Project 3: Visual-Inertial SLAM Solutions

Problems

In square brackets are the points assigned to each problem.

1. [26 pts] Consider the bicycle shown in Fig. 1, in which the diameter of the front wheel is twice that of the rear wheel. Frames $\{a\}$ and $\{b\}$ are attached, respectively, to the centers of the wheels with the axes $\hat{y}_a$ and $\hat{y}_b$ aligned. Frame $\{c\}$ is attached to the top of the front wheel and its distance to frame $\{b\}$ is D in the $\hat{x}$ direction. Assuming that the bike moves forward in the $\hat{y}$-direction, find the transformation ${}_{\{a\}}T_{\{c\}}$ from frame $\{c\}$ to frame $\{a\}$ as a function of the front wheel's rotation angle θ. Assume that $\theta = 0$ at the instant shown in Fig. 1.

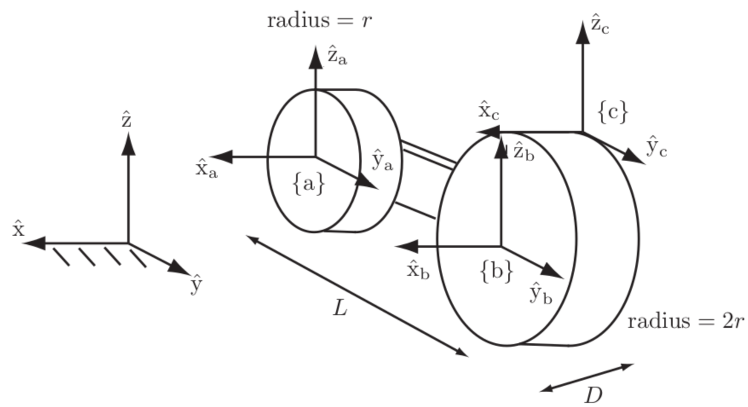

Figure 1: A bicycle with a large front wheel.

Solution
First of all, all the frames, $\{a\}, \{b\}, \{c\}$, are rotating!
${}_{\{a\}}T_{\{c\}} = {}_{\{a\}}T_{\{b\}}\,{}_{\{b\}}T_{\{c\}}$

$$ {}_{\{b\}}T_{\{c\}} = \begin{bmatrix} {}_{\{b\}}R_{\{c\}} & {}_{\{b\}}p_{\{c\}} \\ \mathbf{0}^T & 1 \end{bmatrix} $$

[5 pts for ${}_{\{b\}}R_{\{c\}}$]

$$ {}_{\{b\}}R_{\{c\}} = I_{3\times 3} $$

[5 pts for ${}_{\{b\}}p_{\{c\}}$]

$$ {}_{\{b\}}p_{\{c\}} = [-D \quad 0 \quad 2r]^T $$

$$ {}_{\{a\}}T_{\{b\}} = \begin{bmatrix} {}_{\{b\}}R_{\{c\}} & {}_{\{b\}}p_{\{c\}} \\ \mathbf{0}^T & 1 \end{bmatrix} $$

[5 pts for ${}_{\{a\}}R_{\{b\}}$]
Basically should be a rotation around $\hat{x}_a$:

$$ R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} $$

When the front wheel rotates angle θ, the (smaller) rear wheel rotates angle 2θ. Notice to convert to the right direction of frame $\{a\}$:

$$ \phi = (-\theta) - (-2\theta) = \theta $$

$$
{}_{\{a\}}R_{\{b\}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

[5 pts for ${}_{\{a\}}p_{\{b\}}$]

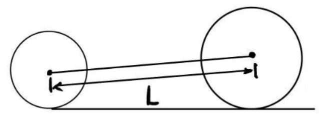

Figure 2: The distance between two centers of the wheels (credit to a student's answer on piazza)

When $\theta = 0$:

$$
{}_{\{a\}}p_{\{b\}}(0) = \begin{bmatrix} 0 & \sqrt{L^2 - r^2} & r \end{bmatrix}^T
$$

