Project 2: Particle Filter SLAM Solutions

Problems

In square brackets are the points assigned to each problem.

1. [26 pts] You lost your green turtle and are not very happy about it. Luckily, you installed the turtle-communication app on your Android and can get some messages from your turtle. Your turtle is not very good at sensing its environment but can tell you (with some noise) how far away it is from the nearest puddle. You have a couple of other handy tools at your disposal – a particle filter from Stanford's Intro to AI course[1] and a Google Maps app showing the city blocks (black squares) and the puddle locations (light blue dots). How can you use the particle filter to localize your turtle?

   (a) Write the equations for the prior probability density function $p_{0|0}(x_0)$, the turtle motion model $p_f(x_{t+1} \mid x_t, u_t)$, in terms of the turtle position (x, y), orientation θ, and velocity v, and the turtle observation model $p_h(z_t \mid x_t)$, in terms of the puddle positions $b_1, \dots, b_{31} \in \mathbb{R}^2$.

   (b) Write the equations for the particle filter prediction and update steps in terms of the models you specified above. Explain the meaning of these equations. What does each step do, and how are the particle positions and weights updated? How is the gray dot estimate generated based on the particles?

   Solution

   (a) (A Gaussian Design) Let $\hat{b}_i = [b_i, 0]^T \;\; \forall i$, $x = [p, \theta]^T$ where $p \in \mathbb{R}^2$, and assume that the observation model has global view of the map. Our observation model gives the distance to the nearest puddle.

$$z_t = h(x_t, m) := \min_i ||b_i - p_t||_2 + \epsilon, \quad i = 1, \dots, 31$$

   where $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$. If the observation model only has a local view of the map then,

$$z_t = \hat{h}(x_t, m) := \begin{cases} h(x_t, m) & h(x_t, m) \le \nu \\ \infty & h(x_t, m) > \nu \end{cases}$$

   where ν is some maximum range. Let the motion model for the turtle be a differential drive model with random orientation.

$$x_{t+1} = f(x_t, u_t) := x_t + \begin{pmatrix} u_t \cos(\theta) \\ u_t \sin(\theta) \\ \dot{\theta} \end{pmatrix} + \eta$$

   where $\dot{\theta} \sim \text{Unif}(0, \pi)$ and $\eta \sim \mathcal{N}(0, \sigma_\eta^2 \mathbf{I}_{3\times3})$. Then given z_0 let our prior be a mixture of Gaussians over the puddle locations,

$$p(\cdot|z_0, m) \sim \frac{1}{31} \sum_{i=1}^{31} \mathcal{N}(\hat{b}_i, 2z_0 \mathbf{I}_{3\times3})$$

   Our models give us Gaussian densities:

$$p_f(\cdot|x_t, u_t) \sim \mathcal{N}(f(x_t, u_t), \sigma_\eta^2 \mathbf{I}_{3\times3})$$

$$p_h(\cdot|x_t) \sim \mathcal{N}(h(x_t, m), \sigma_\epsilon^2)$$

   (b) (Prediction Step & Update Step Description) Given a set of particle with weights, $\mathcal{P} = \left\{ \left(\mu_{t|t}^{(k)}, \alpha_{t|t}^{(k)}\right) \right\}_{k=1}^{N}$,
   Prediction Step:

$$p_{t+1|t}(x_{t+1}) = \sum_{k=1}^{N} \alpha_{t|t}^{(k)} p_f\left(x_{t+1}|\mu_{t|t}^{(k)}, u_t\right) = \sum_{k=1}^{N} \alpha_{t|t}^{(k)} \mathcal{N}(x_{t+1}|f(\mu_{t|t}^{(k)}, u_t), \sigma_\eta^2 \mathbf{I}_{3\times3})$$

[1] https://github.com/mjl/particle_filter_demo

Update Step:

$$p_{t+1|t+1}(x_{t+1}) = \sum_{k=1}^{N} \frac{\alpha_{t+1|t}^{(k)} p_h\left(z_t|\mu_{t+1|t}^{(k)}, \sigma_\epsilon^2\right)}{\sum_{j=1}^{N} \alpha_{t+1|t}^{(j)} p_h\left(z_t|\mu_{t+1|t}^{(j)}, \sigma_\epsilon^2\right)} \delta(x_{t+1}|\mu_{t+1|t}^{(k)})$$
$$= \sum_{k=1}^{N} \frac{\alpha_{t+1|t}^{(k)} \mathcal{N}(z_{t+1}|h(\mu_{t+1|t}^{(k)}), \sigma_\epsilon^2)}{\sum_{j=1}^{N} \alpha_{t+1|t}^{(j)} \mathcal{N}(z_{t+1}|h(\mu_{t+1|t}^{(j)}), \sigma_\epsilon^2)} \delta(x_{t+1}|\mu_{t+1|t}^{(k)})$$

