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COLLEGE PHYSICAL SCIENCES

DEPARTMENT OF MATHEMATICS

MTS 241 (MATHEMATICAL METHOD I)

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MODULE ONE

Real Valued Function of Real Variable

Recall that a function is a rule that establishes the relationship between two or more variables.

If X and Y are an arbitrary sets. For every $x \in X$ there is assigned a unique element $f(x) = y \in Y$, then f is called **a function**. This is written as $f : X \to Y$, where the set X is the **domain of the function** and Y is the **codomain of the function**. The value of f at x is written as $f(x)$ and is called image or range of the function. When the set X is actually a subset of $\mathbb{R}^n$, then we say that $f : \mathbb{R}^n \to Y$ is a function of several variables (even if the dimension n is huge). If the set Y is $\mathbb{R}$, the set of real numbers, then we say that $f : X \to \mathbb{R}$ is a real-valued function. The special cases of real-valued function of a single variable is the primary topic of discussion in calculus I and graphical methods do enhance intuition on the attitude of such functions.

Definition: A function whose range is a set of real numbers is called a real valued function. In other words, it is a function that assigns a real number to each member of its domain.

Pictorial examples of real valued function:

Figure 1: Example of Real valued function.

In the above example, every element in the set D is related to exactly one element in the set $Y \subset \mathbb{R}$

The range $= \{5, 6, 7, 9\}$

The following are said to be real valued function since their range is the set of real numbers or some subset of the real numbers.

(i) $f(x)2$, the range is $\{2\} \subset \mathbb{R}$.

(ii) $f(x) = 11x$, the range is $\mathbb{R}$.

(iii) $f(x) = e^x$, the range is $(0, +\infty) \subset \mathbb{R}$

(iv) $f(x) = \frac{1}{x^2}$, the range is $(0, +\infty) \subset \mathbb{R}$

Now, given the domain of a function, we find the range of the function $g(t) = 6t^2 + 5$ at

(i) $t = 0$, (ii) $t = 2$ and (iii) $t = 3$

Solution

The given function $g(t) = 6t^2 + 5$

At $t = 0, g(0) = 6(0)^2 + 5 = 5$

At $t = 1, g(1) = 6(1)^2 + 5 = 29$

At $t = 3, g(3) = 6(3)^2 + 5 = 59$

Therefore, the domain$= \{0, 2, 3\}$ and the range $= \{5, 29, 59\}$

Types of Functions:

(i) One -to - one (Injective) function: is the type of function that each element in the domain has a distinct image in the co-domain. In another way, a function $f : X \to Y$ is said to be one-to- one or injective if $f(x_1) = f(x_2)$ in Y implies that $x_1 = x_2$ in X or equivalently, $x_1 = x_2$ in X implies that $f(x_1) = f(x_2)$

Figure 2: Example of one –to -one function.

For instance, Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is a set of real numbers) be a function defined by $f(x) = 3x + 1$, is f one -to- one function?

Solution.

$f(x) = f(y)$ implies that $3x + 1 = 3y + 1 = 3x = 3y$ or $x = y$

Hence, f is injective.

(ii) Onto (Surjective)function: is a function that every element in the codomain has at least one pre-image in the domain.

Example : Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is a set of real numbers) be a function defined by $f(x) = 3x + 1$, is f surjective?

Solution.

Let $y \in \mathbb{R}$ be arbitrary. Is there $x \in \mathbb{R}$ such that $3x + 1 = y$

Yes, set $x = \frac{y-1}{3}$, then $3x+1 = \frac{3(y-1)}{3} + 1 = y-1+1 = y$. Hence f is surjective.

(iii) Bijective function: a function $f : X \to Y$ is said to bijective if it is both injective and onto function.

Example: a function defined by $f(x) = 3x+1$ is a bijective function since it is both one to one and onto.

(iv) Many -to- one function: A function is said to be many to one function if there are at least two elements in the domain whose images are the same.

Example : a function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2+1$

Solution.

$f(1) = (1)^2 + 1 = 2$

$f(-1)^2 + 1 = 2$

$f(2)^2 + 1 = 5$

$f(-2)^2 + 1 = 5$ and so on

(v) Constant function: $f : \mathbb{R} \to \mathbb{R}$ defined by $b = f(x) = D.a \in \mathbb{R}$.

Domain is $f = \mathbb{R}$

Range $f = \{D\}$

(vi) Composite function: Let $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ defined by

$f(x) = x^2+1, g(x) = 2x+5$.

Find (i) $(gof)(x)$ (ii) $(gof)(0)$ (iii) $(gof)(2)$ (iv) $(fog)(x)$ (v) $(fog)(3)$

Solution.

(i) $(gof)(x) = g[f(x)] = g[x^2+1] = 2(x^2+1)+5$(by the definition of g)

$\Rightarrow 2x^2+2+5 = 2x^2+7$

(ii) $(gof)(0) = 2 \times 0^2 + 7 = 0+7 = 7$

(iii) $(gof)(2) = 2 \times 2^2 + 7 = +7 = 15$

(iv) $(fog)(x) = f[g[x]] = f[2x+5] = (2x+5)^2 + 1$(by definition of f)

$\Rightarrow 4x^2+20x+25+1 = 4x^2+20x+26$

$(fog)(3) = 4 \times 3^2 + 20 \times 3 + 26 = 36+60+26 = 122$

Exercises

(1). Let X, Y be any two sets, which of the following relations $f : X \to Y$ is a function? When f is a function, determine whether it is injective, surjective or bijective.

(i) $X = \{2,6,8,12\}, Y = \{a,b,c\}.f = \{(2,a),(6,b),(12,c)\}$

(ii) $X = \{a,b,c,d\}, Y = \{5,12,7,10\}.f = \{(a,7),(b,12),(c,10),(d,5)$

(iii) $X = \{6,4,7,11\}, Y = \{a,b,c\}.f = \{(6,a),(4,c),(7,b),(11,c)$

(iv) $X = Y = \mathbb{R}$ (the set of real numbers)$f = \{x, 3x+5 : x \in X$

(v) $X =$ the set of integers, $Y =$the set of positive integers. $f = \{n, n^2 : n \in X\}$

(2). Let $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = x^2,\ if\ x \geq 0\ or\ \frac{1}{2},\ if\ x < 0$

Find (i) $(gof)(0)$ (ii) $(gof)(-1)$, (iii) $(fog)(x)(x > 0)$, (iv)$(gof)(x)(x, 0)$

(3). Let $X = [-1, 1]$ and let $f : X \to \mathbb{R},\ g : X \to \mathbb{R}.\ \ h : X \to \mathbb{R}$ be defined by $f(x) = sinx, g(x) = sin(\pi x), h(x) = logx$

Examine whether

(i) f is injective, surjective

(ii) g is injective, surjective

(iii) h is injective, surjective.

