Problem 24

Solve the equation:

$$(x-3)(x-4)(x-7)(x-8) = 60$$

Solution

Step 1: Group factors strategically

Notice that the factors can be grouped to create a symmetric structure:

$$[(x-3)(x-8)][(x-4)(x-7)] = 60$$

Step 2: Expand each grouped product

First group:

$$(x-3)(x-8) = x^2 - 8x - 3x + 24 = x^2 - 11x + 24$$

Second group:

$$(x-4)(x-7) = x^2 - 7x - 4x + 28 = x^2 - 11x + 28$$

Notice that both have the same linear term $-11x$!

Step 3: Introduce a substitution

Let $u = x^2 - 11x$. Then:

$$(x-3)(x-8) = u + 24$$
$$(x-4)(x-7) = u + 28$$

The equation becomes:

$$(u+24)(u+28) = 60$$

Step 4: Expand and solve for u

$$u^2 + 28u + 24u + 672 = 60$$
$$u^2 + 52u + 672 = 60$$
$$u^2 + 52u + 612 = 0$$

Using the quadratic formula:

$$u = \frac{-52 \pm \sqrt{2704 - 2448}}{2} = \frac{-52 \pm \sqrt{256}}{2} = \frac{-52 \pm 16}{2}$$

Therefore:

$$u = \frac{-52 + 16}{2} = \frac{-36}{2} = -18$$
$$u = \frac{-52 - 16}{2} = \frac{-68}{2} = -34$$

Step 5: Solve for x from each value of u

Case 1: $u = -18$

From $x^2 - 11x = -18$:

$$x^2 - 11x + 18 = 0$$

Using the quadratic formula:

$$x = \frac{11 \pm \sqrt{121 - 72}}{2} = \frac{11 \pm \sqrt{49}}{2} = \frac{11 \pm 7}{2}$$

Therefore:

$$x = \frac{11 + 7}{2} = 9 \quad \text{or} \quad x = \frac{11 - 7}{2} = 2$$

Case 2: $u = -34$

From $x^2 - 11x = -34$:

$$x^2 - 11x + 34 = 0$$

Using the quadratic formula:

$$x = \frac{11 \pm \sqrt{121 - 136}}{2} = \frac{11 \pm \sqrt{-15}}{2} = \frac{11 \pm i\sqrt{15}}{2}$$

These are complex solutions (not real).

Step 6: Complete solution

The real solutions are:

$$\boxed{x = 2 \quad \text{or} \quad x = 9}$$

Verification

For $x = 2$:

$$(2-3)(2-4)(2-7)(2-8) = (-1)(-2)(-5)(-6) = (2)(30) = 60 \quad \checkmark$$

For $x = 9$:

$$(9-3)(9-4)(9-7)(9-8) = (6)(5)(2)(1) = 60 \quad \checkmark$$

Alternative grouping

We could also group as:

$$[(x-3)(x-7)][(x-4)(x-8)] = 60$$

First group:

$$(x-3)(x-7) = x^2 - 10x + 21$$

Second group:

$$(x-4)(x-8) = x^2 - 12x + 32$$

This doesn't give the same linear coefficient, making the substitution less elegant. The original grouping is optimal.

Summary

The key insight was to group the factors as $(x-3)(x-8)$ and $(x-4)(x-7)$, which both simplify to expressions of the form $x^2 - 11x + c$. This allows the substitution $u = x^2 - 11x$, reducing the quartic equation to a quadratic in u.

The strategy works because:

$$3 + 8 = 11 = 4 + 7$$

This symmetry in the sum of paired roots is the key to simplifying the problem.

General principle

For equations of the form $(x-a)(x-b)(x-c)(x-d) = k$, look for groupings where $a+d = b+c$. This allows a substitution that reduces the quartic to a quadratic.

In our case: $3 + 8 = 11 = 4 + 7$, which enabled the elegant solution.

Complete solution set (including complex)

If complex solutions are required:

$$\boxed{x \in \left\{2, 9, \frac{11 \pm i\sqrt{15}}{2}\right\}}$$