2. [26 pts] You are using a particle filter to track the position of a robot moving along the x-axis. The robot is moving according to the following motion model:

$$p_f(x_{t+1} \mid x_t, u_t) := \begin{cases} x_t + u_t & \text{with prob. } 1/2 \\ x_t + u_t + \mathbf{sgn}(u_t) & \text{with prob. } 1/3 \\ x_t + u_t - \mathbf{sgn}(u_t) & \text{with prob. } 1/6 \end{cases} \qquad \mathbf{sgn}(u) := \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -1 & \text{if } u < 0 \end{cases}$$

Your sensor is described by the following observation model:

$$p_h(z_t \mid x_t) = \begin{cases} x_t & \text{with prob. } 1/2 \\ x_t + 1 & \text{with prob. } 1/4 \\ x_t - 1 & \text{with prob. } 1/4 \end{cases}$$

You model the prior distribution over the robot position x_0 using 5 equally weighted particles with positions $\{1, 2, 3, 4, 5\}$. You receive the following observations $z_0 = 2$, $z_1 = 3$, $z_2 = 3$ and also know that the robot applies the controls $u_0 = +1$, $u_1 = -1$, $u_2 = +1$. Evaluate the posterior probability density function $p(\cdot \mid z_{0:2}, u_{0:2})$ at $x_3 = 4$ when (a) no resampling is used and (b) stratified resampling is used after every update step with $u = \frac{1}{2N}$ in line 5 of slide 28 of Lecture 6. Is it better to use or not use resampling in this example?

Solution

(a) (No Resampling)

$$p_{0|-1}(x_0) = \frac{1}{5}$$

$$z_0 = 2 \Rightarrow p_h(2|x_0 = 1) = \frac{1}{4}, p_h(2|x_0 = 2) = \frac{1}{2}, p_h(2|x_0 = 3) = \frac{1}{4}, p_h(2|x_0 = 4) = 0, p_h(2|x_0 = 5) = 0$$

$$\Rightarrow p_{0|0}(x_0) = \begin{cases} 1 & \frac{1}{4} \\ 2 & \frac{1}{2} \\ 3 & \frac{1}{4} \end{cases}$$

$$u_0 = 1 \Rightarrow p_f(x_1|x_0 = 1, u_0) = \begin{cases} 1 & \frac{1}{24} \\ 2 & \frac{1}{8} \\ 3 & \frac{1}{12} \end{cases}, p_f(x_1|x_0 = 2, u_0) = \begin{cases} 2 & \frac{1}{12} \\ 3 & \frac{1}{4} \\ 4 & \frac{1}{6} \end{cases}, p_f(x_1|x_0 = 3, u_0) = \begin{cases} 3 & \frac{1}{24} \\ 4 & \frac{1}{8} \\ 5 & \frac{1}{12} \end{cases}$$

$$\Rightarrow p_{1|0}(x_1) = \begin{cases} 1 & \frac{1}{24} \\ 2 & \frac{5}{24} \\ 3 & \frac{9}{24} \\ 4 & \frac{7}{24} \\ 5 & \frac{2}{24} \end{cases}$$

$$z_1 = 3 \Rightarrow p_h(3|x_1 = 1) = 0, p_h(3|x_1 = 2) = \frac{1}{4}, p_h(3|x_1 = 3) = \frac{1}{2}, p_h(3|x_1 = 4) = \frac{1}{4}, p_h(3|x_1 = 5) = 0$$

$$\Rightarrow p_{1|1}(x_1) = \begin{cases} 2 & \frac{5}{30} \\ 3 & \frac{18}{30} \\ 4 & \frac{7}{30} \end{cases}$$