([-2 pts] if getting the result like $[0 \quad L \quad 0]^T$. How can the rear wheel float in the air?)
If we imagine there is a world frame, then ${}_{\{a\}}p_{\{b\}}$ is not rotating in the world frame, but frame $\{a\}$ is rotating w.r.t. the world frame. So ${}_{\{a\}}p_{\{b\}}$ is rotating in the frame $\{a\}$. And when frame $\{b\}$ rotates θ, frame $\{a\}$ rotates 2θ.

$$
{}_{\{a\}}p_{\{b\}}(\theta) = R_x(2\theta)\begin{bmatrix} 0 & \sqrt{L^2 - r^2} & r \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{L^2 - r^2} \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{L^2 - r^2}\cos(2\theta) - r\sin(2\theta) \\ \sqrt{L^2 - r^2}\sin(2\theta) + r\cos(2\theta) \end{bmatrix}
$$

wrong example caused by floating wheel $[0 \quad L\cos(2\theta) \quad L\sin(2\theta)]^T$

([-3 pts] if no rotation.)

(Another way to get ${}_{\{a\}}T_{\{b\}}$: ${}_{\{a\}}T_{\{b\}} = {}_{\{a\}}T_{\{w\}}{}_{\{w\}}T_{\{b\}}$, where w frame takes the point of rear wheel touching the ground as origin point.)

$$
{}_{\{w\}}R_{\{b\}} = R_x(-\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad {}_{\{w\}}p_{\{b\}} = [0 \quad 2\pi\theta + \sqrt{L^2 - r^2} \quad 2r]^T
$$

$$
{}_{\{w\}}R_{\{a\}} = R_x(-2\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{bmatrix}, \quad {}_{\{a\}}R_{\{w\}} = {}_{\{w\}}R_{\{a\}}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix}
$$

$$
{}_{\{w\}}p_{\{a\}} = [0 \quad 2\pi\theta \quad r]^T, \quad {}_{\{a\}}p_{\{w\}} = -{}_{\{w\}}R_{\{a\}}^T{}_{\{w\}}p_{\{a\}} = [0 \quad -2\pi\theta\cos(2\theta) + r\sin(2\theta) \quad -2\pi\theta\sin(2\theta) - r\cos(2\theta)]^T
$$

$$
{}_{\{a\}}R_{\{b\}} = {}_{\{a\}}R_{\{w\}}{}_{\{w\}}R_{\{b\}} = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

$$
{}_{\{a\}}p_{\{b\}} = {}_{\{a\}}R_{\{w\}}{}_{\{w\}}p_{\{b\}} + {}_{\{a\}}p_{\{w\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} 0 \\ 2\pi\theta + \sqrt{L^2 - r^2} \\ 2r \end{bmatrix} + \begin{bmatrix} 0 \\ -2\pi\theta\cos(2\theta) + r\sin(2\theta) \\ -2\pi\theta\sin(2\theta) - r\cos(2\theta) \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ \sqrt{L^2 - r^2}\cos(2\theta) - r\sin(2\theta) \\ \sqrt{L^2 - r^2}\sin(2\theta) + r\cos(2\theta) \end{bmatrix}
$$

[6 pts]

$$
{}_{\{a\}}T_{\{c\}} = {}_{\{a\}}T_{\{b\}}{}_{\{b\}}T_{\{c\}} = \begin{bmatrix} {}_{\{a\}}R_{\{b\}}{}_{\{b\}}R_{\{c\}} & {}_{\{a\}}R_{\{b\}}{}_{\{b\}}p_{\{c\}} + {}_{\{a\}}p_{\{b\}} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} {}_{\{a\}}R_{\{c\}} & {}_{\{a\}}p_{\{c\}} \\ \mathbf{0}^T & 1 \end{bmatrix}
$$

$$
{}_{\{a\}}R_{\{c\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

$$
{}_{\{a\}}p_{\{c\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -D \\ 0 \\ 2r \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{L^2-r^2}\cos(2\theta) - r\sin(2\theta) \\ \sqrt{L^2-r^2}\sin(2\theta) + r\cos(2\theta) \end{bmatrix} = \begin{bmatrix} -D \\ -2r\sin(\theta) + \sqrt{L^2-r^2}\cos(2\theta) - r\sin(2\theta) \\ 2r\cos(\theta) + \sqrt{L^2-r^2}\sin(2\theta) + r\cos(2\theta) \end{bmatrix}
$$

wrong example caused by floating wheel $[-D \quad -2r\sin(\theta) + L\cos(2\theta) \quad 2r\cos(\theta) + L\sin(2\theta)]^T$

