Problem 25

For which values of the parameter a does exactly one root of the equation

$$(a+1)x^2 - (a^2+a+6)x + 6a = 0$$

lie in the interval $(0,1)$?

Solution

Step 1: Analyze the equation structure

First, note that for a quadratic, we need $a \neq -1$.

Let's try to factor or find special roots. If $x = a$ is a root:

$$(a+1)a^2 - (a^2+a+6)a + 6a = 0$$

$$a^3 + a^2 - a^3 - a^2 - 6a + 6a = 0$$

$$0 = 0 \quad \checkmark$$

So $x = a$ is always a root!

Step 2: Find the other root

Using Vieta's formulas, if $x_1 = a$ and x_2 is the other root:

$$x_1 \cdot x_2 = \frac{6a}{a+1}$$

$$a \cdot x_2 = \frac{6a}{a+1}$$

$$x_2 = \frac{6}{a+1}$$

We can verify: $x_1 + x_2 = a + \frac{6}{a+1} = \frac{a(a+1)+6}{a+1} = \frac{a^2+a+6}{a+1}$

So the two roots are:

$$x_1 = a, \quad x_2 = \frac{6}{a+1}$$

Step 3: Determine conditions for exactly one root in $(0,1)$

For exactly one root to be in $(0,1)$, we have three cases:

Case 1: $x_1 = a \in (0,1)$ and $x_2 = \frac{6}{a+1} \notin (0,1)$

Condition: $0 < a < 1$ and $(\frac{6}{a+1} \leq 0$ or $\frac{6}{a+1} \geq 1)$

Since $0 < a < 1$, we have $a + 1 > 1 > 0$, so $\frac{6}{a+1} > 0$.

For $\frac{6}{a+1} \geq 1$:

$$6 \geq a + 1$$

$$a \leq 5$$

Since we already have $0 < a < 1$, this is automatically satisfied.

For $\frac{6}{a+1} \notin (0,1)$, we need $\frac{6}{a+1} \geq 1$:

$$6 \geq a + 1$$

$$a \leq 5$$

This is satisfied for $0 < a < 1$.

But wait, let's check more carefully. If $0 < a < 1$, then $1 < a + 1 < 2$, so:

$$\frac{6}{2} < \frac{6}{a+1} < \frac{6}{1}$$

$$3 < \frac{6}{a+1} < 6$$

So $x_2 = \frac{6}{a+1} > 3 > 1$, which means $x_2 \notin (0,1)$.

Therefore, for $0 < a < 1$, exactly one root is in $(0,1)$.

Case 2: $x_2 = \frac{6}{a+1} \in (0,1)$ and $x_1 = a \notin (0,1)$

Condition: $0 < \frac{6}{a+1} < 1$ and $(a \leq 0$ or $a \geq 1)$

For $0 < \frac{6}{a+1} < 1$:

$$0 < \frac{6}{a+1} < 1$$

Since the numerator is positive, we need $a + 1 > 0$, i.e., $a > -1$.

For $\frac{6}{a+1} < 1$:

$$6 < a + 1$$

$$a > 5$$

So we need $a > 5$ and $(a \leq 0$ or $a \geq 1)$.

Since $a > 5$, we automatically have $a \geq 1$.

Therefore, for $a > 5$, exactly one root is in $(0,1)$.

Case 3: Check boundary cases

At $a = 1$: $x_1 = 1$, $x_2 = \frac{6}{2} = 3$. Neither in $(0,1)$ (open interval).

At $a = 5$: $x_1 = 5$, $x_2 = \frac{6}{6} = 1$. Neither in $(0,1)$ (open interval).

Step 4: Check for degenerate cases

If $a = -1$, the equation becomes linear (not quadratic).

If $a = 0$: $x_1 = 0$, $x_2 = 6$. Neither in $(0, 1)$.

Step 5: Complete solution

Exactly one root lies in $(0, 1)$ when:

$$\boxed{0 < a < 1 \quad \text{or} \quad a > 5}$$

Or in interval notation:

$$\boxed{a \in (0, 1) \cup (5, \infty)}$$

Verification

For $a = 0.5$ (in $(0, 1)$):

$$x_1 = 0.5 \in (0, 1), \quad x_2 = \frac{6}{1.5} = 4 \notin (0, 1) \quad \checkmark$$

For $a = 2$ (not in our solution):

$$x_1 = 2 \notin (0, 1), \quad x_2 = \frac{6}{3} = 2 \notin (0, 1) \quad \checkmark$$

For $a = 6$ (in $(5, \infty)$):

$$x_1 = 6 \notin (0, 1), \quad x_2 = \frac{6}{7} \approx 0.857 \in (0, 1) \quad \checkmark$$

Summary

The equation $(a + 1)x^2 - (a^2 + a + 6)x + 6a = 0$ has roots $x = a$ and $x = \frac{6}{a+1}$.

For exactly one root to be in the open interval $(0, 1)$:

- When $0 < a < 1$: the root $x = a$ is in $(0, 1)$, while $x = \frac{6}{a+1} > 3$
- When $a > 5$: the root $x = \frac{6}{a+1} < 1$ is in $(0, 1)$, while $x = a > 5$

Therefore: $a \in (0, 1) \cup (5, \infty)$.

Remark

This problem demonstrates the power of recognizing special roots. By testing $x = a$ as a potential root, we discovered that it always satisfies the equation, which allowed us to find the other root using Vieta's formulas. This reduced the problem to a simple inequality analysis rather than dealing with complex discriminant conditions.