Problem 13

Write the expression $y = 1 + \cos 2x + \cos 4x$ in product form, and solve the equation $y = 0$.

Solution

Part 1: Write in product form

Given:

$$y = 1 + \cos 2x + \cos 4x$$

Step 1: Apply sum-to-product formula

We'll use the sum-to-product formula for cosines:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Apply this to $\cos 2x + \cos 4x$:

$$\cos 2x + \cos 4x = 2\cos\left(\frac{2x+4x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)$$

$$= 2\cos(3x)\cos(x)$$

Therefore:

$$y = 1 + 2\cos(3x)\cos(x)$$

Step 2: Factor further using double angle formula

Recall that $\cos(2\theta) = 2\cos^2(\theta) - 1$, which gives us $1 = 2\cos^2(\theta) - \cos(2\theta)$.

Alternatively, we can use: $1 + \cos(2\theta) = 2\cos^2(\theta)$.

Let's write 1 as $2\cos^2(2x) - \cos(4x)$... but this gets complicated.

Instead, let's factor directly. We have:

$$y = 1 + 2\cos(3x)\cos(x)$$

Notice that $1 + 2\cos(3x)\cos(x)$ can be rewritten. Using the product-to-sum formula in reverse:

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

So:

$$2\cos(3x)\cos(x) = \cos(4x) + \cos(2x)$$

This confirms our intermediate step, but we need a product form.

Step 3: Alternative approach - factor completely

Let's use a different strategy. We can write:

$$y = 1 + \cos 2x + \cos 4x$$

Using $\cos 4x = 2\cos^2(2x) - 1$:

$$y = 1 + \cos 2x + 2\cos^2(2x) - 1 = \cos 2x + 2\cos^2(2x)$$

$$= \cos 2x(1 + 2\cos 2x)$$

Wait, let me verify this:

$$\cos 2x(1 + 2\cos 2x) = \cos 2x + 2\cos^2(2x)$$

And using $\cos 4x = 2\cos^2(2x) - 1$:

$$\cos 2x + 2\cos^2(2x) = \cos 2x + \cos 4x + 1$$

Yes! So we have:

$$\boxed{y = \cos 2x(1 + 2\cos 2x)}$$

Or, we can factor $1 + 2\cos 2x$ further. Using $1 + 2\cos\theta = 2\cos^2(\theta/2) + 2\cos^2(\theta/2) - 1 + 2\cos\theta$... this gets messy.

Actually, using the formula $1 + \cos\theta = 2\cos^2(\theta/2)$:

$$1 + 2\cos 2x = 1 + \cos 2x + \cos 2x = 2\cos^2(x) + \cos 2x$$

Hmm, let's try another approach.

Step 4: Complete factorization

Using $\cos 2x = 2\cos^2 x - 1$:

$$1 + 2\cos 2x = 1 + 2(2\cos^2 x - 1) = 1 + 4\cos^2 x - 2 = 4\cos^2 x - 1$$

$$= (2\cos x)^2 - 1 = (2\cos x - 1)(2\cos x + 1)$$

Therefore:

$$y = \cos 2x \cdot (2\cos x - 1)(2\cos x + 1)$$

Using $\cos 2x = 2\cos^2 x - 1 = (\sqrt{2}\cos x)^2 - 1$, but let's keep it as is.

Better yet, using $\cos 2x = 2\cos x\cos x - \sin^2 x - \cos^2 x$... no, let's stick with:

$$\boxed{y = \cos 2x(2\cos x - 1)(2\cos x + 1) = 2\cos 2x(2\cos x - 1)\cos x \cdot \frac{2\cos x + 1}{2\cos x}}$$

Actually, the clearest form is:

$$\boxed{y = \cos 2x(4\cos^2 x - 1)}$$

Or:

$$\boxed{y = \cos 2x(2\cos x - 1)(2\cos x + 1)}$$

Part 2: Solve $y = 0$

From $y = \cos 2x(2\cos x - 1)(2\cos x + 1) = 0$, we have three cases:

Case 1: $\cos 2x = 0$

$$2x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\boxed{x = \frac{\pi}{4} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}}$$

Or in degrees: $x = 45 + 90k$

Case 2: $2\cos x - 1 = 0$

$$\cos x = \frac{1}{2}$$

$$\boxed{x = \pm\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}}$$

Or in degrees: $x = \pm 60 + 360k$

Case 3: $2\cos x + 1 = 0$

$$\cos x = -\frac{1}{2}$$

$$\boxed{x = \pm\frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}}$$

Or in degrees: $x = \pm 120 + 360k$

Complete solution set

$$\boxed{x \in \left\{\frac{\pi}{4} + \frac{k\pi}{2}, \pm\frac{\pi}{3} + 2k\pi, \pm\frac{2\pi}{3} + 2k\pi : k \in \mathbb{Z}\right\}}$$

Or more compactly:

$$x = \frac{\pi}{4} + \frac{k\pi}{2} \quad \text{or} \quad x = \pm\frac{\pi}{3} + 2k\pi \quad \text{or} \quad x = \pm\frac{2\pi}{3} + 2k\pi$$

Verification

Let's verify with $x = \frac{\pi}{4}$ (or 45):

$$y = 1 + \cos(90) + \cos(180) = 1 + 0 + (-1) = 0 \quad \checkmark$$

Let's verify with $x = \frac{\pi}{3}$ (or 60):

$$y = 1 + \cos(120) + \cos(240) = 1 + (-\frac{1}{2}) + (-\frac{1}{2}) = 0 \quad \checkmark$$

Remark

This problem demonstrates the power of sum-to-product formulas and double angle identities in factoring trigonometric expressions. The key steps were:

1. Using $\cos 4x = 2\cos^2(2x) - 1$ to simplify
2. Factoring out $\cos 2x$
3. Recognizing $1 + 2\cos 2x = 4\cos^2 x - 1$ as a difference of squares

The resulting product form makes solving the equation straightforward by setting each factor to zero.