Problem 8

Prove the identity:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} = \sqrt[3]{\frac{9-5\sqrt{3}}{9+5\sqrt{3}}}$$

Solution

Strategy

We will prove this identity by:

1. Simplifying the left side by rationalizing
2. Cubing both sides to eliminate the cube root
3. Verifying that the equality holds

Step 1: Simplify the left side

Rationalize $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ by multiplying by the conjugate:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - 1^2}$$

Expand the numerator:

$$(\sqrt{3}-1)^2 = 3 - 2\sqrt{3} + 1 = 4 - 2\sqrt{3}$$

The denominator:

$$(\sqrt{3})^2 - 1^2 = 3 - 1 = 2$$

Therefore:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{4-2\sqrt{3}}{2} = 2 - \sqrt{3}$$

Step 2: Cube both sides

Let $L = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$ (left side).

We need to show that:

$$(2-\sqrt{3})^3 = \frac{9-5\sqrt{3}}{9+5\sqrt{3}}$$

First, calculate $(2-\sqrt{3})^3$:

$$(2-\sqrt{3})^2 = 4 - 4\sqrt{3} + 3 = 7 - 4\sqrt{3}$$

Then:

$$(2-\sqrt{3})^3 = (2-\sqrt{3})(7-4\sqrt{3})$$
$$= 2\cdot 7 - 2\cdot 4\sqrt{3} - \sqrt{3}\cdot 7 + \sqrt{3}\cdot 4\sqrt{3}$$
$$= 14 - 8\sqrt{3} - 7\sqrt{3} + 4\cdot 3$$
$$= 14 - 15\sqrt{3} + 12$$
$$= 26 - 15\sqrt{3}$$

Step 3: Simplify the right side

Now we need to show that:

$$\frac{9-5\sqrt{3}}{9+5\sqrt{3}} = 26 - 15\sqrt{3}$$

Rationalize $\frac{9-5\sqrt{3}}{9+5\sqrt{3}}$ by multiplying by the conjugate:

$$\frac{9-5\sqrt{3}}{9+5\sqrt{3}} \cdot \frac{9-5\sqrt{3}}{9-5\sqrt{3}} = \frac{(9-5\sqrt{3})^2}{(9)^2-(5\sqrt{3})^2}$$

Expand the numerator:

$$(9-5\sqrt{3})^2 = 81 - 90\sqrt{3} + 25\cdot 3 = 81 - 90\sqrt{3} + 75 = 156 - 90\sqrt{3}$$

The denominator:

$$81 - 25\cdot 3 = 81 - 75 = 6$$

Therefore:

$$\frac{9-5\sqrt{3}}{9+5\sqrt{3}} = \frac{156-90\sqrt{3}}{6} = \frac{156}{6} - \frac{90\sqrt{3}}{6} = 26 - 15\sqrt{3}$$

Step 4: Verify the equality

We have shown that:

- Left side cubed: $(2-\sqrt{3})^3 = 26 - 15\sqrt{3}$
- Right side inside the cube root: $\frac{9-5\sqrt{3}}{9+5\sqrt{3}} = 26 - 15\sqrt{3}$

Since both equal $26 - 15\sqrt{3}$, we have:

$$\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^3 = \frac{9-5\sqrt{3}}{9+5\sqrt{3}}$$

Taking the cube root of both sides:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} = \sqrt[3]{\frac{9-5\sqrt{3}}{9+5\sqrt{3}}}$$

Final result

$$\boxed{\frac{\sqrt{3}-1}{\sqrt{3}+1} = \sqrt[3]{\frac{9-5\sqrt{3}}{9+5\sqrt{3}}}}$$

The identity is proven. ✓

Verification with decimal approximation

Let's verify numerically:

Left side:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{1.732...-1}{1.732...+1} = \frac{0.732...}{2.732...} \approx 0.2679$$

Or using our simplification: $2-\sqrt{3} \approx 2-1.732 = 0.268$

Right side:

$$\frac{9-5\sqrt{3}}{9+5\sqrt{3}} = \frac{9-8.660...}{9+8.660...} = \frac{0.340...}{17.660...} \approx 0.01925$$

The cube root: $\sqrt[3]{0.01925} \approx 0.2679$

Both sides match! ✓

Remark

This identity demonstrates a beautiful relationship between square roots and cube roots. The key insight is that $\frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$, and when this is cubed, it produces exactly the value inside the cube root on the right side.

The proof relies on careful rationalization of both the left side and the expression inside the cube root. Such identities often arise in algebraic number theory and demonstrate the deep connections between different types of radical expressions.