Problem 12

If α and β are acute angles and $\sin(\alpha + \beta) = 2\cos\alpha$, prove that $\alpha > \beta$.

Proof

Given Information

- α and β are acute angles: $0 < \alpha, \beta < 90$ (or $0 < \alpha, \beta < \frac{\pi}{2}$)
- $\sin(\alpha + \beta) = 2\cos\alpha$

We need to prove: $\alpha > \beta$.

Step 1: Use the sine addition formula

Expand $\sin(\alpha + \beta)$:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Substitute into the given equation:

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = 2\cos\alpha$$

Step 2: Rearrange the equation

Rearrange to isolate terms:

$$\sin\alpha\cos\beta = 2\cos\alpha - \cos\alpha\sin\beta$$

$$\sin\alpha\cos\beta = \cos\alpha(2 - \sin\beta)$$

Divide both sides by $\cos\alpha\cos\beta$ (valid since both are positive for acute angles):

$$\frac{\sin\alpha}{\cos\alpha} = \frac{2 - \sin\beta}{\cos\beta}$$

$$\tan\alpha = \frac{2 - \sin\beta}{\cos\beta}$$

Step 3: Analyze the right side

We can rewrite the right side:

$$\tan\alpha = \frac{2}{\cos\beta} - \frac{\sin\beta}{\cos\beta} = \frac{2}{\cos\beta} - \tan\beta$$

Therefore:

$$\tan\alpha + \tan\beta = \frac{2}{\cos\beta}$$

Step 4: Establish bounds

Since β is an acute angle, we have $0 < \beta < 90$, which means:

$$0 < \cos\beta < 1$$

Therefore:

$$\frac{2}{\cos\beta} > 2$$

So:

$$\tan\alpha + \tan\beta > 2$$

Step 5: Prove $\alpha > \beta$

From $\tan\alpha = \frac{2}{\cos\beta} - \tan\beta$, we need to show that $\tan\alpha > \tan\beta$.

This is equivalent to showing:

$$\frac{2}{\cos\beta} - \tan\beta > \tan\beta$$

$$\frac{2}{\cos\beta} > 2\tan\beta$$

$$\frac{2}{\cos\beta} > \frac{2\sin\beta}{\cos\beta}$$

$$2 > 2\sin\beta$$

$$1 > \sin\beta$$

Since β is an acute angle ($0 < \beta < 90$), we have $\sin\beta < 1$, so this inequality is always true.

Since $\tan\alpha > \tan\beta$ and the tangent function is strictly increasing on $(0, 90)$, we conclude:

$$\boxed{\alpha > \beta}$$

Alternative Proof

From the given equation $\sin(\alpha + \beta) = 2\cos\alpha$, we know:

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = 2\cos\alpha$$

Rearrange:

$$\sin\alpha\cos\beta = \cos\alpha(2 - \sin\beta)$$

For this equation to hold with positive angles, we need $2 - \sin\beta > 0$, which gives $\sin\beta < 2$ (always true).

Since $\sin\beta < 1$ for acute angles, we have $2 - \sin\beta > 1$.

From $\sin\alpha\cos\beta = \cos\alpha(2 - \sin\beta)$:

$$\tan\alpha = \frac{2 - \sin\beta}{\cos\beta}$$

We can write:

$$\tan\alpha = \frac{2-\sin\beta}{\cos\beta} = \frac{2}{\cos\beta} - \tan\beta$$

Since $\cos\beta < 1$ for $\beta \in (0, 90)$, we have $\frac{2}{\cos\beta} > 2$.

Also, for acute β, we have $\tan\beta < \frac{2}{\cos\beta} - \tan\beta$, which gives:

$$2\tan\beta < \frac{2}{\cos\beta}$$

$$\tan\beta < \frac{1}{\cos\beta} = \sec\beta$$

This is equivalent to $\sin\beta < 1$, which is true for acute angles.

Therefore, $\tan\alpha > \tan\beta$, and since tangent is increasing, $\alpha > \beta$.

Numerical Verification

Let's verify with $\alpha = 60$ and find the corresponding β.

If $\alpha = 60$:

$$\sin(\alpha+\beta) = 2\cos(60) = 2 \cdot \frac{1}{2} = 1$$

Since $\sin(\alpha+\beta) = 1$, we have $\alpha + \beta = 90$, so:

$$\beta = 90 - 60 = 30$$

Check: $60 > 30$

Verify the original equation:

$$\sin(60+30) = \sin(90) = 1$$

$$2\cos(60) = 2 \cdot \frac{1}{2} = 1$$

Indeed, $1 = 1$

Remark

This problem demonstrates how trigonometric identities and the monotonicity of trigonometric functions can be used to establish inequalities between angles. The key insight is that the condition $\sin(\alpha+\beta) = 2\cos\alpha$ constrains the relationship between α and β in such a way that α must be larger than β for acute angles. This type of problem is common in trigonometry and requires careful manipulation of trigonometric identities combined with knowledge of the behavior of trigonometric functions in specific domains.