Problem 19

Find the real solutions of the equation:

$$\sin^n x + \frac{1}{\cos^m x} = \cos^n x + \frac{1}{\sin^m x}$$

where m and n are odd natural numbers.

Solution

Step 1: Rearrange the equation

Rearrange the equation:

$$\sin^n x - \cos^n x = \frac{1}{\sin^m x} - \frac{1}{\cos^m x}$$
$$\sin^n x - \cos^n x = \frac{\cos^m x - \sin^m x}{\sin^m x \cos^m x}$$

Step 2: Factor using odd power property

Since m and n are odd, we can factor:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

So:

$$\sin^n x - \cos^n x = (\sin x - \cos x)(\sin^{n-1} x + \sin^{n-2} x \cos x + \cdots + \cos^{n-1} x)$$

$$\cos^m x - \sin^m x = (\cos x - \sin x)(\cos^{m-1} x + \cos^{m-2} x \sin x + \cdots + \sin^{m-1} x)$$
$$= -(\sin x - \cos x)(\cos^{m-1} x + \cos^{m-2} x \sin x + \cdots + \sin^{m-1} x)$$

Therefore:

$$(\sin x - \cos x)P_n(x) = \frac{-(\sin x - \cos x)Q_m(x)}{\sin^m x \cos^m x}$$

where $P_n(x)$ and $Q_m(x)$ are positive sums of products.

Step 3: Factor out common term

Factor out $(\sin x - \cos x)$:

$$(\sin x - \cos x)\left[P_n(x) + \frac{Q_m(x)}{\sin^m x \cos^m x}\right] = 0$$

This gives us two cases:

Case 1: $\sin x - \cos x = 0$

$$\sin x = \cos x$$
$$\tan x = 1$$

The general solution is:

$$\boxed{x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}}$$

In degrees: $x = 45 + 180k$

Case 2: $P_n(x) + \frac{Q_m(x)}{\sin^m x \cos^m x} = 0$

This would require:

$$\sin^m x \cos^m x \cdot P_n(x) + Q_m(x) = 0$$

Since both $P_n(x)$ and $Q_m(x)$ are sums of positive terms (products of powers of $\sin x$ and $\cos x$ which are non-negative when both are non-zero), and $\sin^m x \cos^m x > 0$ when both $\sin x \neq 0$ and $\cos x \neq 0$, we have:

$$\sin^m x \cos^m x \cdot P_n(x) + Q_m(x) > 0$$

Therefore, this case yields no real solutions.

Step 4: Verify the solution

Let's verify with $x = \frac{\pi}{4}$ where $\sin x = \cos x = \frac{\sqrt{2}}{2}$:

Left side:

$$\sin^n x + \frac{1}{\cos^m x} = \left(\frac{\sqrt{2}}{2}\right)^n + \frac{1}{(\frac{\sqrt{2}}{2})^m} = \frac{(\sqrt{2})^n}{2^n} + \frac{2^m}{(\sqrt{2})^m}$$
$$= \frac{2^{n/2}}{2^n} + \frac{2^m}{2^{m/2}} = 2^{n/2-n} + 2^{m-m/2} = 2^{-n/2} + 2^{m/2}$$

Right side:

$$\cos^n x + \frac{1}{\sin^m x} = \left(\frac{\sqrt{2}}{2}\right)^n + \frac{1}{(\frac{\sqrt{2}}{2})^m} = 2^{-n/2} + 2^{m/2}$$

Indeed, both sides are equal! ✓

Step 5: Complete solution

$$\boxed{x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}}$$

Or in degrees:

$$\boxed{x = 45 + 180k, \quad k \in \mathbb{Z}}$$

Note: We must exclude values where $\sin x = 0$ or $\cos x = 0$ since these make the original equation undefined. The solution $x = \frac{\pi}{4} + k\pi$ never has $\sin x = 0$ or $\cos x = 0$, so all these values are valid.

Alternative Approach: Direct Factorization

Starting from:

$$\sin^n x - \cos^n x = \frac{\cos^m x - \sin^m x}{\sin^m x \cos^m x}$$

Multiply both sides by $\sin^m x \cos^m x$:

$$\sin^m x \cos^m x(\sin^n x - \cos^n x) = \cos^m x - \sin^m x$$

$$\sin^m x \cos^m x(\sin^n x - \cos^n x) + \sin^m x - \cos^m x = 0$$

Since m is odd:

$$\sin^m x \cos^m x(\sin^n x - \cos^n x) - (\cos^m x - \sin^m x) = 0$$

Factor (using odd power factorization):

$$(\sin x - \cos x)[\sin^m x \cos^m x \cdot S_n + T_m] = 0$$

where S_n and T_m are positive sums. This again gives $\sin x = \cos x$ as the only real solution.

Summary

For the equation $\sin^n x + \frac{1}{\cos^m x} = \cos^n x + \frac{1}{\sin^m x}$ where m and n are odd natural numbers:

The unique family of solutions is:

$$x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

This corresponds to all angles where $\sin x = \cos x$, i.e., angles of 45 and 225 (plus integer multiples of 360).

Remark

This problem demonstrates several important principles:

1. The factorization property of differences of odd powers: $a^n - b^n = (a - b)(\cdots)$ when n is odd

2. When both m and n are odd, the equation simplifies elegantly to $\sin x = \cos x$

3. The solution set is independent of the specific values of m and n, as long as they are odd

4. The symmetry in the equation (swapping sin and cos with appropriate power changes) is broken only when $\sin x = \cos x$

If m or n were even, the factorization would be different (involving $a + b$ or $a - b$ factors), and the solution set would change accordingly. The odd power condition is crucial for this elegant result.