Problem 5

Evaluate the limit:

$$\lim_{x \to 0} \frac{1 - \sqrt{\cos x} \cdot \sqrt[7]{\cos 5x}}{x^2}.$$

Solution

Step 1: Analyze the numerator and denominator

The numerator is $1 - \sqrt{\cos x} \cdot \sqrt[7]{\cos 5x}$, and the denominator is x^2. As $x \to 0$: - $\cos x \to 1$, and $\cos 5x \to 1$, so $\sqrt{\cos x} \cdot \sqrt[7]{\cos 5x} \to 1$, - This makes the numerator $1 - 1 = 0$, and the denominator also $x^2 \to 0$.

Thus, the limit is an indeterminate form $\frac{0}{0}$, and we can proceed by expanding the numerator.

Step 2: Use approximations for $\cos x$ and $\cos 5x$

For small x, we use the Taylor series approximation for $\cos x$:

$$\cos x \approx 1 - \frac{x^2}{2}.$$

Similarly:

$$\cos 5x \approx 1 - \frac{(5x)^2}{2} = 1 - \frac{25x^2}{2}.$$

Substitute these approximations into $\sqrt{\cos x}$ and $\sqrt[7]{\cos 5x}$ using binomial expansions.

Step 3: Expand $\sqrt{\cos x}$

For $\sqrt{\cos x}$, we use the binomial expansion for $(1-u)^{1/2}$:

$$\sqrt{\cos x} \approx \sqrt{1 - \frac{x^2}{2}} \approx 1 - \frac{1}{2} \cdot \frac{x^2}{2} = 1 - \frac{x^2}{4}.$$

Step 4: Expand $\sqrt[7]{\cos 5x}$

For $\sqrt[7]{\cos 5x}$, we use the binomial expansion for $(1-u)^{1/7}$:

$$\sqrt[7]{\cos 5x} \approx \sqrt[7]{1 - \frac{25x^2}{2}} \approx 1 - \frac{1}{7} \cdot \frac{25x^2}{2} = 1 - \frac{25x^2}{14}.$$

Step 5: Multiply $\sqrt{\cos x}$ and $\sqrt[7]{\cos 5x}$

Now multiply the two approximations:

$$\sqrt{\cos x} \cdot \sqrt[7]{\cos 5x} \approx \left(1 - \frac{x^2}{4}\right) \cdot \left(1 - \frac{25x^2}{14}\right).$$

Using the distributive property:

$$\sqrt{\cos x} \cdot \sqrt[7]{\cos 5x} \approx 1 - \frac{x^2}{4} - \frac{25x^2}{14}.$$

Simplify the coefficients:

$$\sqrt{\cos x} \cdot \sqrt[7]{\cos 5x} \approx 1 - \left(\frac{x^2}{4} + \frac{25x^2}{14}\right) = 1 - \frac{7x^2 + 50x^2}{28} = 1 - \frac{57x^2}{28}.$$

Step 6: Simplify the numerator

The numerator is:

$$1 - \sqrt{\cos x} \cdot \sqrt[7]{\cos 5x}.$$

Substitute the expansion for $\sqrt{\cos x} \cdot \sqrt[7]{\cos 5x}$:

$$1 - \sqrt{\cos x} \cdot \sqrt[7]{\cos 5x} \approx 1 - \left(1 - \frac{57x^2}{28}\right) = \frac{57x^2}{28}.$$

Step 7: Simplify the limit

The limit becomes:

$$\lim_{x \to 0} \frac{\frac{57x^2}{28}}{x^2}.$$

Cancel x^2 from the numerator and denominator:

$$\lim_{x \to 0} \frac{\frac{57x^2}{28}}{x^2} = \frac{57}{28}.$$

Simplify the fraction:

$$\frac{57}{28} = \frac{57 \div 7}{28 \div 7} = \frac{57}{28} = \frac{57}{28}.$$

Final Answer

$$\lim_{x \to 0} \frac{1 - \sqrt{\cos x} \cdot \sqrt[7]{\cos 5x}}{x^2} = \frac{57}{28}.$$