Problem 4

Does the following hold for all $x \in \mathbb{R}$:
1° $\sqrt{x^2-4} = \sqrt{x-2}\sqrt{x+2}$?
2° $\sqrt{(2-x)(x-5)} = \sqrt{2-x}\sqrt{x-5}$?
3° $\sqrt{(3-x)^2} = 3-x$?
4° $\sqrt{x(x+1)(x+2)} = \sqrt{x}\sqrt{x+1}\sqrt{x+2}$?

Solution

Part 1°: Does $\sqrt{x^2-4} = \sqrt{x-2}\sqrt{x+2}$ for all $x \in \mathbb{R}$?

Step 1. Analyze the domain of the left side:

$$\sqrt{x^2-4} \text{ is defined when } x^2-4 \geq 0 \implies |x| \geq 2 \implies x \leq -2 \text{ or } x \geq 2.$$

Step 2. Analyze the domain of the right side:

$$\sqrt{x-2}\sqrt{x+2} \text{ requires both } x-2 \geq 0 \text{ and } x+2 \geq 0.$$

$$x \geq 2 \text{ and } x \geq -2 \implies x \geq 2.$$

Step 3. The domains are different! The left side is defined for $x \leq -2$ or $x \geq 2$, but the right side only for $x \geq 2$.

Step 4. Check if equality holds where both are defined ($x \geq 2$):
Using the property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for $a, b \geq 0$:

$$\sqrt{x-2}\sqrt{x+2} = \sqrt{(x-2)(x+2)} = \sqrt{x^2-4}.$$

So equality holds for $x \geq 2$.
Step 5. What about $x \leq -2$? Try $x = -3$:

$$\text{LHS: } \sqrt{(-3)^2-4} = \sqrt{9-4} = \sqrt{5}.$$

$$\text{RHS: } \sqrt{-3-2}\sqrt{-3+2} = \sqrt{-5}\sqrt{-1} \text{ (not real).}$$

Conclusion:

$$\boxed{\text{NO. The identity holds only for } x \geq 2, \text{ not for all } x \in \mathbb{R}.}$$

Part 2°: Does $\sqrt{(2-x)(x-5)} = \sqrt{2-x}\sqrt{x-5}$ for all $x \in \mathbb{R}$?

Step 1. Analyze the domain of the left side:

$$\sqrt{(2-x)(x-5)} \text{ requires } (2-x)(x-5) \geq 0.$$

Since this is a product of two factors, analyze the sign:

- $(2-x) > 0$ when $x < 2$
- $(x-5) > 0$ when $x > 5$

The product $(2-x)(x-5) \geq 0$ when both factors have the same sign or one is zero:

$$x \leq 2 \text{ and } x \leq 5 \text{ (both negative or zero)} \implies x \leq 2.$$

$$x \geq 2 \text{ and } x \geq 5 \text{ (both positive or zero)} \implies x \geq 5.$$

So the domain is $x \leq 2$ or $x \geq 5$.

Wait, let me reconsider. For $(2-x)(x-5) \geq 0$: - If $x < 2$: $(2-x) > 0$ and $(x-5) < 0$, so product is negative. - If $2 \leq x \leq 5$: $(2-x) \leq 0$ and $(x-5) \leq 0$, so product is non-negative. - If $x > 5$: $(2-x) < 0$ and $(x-5) > 0$, so product is negative.

So the domain is $2 \leq x \leq 5$.

Step 2. Analyze the domain of the right side:

$$\sqrt{2-x}\sqrt{x-5} \text{ requires } 2-x \geq 0 \text{ and } x-5 \geq 0.$$

$$x \leq 2 \text{ and } x \geq 5 \implies \text{impossible!}$$

The right side has empty domain (no real x satisfies both conditions).

Step 3. Test a value where left side is defined, say $x = 3$:

$$\text{LHS: } \sqrt{(2-3)(3-5)} = \sqrt{(-1)(-2)} = \sqrt{2}.$$

$$\text{RHS: } \sqrt{2-3}\sqrt{3-5} = \sqrt{-1}\sqrt{-2} \text{ (not real).}$$

Conclusion:

$$\boxed{\text{NO. The right side is never defined in } \mathbb{R}, \text{ so the equality never holds.}}$$

Part 3°: Does $\sqrt{(3-x)^2} = 3 - x$ for all $x \in \mathbb{R}$?

Step 1. Use the property $\sqrt{a^2} = |a|$:

$$\sqrt{(3-x)^2} = |3-x|.$$

Step 2. Evaluate $|3-x|$:

$$|3-x| = \begin{cases} 3-x & \text{if } 3-x \geq 0 \text{ (i.e., } x \leq 3) \\ -(3-x) = x-3 & \text{if } 3-x < 0 \text{ (i.e., } x > 3) \end{cases}$$

Step 3. Test $x = 5$:

$$\text{LHS: } \sqrt{(3-5)^2} = \sqrt{(-2)^2} = \sqrt{4} = 2.$$

$$\text{RHS: } 3-5 = -2.$$

Since $2 \neq -2$, the equality fails.

Conclusion:

$$\boxed{\text{NO. The identity holds only for } x \leq 3. \text{ For } x > 3, \sqrt{(3-x)^2} = x-3 \neq 3-x.}$$

Part 4°: Does $\sqrt{x(x+1)(x+2)} = \sqrt{x}\sqrt{x+1}\sqrt{x+2}$ for all $x \in \mathbb{R}$?

Step 1. Analyze the domain of the left side:

$$\sqrt{x(x+1)(x+2)} \text{ requires } x(x+1)(x+2) \geq 0.$$

The critical points are $x = -2, -1, 0$. Test intervals:

- $x < -2$: all three factors negative, product negative.
- $-2 \leq x < -1$: $(-)(\,)(+) =$ negative.
- $-1 \leq x < 0$: $(+)(+)(+) =$ Wait, let me redo this.
- $x = -2$: product is 0.
- $-2 < x < -1$: $x < 0$, $x+1 > 0$, $x+2 > 0$, so product is negative.
- $x = -1$: product is 0.
- $-1 < x < 0$: $x < 0$, $x+1 > 0$, $x+2 > 0$, so product is negative.
- $x = 0$: product is 0.
- $x > 0$: all positive, product positive.

So the domain is $x \in \{-2, -1, 0\} \cup [0, \infty) = \{-2, -1\} \cup [0, \infty)$.
Step 2. Analyze the domain of the right side:

$$\sqrt{x}\sqrt{x+1}\sqrt{x+2} \text{ requires } x \geq 0, x+1 \geq 0, x+2 \geq 0.$$

$$x \geq 0.$$

Step 3. The domains differ. Where both are defined ($x \geq 0$), check equality:
Using $\sqrt{a}\sqrt{b}\sqrt{c} = \sqrt{abc}$ for $a, b, c \geq 0$:

$$\sqrt{x}\sqrt{x+1}\sqrt{x+2} = \sqrt{x(x+1)(x+2)}.$$

Equality holds for $x \geq 0$.
Conclusion:

$$\boxed{\text{NO. The identity holds only for } x \geq 0, \text{ not for all } x \in \mathbb{R}.}$$