$$u_1 = -1 \Rightarrow p_f(x_2|x_1 = 2, u_1) = \begin{cases} 0 & \frac{5}{90} \\ 1 & \frac{5}{60} \\ 2 & \frac{5}{180} \end{cases}, p_f(x_2|x_1 = 3, u_1) = \begin{cases} 1 & \frac{18}{90} \\ 2 & \frac{18}{60} \\ 3 & \frac{18}{180} \end{cases}, p_f(x_2|x_1 = 4, u_1) = \begin{cases} 2 & \frac{7}{90} \\ 3 & \frac{7}{60} \\ 4 & \frac{7}{180} \end{cases}$$

$$\Rightarrow p_{2|1}(x_2) = \begin{cases} 0 & \frac{10}{180} \\ 1 & \frac{51}{180} \\ 2 & \frac{73}{180} \\ 3 & \frac{39}{180} \\ 4 & \frac{7}{180} \end{cases}$$

$$z_2 = 3 \Rightarrow p_h(3|x_2 = 0) = 0, p_h(3|x_2 = 1) = 0, p_h(3|x_2 = 2) = \frac{1}{4}, p_h(3|x_2 = 3) = \frac{1}{2}, p_h(3|x_2 = 4) = \frac{1}{4}$$

$$\Rightarrow p_{2|2}(x_2) = \begin{cases} 2 & \frac{73}{158} \\ 3 & \frac{78}{158} \\ 4 & \frac{7}{158} \end{cases}$$

$$u_1 = 1 \Rightarrow p_f(x_3|x_2 = 2, u_1) = \begin{cases} 2 & \frac{73}{948} \\ 3 & \frac{73}{316} \\ 4 & \frac{73}{474} \end{cases}, p_f(x_3|x_2 = 3, u_1) = \begin{cases} 3 & \frac{78}{948} \\ 4 & \frac{78}{316} \\ 5 & \frac{78}{474} \end{cases}, p_f(x_3|x_2 = 4, u_1) = \begin{cases} 4 & \frac{7}{948} \\ 5 & \frac{7}{316} \\ 6 & \frac{7}{474} \end{cases}$$

$$\Rightarrow p_{3|2}(x_3) = \begin{cases} 2 & \frac{73}{948} \\ 3 & \frac{297}{948} \\ 4 & \frac{387}{948} \\ 5 & \frac{177}{949} \\ 6 & \frac{14}{948} \end{cases} \Rightarrow p_{3|2}(x_3 = 4) = \frac{387}{948}$$

(b) (Stratified Resampling with $u = \frac{1}{2N}$)
Starting at $p_{0|0}(x_0)$,

$$p_{0|0}(x_0) = \begin{cases} 1 & \frac{1}{4} \\ 2 & \frac{1}{2} \\ 3 & \frac{1}{4} \end{cases}$$

$$u^{(1)} = \frac{1}{6}, u^{(2)} = \frac{3}{6}, u^{(3)} = \frac{5}{6}, c_1 = \left(0, \frac{1}{4}\right], c_2 = \left(\frac{1}{4}, \frac{1}{2}\right], c_3 = \left(\frac{1}{2}, 1\right]$$

$$\Rightarrow p_{0|0}(x_0) = \begin{cases} 1 & \frac{1}{3} \\ 2 & \frac{1}{3} \\ 3 & \frac{1}{3} \end{cases}$$

$$u_0 = 1 \Rightarrow p_f(x_1|x_0 = 1, u_0) = \begin{cases} 1 & \frac{1}{18} \\ 2 & \frac{1}{6} \\ 3 & \frac{1}{9} \end{cases}, p_f(x_1|x_0 = 2, u_0) = \begin{cases} 2 & \frac{1}{18} \\ 3 & \frac{1}{6} \\ 4 & \frac{1}{9} \end{cases}, p_f(x_1|x_0 = 3, u_0) = \begin{cases} 3 & \frac{1}{18} \\ 4 & \frac{1}{6} \\ 5 & \frac{1}{9} \end{cases}$$

$$\Rightarrow p_{1|0}(x_1) = \begin{cases} 1 & \frac{1}{18} \\ 2 & \frac{4}{18} \\ 3 & \frac{6}{18} \\ 4 & \frac{5}{18} \\ 5 & \frac{2}{18} \end{cases}$$

$$z_1 = 3 \Rightarrow p_h(3|x_1 = 1) = 0, p_h(3|x_1 = 2) = \frac{1}{4}, p_h(3|x_1 = 3) = \frac{1}{2}, p_h(3|x_1 = 4) = \frac{1}{4}, p_h(3|x_1 = 5) = 0$$

