Problem 7

Evaluate the sum S for the series:

$$S = \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right).$$

Solution

Step 1: Decomposing the General Term

The general term of the series is $a_n = \ln\left(1 + \frac{1}{n}\right)$. We simplify the expression inside the logarithm:

$$a_n = \ln\left(\frac{n+1}{n}\right).$$

Using the property of logarithms $\ln(a/b) = \ln a - \ln b$:

$$a_n = \ln(n+1) - \ln(n).$$

This expression is in the form of a difference, $a_n = b_{n+1} - b_n$, where $b_n = \ln(n)$.

Step 2: Determine the Partial Sum S_N

The series is a **telescoping series**. The N-th partial sum S_N is the sum of the first N terms:

$$S_N = \sum_{n=1}^{N} \left(\ln(n+1) - \ln(n)\right).$$

Writing out the terms:

$$\begin{aligned} S_N = &\left(\ln(2) - \ln(1)\right) + \left(\ln(3) - \ln(2)\right) + \left(\ln(4) - \ln(3)\right) + \cdots \\ &+ \left(\ln(N) - \ln(N-1)\right) + \left(\ln(N+1) - \ln(N)\right). \end{aligned}$$

The intermediate terms cancel out. The simplified partial sum is left with the last positive term and the first negative term:

$$S_N = \ln(N+1) - \ln(1).$$

Since $\ln(1) = 0$:

$$S_N = \ln(N+1).$$

Step 3: Calculate the Sum S

The sum of the series S is the limit of the partial sum S_N as $N \to \infty$:

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln(N+1).$$

Since the natural logarithm function $\ln(x)$ tends to infinity as $x \to \infty$:

$$\lim_{N\to\infty} \ln(N+1) = \infty.$$

The series diverges.

Final Answer

Partial Sum S_N

The N-th partial sum is:

$$\boxed{S_N = \ln(N+1).}$$

Sum S

The series **diverges**, and the sum is:

$$\boxed{S = \infty.}$$