Problem 162

Evaluate the integral:

$$\int \sin(\ln x)\, dx.$$

Solution

We solve the integral $I = \int \sin(\ln x)\, dx$ using the substitution method.

Step 1: Substituting $t = \ln x$ Let:

$$t = \ln x, \quad dt = \frac{dx}{x}.$$

Rewriting the integral:

$$\begin{aligned} I &= \int \sin(\ln x)\, dx \\ &= \int \sin t \cdot e^t \, dt. \end{aligned}$$

Step 2: Integration by Parts Using integration by parts where:

$$u = \sin t, \qquad dv = e^t \, dt.$$

Computing derivatives and integrals:

$$du = \cos t \, dt, \qquad v = e^t.$$

Applying integration by parts formula:

$$I = \sin t \cdot e^t - \int e^t \cos t \, dt.$$

Step 3: Solving $\int e^t \cos t \, dt$ Using integration by parts again with:

$$u = \cos t, \qquad dv = e^t \, dt.$$

Computing derivatives and integrals:

$$du = -\sin t \, dt, \qquad v = e^t.$$

Applying integration by parts again:

$$\int e^t \cos t \, dt = \cos t \cdot e^t + \int e^t \sin t \, dt.$$

We now have:

$$I = \sin t \cdot e^t - (\cos t \cdot e^t + I).$$

Rearranging for I:

$$2I = e^t(\sin t - \cos t),$$

$$I = \frac{e^t}{2}(\sin t - \cos t) + C.$$

Step 4: Substituting Back $t = \ln x$

$$I = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C.$$

Final Answer:

$$\int \sin(\ln x)\, dx = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C.$$