(This is **not** recommended but if you did assume that $\{a\}, \{b\}$ don't rotate and only $\{c\}$ rotate)

$$
{}_{\{b\}}R_{\{c\}} = R_x(-\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad {}_{\{b\}}p_{\{c\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -D \\ 0 \\ 2r \end{bmatrix} = \begin{bmatrix} -D \\ 2r\sin\theta \\ 2r\cos\theta \end{bmatrix}
$$

$$
{}_{\{a\}}R_{\{b\}} = I_{3\times3}, \quad {}_{\{a\}}p_{\{b\}} = \begin{bmatrix} 0 \\ \sqrt{L^2-r^2} \\ r \end{bmatrix}
$$

$$
{}_{\{a\}}R_{\{c\}} = {}_{\{a\}}R_{\{b\}\,\{b\}}R_{\{c\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}
$$

$$
{}_{\{a\}}p_{\{c\}} = {}_{\{a\}}R_{\{b\}\,\{b\}}p_{\{c\}} + {}_{\{a\}}p_{\{b\}} = \begin{bmatrix} -D \\ \sqrt{L^2-r^2} + 2r\sin\theta \\ r + 2r\cos\theta \end{bmatrix}
$$

2. [26 pts] In class, we developed expressions for the matrix exponential for spatial motions mapping elements from $\mathfrak{so}(3)$ to $SO(3)$ and elements of $\mathfrak{se}(3)$ to $SE(3)$. Similarly, we showed how to compute the matrix logarithm going the other direction. In this problem, your objective is to write down the explicit matrix exponential from $\mathfrak{so}(2)$ to $SO(2)$ and from $\mathfrak{se}(2)$ to $SE(2)$ and the logarithm function going back. For the $\mathfrak{so}(2)$ to $SO(2)$ case there is a single exponential coordinate, while for the $\mathfrak{se}(2)$ to $SE(2)$ case there are three exponential coordinates, corresponding to a planar translation and angle of rotation.

Solution

(a) $\mathfrak{so}(2)$ and $SO(2)$

$\mathfrak{so}(2) = \{\theta_\times = \theta G \in \mathbb{R}^{2\times2} | \theta \in \mathbb{R}, G = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\}$

$SO(2) = \{R \in \mathbb{R}^{2\times2} | R^T R = I, \det(R) = 1\}$

$\mathfrak{so}(2) \to SO(2)$:

$$
\exp(\theta_\times) = \exp\left(\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}\right) = I + \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} + \tfrac{1}{2!}\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^2 + \cdots + \tfrac{1}{n!}\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^n + \ldots
$$

$$
\left(\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^2 = \begin{bmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{bmatrix}, \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & \theta^3 \\ -\theta^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^4 = \begin{bmatrix} \theta^4 & 0 \\ 0 & \theta^4 \end{bmatrix}, \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}^5 = \begin{bmatrix} 0 & -\theta^5 \\ \theta^5 & 0 \end{bmatrix}\right)
$$

$$
\exp(\theta_\times) = \begin{bmatrix} 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \ldots & -\theta + \frac{1}{3!}\theta^3 - \ldots \\ \theta - \frac{1}{3!}\theta^3 + \ldots & 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \ldots \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}
$$

$SO(2) \to \mathfrak{so}(2)$: (simply inverse the mapping from above)

$$
\log(R) = \begin{bmatrix} 0 & -\arccos(R_{11}) \\ \arccos(R_{11}) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\text{atan2}(R_{21}, R_{11}) \\ \text{atan2}(R_{21}, R_{11}) & 0 \end{bmatrix}
$$

