Problem 8

Evaluate the sum S for the series:

$$S = \sum_{n=1}^{\infty} (\sqrt[n]{a} - \sqrt[n+1]{a}), \quad a > 0.$$

Solution

Step 1: Recognizing the Telescoping Form

The general term of the series, $a_n = \sqrt[n]{a} - \sqrt[n+1]{a}$, is already in the form of a difference, $b_n - b_{n+1}$, where $b_n = \sqrt[n]{a}$.

We can rewrite the roots using fractional exponents:

$$a_n = a^{1/n} - a^{1/(n+1)}.$$

Step 2: Determine the Partial Sum S_N

The N-th partial sum S_N is the sum of the first N terms:

$$S_N = \sum_{n=1}^{N} \left(a^{1/n} - a^{1/(n+1)}\right).$$

Writing out the terms:

$$\begin{aligned} S_N = &\left(a^{1/1} - a^{1/2}\right) + \left(a^{1/2} - a^{1/3}\right) + \left(a^{1/3} - a^{1/4}\right) + \cdots \\ &+ \left(a^{1/(N-1)} - a^{1/N}\right) + \left(a^{1/N} - a^{1/(N+1)}\right). \end{aligned}$$

All intermediate terms cancel out. The simplified partial sum is left with the first term of the first parenthesis and the last term of the last parenthesis:

$$S_N = a^{1/1} - a^{1/(N+1)} = a - a^{1/(N+1)}.$$

Step 3: Calculate the Sum S

The sum of the series S is the limit of the partial sum S_N as $N \to \infty$:

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(a - a^{1/(N+1)}\right).$$

We evaluate the limit of the second term:

$$\lim_{N\to\infty} \frac{1}{N+1} = 0.$$

Since the exponential function is continuous, we have:

$$\lim_{N\to\infty} a^{1/(N+1)} = a^0 = 1, \quad \text{for } a > 0.$$

Substituting this back into the expression for S:

$$S = a - 1.$$

Since the limit of the partial sums is finite, the series converges.

Final Answer

Partial Sum S_N

The N-th partial sum is:

$$\boxed{S_N = a - a^{1/(N+1)}.}$$

Sum S

The sum of the series is:

$$\boxed{S = a - 1.}$$