# Problem 7

Find the general solution of the differential equation whose solution is given by

$$cy = \sin(cx).$$

State the differential equation, solve it (showing steps), and give any domain restrictions.

# Solution

## 1. Determine the differential equation

Assume the solution has the form

$$cy(x) = \sin(cx),$$

where $c$ is an arbitrary constant. Differentiate both sides with respect to $x$:

$$c\,\frac{dy}{dx} = \cos(cx) \cdot c = c\cos(cx).$$

Simplifying (assuming $c \neq 0$):

$$\frac{dy}{dx} = \cos(cx).$$

To eliminate the arbitrary constant $c$, we need another relation. From the original equation:

$$cy = \sin(cx).$$

Square both sides:

$$c^2y^2 = \sin^2(cx).$$

From the derivative, we have:

$$y' = \cos(cx).$$

Square this as well:

$$(y')^2 = \cos^2(cx).$$

Using the Pythagorean identity $\sin^2(cx) + \cos^2(cx) = 1$:

$$c^2y^2 + (y')^2 = 1.$$

Thus the differential equation satisfied by $y$ is

$$\boxed{(y')^2 + c^2y^2 = 1.}$$

However, since $c$ still appears, we need to eliminate it further. From $c^2y^2 = 1 - (y')^2$:

$$c^2 = \frac{1 - (y')^2}{y^2}, \qquad y \neq 0.$$

Differentiate $(y')^2 + c^2y^2 = 1$ with respect to $x$:

$$2y'\,y'' + c^2 \cdot 2y \cdot y' = 0.$$

Factor out $2y'$:

$$2y'(y'' + c^2 y) = 0.$$

Assuming $y' \neq 0$ at generic points:

$$y'' + c^2 y = 0.$$

From $c^2 = \frac{1-(y')^2}{y^2}$:

$$y'' + \frac{1-(y')^2}{y^2} \cdot y = 0,$$

$$y'' + \frac{1-(y')^2}{y} = 0.$$

Multiply through by $y$:

$$\boxed{yy'' + 1 - (y')^2 = 0,}$$

or equivalently:

$$\boxed{yy'' = (y')^2 - 1.}$$

## 2. Solve the differential equation

We solve the second-order ODE

$$yy'' = (y')^2 - 1.$$

This can be rewritten as:

$$yy'' - (y')^2 = -1.$$

Notice that:

$$\frac{d}{dx}\left[\frac{(y')^2}{2}\right] = y'y'',$$

and

$$\frac{d}{dx}\left[\frac{y^2}{2}\right] = yy'.$$

Let $p = y'$. Then $y'' = p\,\frac{dp}{dy}$, and the equation becomes:

$$y \cdot p\,\frac{dp}{dy} = p^2 - 1.$$

This is separable:

$$\frac{p\,dp}{p^2-1} = \frac{dy}{y}.$$

Integrate both sides. For the left side, use substitution $u = p^2 - 1$, $du = 2p\,dp$:

$$\int \frac{p\,dp}{p^2-1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|p^2-1|.$$

For the right side:

$$\int \frac{dy}{y} = \ln|y|.$$

Equating:

$$\frac{1}{2}\ln|p^2-1| = \ln|y| + C_1.$$

Multiply by 2:

$$\ln|p^2-1| = 2\ln|y| + 2C_1 = \ln(y^2) + 2C_1.$$

Exponentiate:

$$|p^2-1| = e^{2C_1}y^2.$$

Let $K = e^{2C_1}$, so:

$$p^2 - 1 = \pm Ky^2.$$

Thus:

$$(y')^2 = 1 \pm Ky^2.$$

Comparing with our earlier result $(y')^2 + c^2y^2 = 1$, we identify $c^2 = K$ (taking the minus sign):

$$(y')^2 = 1 - c^2y^2.$$

Separating variables:

$$\frac{dy}{\sqrt{1-c^2y^2}} = \pm dx.$$

Integrate:

$$\int \frac{dy}{\sqrt{1-c^2y^2}} = \pm \int dx.$$

The left side gives $\frac{1}{c}\arcsin(cy)$, and the right side gives $\pm x + C_2$:

$$\frac{1}{c}\arcsin(cy) = \pm x + C_2.$$

Thus:

$$\arcsin(cy) = \pm cx + cC_2.$$

Taking the sine of both sides:

$$cy = \sin(\pm cx + cC_2).$$

Using $\sin(-\theta) = -\sin(\theta)$ and absorbing constants:

$$\boxed{cy = \sin(cx + \phi),}$$

where $\phi$ is an arbitrary constant. Setting $\phi = 0$ recovers the original form:

$$\boxed{cy = \sin(cx).}$$

## 3. Verification

Differentiate $cy = \sin(cx)$:

$$cy' = c\cos(cx).$$

Thus $y' = \cos(cx)$.

Differentiate again:

$$y'' = -c\sin(cx) = -c \cdot cy = -c^2y.$$

Now check:

$$yy'' = y \cdot (-c^2 y) = -c^2 y^2.$$

$$(y')^2 - 1 = \cos^2(cx) - 1 = -\sin^2(cx) = -(cy)^2 = -c^2 y^2.$$

Thus $yy'' = (y')^2 - 1$, confirming that the function satisfies the derived ODE.

**Remark.** The solution $cy = \sin(cx)$ is valid for $|cy| \leq 1$, which requires $|y| \leq \frac{1}{|c|}$. The constant $c$ determines both the frequency and amplitude of the sinusoidal relationship. When $c = 0$, the original equation becomes $0 = 0$, which is satisfied by any function $y(x)$, but the differential equation derivation assumes $c \neq 0$.