(b) $\mathfrak{se}(2)$ and $SE(2)$

$$
\mathfrak{se}(2) = \left\{ \begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} | \theta_\times \in \mathfrak{so}(2), \rho \in \mathbb{R}^2 \right\}
$$

$SE(2) = \{T = \begin{bmatrix} R & p \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} | R \in SO(2), p \in \mathbb{R}^2\}$
$\mathfrak{se}(2) \to SE(2)$:
$\exp\left(\begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix}\right) = I + \begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix}^2 + \cdots + \frac{1}{n!}\begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix}^n + \ldots$
$\begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix}^2 = \begin{bmatrix} (\theta_\times)^2 & \theta_\times \rho \\ \mathbf{0}^T & 0 \end{bmatrix}, \begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix}^3 = \begin{bmatrix} (\theta_\times)^3 & (\theta_\times)^2 \rho \\ \mathbf{0}^T & 0 \end{bmatrix}$
$\exp\left(\begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 1 \end{bmatrix}\right) = \begin{bmatrix} \exp(\theta_\times) & J_L(\theta_\times)\rho \\ \mathbf{0}^T & 1 \end{bmatrix}$
$J_L(\theta_\times) = I + \frac{1}{2!}\theta_\times + \frac{1}{3!}(\theta_\times)^2 + \cdots = \sum_{i=0}^{\infty} \frac{1}{(i+1)!}(\theta_\times)^i = \sum_{i=0}^{\infty}\left(\frac{(\theta_\times)^{2i}}{(2i+1)!} + \frac{(\theta_\times)^{2i+1}}{(2i+2)}\right)$
$(\theta_\times)^{2i} = \theta^{2i}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{2i} = (-1)^i \theta^{2i}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$(\theta_\times)^{2i+1} = \theta^{2i+1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{2i+1} = (-1)^i \theta^{2i+1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$J_L(\theta_\times) = \left(\sum_{i=0}^{\infty} \frac{(-1)^i\theta^{2i}}{(2i+1)!}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\sum_{i=0}^{\infty} \frac{(-1)^i\theta^{2i+1}}{(2i+2)!}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$= \left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \ldots\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{\theta}{2!} - \frac{\theta^3}{4!} + \frac{\theta^5}{6!} - \ldots\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$= \frac{1}{\theta}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{\theta}\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$= \frac{1}{\theta}\begin{bmatrix} \sin\theta & -(1-\cos\theta) \\ (1-\cos\theta) & \sin\theta \end{bmatrix}$
$SE(2) \to \mathfrak{se}(2)$:
$\log(T) = \log\left(\begin{bmatrix} R & p \\ \mathbf{0}^T & 1 \end{bmatrix}\right) = \begin{bmatrix} \theta_\times & \rho \\ \mathbf{0}^T & 0 \end{bmatrix}$
$\theta_\times = \log(R) = \begin{bmatrix} 0 & -\arccos(R_{11}) \\ \arccos(R_{11}) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\text{atan2}(R_{21}, R_{11}) \\ \text{atan2}(R_{21}, R_{11}) & 0 \end{bmatrix}$
$\theta = \theta_{\times 21} = \arccos(R_{11})$ (recover the orientation for calculating ρ)
$\rho = (J_L(\theta_\times))^{-1}p = \frac{\theta^2}{(\sin\theta)^2 + (1-\cos\theta)^2}\frac{1}{\theta}\begin{bmatrix} \sin\theta & (1-\cos\theta) \\ -(1-\cos\theta) & \sin\theta \end{bmatrix} p = \frac{\theta}{2(1-\cos\theta)}\begin{bmatrix} \sin\theta & (1-\cos\theta) \\ -(1-\cos\theta) & \sin\theta \end{bmatrix} p$

