Problem 2

Find the differential equation whose general solution is given by:

$$y = e^{cx}$$

where c is an arbitrary constant.

Explanation of Method

To find the differential equation from its general solution, we must **eliminate the arbitrary constant** (c) using differentiation. Since there is only one arbitrary constant, a first-order differential equation is expected, requiring only a single differentiation step.

1. **Differentiate** the solution with respect to x.
2. **Eliminate** the constant c by substitution or algebraic manipulation.

Solution

Step 1: Differentiate the General Solution

The given general solution is:

$$y = e^{cx} \quad \text{(i)}$$

Differentiating both sides with respect to x (using the Chain Rule):

$$\frac{dy}{dx} = c \cdot e^{cx} \quad \text{(ii)}$$

Step 2: Substitute and Eliminate e^{cx}

From equation (i), we know that $e^{cx} = y$. Substituting this into equation (ii):

$$\frac{dy}{dx} = c \cdot y$$

Step 3: Eliminate the Constant c

This equation can be rearranged to express c:

$$c = \frac{1}{y}\frac{dy}{dx}$$

Now, to eliminate c completely, we substitute c back into the original solution (i). However, it is algebraically simpler to first take the natural logarithm of (i):

$$\ln y = \ln(e^{cx})$$

$$\ln y = cx$$

Substitute the expression for c into this logarithmic equation:

$$\ln y = \left(\frac{1}{y}\frac{dy}{dx}\right) x$$

Result

Rearranging the final expression to present the differential equation in a standard form ($\frac{dy}{dx}$ isolated):

$$x\frac{dy}{dx} = y \ln y$$

$$\frac{dy}{dx} = \frac{y \ln y}{x}$$