Problem 5

Evaluate the sum S for the series:

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}.$$

Solution

Step 1: Decomposing the General Term

The general term of the series is $a_n = \frac{2n+1}{n^2(n+1)^2}$. We aim to express this term as a difference of two consecutive terms, b_n and b_{n+1}, to form a telescoping series.

We notice that the numerator, $2n+1$, can be manipulated as $(n+1)^2 - n^2$:

$$(n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1.$$

Substituting this into the expression for a_n:

$$a_n = \frac{(n+1)^2 - n^2}{n^2(n+1)^2}.$$

We now separate the fraction:

$$\begin{aligned} a_n &= \frac{(n+1)^2}{n^2(n+1)^2} - \frac{n^2}{n^2(n+1)^2} \\ &= \frac{1}{n^2} - \frac{1}{(n+1)^2}. \end{aligned}$$

This successfully decomposes a_n into the required difference form, $a_n = b_n - b_{n+1}$, where $b_n = \frac{1}{n^2}$.

Step 2: Determine the Partial Sum S_N

The N-th partial sum S_N is the sum of the first N terms:

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right).$$

Writing out the terms shows the telescoping cancellation:

$$\begin{aligned} S_N = &\left(\frac{1}{1^2} - \frac{1}{2^2} \right) + \left(\frac{1}{2^2} - \frac{1}{3^2} \right) + \left(\frac{1}{3^2} - \frac{1}{4^2} \right) + \cdots \\ &+ \left(\frac{1}{(N-1)^2} - \frac{1}{N^2} \right) + \left(\frac{1}{N^2} - \frac{1}{(N+1)^2} \right). \end{aligned}$$

After cancellation, the simplified partial sum is left with the first term of the first parenthesis and the last term of the last parenthesis:

$$S_N = \frac{1}{1^2} - \frac{1}{(N+1)^2} = 1 - \frac{1}{(N+1)^2}.$$

Step 3: Calculate the Sum S

The sum of the series S is the limit of the partial sum S_N as $N \to \infty$:

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{(N+1)^2} \right).$$

Since $\lim_{N\to\infty} \frac{1}{(N+1)^2} = 0$:

$$S = 1 - 0 = 1.$$

Final Answer

Partial Sum S_N

The N-th partial sum is:

$$\boxed{S_N = 1 - \frac{1}{(N+1)^2}.}$$

Sum S

The sum of the series is:

$$\boxed{S = 1.}$$