MODULE ONE

Review of Differentiation

The Gradient Function

Consider the curve whose equation is given by $y = f(x)$

Let m be the gradient of the secant line AB, then

$m = \frac{increase\ in\ y}{increase\ in\ x}$

From the idea of gradient of a line, we have that

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Suppose we put $x_2 - x_1 = \triangle x$ ($\triangle\ x \neq 0,$ Since $x_2 \neq x_1$) we have $x_2 = x_1 + \triangle x$

$y_1 = f(x_1),\ y_2 = f(x_2) = f(x_1 + \triangle x)$

Therefore $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$

Let the point A remain fixed, and let the point B move closer to A along the curve. As B moves closer to A along the curve, $\triangle x$ gets closer to zero. $\triangle x$ is called the increment of x, and is read 'δx'. As B approaches A along the curve, the slope of the secant approaches the slope of the tangent at A.

If $\triangle x$ is made to approach zero, the function

$m = \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$ is called the gradient function of $y = f(x)$

The Derivative of a function

Recall that the gradient of the curve $y = f(x)$ is given by

$m_1 = \frac{f(x + \triangle x) - f(x)}{\triangle x}$

The expression m_1 is the gradient of the secant AB. As B moves closer to A, $\triangle x$ becomes smaller and it tends to zero. Let m_2 be the gradient of the tangent at a point A. As B approaches A along the curve, m_1 tends to m_2.

In other words, the gradient of the secant AB tends to the gradient of the tangent to the curve at A. We say that m_2 is the limiting value of m_1 as $\triangle x$ tends to zero. This limiting value is written as

$\lim\limits_{\triangle x \to 0} \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$

and it is denoted simply as $f'(x)$. Thus

$f'(x) = \lim\limits_{\triangle x \to 0} \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$

Consider again the function $y = f(x)$. A small increment in x will cause a small increment in y. If we represent the small increment in x by $\triangle x$, then we can also represent a small increment in y by $\triangle y$. Thus if $y = f(x)$, then $y + \triangle y = f(x + \triangle x)$

As B moves closer to $A, \triangle x \to 0$.

If m_1 is the gradient of the secant AB and m_2 is the gradient of the tangent at A. m_2 is the limiting value of m_1 as $\triangle x \to 0$. $m_2 = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$

The expression $m_2 = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$ is denoted by $\frac{dy}{dx}$, which is read $'dee\ y\ dee\ x'$. $\frac{dy}{dx}$ is the limiting value of the slope of the secant AB as B gets closer to A.

The process of finding the derivative of a function is called **differentiation**. The notations which are commonly used for the derivative of a function are:

$f'(x)$ read $f-$prime of x;

$\frac{d}{dx}f$ read $dee-deex$ of f;

$\frac{df}{dx}$ read $dee-f-by-deex$;

If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$. The notations $\frac{dy}{dx}$ and $f'(x)$ will often be used interchangeably. Sometimes, $\frac{dy}{dx}$ is called the **differential coefficient of** y **with respect to (wrt)** x.

As we proceed, different meanings associated with $\frac{dy}{dx}$ will be unfolded, but for the time being $\frac{dy}{dx}$ is either the gradient function of the curve $y = f(x)$ or the rate of change of y with respect to x.

Differentiation by First Principle

The technique adopted in the last section in finding the derivative of a function from the consideration of the limiting value is called **differentiation from first principle**.

Examples

(i)Find the derivative of $f(x) = x^2$ from the first principle.

Solution

$f(x) = x^2$

$f(x+\triangle x) = (x+\triangle x)^2 = x^2 + 2x\triangle x + (\triangle x)^2$

$f(x+\triangle x) - f(x) = x^2 + 2x\triangle x + (\triangle x)^2 - x^2 = 2x\triangle x + (\triangle x)^2$

Thus

$\frac{f(x+\triangle x)-f(x)}{\triangle x} = 2x + \triangle x$

$\lim\limits_{\triangle x \to 0} \frac{f(x+\triangle x)-f(x)}{\triangle x} = 2x$

Therefore $f'(x) = 2x$.

(ii) Find the derivative of $y = x^3$ from the first principle

Solution.

$y = x^3$

$y + \triangle y = (x+\triangle x)^3 = x^3 + 3x^2\triangle x + 3x(\triangle x)^2 + (\triangle x)^3$

$\triangle y = (x+\triangle x)^3 - y = (x+\triangle x)^3 - x^3$

$= 3x^2 \triangle x + 3x(\triangle x)^2 + (\triangle x)^3$

$\frac{\triangle y}{\triangle x} = \frac{3x^2\triangle x + 3x(\triangle x)^2 + (\triangle x)^3}{\triangle x} = 3x^2 + 3x\triangle x + (\triangle x)^2$

Therefore, $\frac{dy}{dx} = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = 3x^2$

Hence, $\frac{dy}{dx} = 3x^2$.

(iii) Find from the first principle, the derivative of $y = ax + b$, a and b are constants.

Solution.

Given $y = ax + b$

$y + \triangle y = a(x+\triangle x) + b = ax + a\triangle x + b$

$\triangle y = ax + a\triangle x + b - (ax+b)$

$= ax + a\triangle x + b - ax - b = a\triangle x$

Therefore $\frac{\triangle y}{\triangle x} = a$

So that $\frac{dy}{dx} = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = a \Rightarrow \frac{dy}{dx} = a$

(iv) Find the derivative of $y = \frac{1}{x}$

Solution.

$y = \frac{1}{x}$

$y + \triangle y = \frac{1}{x+\triangle x}$

$\triangle y = \frac{1}{x+\triangle x} - y = \frac{1}{x+\triangle x} - \frac{1}{x}$

$= \frac{x-(x+\triangle x)}{(x+\triangle x)x} = \frac{-\triangle x}{x(x+\triangle x)}$

$\lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = -\frac{1}{x^2}$

Hence, $\frac{dy}{dx} = \frac{1}{x^2}$

In what follows, we are going to prove some theorems that gives basic formulas for differentiation.