$$\Rightarrow p_{1|1}(x_1) = \begin{cases} 2 & \frac{4}{21} \\ 3 & \frac{12}{21} \\ 4 & \frac{5}{21} \end{cases}$$

$$u^{(1)} = \frac{1}{6}, u^{(2)} = \frac{3}{6}, u^{(3)} = \frac{5}{6}, c_1 = \left(0, \frac{4}{21}\right], c_2 = \left(\frac{4}{21}, \frac{16}{21}\right], c_3 = \left(\frac{16}{21}, 1\right]$$

$$\Rightarrow p_{1|1}(x_1) = \begin{cases} 2 & \frac{1}{3} \\ 3 & \frac{1}{3} \\ 4 & \frac{1}{3} \end{cases}$$

$$u_1 = -1 \Rightarrow p_f(x_2|x_1 = 2, u_1) = \begin{cases} 0 & \frac{1}{9} \\ 1 & \frac{1}{6} \\ 2 & \frac{1}{18} \end{cases}, p_f(x_2|x_1 = 3, u_1) = \begin{cases} 1 & \frac{1}{9} \\ 2 & \frac{1}{6} \\ 3 & \frac{1}{18} \end{cases}, p_f(x_2|x_1 = 4, u_1) = \begin{cases} 2 & \frac{1}{9} \\ 3 & \frac{1}{6} \\ 4 & \frac{1}{18} \end{cases}$$

$$\Rightarrow p_{2|1}(x_2) = \begin{cases} 0 & \frac{2}{18} \\ 1 & \frac{5}{18} \\ 2 & \frac{6}{18} \\ 3 & \frac{4}{18} \\ 4 & \frac{1}{18} \end{cases}$$

$$z_2 = 3 \Rightarrow p_h(3|x_2 = 0) = 0, p_h(3|x_2 = 1) = 0, p_h(3|x_2 = 2) = \frac{1}{4}, p_h(3|x_2 = 3) = \frac{1}{2}, p_h(3|x_2 = 4) = \frac{1}{4}$$

$$\Rightarrow p_{2|2}(x_2) = \begin{cases} 2 & \frac{6}{15} \\ 3 & \frac{8}{15} \\ 4 & \frac{1}{15} \end{cases}$$

$$u^{(1)} = \frac{1}{6}, u^{(2)} = \frac{3}{6}, u^{(3)} = \frac{5}{6}, c_1 = \left(0, \frac{6}{15}\right], c_2 = \left(\frac{6}{15}, \frac{14}{15}\right], c_3 = \left(\frac{14}{15}, 1\right]$$

$$\Rightarrow p_{2|2}(x_2) = \begin{cases} 2 & \frac{1}{3} \\ 3 & \frac{2}{3} \end{cases}$$

$$u_1 = 1 \Rightarrow p_f(x_3|x_2 = 2, u_1) = \begin{cases} 2 & \frac{1}{18} \\ 3 & \frac{1}{6} \\ 4 & \frac{1}{9} \end{cases}, p_f(x_3|x_2 = 3, u_1) = \begin{cases} 3 & \frac{2}{18} \\ 4 & \frac{2}{6} \\ 5 & \frac{2}{9} \end{cases}$$

$$\Rightarrow p_{3|2}(x_3) = \begin{cases} 2 & \frac{1}{18} \\ 3 & \frac{5}{18} \\ 4 & \frac{8}{18} \\ 5 & \frac{4}{18} \end{cases} \Rightarrow p_{3|2}(x_3 = 4) = \frac{8}{18}$$

(c) (Comparison)
When evaluation the predictive distribution at $x_3 = 4$ the filter with resampling produces a higher probability because during the 3^{rd} resampling step the 4^{th} particle has such a low probability it doesn't get chosen. This reduces the variance of the sample distribution allowing for a more concentrated prediction step. Furthermore, the systematic resampling reduces the computational cost for the person making it overall better to use in this example.

