problem reducible to analyzing a quadratic for the remaining two roots.