Problem 28

In the equation $4x^2 - 4kx + k^2 - 4 = 0$, determine the bounds within which k can vary so that both roots of the equation lie between -3 and 4.

Solution

Step 1: Conditions for both roots in $(-3, 4)$

For a quadratic $f(x) = ax^2 + bx + c$ with $a > 0$, both roots lie in the interval (α, β) if and only if:

1. The discriminant $\Delta \geq 0$ (real roots exist)
2. $f(\alpha) > 0$ (parabola is positive at left endpoint)
3. $f(\beta) > 0$ (parabola is positive at right endpoint)
4. The vertex lies in (α, β): $\alpha < x_v < \beta$ where $x_v = -\frac{b}{2a}$

In our case: $f(x) = 4x^2 - 4kx + k^2 - 4$ with $a = 4 > 0$, $\alpha = -3$, $\beta = 4$.

Step 2: Condition 1 - Real roots ($\Delta \geq 0$)

$$\Delta = (-4k)^2 - 4(4)(k^2 - 4) = 16k^2 - 16k^2 + 64 = 64 > 0$$

This is always satisfied!

Step 3: Condition 2 - $f(-3) > 0$

$$\begin{aligned} f(-3) &= 4(-3)^2 - 4k(-3) + k^2 - 4 \\ &= 36 + 12k + k^2 - 4 \\ &= k^2 + 12k + 32 \end{aligned}$$

We need $k^2 + 12k + 32 > 0$.
Factor or use the discriminant:

$$k^2 + 12k + 32 = 0$$

$$k = \frac{-12 \pm \sqrt{144 - 128}}{2} = \frac{-12 \pm \sqrt{16}}{2} = \frac{-12 \pm 4}{2}$$

So $k = -4$ or $k = -8$.
Since the parabola opens upward, $k^2 + 12k + 32 > 0$ when:

$$k < -8 \quad \text{or} \quad k > -4$$

Step 4: Condition 3 - $f(4) > 0$

$$\begin{aligned} f(4) &= 4(4)^2 - 4k(4) + k^2 - 4 \\ &= 64 - 16k + k^2 - 4 \\ &= k^2 - 16k + 60 \end{aligned}$$

We need $k^2 - 16k + 60 > 0$.
Find the roots:

$$k = \frac{16 \pm \sqrt{256 - 240}}{2} = \frac{16 \pm \sqrt{16}}{2} = \frac{16 \pm 4}{2}$$

So $k = 10$ or $k = 6$.
Since the parabola opens upward, $k^2 - 16k + 60 > 0$ when:

$$k < 6 \quad \text{or} \quad k > 10$$

Step 5: Condition 4 - Vertex in $(-3, 4)$

The vertex is at:

$$x_v = -\frac{-4k}{2(4)} = \frac{4k}{8} = \frac{k}{2}$$

We need:

$$\begin{aligned} -3 < \frac{k}{2} < 4 \\ -6 < k < 8 \end{aligned}$$

Step 6: Find the intersection of all conditions

We need all four conditions to hold simultaneously:

1. $\Delta \geq 0$: always true
2. $f(-3) > 0$: $k < -8$ or $k > -4$
3. $f(4) > 0$: $k < 6$ or $k > 10$
4. Vertex condition: $-6 < k < 8$

Let's find the intersection:
From conditions 2 and 4: $(k < -8 \text{ or } k > -4) \cap (-6 < k < 8)$

$$= (-6 < k < -4) \cup (-4 < k < 8) = (-6, -4) \cup (-4, 8)$$

Wait, let me be more careful. Since we need $k > -4$ or $k < -8$, and also $-6 < k < 8$: - If $k < -8$: this doesn't intersect with $-6 < k < 8$ - If $k > -4$: this gives $-4 < k < 8$ (intersecting with $-6 < k < 8$)

So from conditions 2 and 4: $-4 < k < 8$.

Now intersect with condition 3: $(k < 6 \text{ or } k > 10) \cap (-4 < k < 8)$ - If $k < 6$: this gives $-4 < k < 6$ - If $k > 10$: this doesn't intersect with $k < 8$

Therefore: $-4 < k < 6$.

Step 7: Verify boundary behavior

At $k = -4$: $f(-3) = 16 - 48 + 32 = 0$, so one root is at $x = -3$ (boundary).

At $k = 6$: $f(4) = 36 - 96 + 60 = 0$, so one root is at $x = 4$ (boundary).

For the open interval (both roots strictly between -3 and 4):

$$\boxed{-4 < k < 6}$$

For the closed interval (roots can be at boundaries):

$$\boxed{-4 \leq k \leq 6}$$

Verification

For $k = 0$ (in the interval):

$$4x^2 - 4 = 0 \implies x = \pm 1$$

Both roots are in $(-3, 4)$.

For $k = 3$ (in the interval):

$$4x^2 - 12x + 5 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 80}}{8} = \frac{12 \pm 8}{8}$$

$x = \frac{5}{2}$ or $x = \frac{1}{2}$. Both in $(-3, 4)$.

For $k = 7$ (outside the interval):

$$4x^2 - 28x + 45 = 0$$

$$x = \frac{28 \pm \sqrt{784 - 720}}{8} = \frac{28 \pm 8}{8}$$

$x = 4.5$ or $x = 2.5$. One root (4.5) is outside $(-3, 4)$.

Summary

For both roots of $4x^2 - 4kx + k^2 - 4 = 0$ to lie in the interval $(-3, 4)$:

$$\boxed{-4 < k < 6}$$

Or if boundary inclusion is allowed:

$$\boxed{-4 \leq k \leq 6}$$

Remark

This problem demonstrates the systematic approach to root location problems:

1. Ensure real roots exist (discriminant condition)
2. Ensure the parabola has the correct sign at interval endpoints
3. Ensure the vertex (where the minimum occurs for $a > 0$) lies within the interval

The key insight is that for both roots to be in (α, β) with $a > 0$, the parabola must be positive at both endpoints and have its minimum between them.