Theorem 1: If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

Proof. Let $y = x^n$

$y + \triangle y = (x+\triangle x)^n$

$= x^n + nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^n$

$\triangle y = x^n + nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^n - x^n$

$= nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^n$

$\frac{\triangle y}{\triangle x} = nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^{n-1}$

Therefore, $\frac{dy}{dx} = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = nx^{n-1}$

Hence if, $y = x_n$, then $\frac{dy}{dx} = nx^{n-1}$

Examples

(i) $y = ax^n$:

$\frac{dy}{dx} = nax^{n-1}$

(ii) $y = -2x^3$:

$\frac{dy}{dx} = -2.3x^{3-1} = -6x^2$

(iii) $\frac{1}{x^3} = x^{-3}$: $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$

Theorem 2: (The Derivative of a sum) Let f, U and V be differential functions of x such that $f(x) = U(x) + V(x)$ then

$f'(x) = u'(x) + v'(x)$

In other words, If $y = u + v$, where u and v are differential functions of x

then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Hence, the derivative of sum is the sum of the derivatives.

Proof. Let $y = u(x) + v(x)$, where u and v are differentiable functions of x. Then applying first principle rule gives the desired result.

Examples. Find the derivative of each of the following:

(i) $2x^3 - 5x^2 + 2$ (ii) $3x^2 + \frac{1}{x}$ (iii) $\frac{x^3+2x^2+1}{x}$

Solutions.

(i)Let $y = 2x^3 - 5x^2 + 2$

Therefore, $\frac{dy}{dx} = 6x^2 - 10x$

(ii) Let $y = 3x^2 + \frac{1}{x} = 3x^2 + x^{-1}$

$\frac{dy}{dx} = 6x - \frac{1}{x^2}$

(iii) Let $y = \frac{1}{x^3} = x^{-3}$: $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4} = x^2 + 2x + x^{-1}$

$\frac{dy}{dx} = 2x + 2 - \frac{1}{x^2}$

Derivative of function of a function (Composite function).

Theorem 3: Suppose $y = f(u)$ and $u = g(x)$, then

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

The above formula is called **Chain Rule** and the prove is left as Exercise for the reader.

Examples. Find the derivative of each of the following:

(i) $y = (3x^2 - 2)^3$ (ii) $y = \sqrt{(1 - 2x^3)}$ (iii) $y = \frac{5}{(6-x^2)^3}$.

Solutions.

(i) Given $y = (3x^2 - 2)^3$, Let $u = 3x^2 - 2$, then $y = u^3$

$\frac{dy}{du} = 3u^2$, $\frac{du}{dx} = 6x$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 3u^2.6x = 18xu^2$

Substituting for u, we have

$\frac{dy}{dx} = 18x(3x^2-2)^2$

(ii) Given $y = \sqrt{(1-2x^3)}$. Let $u = 1-2x^3$. Then $y = u\frac{1}{2}$

$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$, $\frac{du}{dx} = -6x^2$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2\sqrt{u}}, -6x^2 = \frac{3x^2}{\sqrt{u}}$

Hence, substituting for u, gives

$\frac{dy}{dx} = -\frac{3x^2}{\sqrt{1-2x^3}}$

(iii) $y = \frac{5}{(6-x^2)^3}$.

Let $u = 6-x^2$, Then $y = \frac{5}{u^3} = 5u^{-3}$.

$\frac{dy}{dx} = -15u^{-4}$, $\frac{du}{dx} = -2x$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -15u^{-4}.-2x = 30xu^{-4} = \frac{30x}{u^4}$

Hence, substituting for u, you have

$\frac{dy}{dx} = \frac{30x}{(6-x^2)^4}$

The Derivative of a Product

We shall now consider the derivative of $y = uv$ where u and v are functions of x.

Theorem 4: Suppose $y = uv$ where u and v are functions of x, then

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

The above formula is called **Product Rule**. **Examples:** Find the derivative of the following;

(i) $y = (3+2x)(1-x)$ (ii) $y = (1-2x+3x^2)(4-5x^2)$ (iii) $y = \sqrt{x}(1+2x)^2$

Solutions.

(i) $y = (3+2x)(1-x)$. Let $u = 3+2x,\ v = 1-x$

$\frac{du}{dx} = 2,\ \frac{dv}{dx} = -1$

Therefore, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (3+2x)(-1) + (1-x)(2)$

$= -(3+2x) + 2(1-x)$

$= -3-2x+2-2x$

$\frac{dy}{dx} = -1-4x$

(ii) $y = (1-2x+3x^2)(4-5x^2)$. Let $u = 1-2x+3x^2,\ v = 4-5x^2$

$\frac{du}{dx} = -2+6x,\ \frac{dv}{dx} = -10x$

Therefore

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (1-2x+3x^2)(-10x) + (4-5x^2)(-2+6x)$

$= -10x(1-2x+3x^2) + (6x-2)(4-x^2)$

(iii)$y = \sqrt{x}(1+2x)^2$. Let $u = \sqrt{x},\ v = (1+2x)^2$.

$\frac{du}{dx} = \frac{1}{2\sqrt{x}},\ \frac{dv}{dx} = 4(1+2x)$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= \sqrt{x}.\ 4(1+2x) + (1+2x)^2.\frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = 4\sqrt{x}(1+2x) + \frac{1}{2\sqrt{x}}(1+2x)^2$

The Derivative of a Quotient

Theorem 6: Let $y = \frac{u}{v}$ where u and v are functions of x and $v(x) \neq 0$ for all x, then

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}$

Let $y = \frac{u}{v}$ where u and v are functions of x and $v(x) \neq 0$ for all x

The above formula is called **Quotient Rule**.

Examples. Find the derivative of each of the following;

(i) $\frac{1+x^2}{1-x^2}$ (ii) $\frac{3+2x-x^2}{\sqrt{1+x}}$ (iii) $\frac{2+x}{x^2+2x+7}$

Solutions.

(i) $y = \frac{1+x^2}{1-x^2}$. Let $u = 1+x^2,\ v = 1-x^2$

$\frac{du}{dx} = 2x,\ \frac{dv}{dx} = -2x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}$

$= \frac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2}$

$= \frac{2x-2x^3+2x+2x^3}{(1-x^2)^2}$

$= \frac{4x}{(1-x^2)^2}$

(ii) $y = \frac{3+2x-x^2}{\sqrt{1+x}}$. Let $u = 3+2x-x^2,\ v = (1+x)^{\frac{1}{2}}$.