3. [26 pts] Consider a mobile robot moving around on the xy-plane with motion model:

$$\mathbf{s}_{t+1} := \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \tau \begin{bmatrix} \cos\theta_t & 0 \\ \sin\theta_t & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} v_t \\ \omega_t \end{bmatrix} + \mathbf{w}_t\right)$$

where $\mathbf{s}_t \in SE(2)$ is the robot state at time t, $\mathbf{u}_t := [v_t,\ \omega_t]^T \in \mathbb{R}^2$ is the control input, $\tau > 0$ is the time discretization, and $\mathbf{w}_t \sim \mathcal{N}(0, \mathbf{W})$ is Gaussian control noise with covariance $\mathbf{W} \in \mathbb{R}^{2\times 2}$. Suppose that the robot is equipped with a sensor that measures the range and bearing to the origin, according to the observation model:

$$\mathbf{z}_t := \begin{bmatrix} r_t \\ \phi_t \end{bmatrix} = \begin{bmatrix} \sqrt{x_t^2 + y_t^2} \\ \mathbf{atan2}(-y_t, -x_t) - \theta_t \end{bmatrix} + \boldsymbol{\eta}_t$$

where $\mathbf{z}_t$ is the sensor observation at time t and $\boldsymbol{\eta}_t \sim \mathcal{N}(0, \mathbf{V})$ is Gaussian measurement noise with covariance $\mathbf{V} \in \mathbb{R}^{2\times 2}$. Let the prior distribution of the robot state at time t be $\mathcal{N}\left(\boldsymbol{\mu}_{t|t}, \boldsymbol{\Sigma}_{t|t}\right)$ and assume that the robot moves before observing, i.e., the motion model is applied before the observation model.

(a) Work out the equations of the prediction step of the Extended Kalman Filter, needed to compute the distribution $\mathcal{N}\left(\boldsymbol{\mu}_{t+1|t}, \boldsymbol{\Sigma}_{t+1|t}\right)$. Make sure that the equations you write are specific to the robot motion model specified above.

(b) Work out the equations of the update step of the Extended Kalman Filter, needed to compute the distribution $\mathcal{N}\left(\boldsymbol{\mu}_{t+1|t+1}, \boldsymbol{\Sigma}_{t+1|t+1}\right)$. Make sure that the equations you write are specific to the robot observation model specified above.

(c) Derive the prediction and update steps for the information matrix. In other words, let $\boldsymbol{\Omega}_{t|t} := \boldsymbol{\Sigma}_{t|t}^{-1}$ and derive the prediction equations for $\boldsymbol{\Omega}_{t+1|t}$ as well as the update equations for $\boldsymbol{\Omega}_{t+1|t+1}$.

Solution

(a) Motion Model :

$$\mathbf{s}_{t+1} := \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} + \tau \begin{bmatrix} \cos\theta_t & 0 \\ \sin\theta_t & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} v_t \\ \omega_t \end{bmatrix} + \mathbf{w}_t \right)$$

$$\mathbf{s}_{t+1} = f(\mathbf{s}_t, \mathbf{u}_t, \mathbf{w}_t) = \begin{bmatrix} x_t + \tau v_t \cos\theta_t \\ y_t + \tau v_t \sin\theta_t \\ \theta_t + \tau\omega_t \end{bmatrix} + \tau \begin{bmatrix} \cos\theta_t & 0 \\ \sin\theta_t & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}_t, \qquad \mathbf{w}_t \sim \mathcal{N}(0, \mathbf{W})$$

Observation Model :

$$\mathbf{z}_t = h(\mathbf{s}_t, \boldsymbol{\eta}_t) = \begin{bmatrix} \sqrt{x_t^2 + y_t^2} \\ \mathbf{atan2}\,(-y_t, -x_t) - \theta_t \end{bmatrix} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim \mathcal{N}(0, \mathbf{V})$$

