

Problem 3

Evaluate the partial sum S_n and the sum $S = \lim_{n \rightarrow \infty} S_n$ for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Solution

Step 1: Partial Fraction Decomposition

The general term of the series, $a_n = \frac{1}{n(n+1)}$, can be decomposed using partial fractions:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}.$$

Multiplying by $n(n+1)$:

$$1 = A(n+1) + Bn.$$

- Set $n = 0$: $1 = A(1) + 0 \implies A = 1$.
- Set $n = -1$: $1 = 0 + B(-1) \implies B = -1$.

Thus, the general term is:

$$a_n = \frac{1}{n} - \frac{1}{n+1}.$$

Step 2: Determine the Partial Sum S_n

The series is a **telescoping series**. The n -th partial sum S_n is the sum of the first n terms:

$$S_n = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Writing out the terms:

$$S_n = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

The intermediate terms cancel out in pairs (the negative part of one term cancels the positive part of the next term). This is the characteristic of a telescoping series. The simplified partial sum is left with only the first positive term and the last negative term:

$$S_n = \frac{1}{1} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Step 3: Calculate the Sum S

The sum of the series S is the limit of the partial sum S_n as $n \rightarrow \infty$:

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right).$$

Since $\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$:

$$S = 1 - 0 = 1.$$

Final Answer

Partial Sum S_n

The n -th partial sum is:

$$S_n = 1 - \frac{1}{n+1}.$$

Sum S

The sum of the series is:

$$S = 1.$$