$\frac{du}{dx} = 2-2x = 2(1-x),\ \frac{dv}{dx} = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2(1+x)^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{(\sqrt{1+x}).2(1-x)-(3+2x-x^2).\frac{1}{2(1+x)^{\frac{1}{2}}}}{1+x}$

$= \frac{4(1+x)(1-x)-(3+2x-x^2)}{2(1+x)(1+x)^{\frac{1}{2}}} = \frac{4-4x^2-3-2x+x^2}{2(1+x)^{\frac{3}{2}}} = \frac{1-2x-3x^2}{2(1+x)^{\frac{3}{2}}}$

(iii) $y = \frac{2+x}{x^2+2x+7}$, Put $u = 2+x,\ v = x^2+2x+7$

$\frac{du}{dx} = 1,\ \frac{dv}{dx} = 2x+2$

$\frac{dy}{dx} = \frac{(x^2+2x+7).1-(2+x)(2x+2)}{(x^2+2x+7)^2}$

$= \frac{x^2+2x+7-2(x+2)(x+1)}{(x^2+2x+7)^2} = \frac{x^2+2x+7-2x^2-6x-4}{(x^2+2x+7)^2} = \frac{-x^2-4x+3}{(x^2+2x+7)^2}$

$\frac{dy}{dx} = \frac{3-4x-x^2}{(x^2+2x+7)^2}$

Derivative of Trigonometric, logarithmic, Exponential and Hyperbolic Functions are summarized in the table below;

S/N	y	$\frac{dy}{dx}$
1	$cosx$	$-sinx$
2	$sinx$	$cosx$
3	$tanx$	sec^2x
4	$secx$	$secx\ tanx$
5	$cosecx$	$-cosecx\ cotx$
6	$cotx$	$-cosec^2x$
7	log_ax	$\frac{1}{x}log_ae$
8	$ln\ x,$	$\frac{1}{x}$
9	a^x	a^xlna
10	e^x	e^x
11	e^{ax}	ae^{ax}
12	$cosh\ x$	$sinh\ x$
13	$sinhx$	$coshx$
14	$tanhx$	$sech^2x$
15	$tan^{-1}x$	$\frac{1}{1+x^2}$
16	$cot^{-1}x$	$\frac{-1}{1+x^2}$
17	$cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$
18	$sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
19	$sec^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$
20	$tanh^{-1}x$	$\frac{1}{1-x^2}$
21	$cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}$
22	$sinh^{-1}x$	$\frac{1}{\sqrt{x^2+1}}$

Table 1.0

Examples: Find the derivative of each of the following;

(a) $cos\ 2x$ (b) $sec\ 6x$ (c) $cosec\ \frac{3}{4}\ x^2$

Solutions.

(a) Given that $y = cos\ 2x$, let $u = 2x, \Rightarrow y = cos\ u$.

Therefore by chain rule,

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -sin\ u.2 = -2sin\ 2x$

(b) Let $y = sin\frac{1}{3}x$

$\frac{dy}{dx} = cos\frac{1}{3}x.\frac{d}{dx}(\frac{1}{3}x) = \frac{1}{3}cos\frac{1}{3}x$

(c) Let $y = cosec\ \frac{3}{4}x^2$. Put $u = \frac{3}{4}$, So $y = cosec\ u$

$\frac{dy}{du} = -cosec\ u.cot\ u$ and $\frac{du}{dx} = \frac{3}{2}x$

$\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = -\frac{3}{2}x\ cot\ \frac{3}{4}x^2.cosec\ \frac{3}{4}x^2.$

(d) $log_a(5x+1)$

Solution

Given that $y = log_a(5x+1)$, let $u = 5x+1$, then $y = log_a u$

$\frac{dy}{du} = \frac{1}{u}log_a e,\ \frac{du}{dx} = 5$

Thus $\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = 5.\frac{1}{u}log_a e = \frac{5}{5x+1}log_a e$

(e) $y = sinh\ 3x$

$\frac{dy}{dx} = cosh\ 3x\frac{d}{dx}(3x) = 3cosh\ 3x$

Application of Differentiation

The differential Calculus is a powerful tool for solving problems or determining the maximum or minimum values of a function. The business man wants to maximize profit and minimize cost, the government determines the tax rate that optimize tax revenue. Hence, the nature determines many maximum or minimum problems.

Definition: Turning points are those points along the graph of the function which have a zero gradient or the points with horizontal tangents to the graph of the functions.

It is where the gradient of the curve vanishes. i.e, where $\frac{dy}{dx} = 0$

Maximum and Minimum Values

Definition: A function f is said to have a local (or relative) maximum at a $x = x_0$, if $f(x0) \geq f(x_0 + h)$ for all positive and negative values of h sufficiently near zero, and it is said to have a local (or relative)minimum at a point $x = x_0$, if $f(x_0) \leq f(x_0 + h)$ for all values of h near zero.

The conditions for the function to attain their minimum or maximum values are as follows

(1) For minimum, $\frac{d^2y}{dx^2} > 0$

(2) For maximum, $\frac{d^2y}{dx^2} < 0$

Example: A monopolist's revenue function is $R = 240x + 57x^2 - x^3$

Find the output level that gives maximum revenue. What is the maximum revenue?

Solution.

$R = 240x + 57x^2 - x^3$

$\frac{dR}{dx} = 240 + 114x - 3x^2$

At turning point $\frac{dR}{dx} = 0$ i.e,$x^2 - 38x - 80 = 0 \Rightarrow (x-40)(x+2) = 0 \Rightarrow x = 40$ or $x = -2$

The output level that gives maximum revenue is 40. Thus

Maximum Revenue $R = 240x + 57x^2 - x^3 = 240(40) + 57(40) - 40^3 = 36,800$

Therefore, the maximum revenue is $36,800$.

Integration

Integration is the opposite of differentiation.

Recall that during differentiation, we multiplied the power by the function and subtract 1 from the power. But in integration, we add 1 to the power and divide the function by the addition of power and 1. i.e.,$\int x^n = \frac{x^{n+1}}{n+1}$

Examples : Evaluate $\int 3x^5, \Rightarrow 3.\frac{x^5+1}{5+1} = 3.\frac{x^6}{6} = \frac{x^6}{2} = \frac{1}{2}x^6$

Definition: A function F is called the **Integral** (anti-derivative) of a function f on an interval I if $F'(x) = f(x)$ for every value of $x \in I$.

Remark: If F is defined by $F(x) = 4x^3 + x^2 + 5$, then $F'(x) = 12x^2 + 2x$.

Thus, if f is the function defined by $f(x) = 12x^2 + 2x$, we say that f is the derivative of F and that F is an integral of f.

In general, if a function F is an integral of a functions f on an interval I and G is defined by

$G(x) = F(x) + C$, where C is an arbitrary constant, then

$G'(x) = F'(x) = f(x)$

and G is also an anti-derivative of f on the interval I. Infact, we shall show below that if F is any particular integral of f on an interval I, then all possible integral of f on I are defined by $F(x) + C$ where C is an arbitrary constant.

