Problem 18

Evaluate the following limit:

$$\lim_{n \to +\infty} \cos^n \left(\frac{x}{\sqrt{n}} \right)$$

Solution

As $n \to +\infty$, the term $\frac{x}{\sqrt{n}}$ approaches 0 for any finite value of x. Therefore:

$$\cos \left(\frac{x}{\sqrt{n}} \right) \to \cos(0) = 1.$$

Taylor Expansion of $\cos \left(\frac{x}{\sqrt{n}} \right)$

Using the Taylor series expansion of $\cos(z)$ around 0:

$$\cos(z) = 1 - \frac{z^2}{2} + O(z^4),$$

we substitute $z = \frac{x}{\sqrt{n}}$ to get:

$$\cos \left(\frac{x}{\sqrt{n}} \right) = 1 - \frac{x^2}{2n} + O \left(\frac{1}{n^2} \right).$$

Expression for $\cos^n \left(\frac{x}{\sqrt{n}} \right)$

Now, we raise the expression to the power of n:

$$\cos^n \left(\frac{x}{\sqrt{n}} \right) = \left(1 - \frac{x^2}{2n} + O \left(\frac{1}{n^2} \right) \right)^n.$$

Using the approximation:

$$\left(1 + \frac{a}{n} \right)^n \to e^a \quad \text{as} \quad n \to \infty,$$

with $a = -\frac{x^2}{2}$, we obtain:

$$\cos^n \left(\frac{x}{\sqrt{n}} \right) \to e^{-\frac{x^2}{2}} \quad \text{as} \quad n \to \infty.$$

Conclusion

Therefore, the value of the limit is:

$$\lim_{n \to +\infty} \cos^n \left(\frac{x}{\sqrt{n}} \right) = e^{-\frac{x^2}{2}}.$$