Problem 14

Given the equation: $\tan x - \tan 3x = m \tan 2x$, where m is a given constant.

1. Solve the equation when: a) $m = 0$, b) $m = -1$.
2. Show that the equation can be written in the form: $\sin 2x = 0$ or $2m \cos^2 2x + (m + 2) \cos 2x - m = 0$.

Solution

Part 1°: Solve for specific values of m

Case a) $m = 0$

The equation becomes:

$$\tan x - \tan 3x = 0$$

$$\tan x = \tan 3x$$

The general solution to $\tan A = \tan B$ is:

$$A = B + k\pi, \quad k \in \mathbb{Z}$$

Therefore:

$$x = 3x + k\pi$$

$$-2x = k\pi$$

$$\boxed{x = -\frac{k\pi}{2} = \frac{n\pi}{2}, \quad n \in \mathbb{Z}}$$

Note: We must exclude values where $\tan x$, $\tan 2x$, or $\tan 3x$ are undefined.

Case b) $m = -1$

The equation becomes:

$$\tan x - \tan 3x = -\tan 2x$$

$$\tan x - \tan 3x + \tan 2x = 0$$

We'll use the tangent addition formula. Recall:

$$\tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

This approach gets complicated. Let's use Part 2 to help us solve this.
From Part 2 (which we'll prove shortly), when $m = -1$:

$$2(-1)\cos^2 2x + (-1 + 2)\cos 2x - (-1) = 0$$

$$-2\cos^2 2x + \cos 2x + 1 = 0$$
$$2\cos^2 2x - \cos 2x - 1 = 0$$

This is a quadratic in $\cos 2x$. Using the quadratic formula:

$$\cos 2x = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4}$$

So $\cos 2x = 1$ or $\cos 2x = -\frac{1}{2}$.
Subcase 1: $\cos 2x = 1$

$$2x = 2k\pi, \quad k \in \mathbb{Z}$$
$$x = k\pi$$

Subcase 2: $\cos 2x = -\frac{1}{2}$

$$2x = \pm\frac{2\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$
$$x = \pm\frac{\pi}{3} + k\pi$$

Also from Part 2, we have solutions from $\sin 2x = 0$:

$$2x = k\pi \implies x = \frac{k\pi}{2}$$

Combining all solutions:

$$\boxed{x \in \left\{k\pi, \pm\frac{\pi}{3} + k\pi, \frac{k\pi}{2} : k \in \mathbb{Z}\right\}}$$

Note: Some of these may overlap (e.g., $x = k\pi$ is a subset of $x = \frac{k\pi}{2}$ when k is even).

Part 2°: Derive the alternative form

We need to show that $\tan x - \tan 3x = m \tan 2x$ is equivalent to:

$$\sin 2x = 0 \quad \text{or} \quad 2m\cos^2 2x + (m+2)\cos 2x - m = 0$$

Step 1: Convert to sines and cosines

$$\tan x - \tan 3x = m \tan 2x$$
$$\frac{\sin x}{\cos x} - \frac{\sin 3x}{\cos 3x} = m\frac{\sin 2x}{\cos 2x}$$

Multiply through by $\cos x \cos 3x \cos 2x$ (assuming these are non-zero):

$$\sin x \cos 3x \cos 2x - \sin 3x \cos x \cos 2x = m \sin 2x \cos x \cos 3x$$

$$\cos 2x(\sin x \cos 3x - \sin 3x \cos x) = m \sin 2x \cos x \cos 3x$$

Using the sine difference formula $\sin(A-B) = \sin A\cos B - \cos A\sin B$:

$$\cos 2x\sin(x-3x) = m\sin 2x\cos x\cos 3x$$
$$\cos 2x\sin(-2x) = m\sin 2x\cos x\cos 3x$$
$$-\cos 2x\sin 2x = m\sin 2x\cos x\cos 3x$$

Factor out $\sin 2x$:

$$\sin 2x(-\cos 2x - m\cos x\cos 3x) = 0$$

This gives us:

$$\boxed{\sin 2x = 0} \quad \text{or} \quad \cos 2x + m\cos x\cos 3x = 0$$

Step 2: Simplify the second condition

For the second condition:

$$\cos 2x + m\cos x\cos 3x = 0$$

Use the product-to-sum formula: $\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$:

$$\cos x\cos 3x = \frac{1}{2}[\cos(3x-x) + \cos(3x+x)] = \frac{1}{2}[\cos 2x + \cos 4x]$$

Therefore:

$$\cos 2x + \frac{m}{2}(\cos 2x + \cos 4x) = 0$$
$$\cos 2x + \frac{m}{2}\cos 2x + \frac{m}{2}\cos 4x = 0$$
$$\cos 2x\left(1 + \frac{m}{2}\right) + \frac{m}{2}\cos 4x = 0$$

Multiply by 2:

$$(2+m)\cos 2x + m\cos 4x = 0$$

Using $\cos 4x = 2\cos^2 2x - 1$:

$$(2+m)\cos 2x + m(2\cos^2 2x - 1) = 0$$
$$(2+m)\cos 2x + 2m\cos^2 2x - m = 0$$
$$\boxed{2m\cos^2 2x + (m+2)\cos 2x - m = 0}$$

This completes the proof! The equation $\tan x - \tan 3x = m\tan 2x$ is equivalent to:

$$\sin 2x = 0 \quad \text{or} \quad 2m\cos^2 2x + (m+2)\cos 2x - m = 0$$

Summary

Part 1:

- When $m = 0$: $x = \frac{n\pi}{2}$, $n \in \mathbb{Z}$
- When $m = -1$: $x \in \{k\pi, \pm\frac{\pi}{3} + k\pi, \frac{k\pi}{2}\}$, $k \in \mathbb{Z}$

Part 2: We proved that the equation can be written as:

$$\sin 2x = 0 \quad \text{or} \quad 2m\cos^2 2x + (m+2)\cos 2x - m = 0$$

Remark

This problem demonstrates how complex trigonometric equations can be transformed into simpler forms through careful algebraic manipulation and the use of trigonometric identities. The key steps were:

1. Converting tangents to sines and cosines
2. Using the sine difference formula
3. Applying product-to-sum formulas
4. Using the double angle formula for cosine

The resulting form separates the solutions into two cases: those where $\sin 2x = 0$ (which are independent of m) and those satisfying a quadratic in $\cos 2x$ (which depend on the parameter m).