We have various integration functions as shown below;

Indefinite Integral: We know that after the integration of an indefinite integral, there is need to add a

constant of integration as follows;

$\int x^n = \frac{x^n+1}{n+1} + C$ where C is a constant

Since a constant can be arbitrarily lost after differentiation, and we know that integration is just an opposite of differentiation.

Lecture 3

Changing variables & Jacobians

3.1 Introduction

The slope of the $f = f(x, y)$ surface depends on the direction in which one moves from the point (x, y), and in Lecture 1 we learnt that the partial derivative must be defined in terms of the change along a particular direction or axis.

For a function $f(x, y)$ we decided that the obvious directions to choose are along the x and y axes, keeping y and x fixed, respectively.

The question now is are these *really* the obvious directions?

For example, consider the function

$$f(x, y) = e^{-(x^2+y^2)} \cos(4(x^2 + y^2))$$

shown in Figure 3.1. Does it really make sense to impose a square "x-constant, y-constant" cake-rack mesh onto this function? Probably not! It would be in more sympathy with the function to use a mesh with radial symmetry as in Figure 3.2.

Figure 3.1: The function $\exp -(x^2 + y^2) \cos(4(x^2 + y^2))$.

Figure 3.2: A radial "dartboard" mesh reflects the symmetry rather better than the x, y constant cake rack.

To achieve this we must make an appropriate **transformation** to a **new set of variables**. This raises the questions of

- How to choose the new variables and thus describe the transformation.
- How to describe the function in the new variables.
- How to find the partial derivatives with respect to these new variables.

3.1.1 The general problem

The more general problem is illustrated in Figure 3.3. Subfigure (a) shows a surface $f(x, y)$ with lines of constant x and y underneath. The partial derivatives wrt x and y are found by slicing or moving along these directions.

Figure 3.3(b) shows the exactly the same surface with lines of constant u and v underneath. The partial derivatives wrt u and v are found by slicing along these new directions.

Figure 3.3: (a) Surface $f(x, y)$ above a set of x-constant and y-constant lines. (b) Surface $F(u, v)$ above a set of u-constant and v-constant curves but plotted in the x, y frame. The surface has exactly the same shape.

3.1.2 Choosing the new variables

There is no routine method for doing this, but usually the problem symmetry drops very large hints as to the best transformation.

Later we will consider some of the most standard transformations, viz between Cartesian and

- Plane Polar Coordinates (2D): for radial symmetry
- Spherical Polar Coordinates (3D): for spherical symmetry, and
- Cylindrical Polars Coordinates (3D): for cylindrical symmetry.

We also consider what would make a bad transformation.

3.2 Rewriting the function and finding the derivatives

First though we should consider the general problems of rewriting the function and finding the partial derivatives with respect to arbitrary new variables (u, v).

There are two main cases to consider:

1. The straightforward case. **OLD variables in terms of NEW**.

$$x = x(u, v) \quad \text{and} \quad y = y(u, v) \quad . \tag{3.1}$$

2. The more complicated case. **NEW variables in terms of OLD**.

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \quad . \tag{3.2}$$

3.2.1 CASES 1A and 1B: OLD variables in terms of NEW

We are transforming from (x, y) to (u, v) coordinates, and the transformation is given as "old in terms of new" variables. That is:

$$x = x(u, v) \quad \text{and} \quad y = y(u, v) \quad . \tag{3.3}$$

Case 1A:
If we know what the function $f(x, y)$ is explicitly, we can replace x and y in the function to get the new function as

$$f = f(x(u, v), y(u, v)) = F(u, v) \quad . \tag{3.4}$$

Then one can work out the partial derivatives with respect to u and v directly from the new function $F(u, v)$.

♣ Example of Case 1A.

Let $f(x, y) = x/y$, and let the transformation be $x = u$ and $y = u/v$.

$$\Rightarrow f = F(u, v) = \frac{u}{(u/v)} = v \quad \text{and} \quad \Rightarrow \frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial v} = 1 \ . \tag{3.5}$$

Case 1B: In this case we don't have an explicit form for $f(x, y)$, and, therefore, cannot find an explicit form for function $F(u, v)$.

But we can still write expressions for the partial derivatives using the Chain Rule for partials from Lecture 2:

Case 1B: Use the Chain Rule for partials to obtain the derivatives

$$\left(\frac{\partial f}{\partial u}\right) = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) \tag{3.6}$$

$$\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right) \tag{3.7}$$

♣ Example of Case 1B.

We are told $f = f(x, y)$, and $x = u$ and $y = u/v$.

Given this transformation, the best we can do is to write for **any** function $f(x, y)$

$$\left(\frac{\partial f}{\partial u}\right) = \left(\frac{\partial f}{\partial x}\right)(1) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{1}{v}\right) = \left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\frac{1}{v} \tag{3.8}$$

$$\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)(0) + \left(\frac{\partial f}{\partial y}\right)\left(-\frac{u}{v^2}\right) = -\left(\frac{\partial f}{\partial y}\right)\frac{u}{v^2} \ . \tag{3.9}$$

To check this result, let's use (as in the example of Case 1A) $f(x, y) = x/y$. Then

$$\left(\frac{\partial f}{\partial u}\right) = \frac{1}{y}(1) + \left(-\frac{x}{y^2}\right)\left(\frac{1}{v}\right) = \left(\frac{v}{u}\right) - \left(\frac{u}{v}\right)\left(\frac{v^2}{u^2}\right) = \left(\frac{v}{u}\right) - \left(\frac{v}{u}\right) = 0 \tag{3.10}$$

$$\left(\frac{\partial f}{\partial v}\right) = \frac{1}{y}(0) + \left(-\frac{x}{y^2}\right)\left(-\frac{u}{v^2}\right) = \frac{u^2 v}{u^2 v} = 1 \tag{3.11}$$

This is exactly as found earlier at Equation 3.5.

3.2.2 A source of confusion introduced, and cleared up

Before continuing, it is worth checking that you are happy that the functions $f()$ and $F()$ have different names. They work differently on their parameters, so should be named differently.

But notice that the chain rule for partials seems to deliver an expression for $\partial f/\partial u$ and $\partial f/\partial v$.

But you might complain that f is a function of (x, y) NOT (u, v), and so we should have written $\partial F/\partial u$ and $\partial F/\partial v$. You are correct. So how are we getting away with it?

