Problem 6

Evaluate the following limit:

$$\lim_{x \to 0^+} x^2 \ln^3(x)$$

Solution

We evaluate the limit step by step:

1. Analyze the behavior of individual terms:
   - $x^2 \to 0$ as $x \to 0^+$.
   - $\ln^3(x) \to -\infty$ as $x \to 0^+$.

   The product $x^2 \ln^3(x)$ results in the indeterminate form $0 \cdot (-\infty)$.

2. Rewrite the product as a fraction:
$$x^2 \ln^3(x) = \frac{\ln^3(x)}{\frac{1}{x^2}}.$$
   Here, $\ln^3(x) \to -\infty$ and $\frac{1}{x^2} \to \infty$, resulting in an indeterminate form $\frac{-\infty}{\infty}$.

3. Apply L'Hôpital's Rule: Using $g(x) = \ln^3(x)$ and $h(x) = \frac{1}{x^2}$, we differentiate the numerator and denominator:
$$\lim_{x \to 0^+} \frac{\ln^3(x)}{\frac{1}{x^2}} = \lim_{x \to 0^+} \frac{\frac{d}{dx} \ln^3(x)}{\frac{d}{dx} \frac{1}{x^2}}.$$
   Differentiate each term:
   - $\frac{d}{dx} \ln^3(x) = 3 \ln^2(x) \cdot \frac{1}{x}$.
   - $\frac{d}{dx} \frac{1}{x^2} = -\frac{2}{x^3}$.

4. Simplify the new fraction:
$$\lim_{x \to 0^+} \frac{3 \ln^2(x) \cdot \frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x \to 0^+} \frac{-3 \ln^2(x) \cdot x^2}{2}.$$
   As $x \to 0^+$, $x^2 \to 0$ faster than $\ln^2(x) \to \infty$, so the entire expression tends to 0.

Final Answer

$$\boxed{\lim_{x \to 0^+} x^2 \ln^3(x) = 0}$$