Problem 62

We are given a piecewise function $f(x)$ defined as:

$$f(x) = \begin{cases} kx^2 + 1, & \text{if } x < 2, \\ 5, & \text{if } x = 2, \\ x + 3, & \text{if } x > 2. \end{cases}$$

The task is to analyze the continuity of the function $f(x)$ at $x = 2$ and determine the value of k (if possible) such that $f(x)$ is continuous at $x = 2$.

Solution

A function $f(x)$ is continuous at a point $x = 2$ if the following conditions hold:

1. $f(2)$ is defined.
2. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$.
3. The value of the function at $x = 2$ matches the limit, i.e., $f(2) = \lim_{x \to 2} f(x)$.

Step 1: Value of $f(2)$

From the piecewise definition, $f(2) = 5$.

Step 2: Left-hand limit $(x \to 2^-)$

For $x < 2$, the function is $f(x) = kx^2 + 1$. The left-hand limit as $x \to 2^-$ is:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (kx^2 + 1) = k(2^2) + 1 = 4k + 1.$$

Step 3: Right-hand limit $(x \to 2^+)$

For $x > 2$, the function is $f(x) = x + 3$. The right-hand limit as $x \to 2^+$ is:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + 3) = 2 + 3 = 5.$$

Step 4: Continuity Condition

For the function to be continuous at $x = 2$, the left-hand limit, the right-hand limit, and the value of the function at $x = 2$ must all be equal. Therefore, we require:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2).$$

Substituting the values:

$$4k + 1 = 5 \quad \text{and} \quad 5 = 5.$$

Solve for k:

$$4k + 1 = 5 \implies 4k = 4 \implies k = 1.$$

Step 5: Final Answer

The function $f(x)$ is continuous at $x = 2$ if $k = 1$.