Problem 23

For which values of the parameter a (where a is an integer) does the equation:

$$(a^2 - 1)x^2 - 6(3a - 1)x + 72 = 0$$

have:

1. negative, integer roots;
2. opposite sign, integer roots;
3. positive, distinct, integer roots;
4. equal roots;
5. one integer root, one fractional root;
6. a root equal to the parameter.

Solution

Preliminary analysis

For the equation $(a^2 - 1)x^2 - 6(3a - 1)x + 72 = 0$:

Coefficients:

- $A = a^2 - 1$
- $B = -6(3a - 1) = -18a + 6$
- $C = 72$

Vieta's formulas: If x_1 and x_2 are roots, then:

$$x_1 + x_2 = \frac{18a - 6}{a^2 - 1}, \quad x_1 x_2 = \frac{72}{a^2 - 1}$$

Note: $a^2 - 1 \neq 0$, so $a \neq \pm 1$.

Part 1°: Negative, integer roots

For both roots to be negative integers, we need:

- $x_1 + x_2 < 0$
- $x_1 x_2 > 0$
- Both roots are integers

From $x_1x_2 = \frac{72}{a^2-1} > 0$, we need $a^2 - 1 > 0$, so $|a| > 1$.

Since we want negative roots and $x_1x_2 > 0$, both must be negative.

From $x_1 + x_2 = \frac{18a-6}{a^2-1} < 0$:

- If $a > 1$: need $18a - 6 < 0$, so $a < \frac{1}{3}$. No solution.
- If $a < -1$: need $18a - 6 > 0$, so $a > \frac{1}{3}$. No solution.

Wait, let me reconsider. If $a > 1$, then $a^2 - 1 > 0$. For the sum to be negative: $18a - 6 < 0 \implies a < \frac{1}{3}$, which contradicts $a > 1$.

If $a < -1$, then $a^2 - 1 > 0$. For the sum to be negative: $18a - 6 < 0 \implies a < \frac{1}{3}$, which is satisfied.

So we need $a < -1$ and $\frac{72}{a^2-1}$ to be the product of two negative integers.

Let's try $a = -2$: $a^2 - 1 = 3$

$$3x^2 - 6(-6-1)x + 72 = 0 \implies 3x^2 + 42x + 72 = 0 \implies x^2 + 14x + 24 = 0$$

$$x = \frac{-14 \pm \sqrt{196 - 96}}{2} = \frac{-14 \pm 10}{2}$$

$x = -2$ or $x = -12$. Both negative integers!

Try $a = -3$: $a^2 - 1 = 8$

$$8x^2 + 60x + 72 = 0 \implies 2x^2 + 15x + 18 = 0$$

$$x = \frac{-15 \pm \sqrt{225 - 144}}{4} = \frac{-15 \pm 9}{4}$$

$x = -\frac{3}{2}$ or $x = -6$. Not both integers.

$$\boxed{a = -2}$$

Part 2°: Opposite sign, integer roots

For opposite signs: $x_1x_2 < 0$, so $\frac{72}{a^2-1} < 0$.

This requires $a^2 - 1 < 0$, so $|a| < 1$.

Since a is an integer: $a = 0$.

For $a = 0$:

$$-x^2 + 6x + 72 = 0 \implies x^2 - 6x - 72 = 0$$

$$x = \frac{6 \pm \sqrt{36 + 288}}{2} = \frac{6 \pm 18}{2}$$

$x = 12$ or $x = -6$. Both integers, opposite signs!

$$\boxed{a = 0}$$

Part 3°: Positive, distinct, integer roots

Need: $x_1, x_2 > 0$, $x_1 \neq x_2$, both integers.

This requires:

- $x_1 x_2 > 0$ and $x_1 + x_2 > 0$
- Discriminant > 0

From $x_1 x_2 = \frac{72}{a^2-1} > 0$: need $a^2 > 1$.
From $x_1 + x_2 = \frac{18a-6}{a^2-1} > 0$:

- If $a > 1$: need $18a - 6 > 0$, so $a > \frac{1}{3}$
- If $a < -1$: need $18a - 6 < 0$, so $a < \frac{1}{3}$

So $a > 1$ or $a < -1$.
Try $a = 2$: $a^2 - 1 = 3$

$$3x^2 - 30x + 72 = 0 \implies x^2 - 10x + 24 = 0$$

$x = \frac{10 \pm 2}{2}$, so $x = 6$ or $x = 4$. Both positive integers!
Try $a = 3$: $a^2 - 1 = 8$

$$8x^2 - 48x + 72 = 0 \implies x^2 - 6x + 9 = 0$$

$(x-3)^2 = 0$, so $x = 3$ (double root). Not distinct.
Try $a = 4$: $a^2 - 1 = 15$

$$15x^2 - 66x + 72 = 0$$

$$x = \frac{66 \pm \sqrt{4356 - 4320}}{30} = \frac{66 \pm 6}{30}$$

$x = \frac{12}{5}$ or $x = \frac{12}{5}$... wait, $\frac{72}{30} = \frac{12}{5}$ and $\frac{60}{30} = 2$.
So $x = \frac{12}{5}$ or $x = 2$. Not both integers.

$$\boxed{a = 2}$$

Part 4°: Equal roots

Discriminant $= 0$:

$$\Delta = 36(3a-1)^2 - 4(a^2-1)(72) = 0$$

$$36(9a^2 - 6a + 1) - 288(a^2 - 1) = 0$$

$$324a^2 - 216a + 36 - 288a^2 + 288 = 0$$

$$36a^2 - 216a + 324 = 0$$

$$a^2 - 6a + 9 = 0$$

$$(a-3)^2 = 0$$

$$\boxed{a = 3}$$

Part 5°: One integer, one fractional root

This occurs when the discriminant is a perfect square but not all coefficients allow integer roots.

From earlier, $a = 4$ gave $x = 2$ (integer) and $x = \frac{12}{5}$ (fraction).

$$\boxed{a = 4}$$

Part 6°: One root equals a

Substitute $x = a$:

$$(a^2 - 1)a^2 - 6(3a - 1)a + 72 = 0$$

$$a^4 - a^2 - 18a^2 + 6a + 72 = 0$$

$$a^4 - 19a^2 + 6a + 72 = 0$$

Test small integer values: $a = 2$: $16 - 76 + 12 + 72 = 24 \neq 0$ $a = 3$: $81 - 171 + 18 + 72 = 0$

$$\boxed{a = 3}$$

Summary

1. Negative, integer roots: $a = -2$
2. Opposite sign, integer roots: $a = 0$
3. Positive, distinct, integer roots: $a = 2$
4. Equal roots: $a = 3$
5. One integer, one fractional root: $a = 4$
6. Root equal to parameter: $a = 3$