3. [26 pts] Consider a picture frame, represented by four line segments with vertices $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, (\mathbf{e}_1 + \mathbf{e}_3) \in \mathbb{R}^3$, where $\mathbf{e}_i$ is the ith standard basis vector. Suppose that the picture frame is moving with constant velocity $\mathbf{e}_1$ along the x-axis. Your robot is observing the picture frame with a camera located at position $p \in \mathbb{R}^3$ and orientation, specified by a quaternion q (Hamilton convention):

$$p = \begin{pmatrix} -\frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 3 \end{pmatrix} \qquad q = \begin{pmatrix} \cos(\frac{45^\circ}{2})\cos(-\frac{30^\circ}{2}) \\ -\sin(\frac{45^\circ}{2})\sin(-\frac{30^\circ}{2}) \\ \cos(\frac{45^\circ}{2})\sin(-\frac{30^\circ}{2}) \\ \sin(\frac{45^\circ}{2})\cos(-\frac{30^\circ}{2}) \end{pmatrix}$$

(a) Compute the projections of the 4 picture frame vertices in the image plane.

(b) Compute the velocities of the 4 projected points in the image plane. Which point appears to move fastest?

(c) Compute the projected lengths of the 4 line segments.

(d) Compute the rate of change of these projected lengths. Are they growing or shrinking? Which projected line segment is changing length the fastest?

Solutions

(a) Compute $R = \mathbf{E}[q]\mathbf{G}[q]^T$ or $R = \exp(\theta\hat{\eta})$ and use $R_{oc}R_{wc}^T(m - p)$

$$\begin{aligned}
q &= [0.8923, 0.099, -0.2391, 0.3696] \\
\theta &= 0.9363 \\
\eta &= [0.2193, -0.5297, 0.8188]^T \\
\theta\hat{\eta} &= \begin{bmatrix} 0 & -0.7667 & -0.4959 \\ 0.7667 & 0 & -0.2053 \\ 0.4959 & 0.2053 & 0 \end{bmatrix} \\
R_{wc} = \mathbf{E}[q]\mathbf{G}[q]^T = \exp(\theta\hat{\eta}) &= \begin{bmatrix} 0.6127 & -0.7069 & -0.3535 \\ 0.6123 & 0.7073 & -0.3532 \\ 0.4997 & 0 & 0.8662 \end{bmatrix}
\end{aligned}$$

The projected vertices are:
For vertex $\mathbf{0}$: [4.999, 2.5996 -1.497]
For vertex $\mathbf{e}_1$: [5.7061, 2.9532, -0.8849]
For vertex $\mathbf{e}_3$: [4.999, 1.7335, -0.9979]
For vertex $\mathbf{e}_1 + \mathbf{e}_3$: [5.7061, 2.08696, -0.3853]

(b) Project velocities of vertices to image plane.
Let p = point transformed to image plane.
v = velocity at that point.
Therefore using,

$$v = \hat{w}p + e_1$$

Projected velocities are:
For vertex $\mathbf{0}$: [-0.2505, 4.14036, 3.0128]
For vertex $\mathbf{e}_1$: [-0.8253, 4.5565, 3.4359]
For vertex $\mathbf{e}_3$: [0.1658, 4.0377, 2.8349]
For vertex $\mathbf{e}_1 + \mathbf{e}_3$: [-0.409, 4.4539, 3.258]

norm of v0 = 5.1267
norm of v1 = 5.766

norm of v3 = 4.936
norm of v13 = 5.53
Fastest is e1 vertex. (Just one way of arriving at this conclusion.)

(c) Compute norm of differences between projected vertices. The line segments' lengths are:
For $\mathbf{0}$ - $\mathbf{e}_1$: 1.00003
For $\mathbf{e}_1$ - $\mathbf{e}_3$: 1.414
For $\mathbf{e}_3$ - $(\mathbf{e}_1 + \mathbf{e}_3)$: 1.00003
For $\mathbf{0}$ - $(\mathbf{e}_1 + \mathbf{e}_3)$: 1.4142

(d) Rate of Change = norm(Projected Lengths / Projected Velocities(Relative))
For $\mathbf{0}$ - $\mathbf{e}_1$: 3.7928
For $\mathbf{e}_1$ - $\mathbf{e}_3$: 3.8737
For $\mathbf{e}_3$ - $(\mathbf{e}_1 + \mathbf{e}_3)$: 3.7928
For $\mathbf{0}$ - $(\mathbf{e}_1 + \mathbf{e}_3)$: 11.5387
The final line segment is growing the fastest. (Could be arrived at using a different way. But final observation is the same.)