Problem 1

Let $x = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}$ and $y = \sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}}$.
1° Find x^2 and y^2, then find x and y in simpler form than given.
2° Show that $7+4\sqrt{3}$ can be written as the square of the sum of an integer and an irrational number.
3° Calculate the value of the expression:

$$A = \frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}.$$

Solution

Part 1°: Find x^2, y^2, x, and y

Step 1. Let $a = \sqrt{7+4\sqrt{3}}$ and $b = \sqrt{7-4\sqrt{3}}$ for convenience. Then:

$$x = a+b, \quad y = a-b.$$

Step 2. Calculate x^2:

$$\begin{aligned} x^2 &= (a+b)^2 = a^2 + 2ab + b^2 \\ &= (7+4\sqrt{3}) + 2\sqrt{(7+4\sqrt{3})(7-4\sqrt{3})} + (7-4\sqrt{3}). \end{aligned}$$

Step 3. Simplify the middle term using difference of squares:

$$(7+4\sqrt{3})(7-4\sqrt{3}) = 7^2 - (4\sqrt{3})^2 = 49 - 48 = 1.$$

Therefore:

$$\begin{aligned} x^2 &= (7+4\sqrt{3}) + 2\sqrt{1} + (7-4\sqrt{3}) \\ &= 7+4\sqrt{3}+2+7-4\sqrt{3} \\ &= 16. \end{aligned}$$

Step 4. Since $x = a+b > 0$, we have:

$$\boxed{x = 4}.$$

Step 5. Calculate y^2:

$$\begin{aligned} y^2 &= (a-b)^2 = a^2 - 2ab + b^2 \\ &= (7+4\sqrt{3}) - 2\sqrt{1} + (7-4\sqrt{3}) \\ &= 7+4\sqrt{3}-2+7-4\sqrt{3} \\ &= 12. \end{aligned}$$

Step 6. Since $a = \sqrt{7+4\sqrt{3}} > b = \sqrt{7-4\sqrt{3}}$ (because $7+4\sqrt{3} > 7-4\sqrt{3}$), we have $y = a-b > 0$:

$$\boxed{y = 2\sqrt{3}}.$$

Part 2°: Express $7+4\sqrt{3}$ as a perfect square

Step 1. We want to find integers or simple radicals p and q such that:

$$7+4\sqrt{3}=(p+q\sqrt{3})^2.$$

Step 2. Expand the right side:

$$(p+q\sqrt{3})^2=p^2+2pq\sqrt{3}+3q^2=(p^2+3q^2)+2pq\sqrt{3}.$$

Step 3. Match coefficients:

$$p^2+3q^2=7, \quad 2pq=4.$$

From the second equation: $pq=2$, so $q=\frac{2}{p}$.
Step 4. Substitute into the first equation:

$$p^2+3\left(\frac{2}{p}\right)^2=7 \implies p^2+\frac{12}{p^2}=7.$$

Multiply by p^2:

$$p^4+12=7p^2 \implies p^4-7p^2+12=0.$$

Step 5. Let $u=p^2$:

$$u^2-7u+12=0 \implies (u-3)(u-4)=0.$$

So $u=3$ or $u=4$, giving $p^2=3$ or $p^2=4$.
Step 6. Take $p=2$ (from $p^2=4$), then $q=\frac{2}{2}=1$. Verify:

$$(2+\sqrt{3})^2=4+4\sqrt{3}+3=7+4\sqrt{3}. \quad \checkmark$$

Therefore:

$$\boxed{7+4\sqrt{3}=(2+\sqrt{3})^2}.$$

This shows $7+4\sqrt{3}$ is the square of the sum of the integer 2 and the irrational number $\sqrt{3}$.

Part 3°: Calculate A

Step 1. From Part 1, we have $x=4$ and $y=2\sqrt{3}$. Calculate:

$$x+y=4+2\sqrt{3},$$
$$x-y=4-2\sqrt{3}.$$

Step 2. From Part 2, we know $(2+\sqrt{3})^2=7+4\sqrt{3}$. Similarly:

$$(2-\sqrt{3})^2=4-4\sqrt{3}+3=7-4\sqrt{3}.$$

Step 3. Notice that:

$$\begin{aligned} x + y &= 4 + 2\sqrt{3} = 2(2 + \sqrt{3}), \\ x - y &= 4 - 2\sqrt{3} = 2(2 - \sqrt{3}). \end{aligned}$$

Step 4. Calculate the square roots:

$$\begin{aligned} \sqrt{x + y} &= \sqrt{2(2 + \sqrt{3})} = \sqrt{2}\sqrt{2 + \sqrt{3}}, \\ \sqrt{x - y} &= \sqrt{2(2 - \sqrt{3})} = \sqrt{2}\sqrt{2 - \sqrt{3}}. \end{aligned}$$

Step 5. Substitute into the expression for A:

$$\begin{aligned} A &= \frac{\sqrt{2}\sqrt{2 + \sqrt{3}} + \sqrt{2}\sqrt{2 - \sqrt{3}}}{\sqrt{2}\sqrt{2 + \sqrt{3}} - \sqrt{2}\sqrt{2 - \sqrt{3}}} \\ &= \frac{\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}}}{\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}}. \end{aligned}$$

Step 6. Rationalize by multiplying numerator and denominator by the conjugate:

$$\begin{aligned} A &= \frac{(\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}})^2}{(\sqrt{2 + \sqrt{3}})^2 - (\sqrt{2 - \sqrt{3}})^2} \\ &= \frac{(2 + \sqrt{3}) + 2\sqrt{(2 + \sqrt{3})(2 - \sqrt{3})} + (2 - \sqrt{3})}{(2 + \sqrt{3}) - (2 - \sqrt{3})}. \end{aligned}$$

Step 7. Simplify:

$$\begin{aligned} (2 + \sqrt{3})(2 - \sqrt{3}) &= 4 - 3 = 1, \\ \text{Numerator} &= 4 + 2\sqrt{1} = 4 + 2 = 6, \\ \text{Denominator} &= 2\sqrt{3}. \end{aligned}$$

Therefore:

$$\boxed{A = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}}.$$