## Problem 9

Evaluate the sum $S$ for the series:

$$S = \sum_{n=2}^{\infty} \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}},$$

where the positive numbers $a_n$ form an arithmetic progression.

## Solution

### Step 1: Properties of the Arithmetic Progression

Let the arithmetic progression be defined by the first term $a_1$ and the common difference $d$. By definition:

$$a_n - a_{n-1} = d.$$

Since all terms $a_n$ are positive, we must have $a_1 > 0$.

### Step 2: Decomposing the General Term

The general term of the series, starting from $n = 2$, is $b_n = \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$. We use the technique of rationalizing the denominator by multiplying the numerator and denominator by the conjugate $\left(\sqrt{a_n} - \sqrt{a_{n-1}}\right)$:

$$\begin{aligned} b_n &= \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} \cdot \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{\sqrt{a_n} - \sqrt{a_{n-1}}} \\ &= \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{(\sqrt{a_n})^2 - (\sqrt{a_{n-1}})^2} \\ &= \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{a_n - a_{n-1}}. \end{aligned}$$

Since $a_n - a_{n-1} = d$, we substitute the common difference $d$ (assuming $d \neq 0$):

$$b_n = \frac{1}{d}\left(\sqrt{a_n} - \sqrt{a_{n-1}}\right).$$

This expression is a difference of two consecutive terms, $b_n = \frac{1}{d}(\sqrt{a_n} - \sqrt{a_{n-1}})$, which is characteristic of a telescoping series.

### Step 3: Determine the Partial Sum $S_N$

The $N$-th partial sum $S_N$ is the sum from $n = 2$ to $N$:

$$S_N = \sum_{n=2}^{N} \frac{1}{d}\left(\sqrt{a_n} - \sqrt{a_{n-1}}\right) = \frac{1}{d}\sum_{n=2}^{N}\left(\sqrt{a_n} - \sqrt{a_{n-1}}\right).$$

Writing out the terms:

$$\begin{aligned} S_N = \frac{1}{d}\big[&(\sqrt{a_2} - \sqrt{a_1}) + (\sqrt{a_3} - \sqrt{a_2}) + (\sqrt{a_4} - \sqrt{a_3}) + \cdots \\ &+ \left(\sqrt{a_{N-1}} - \sqrt{a_{N-2}}\right) + \left(\sqrt{a_N} - \sqrt{a_{N-1}}\right)\big]. \end{aligned}$$

The simplified partial sum is left with the last positive term and the first negative term:

$$S_N = \frac{1}{d}\left(\sqrt{a_N} - \sqrt{a_1}\right).$$

### Step 4: Calculate the Sum $S$

The sum of the series $S$ is the limit of the partial sum $S_N$ as $N \to \infty$:

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{d}\left(\sqrt{a_N} - \sqrt{a_1}\right).$$

The behavior of $\sqrt{a_N}$ as $N \to \infty$ depends on $d$. Since $a_N = a_1 + (N-1)d$:

- **Case 1:** $d > 0$ (Increasing A.P.):

$$\lim_{N\to\infty} a_N = \lim_{N\to\infty} (a_1 + (N-1)d) = \infty.$$

  Thus, $S = \frac{1}{d}(\infty - \sqrt{a_1}) = \infty$. The series diverges.

- **Case 2:** $d < 0$ (Decreasing A.P.): Since $a_N > 0$ for all $N$, the sequence must stop before reaching or crossing zero. The maximum index for which $a_N$ is positive is $N_{\max}$ such that $a_1 + (N_{\max} - 1)d \geq 0$. An infinite series requires $a_N \to L$ (where $L$ may be $\pm\infty$ or a finite number). If $d < 0$, $\lim_{N\to\infty} a_N = -\infty$, which violates the condition that $a_n$ are positive numbers. Therefore, the progression must be finite, contradicting the problem's request for an infinite sum. We only consider the case where an infinite sum is possible.

- **Case 3:** $d = 0$ (Constant A.P.): If $d = 0$, $a_n = a_1$. The original general term $b_n$ is $\frac{1}{\sqrt{a_1}+\sqrt{a_1}} = \frac{1}{2\sqrt{a_1}}$. The sum is $S = \sum_{n=2}^{\infty} \frac{1}{2\sqrt{a_1}}$, which is a sum of infinite constant terms, so $S = \infty$. The series diverges.

Since the problem asks for the sum $S$, and the series diverges for all possible infinite positive arithmetic progressions ($d \geq 0$), the formal solution is $\infty$.

## Final Answer

### Partial Sum $S_N$

For $d \neq 0$, the $N$-th partial sum is:

$$\boxed{S_N = \frac{1}{d}\left(\sqrt{a_N} - \sqrt{a_1}\right).}$$

### Sum $S$

Assuming an infinite arithmetic progression of positive numbers ($a_n > 0$) must have $d \geq 0$ (for $N \to \infty$), the series diverges.

$$\boxed{S = \infty.}$$