Problem 11

If the numbers a, b, and c are positive and satisfy $a^2 = b^2 + c^2$:

1. Compare a with $b + c$, b, and c.
2. Compare a^3 and $b^3 + c^3$.

Also investigate: Does the converse hold?

Solution

Part $1°$: Compare a with $b + c$, b, and c

Given: $a^2 = b^2 + c^2$ where $a, b, c > 0$.

Compare a with b

Since $c > 0$, we have $c^2 > 0$. Therefore:

$$a^2 = b^2 + c^2 > b^2$$

Since both a and b are positive, taking square roots preserves the inequality:

$$\boxed{a > b}$$

Compare a with c

Similarly, since $b > 0$, we have $b^2 > 0$. Therefore:

$$a^2 = b^2 + c^2 > c^2$$

Taking square roots:

$$\boxed{a > c}$$

Compare a with $b + c$

We need to compare a with $b + c$. Square both sides (valid since both are positive):

$$a^2 \stackrel{?}{<} (b + c)^2$$

Expand the right side:

$$a^2 \stackrel{?}{<} b^2 + 2bc + c^2$$

Substitute $a^2 = b^2 + c^2$:

$$b^2 + c^2 \stackrel{?}{<} b^2 + 2bc + c^2$$

Simplify:

$$0 < 2bc$$

Since $b, c > 0$, this is always true. Therefore:

$$a^2 < (b+c)^2$$

Taking square roots:

$$\boxed{a < b + c}$$

Summary of Part 1:

$$\boxed{b < a < b + c \quad \text{and} \quad c < a < b + c}$$

More precisely: $\max(b, c) < a < b + c$.

Part 2°: Compare a^3 and $b^3 + c^3$

We need to determine whether $a^3 < b^3 + c^3$, $a^3 = b^3 + c^3$, or $a^3 > b^3 + c^3$.

Method: Test with specific values

Let's test with a 3-4-5 right triangle where $a = 5$, $b = 4$, $c = 3$:

$$a^2 = 25 = 16 + 9 = b^2 + c^2 \quad \checkmark$$

Compare cubes:

$$a^3 = 125, \quad b^3 + c^3 = 64 + 27 = 91$$

So $125 > 91$, which means $a^3 > b^3 + c^3$.
Let's verify with another example: $a = \sqrt{2}$, $b = 1$, $c = 1$:

$$a^2 = 2 = 1 + 1 = b^2 + c^2 \quad \checkmark$$

Compare cubes:

$$a^3 = (\sqrt{2})^3 = 2\sqrt{2} \approx 2.828, \quad b^3 + c^3 = 1 + 1 = 2$$

So $2.828 > 2$, which means $a^3 > b^3 + c^3$.

General proof that $a^3 > b^3 + c^3$

We will prove that $a^3 > b^3 + c^3$ using the fact that $a^2 = b^2 + c^2$.
From $a^2 = b^2 + c^2$, we have $a > b$ and $a > c$ (proven in Part 1).
Consider:

$$a^3 - b^3 - c^3 = a^3 - (b^3 + c^3)$$

We can write:

$$a^3 = a \cdot a^2 = a(b^2 + c^2)$$

So:

$$a^3 - b^3 - c^3 = a(b^2 + c^2) - b^3 - c^3 = ab^2 + ac^2 - b^3 - c^3$$
$$= b^2(a - b) + c^2(a - c)$$

Since $a > b$ and $a > c$ (from Part 1), we have:

$$a - b > 0 \quad \text{and} \quad a - c > 0$$

Therefore:

$$b^2(a - b) > 0 \quad \text{and} \quad c^2(a - c) > 0$$

Thus:

$$a^3 - b^3 - c^3 = b^2(a - b) + c^2(a - c) > 0$$

This proves:

$$\boxed{a^3 > b^3 + c^3}$$

Does the converse hold?

We need to investigate: If $a^3 > b^3 + c^3$, does it follow that $a^2 = b^2 + c^2$?

Answer: No, the converse does not hold.

Counterexample

Let $a = 10$, $b = 1$, $c = 1$.

Check if $a^3 > b^3 + c^3$:

$$a^3 = 1000, \quad b^3 + c^3 = 1 + 1 = 2$$

So $1000 > 2$ is true.

Check if $a^2 = b^2 + c^2$:

$$a^2 = 100, \quad b^2 + c^2 = 1 + 1 = 2$$

So $100 \neq 2$.

Therefore, $a^3 > b^3 + c^3$ does not imply $a^2 = b^2 + c^2$.

Summary

Given $a^2 = b^2 + c^2$ with $a, b, c > 0$:

1. **Comparisons:**
   - $a > b$ and $a > c$ (each leg is less than the hypotenuse)
   - $a < b + c$ (triangle inequality)
2. **Cube comparison:** $a^3 > b^3 + c^3$
3. **Converse:** The converse does NOT hold. Having $a^3 > b^3 + c^3$ does not imply $a^2 = b^2 + c^2$.

Geometric Interpretation

The condition $a^2 = b^2 + c^2$ is the Pythagorean theorem, so a, b, c represent the sides of a right triangle with a as the hypotenuse.

The result $a < b + c$ is the triangle inequality: the sum of any two sides of a triangle exceeds the third side.

The result $a^3 > b^3 + c^3$ shows that the Pythagorean relationship for squares does not extend to cubes. In fact, for any $n > 2$, we have $a^n > b^n + c^n$ when $a^2 = b^2 + c^2$. This is related to Fermat's Last Theorem, which states that $a^n = b^n + c^n$ has no positive integer solutions for $n > 2$.