Problem 22

Solve the equation:

$$\left(x+\frac{1}{x}-\frac{4}{x+\frac{1}{x}}\right)^2=\frac{81}{100}$$

Solution

Step 1: Introduce a substitution

Let $t = x + \frac{1}{x}$. Then the equation becomes:

$$\left(t-\frac{4}{t}\right)^2=\frac{81}{100}$$

Step 2: Take square roots

Taking square roots of both sides:

$$t-\frac{4}{t}=\pm\frac{9}{10}$$

This gives us two cases.

Step 3: Solve Case 1: $t - \frac{4}{t} = \frac{9}{10}$

Multiply by t:

$$t^2-4=\frac{9t}{10}$$

$$10t^2-40=9t$$

$$10t^2-9t-40=0$$

Using the quadratic formula:

$$t=\frac{9\pm\sqrt{81+1600}}{20}=\frac{9\pm\sqrt{1681}}{20}=\frac{9\pm 41}{20}$$

Therefore:

$$t=\frac{9+41}{20}=\frac{50}{20}=\frac{5}{2}\quad\text{or}\quad t=\frac{9-41}{20}=\frac{-32}{20}=-\frac{8}{5}$$

Step 4: Solve Case 2: $t - \frac{4}{t} = -\frac{9}{10}$

Multiply by t:

$$t^2-4=-\frac{9t}{10}$$

$$10t^2-40=-9t$$

$$10t^2+9t-40=0$$

Using the quadratic formula:

$$t = \frac{-9 \pm \sqrt{81 + 1600}}{20} = \frac{-9 \pm \sqrt{1681}}{20} = \frac{-9 \pm 41}{20}$$

Therefore:

$$t = \frac{-9 + 41}{20} = \frac{32}{20} = \frac{8}{5} \quad \text{or} \quad t = \frac{-9 - 41}{20} = \frac{-50}{20} = -\frac{5}{2}$$

Step 5: Solve for x from each value of t

Recall that $t = x + \frac{1}{x}$, which gives us:

$$x + \frac{1}{x} = t$$

$$x^2 - tx + 1 = 0$$

Using the quadratic formula:

$$x = \frac{t \pm \sqrt{t^2 - 4}}{2}$$

For $t = \frac{5}{2}$:

$$t^2 - 4 = \frac{25}{4} - 4 = \frac{25 - 16}{4} = \frac{9}{4}$$

$$x = \frac{\frac{5}{2} \pm \frac{3}{2}}{2} = \frac{5 \pm 3}{4}$$

So:

$$x = \frac{5 + 3}{4} = 2 \quad \text{or} \quad x = \frac{5 - 3}{4} = \frac{1}{2}$$

For $t = -\frac{8}{5}$:

$$t^2 - 4 = \frac{64}{25} - 4 = \frac{64 - 100}{25} = -\frac{36}{25}$$

Since $t^2 - 4 < 0$, we have complex solutions:

$$x = \frac{-\frac{8}{5} \pm i\frac{6}{5}}{2} = \frac{-8 \pm 6i}{10} = \frac{-4 \pm 3i}{5}$$

For $t = \frac{8}{5}$:

$$t^2 - 4 = \frac{64}{25} - 4 = \frac{64 - 100}{25} = -\frac{36}{25}$$

Similarly, complex solutions:

$$x = \frac{\frac{8}{5} \pm i\frac{6}{5}}{2} = \frac{8 \pm 6i}{10} = \frac{4 \pm 3i}{5}$$

For $t = -\frac{5}{2}$:

$$t^2 - 4 = \frac{25}{4} - 4 = \frac{9}{4}$$

$$x = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2} = \frac{-5 \pm 3}{4}$$

So:

$$x = \frac{-5+3}{4} = -\frac{1}{2} \quad \text{or} \quad x = \frac{-5-3}{4} = -2$$

Step 6: Real solutions

The real solutions are:

$$\boxed{x \in \left\{-2, -\frac{1}{2}, \frac{1}{2}, 2\right\}}$$

Verification

Let's verify with $x = 2$:

$$x + \frac{1}{x} = 2 + \frac{1}{2} = \frac{5}{2}$$

$$x + \frac{1}{x} - \frac{4}{x + \frac{1}{x}} = \frac{5}{2} - \frac{4}{\frac{5}{2}} = \frac{5}{2} - \frac{8}{5} = \frac{25-16}{10} = \frac{9}{10}$$

$$\left(\frac{9}{10}\right)^2 = \frac{81}{100} \quad \checkmark$$

Let's verify with $x = -\frac{1}{2}$:

$$x + \frac{1}{x} = -\frac{1}{2} + \frac{1}{-\frac{1}{2}} = -\frac{1}{2} - 2 = -\frac{5}{2}$$

$$x + \frac{1}{x} - \frac{4}{x + \frac{1}{x}} = -\frac{5}{2} - \frac{4}{-\frac{5}{2}} = -\frac{5}{2} + \frac{8}{5} = \frac{-25+16}{10} = -\frac{9}{10}$$

$$\left(-\frac{9}{10}\right)^2 = \frac{81}{100} \quad \checkmark$$

Complete solution set (including complex)

If complex solutions are required:

$$\boxed{x \in \left\{-2, -\frac{1}{2}, \frac{1}{2}, 2, \frac{-4 \pm 3i}{5}, \frac{4 \pm 3i}{5}\right\}}$$

Summary

The equation was solved by:

1. Substituting $t = x + \frac{1}{x}$ to simplify the nested fraction
2. Taking square roots to get two linear-fractional equations in t
3. Solving two quadratic equations for t, yielding four values: $\pm\frac{5}{2}, \pm\frac{8}{5}$
4. For each value of t, solving $x + \frac{1}{x} = t$ by converting to $x^2 - tx + 1 = 0$
5. Obtaining 4 real solutions and 4 complex solutions

The real solutions form two pairs of reciprocals: $(2, \frac{1}{2})$ and $(-2, -\frac{1}{2})$, which is expected from the symmetry of the equation.