Problem 6: Identity Proof

Prove that:

$$\sqrt{x+6\sqrt{x-9}}+\sqrt{x-6\sqrt{x-9}} \equiv \begin{cases} 2\sqrt{x-9}, & x>18 \\ 6, & 9<x<18 \end{cases}$$

Proof

Step 1: Define the expression

Let:

$$S=\sqrt{x+6\sqrt{x-9}}+\sqrt{x-6\sqrt{x-9}}$$

Note that for S to be real, we need $x \geq 9$ (so that $\sqrt{x-9}$ is real), and we need:

$$x+6\sqrt{x-9} \geq 0 \quad \text{and} \quad x-6\sqrt{x-9} \geq 0$$

The first condition is always satisfied for $x \geq 9$. The second requires:

$$x \geq 6\sqrt{x-9}$$

Step 2: Square the expression

To find S, we square it:

$$S^2=\left(\sqrt{x+6\sqrt{x-9}}+\sqrt{x-6\sqrt{x-9}}\right)^2$$

Expanding:

$$\begin{aligned} S^2 &= \left(x+6\sqrt{x-9}\right)+2\sqrt{(x+6\sqrt{x-9})(x-6\sqrt{x-9})}+\left(x-6\sqrt{x-9}\right) \\ &= 2x+2\sqrt{(x+6\sqrt{x-9})(x-6\sqrt{x-9})} \end{aligned}$$

Step 3: Simplify the product under the square root

$$\begin{aligned} (x+6\sqrt{x-9})(x-6\sqrt{x-9}) &= x^2-(6\sqrt{x-9})^2 = x^2-36(x-9) \\ &= x^2-36x+324=(x-18)^2 \end{aligned}$$

Therefore:

$$\sqrt{(x+6\sqrt{x-9})(x-6\sqrt{x-9})}=|x-18|$$

Step 4: Apply the absolute value

$$S^2 = 2x + 2|x - 18|$$

Now we need to consider two cases:
Case 1: $x > 18$
When $x > 18$, we have $|x - 18| = x - 18$:

$$S^2 = 2x + 2(x - 18) = 2x + 2x - 36 = 4x - 36 = 4(x - 9)$$

Since $S > 0$:

$$S = \sqrt{4(x - 9)} = 2\sqrt{x - 9}$$

Case 2: $9 < x < 18$
When $9 < x < 18$, we have $|x - 18| = 18 - x$:

$$S^2 = 2x + 2(18 - x) = 2x + 36 - 2x = 36$$

Since $S > 0$:

$$S = \sqrt{36} = 6$$

Step 5: Verify the domain condition for Case 2

For $9 < x < 18$, we need to verify that $x - 6\sqrt{x - 9} \geq 0$.
At $x = 18$: $18 - 6\sqrt{9} = 18 - 18 = 0$
At $x = 9$: $9 - 6\sqrt{0} = 9 > 0$
For $9 < x < 18$, we check if $x \geq 6\sqrt{x - 9}$:

$$x \geq 6\sqrt{x - 9}$$

Squaring both sides (valid since both sides are positive):

$$x^2 \geq 36(x - 9)$$

$$x^2 - 36x + 324 \geq 0$$

$$(x - 18)^2 \geq 0$$

This is always true, with equality at $x = 18$. So $x - 6\sqrt{x - 9} \geq 0$ for all $x \geq 9$.

Step 6: Boundary check at $x = 18$

At $x = 18$:

- From Case 1: $S = 2\sqrt{18 - 9} = 2\sqrt{9} = 6$
- From Case 2: $S = 6$

Both cases agree at the boundary, confirming continuity.

Conclusion

We have proven that:

$$\boxed{\sqrt{x+6\sqrt{x-9}}+\sqrt{x-6\sqrt{x-9}}=\begin{cases}2\sqrt{x-9}, & x>18\\ 6, & 9<x\leq 18\end{cases}}$$

Note: At $x=18$, both expressions equal 6, so we can include or exclude it from either interval.

Remark. This identity demonstrates a piecewise behavior where a seemingly continuous radical expression simplifies to different forms depending on whether x is above or below the critical value 18. The key insight is recognizing that $(x+6\sqrt{x-9})(x-6\sqrt{x-9}) = (x-18)^2$, which introduces the absolute value $|x-18|$ that creates the piecewise structure. Such identities often arise in simplification problems and in the study of nested radicals.