We are using f as a "go-between" value $f = f(x, y)$ and $f = F(u, v)$. This is correct because at corresponding values of (x, y) and (u, v) the functions do return exactly the same value. (Look back at Figure 3.3.)

Some books introduce another symbol (z, say) as the "go-between". They write $z = f(x, y)$ and $z = F(u, v)$ and then write

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \ , \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \ , \tag{3.12}$$

and so on — but this seems over-fussy.

3.2.3 CASES 2A and 2B: NEW variables in terms of OLD

We still wish to transform the function $f = f(x, y)$ to (u, v) coordinates. But now the transformation is given as "new in terms of old" variables, that is:

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \ . \tag{3.13}$$

One's first thought is that this should be the better way of defining a transformation, but it turns out to be really inconvenient!

New-ito-Old prevents us from finding either $F(u, v)$ or using the Chain Rule for Partials. We cannot immediately work out $\partial x/\partial u$, etc.

We proceed in one of two ways.

Case 2A: Try to invert the transformation.
Use $u = u(x, y)$ and $v = v(x, y)$ as simultaneous equations from which to find $x = x(u, v)$ and $y = y(u, v)$.
If you can invert, the problem becomes a Case 1 problem.

♣ Example of Case 2A

Suppose you are told that

$$u = x \quad \text{and} \quad v = x/y \ . \tag{3.14}$$

These are new in terms of old. But it is obvious that

$$x = u, \quad \text{and} \quad \Rightarrow v = u/y, \quad \Rightarrow y = u/v \ . \tag{3.15}$$

We now have Old-in-terms-of-New, and can move to Case 1 above.

Case 2B: When the transformation is not invertible Write down the Chain Rule for Partials the "wrong" way round! That is, we want $\partial f/\partial u$, so start with $\partial f/\partial x$.

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \tag{3.16}$$

$$\frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial y} \tag{3.17}$$

Treat these as simultaneous equations in the unknown $\partial f/\partial u$ and $\partial f/\partial v$ and re-arrange. Ie:

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial v}{\partial x}\right) / \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.18}$$

$$\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial u}{\partial y}\right) / \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.19}$$

So we can recover the partial derivatives with respect to the new variables without an explicit expression for the function in the new variables.

3.2.4 SUMMARY

To summarize:

1: If you have the transformation as **OLD variables in terms of NEW** either

- **1A:** find the function explicity and find the partials from it directly, or
- **1B:** find just the partials using the Chain-rule-for-partials directly.

2: If you have the transformation as **NEW variables in terms of OLD** either

- **2A:** invert the given transformation to get OLD variables in terms of NEW, and then go to CASE 1, or
- **2B:** use the Chain-rule-for-partials for the OLD variables to give simultaneous equations for the partial derivatives with respect to to the NEW variables.

♣ Example #1

Q: The elevation of a hill is described by

$$z = z(x, y) = \exp\left\{-(x^2 + y^2)\right\} \ . \tag{3.20}$$

By deriving the function in plane polars, find $\partial z/\partial r$ and $\partial z/\partial \phi$ and comment on their values. Determine the value of r where the hill is steepest.

A: Plane polars are (r, ϕ) where $x = r\cos\phi$, $y = r\sin\phi$.

This is old in terms of new, and we know the function, ⇒CASE 1A.

The hill is therefore

$$z = \exp\left\{-r^2\right\} \ . \tag{3.21}$$

So

$$\frac{\partial z}{\partial r} = -2r\exp\left\{-r^2\right\} \quad \text{and} \quad \frac{\partial z}{\partial \phi} = 0 \ . \tag{3.22}$$

The change of height is all in the radial direction. $\partial z/\partial \phi = 0$ means that if you move at constant r (that is, "round" the hill) you will not change height at all.

The gradient is a function of r alone, so we can find the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}r}\text{Slope} = \frac{\mathrm{d}}{\mathrm{d}r}(-2r\exp\left\{-r^2\right\}) = (-2 + 4r^2)\exp\left\{-r^2\right\} \tag{3.23}$$

This is zero when $r = 1/\sqrt{2}$ and Slope $= -\sqrt{2}\exp\{-1/2\}$.

♣ Example #2.

Here is one that can be solved using all approaches.

Q: Find the partial derivatives $\partial f/\partial u$ and $\partial f/\partial v$ when

$$f = f(x, y) = x^2 - y^2 \quad \text{and} \quad u = (x + y) \ \ v = (x - y) \ . \tag{3.24}$$

A: Transformation is new in terms of old. ⇒CASE 2.

Try CASE 2A: Can we invert the transformation? Yes! Adding then substracting we find

$$x = \frac{1}{2}(u + v) \quad y = \frac{1}{2}(u - v) \ . \tag{3.25}$$

Now goto CASE 1. We know the function, ⇒**CASE 1A.**

$$f = F(u, v) = x^2 - y^2 = \frac{1}{4}\left((u + v)^2 - (u - v)^2\right) = uv \tag{3.26}$$

So the derivatives are

$$\frac{\partial f}{\partial u} = v \quad \text{and} \quad \frac{\partial f}{\partial v} = u \tag{3.27}$$

That's the problem solved. But now notice that if **CASE *A** works, we can always find **just the derivatives** using **CASE *B**.

For example, having inverted the transformation to get

$$x = \frac{1}{2}(u+v) \quad y = \frac{1}{2}(u-v) \quad .$$

we could use CASE 1B to find just the derivatives:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = (2x).\frac{1}{2} + (-2y).\frac{1}{2} = x - y = v \tag{3.28}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = (2x).\frac{1}{2} + (-2y).\frac{-1}{2} = x + y = u \tag{3.29}$$

Now pretend that we did not (or could not) invert the transformation. We could still find the derivative using Case 2B.

Write the chain rule for partials the other way around:

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \tag{3.30}$$

$$\frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial y} \tag{3.31}$$

and for simplicity fill in the values you know

$$2x = \left(\frac{\partial f}{\partial u}\right)(1) + \left(\frac{\partial f}{\partial v}\right)(1) \tag{3.32}$$

$$-2y = \left(\frac{\partial f}{\partial u}\right)(1) + \left(\frac{\partial f}{\partial v}\right)(-1) \tag{3.33}$$

Now use as simultaneous equations for $\partial f/\partial u$ and $\partial f/\partial v$:

$$2x - 2y = 2\left(\frac{\partial f}{\partial u}\right) \quad \Rightarrow \quad \left(\frac{\partial f}{\partial u}\right) = (x-y) = v \quad , \tag{3.34}$$

and subtracting gives, once again, the same results:

$$2x + 2y = 2\left(\frac{\partial f}{\partial v}\right) \quad \Rightarrow \quad \left(\frac{\partial f}{\partial v}\right) = (x+y) = u \quad . \tag{3.35}$$

♣ **Example #3.** Harder!
Q: A quantity $V = V(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \ . \tag{3.36}$$

Express this equation in plane polar coordinates (r, ϕ) where $x = r\cos\phi$, $y = r\sin\phi$.

