Problem 3

$$\lim_{x \to 0} \left(\frac{\arctan x}{x} \right)^{\frac{1}{x^2}}$$

Solution

Using the approximation $\arctan x = x - \frac{x^3}{3} + O(x^5)$, we get:

$$\frac{\arctan x}{x} = 1 - \frac{x^2}{3} + O(x^4)$$

Taking the logarithm of the expression:

$$\ln L = \lim_{x \to 0} \frac{1}{x^2} \left(-\frac{x^2}{3} + O(x^4) \right)$$

Thus:

$$\ln L = -\frac{1}{3}$$

Exponentiating both sides:

$$L = e^{-\frac{1}{3}}$$

Thus, the limit is:

$$\lim_{x \to 0} \left(\frac{\arctan x}{x} \right)^{\frac{1}{x^2}} = e^{-\frac{1}{3}}$$