Jacobians for the Motion Model:

$$F_t = \frac{\partial f}{\partial \mathbf{s}}(\boldsymbol{\mu}_{t|t}, \mathbf{u}_t, 0) = \begin{bmatrix} \partial x_{t+1}/\partial x_t & \partial x_{t+1}/\partial y_t & \partial x_{t+1}/\partial \theta_t \\ \partial y_{t+1}/\partial x_t & \partial y_{t+1}/\partial y_t & \partial y_{t+1}/\partial \theta_t \\ \partial \theta_{t+1}/\partial x_t & \partial \theta_{t+1}/\partial y_t & \partial \theta_{t+1}/\partial \theta_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\tau v_{t|t} \sin\theta_{t|t} \\ 0 & 1 & \tau v_{t|t} \cos\theta_{t|t} \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_t = \frac{\partial f}{\partial \mathbf{w}}(\boldsymbol{\mu}_{t|t}, \mathbf{u}_t, 0) = \tau \begin{bmatrix} \partial x_{t+1}/\partial \mathbf{w}_{t1} & \partial x_{t+1}/\partial \mathbf{w}_{t2} \\ \partial y_{t+1}/\partial \mathbf{w}_{t1} & \partial y_{t+1}/\partial \mathbf{w}_{t2} \\ \partial \theta_{t+1}/\partial \mathbf{w}_{t1} & \partial \theta_{t+1}/\partial \mathbf{w}_{t2} \end{bmatrix} = \tau \begin{bmatrix} \cos\theta_{t|t} & 0 \\ \sin\theta_{t|t} & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{where} \qquad \boldsymbol{\mu}_{t|t} = \begin{bmatrix} x_{t|t} \\ y_{t|t} \\ \theta_{t|t} \end{bmatrix}$$

Jacobians for the Observation Model:

$$H_{t+1} = \frac{\partial h}{\partial \mathbf{s}}(\boldsymbol{\mu}_{t+1|t}, 0) = \begin{bmatrix} \partial \mathbf{z}_{t1}/\partial x_t & \partial \mathbf{z}_{t1}/\partial y_t & \partial \mathbf{z}_{t1}/\partial \theta_t \\ \partial \mathbf{z}_{t2}/\partial x_t & \partial \mathbf{z}_{t2}/\partial y_t & \partial \mathbf{z}_{t2}/\partial \theta_t \end{bmatrix} = \begin{bmatrix} \frac{x_{t+1|t}}{\sqrt{x_{t+1|t}^2 + y_{t+1|t}^2}} & \frac{y_{t+1|t}}{\sqrt{x_{t+1|t}^2 + y_{t+1|t}^2}} & 0 \\ \frac{-y_{t+1|t}}{x_{t+1|t}^2 + y_{t+1|t}^2} & \frac{x_{t+1|t}}{x_{t+1|t}^2 + y_{t+1|t}^2} & -1 \end{bmatrix}$$

$$R_{t+1} = \frac{\partial h}{\partial \boldsymbol{\eta}}(\boldsymbol{\mu}_{t+1|t}, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Prediction Equations are:

$$\boldsymbol{\mu}_{t+1|t} = f(\boldsymbol{\mu}_{t|t}, \mathbf{u}_t, 0) = \begin{bmatrix} x_{t|t} + \tau v_t \cos\theta_{t|t} \\ y_{t|t} + \tau v_t \sin\theta_{t|t} \\ \theta_{t|t} + \tau\omega_t \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{t+1|t} = F_t \boldsymbol{\Sigma}_{t|t} F_t^T + Q_t \mathbf{W} Q_t^T$$

(Final answer should be expanded using the defined matrices for F_t and Q_t.)

(b) The Update Equations are :-
Kalman Gain:

$$K_{t+1|t} = \boldsymbol{\Sigma}_{t+1|t} H_{t+1}^T (H_{t+1} \boldsymbol{\Sigma}_{t+1|t} H_{t+1}^T + \mathbf{V})$$

(Final step can have H_{t+1} matrix expanded.)