A: We want first to write down expressions for $\partial V/\partial x$ and $\partial V/\partial y$ which will involve $\partial r/\partial x$, $\partial r/\partial y$, $\partial\phi/\partial x$, and $\partial\phi/\partial y$. This is the wrong way round for the transformation. So

- Either invert the transformation (Case 2A), or
- Write the CRfP the "other way round" (Case 2B) and use sim eqs.

Both will work, but let's use 2B ...

$$\frac{\partial V}{\partial r} = \left(\frac{\partial V}{\partial x}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial V}{\partial y}\right)\frac{\partial y}{\partial r} = \left(\frac{\partial V}{\partial x}\right)\cos\phi \qquad + \left(\frac{\partial V}{\partial y}\right)\sin\phi \tag{3.37}$$

$$\frac{\partial V}{\partial \phi} = \left(\frac{\partial V}{\partial x}\right)\frac{\partial x}{\partial \phi} + \left(\frac{\partial V}{\partial y}\right)\frac{\partial y}{\partial \phi} = \left(\frac{\partial V}{\partial x}\right)(-r\sin\phi) + \left(\frac{\partial V}{\partial y}\right)(r\cos\phi) \tag{3.38}$$

Sorting out the simultaneous equations (do check!) gives

$$\frac{\partial V}{\partial x} = \cos\phi\frac{\partial V}{\partial r} - \frac{\sin\phi}{r}\frac{\partial V}{\partial \phi} \tag{3.39}$$

$$\frac{\partial V}{\partial y} = \sin\phi\frac{\partial V}{\partial r} + \frac{\cos\phi}{r}\frac{\partial V}{\partial \phi} \tag{3.40}$$

We can drop the V and consider $\frac{\partial}{\partial x}$ as an *operator*

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \tag{3.41}$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \tag{3.42}$$

So the operator

$$\frac{\partial^2}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) \tag{3.43}$$

$$= \cos^2\phi\frac{\partial^2}{\partial r^2} - \cos\phi\sin\phi\left[-\frac{1}{r^2}\frac{\partial}{\partial\phi} + \frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right] \tag{3.44}$$

$$- \frac{1}{r}\sin\phi\left[-\sin\phi\frac{\partial}{\partial r} + \cos\phi\frac{\partial^2}{\partial\phi\partial r}\right] + \frac{-\sin\phi}{r}\left[-\frac{\cos\phi}{r}\frac{\partial}{\partial\phi} - \frac{\sin\phi}{r}\frac{\partial^2}{\partial\phi^2}\right]$$

which can be tidied up a little. Note how the operators move through to the right, operating as they go.

Let's follow through a couple of terms. The following is straightforward:

$$\left(\cos\phi\frac{\partial}{\partial r}\right)\left(\cos\phi\frac{\partial}{\partial r}\right) = \cos^2\phi\frac{\partial^2}{\partial r^2} \tag{3.45}$$

but this requires the product rule

$$\left(\cos\phi\frac{\partial}{\partial r}\right)\left(-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) = -\cos\phi\sin\phi\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial\phi} - \cos\phi\sin\phi\left(\frac{1}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \tag{3.46}$$

The tidied expression for y is (you should check!)

$$\frac{\partial^2}{\partial y^2} = \sin\phi^2\frac{\partial^2}{\partial r^2} - \cos\phi\sin\phi\left[\frac{2}{r^2}\frac{\partial}{\partial\phi} - \frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right] \tag{3.47}$$

$$+\frac{1}{r}\cos\phi\left[\cos\phi\frac{\partial}{\partial r} + \sin\phi\frac{\partial^2}{\partial\phi\partial r}\right] + \frac{1}{r^2}\cos\phi^2\frac{\partial^2}{\partial\phi^2} \tag{3.48}$$

Adding them up:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]. \tag{3.49}$$

So Laplace's equation in plane polar coordinates is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} = 0. \tag{3.50}$$

3.3 Jacobians (Jacobian determinants)

Recall the equations that appeared from solving the simultaneous equations in case 2B:

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial v}{\partial x}\right) \Big/ \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.51}$$

$$\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial u}{\partial y}\right) \Big/ \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.52}$$

Notice that the denominator is the same in both expressions. It has special significance, and is called the Jacobian.

It is common to write the Jacobian as a determinant, but there is also another useful notation.

$$J = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} \tag{3.53}$$

As we shall see later, the Jacobian is especially useful when changing variables from (x, y) to (u, v) in multiple integrals.

3.3.1 Multiple definitions of the Jacobian determinant?

In some books one reads that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} \tag{3.54}$$

while in others

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad . \tag{3.55}$$

This is puzzling, until you realize that they are **identical**! However, perhaps the first way is to be preferred for reasons we see now.

3.3.2 Jacobians and their inverses

In examples given later you will see that

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)} \tag{3.56}$$

We shall gain an intuitive understanding of why this must be so when we come to use Jacobians in multiple integration. However it is quite easy to prove using the **Jacobian matrix**. This is not on the syllabus.

The Chain rule for partials says

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \tag{3.57}$$

or, switching the order of the products,

$$\frac{\partial f}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial f}{\partial y} \tag{3.58}$$

We can treat these as operator equations: in steps ...

$$\frac{\partial}{\partial u} f = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} f \qquad \frac{\partial}{\partial v} f = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} f \tag{3.59}$$

$$\Rightarrow \frac{\partial}{\partial u} f \equiv \frac{\partial x}{\partial u}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} f \qquad \frac{\partial}{\partial v} f \equiv \frac{\partial x}{\partial v}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} f \tag{3.60}$$

In matrix notation, this operator is

$$\begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \begin{bmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{bmatrix} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \tag{3.61}$$

The matrix $\boldsymbol{J}_{\frac{x,y}{u,v}}$ is a **Jacobian matrix**.

