Problem 21

Factor the polynomial into irreducible factors with integer coefficients:

$$P(x) = x^5 + x^4 + x^3 + x^2 + x + 1$$

Solution

Step 1: Recognize as a geometric series

Notice that this is a geometric series:

$$P(x) = x^5 + x^4 + x^3 + x^2 + x + 1 = \sum_{k=0}^{5} x^k$$

Using the formula for the sum of a geometric series:

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}$$

Therefore:

$$P(x) = \frac{x^6 - 1}{x - 1}$$

Step 2: Factor $x^6 - 1$

We can factor $x^6 - 1$ as a difference of squares:

$$x^6 - 1 = (x^3)^2 - 1^2 = (x^3 - 1)(x^3 + 1)$$

Factor each cubic:

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$
$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

Therefore:

$$x^6 - 1 = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$$

Step 3: Divide by $(x - 1)$

Since $P(x) = \frac{x^6-1}{x-1}$:

$$P(x) = \frac{(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)}{x - 1}$$

$$P(x) = (x^2 + x + 1)(x + 1)(x^2 - x + 1)$$

Step 4: Multiply out to verify and reorder

We can verify by expanding, but first let's reorder the factors:

$$P(x) = (x + 1)(x^2 + x + 1)(x^2 - x + 1)$$

Let's check if we can factor further. We need to verify that $(x^2 + x + 1)$ and $(x^2 - x + 1)$ are irreducible over the integers.

For $x^2 + x + 1$:

Discriminant: $\Delta = 1^2 - 4(1)(1) = 1 - 4 = -3 < 0$

Since the discriminant is negative, this quadratic has no real roots and is irreducible over $\mathbb{R}$ (and hence over $\mathbb{Z}$).

For $x^2 - x + 1$:

Discriminant: $\Delta = (-1)^2 - 4(1)(1) = 1 - 4 = -3 < 0$

Similarly, this is irreducible over $\mathbb{Z}$.

Step 5: Final factorization

$$\boxed{P(x) = (x + 1)(x^2 + x + 1)(x^2 - x + 1)}$$

Verification

Let's verify by expanding:

$$(x + 1)(x^2 + x + 1) = x^3 + x^2 + x + x^2 + x + 1 = x^3 + 2x^2 + 2x + 1$$

Now multiply by $(x^2 - x + 1)$:

$$(x^3 + 2x^2 + 2x + 1)(x^2 - x + 1)$$

Expanding:

$$\begin{aligned}
&= x^3(x^2 - x + 1) + 2x^2(x^2 - x + 1) + 2x(x^2 - x + 1) + 1(x^2 - x + 1) \\
&= x^5 - x^4 + x^3 + 2x^4 - 2x^3 + 2x^2 + 2x^3 - 2x^2 + 2x + x^2 - x + 1 \\
&= x^5 + (-1 + 2)x^4 + (1 - 2 + 2)x^3 + (2 - 2 + 1)x^2 + (2 - 1)x + 1 \\
&= x^5 + x^4 + x^3 + x^2 + x + 1 \quad \checkmark
\end{aligned}$$

Alternative approach: Grouping

We can also factor by grouping:

$$P(x) = x^5 + x^4 + x^3 + x^2 + x + 1$$

Group pairs:

$$\begin{aligned}
&= (x^5 + x^4) + (x^3 + x^2) + (x + 1) \\
&= x^4(x + 1) + x^2(x + 1) + (x + 1)
\end{aligned}$$

$$= (x+1)(x^4+x^2+1)$$

Now we need to factor x^4+x^2+1. This can be done by completing the square:

$$x^4+x^2+1 = x^4+2x^2+1-x^2 = (x^2+1)^2-x^2$$

$$= (x^2+1-x)(x^2+1+x) = (x^2-x+1)(x^2+x+1)$$

This gives us the same factorization:

$$P(x) = (x+1)(x^2-x+1)(x^2+x+1)$$

Connection to cyclotomic polynomials

This polynomial is related to the 6th cyclotomic polynomial. The n-th roots of unity satisfy $x^n = 1$, and:

$$x^6 - 1 = \prod_{d|6} \Phi_d(x)$$

where $\Phi_d(x)$ is the d-th cyclotomic polynomial.
The divisors of 6 are: 1, 2, 3, 6, and:

- $\Phi_1(x) = x - 1$
- $\Phi_2(x) = x + 1$
- $\Phi_3(x) = x^2 + x + 1$
- $\Phi_6(x) = x^2 - x + 1$

So $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$, and:

$$P(x) = \frac{x^6-1}{x-1} = (x+1)(x^2+x+1)(x^2-x+1) = \Phi_2(x)\cdot\Phi_3(x)\cdot\Phi_6(x)$$

Summary

The complete factorization into irreducible factors with integer coefficients is:

$$\boxed{x^5+x^4+x^3+x^2+x+1 = (x+1)(x^2+x+1)(x^2-x+1)}$$

All three factors are irreducible over $\mathbb{Z}$:

- $(x+1)$ is linear
- (x^2+x+1) has discriminant $-3 < 0$
- (x^2-x+1) has discriminant $-3 < 0$