Updated mean:

$$\boldsymbol{\mu}_{t+1|t+1} = \boldsymbol{\mu}_{t+1|t} + K_{t+1|t}(\mathbf{z}_{t+1} - h(\boldsymbol{\mu}_{t+1|t}, 0))$$

where,

$$h(\boldsymbol{\mu}_{t+1|t}, 0) = \begin{bmatrix} \sqrt{x_{t+1|t}^2 + y_{t+1|t}^2} \\ \mathbf{atan2}\left(-y_{t+1|t}, -x_{t+1|t}\right) - \theta_{t+1|t} \end{bmatrix}$$

Updated Covariance:

$$\boldsymbol{\Sigma}_{t+1|t+1} = (I - K_{t+1|t}H_{t+1})\boldsymbol{\Sigma}_{t+1|t}$$

(c) matrix inversion lemma / Woodbury matrix identity:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}$$

$$\left(M^{-1} + QWQ^T\right)^{-1} = M - MQ\left(W^{-1} + Q^TMQ\right)^{-1}Q^TM$$

Prediction step equation:
Let $M_t = (F_t\Sigma_{t|t}F_t^T)^{-1} = (F_t^T)^{-1}\Sigma_{t|t}^{-1}F_t^{-1} = (F_t^T)^{-1}\Omega_{t|t}F_t^{-1}$

$$\begin{aligned}
\Sigma_{t+1|t} &= F_t\Sigma_{t|t}F_t^T + Q_tWQ_t^T \\
\Omega_{t+1|t} &= \Sigma_{t+1|t}^{-1} \\
&= (F_t\Sigma_{t|t}F_t^T + Q_tWQ_t^T)^{-1} \\
&= (F_t\Omega_{t|t}^{-1}F_t^T + Q_tWQ_t^T)^{-1}
\end{aligned}$$

Update step equation:

$$K_{t+1|t} = \Sigma_{t+1|t}H_{t+1}^T\left(H_{t+1}\Sigma_{t+1|t}H_{t+1}^T + R_{t+1}VR_{t+1}^T\right)^{-1}$$

$$K_{t+1|t}^{-1} = \left(H_{t+1}\Sigma_{t+1|t}H_{t+1}^T + R_{t+1}VR_{t+1}^T\right)(H_{t+1}^T)^{-1}\Sigma_{t+1|t}^{-1} = H_{t+1} + R_{t+1}VR_{t+1}^T(H_{t+1}^T)^{-1}\Sigma_{t+1|t}^{-1}$$

$$\begin{aligned}
\Omega_{t+1|t+1} &= \Sigma_{t+1|t+1}^{-1} \\
&= \Sigma_{t+1|t}^{-1}(I - K_{t+1|t}H_{t+1})^{-1} \\
&= \Omega_{t+1|t}(I - K_{t+1|t}H_{t+1})^{-1} \\
&= \Omega_{t+1|t}(K_{t+1|t}(K_{t+1|t}^{-1} - H_{t+1}))^{-1} \\
&= \Omega_{t+1|t}(K_{t+1|t}(\left(H_{t+1}\Sigma_{t+1|t}H_{t+1}^T + R_{t+1}VR_{t+1}^T\right)(H_{t+1}^T)^{-1}\Sigma_{t+1|t}^{-1} - H_{t+1}))^{-1} \\
&= \Omega_{t+1|t}(K_{t+1|t}R_{t+1}VR_{t+1}^T(H_{t+1}^T)^{-1}\Omega_{t+1|t})^{-1} \\
&= H_{t+1}^T(R_{t+1}^T)^{-1}V^{-1}R_{t+1}^{-1}K_{t+1|t}^{-1} \\
&= H_{t+1}^T(R_{t+1}^T)^{-1}V^{-1}R_{t+1}^{-1}\left(H_{t+1}\Sigma_{t+1|t}H_{t+1}^T + R_{t+1}VR_{t+1}^T\right)(H_{t+1}^T)^{-1}\Sigma_{t+1|t}^{-1} \\
&= H_{t+1}^T(R_{t+1}^T)^{-1}V^{-1}R_{t+1}^{-1}(H_{t+1} + R_{t+1}VR_{t+1}^T(H_{t+1}^T)^{-1}\Sigma_{t+1|t}^{-1}) \\
&= \Omega_{t+1|t} + H_{t+1}^T(R_{t+1}^T)^{-1}V^{-1}R_{t+1}^{-1}H_{t+1} \\
\Omega_{t+1|t+1} &= \Omega_{t+1|t} + H_{t+1}^T(R_{t+1}^T)^{-1}V^{-1}R_{t+1}^{-1}H_{t+1}
\end{aligned}$$