Now repeat with the Chain Rule around the other way. You will arrive at the following

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \begin{bmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{bmatrix} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \boldsymbol{J}_{\frac{u,v}{x,y}} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} \tag{3.62}$$

But it follows that

$$\begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \boldsymbol{J}_{\frac{u,v}{x,y}} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} \tag{3.63}$$

so that

$$\boldsymbol{J}_{\frac{x,y}{u,v}} \boldsymbol{J}_{\frac{u,v}{x,y}} = \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{J}_{\frac{x,y}{u,v}} = \left[\boldsymbol{J}_{\frac{u,v}{x,y}}\right]^{-1} . \tag{3.64}$$

But a result from linear algebra tells us that, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square, $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$, and hence $|\boldsymbol{A}||\boldsymbol{A}^{-1}| = 1$. The result for Jacobian determinants follows immediately.

3.4 Three standard transformations

There are certain transformations which occur very frequently, viz:
(1) Cartesian to plane polar coordinates;
(2) Cartesian to spherical polar coordinates; and
(3) Cartesian to cylindrical polar coordinates.

3.4.1 Cartesian to plane polars

This is useful for problems with radial symmetry in the plane.

Figure 3.4: Cartesian to plane polars. Many texts use θ rather than ϕ.

The transformation is

$$x = r\cos\phi,\ \ y = r\sin\phi \tag{3.65}$$

$$\Rightarrow \frac{\partial x}{\partial r} = \cos\phi;\ \ \frac{\partial x}{\partial \phi} = -r\sin\phi;\ \ \frac{\partial y}{\partial r} = \sin\phi;\ \ \frac{\partial y}{\partial \phi} = r\cos\phi \tag{3.66}$$

Writing the CRfP in operator form gives

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \tag{3.67}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y}\ . \tag{3.68}$$

Using these are simutaneous equations gives

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\ . \tag{3.69}$$

(You could also find these by inverting the transformation.)

The Jacobian

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = r\cos^2\phi + r\sin^2\phi = r\ . \tag{3.70}$$

and

$$\frac{\partial(r,\phi)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix} = (\cos\phi)(\frac{\cos\phi}{r}) - (\sin\phi)(-\frac{\sin\phi}{r}) = \frac{1}{r}\ . \tag{3.71}$$

3.4.2 3D Cartesian to cylindrical polars

Cylindrical polars add a z-axis, perpendicular to the plane, which is identical to the Cartesian z-axis.

Figure 3.5: Cartesian x, y, z to cylindrical polars r, ϕ, z

The transformation is

$$x = r\cos\phi; \quad y = r\sin\phi; \quad z = z. \tag{3.72}$$

So

$$\frac{\partial}{\partial r} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \tag{3.73}$$

$$\frac{\partial}{\partial \phi} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y} \qquad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{3.74}$$

and using these as simultaneous equations (or inverting the transformation):

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \tag{3.75}$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \qquad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{3.76}$$

The Jacobian is now a 3×3 determinant

$$\frac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -r\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r, \tag{3.77}$$

and (do check!)

$$\frac{\partial(r,\phi,z)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \phi}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \phi}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \frac{1}{r}\ . \tag{3.78}$$

Most entries can be copied over from the plane polar case.

3.4.3 Cartesian to spherical polars

Figure 3.6: Cartesian to spherical polars

The transformation involves the radius r of a sphere $r^2 = x^2 + y^2 + z^2$, the polar angle $0 \leq \theta \leq \pi$ and the azimuthal angle $0 \leq \phi \leq 2\pi$

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi; \quad z = r\cos\theta. \tag{3.79}$$

Hence

$$\frac{\partial}{\partial r} = \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z} \tag{3.80}$$

$$\frac{\partial}{\partial \theta} = r\cos\theta\cos\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z} \tag{3.81}$$

$$\frac{\partial}{\partial \phi} = -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} \tag{3.82}$$

and

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.83}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.84}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{3.85}$$

The Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

(3.86)

and as an exercise you should show that

$$\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \frac{1}{r^2 \sin\theta} \ . \tag{3.87}$$

3.5 A "good" transformation

Cylindrical and spherical polars are particularly useful when solving problems involving cylindrical and spherical regions. But what in general makes a "good" transformation.

If you were asked what is the nicest region shape to fit into Cartesian coordinates Oxy you would answer a rectangle.

Similarly, if you drew your Ouv coordinate system at right angles and asked the same question, you would again answer a square.

So given some arbitrary region shape in the xy plane, the best sort of transformation is one which maps it onto a rectangle in the uv plane, as illustrated in Figure 3.7(a). A specific example is given in Figure 3.7(b) for plane polars. The quadrant's extent is awkard to describe in Cartesians, but easy in describe in plane polars.

Figure 3.7: Useful transformations tend to take an awkward region shape in Oxy into a rectangle in Ouv.

3.5.1 Will any transformation work? When is one bad?

Another question is "will any transformation work?" The answer is no. If we start with two (or n) independent variables (x, y), we must ensure that the new two (or n) are also independent.

The test for independence is rather simple.

If $u(x, y)$ and $v(x, y)$ are functionally INDEPENDENT then

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0. \tag{3.88}$$

Recall that the Jacobian is the denominator in a relationship relating partial derivatives with respect to different sets of variables. Trouble is expected if this is zero.

Proof: For reading only

The proof is straightforward. If $u(x, y)$ and $v(x, y)$ are functionally dependent, then $u = u(v)$, which in turn means that a function $z = z(u, v) = 0$ exists. Differentiating z with respect to x and y gives:

$$\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{3.89}$$

$$\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{3.90}$$

For consistency between these two equations we must have $u_x = \beta u_y$ and $v_x = \beta v_y$, where β is some constant. Thus

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = (u_x v_y - v_x u_y) = (\beta u_y v_y - \beta v_y u_y) = 0 \quad . \tag{3.91}$$

♣ Example

Q: Test whether there is a functional dependence between the variables (u, v) if

$$u = x^2 + y + 1 \quad \text{and} \quad v = x^4 + 2x^2 y + y^2 + x^2 - y. \tag{3.92}$$

A:

$$\begin{aligned} \frac{\partial(u, v)}{\partial(x, y)} &= (u_x v_y - v_x u_y) \\ &= (2x)(2x^2 + 2y - 1) - (4x^3 + 4xy + 2x)(1) \\ &= (4x^3 + 4xy - 2x) - (4x^3 + 4xy + 2x) \\ &= 0 \end{aligned} \tag{3.93}$$

So, yes the variables are not independent. With a little work you'll spot that the dependence is $v = u^2 - 3u + 2$.

12

MULTIPLE INTEGRALS

The first seven sections of this chapter develop the double and triple integral. They depend on Sections 11.1 and 11.2 on surfaces and continuous functions, but are independent of Chapter 10 on vectors.

Sections 8 through 10 of this chapter discuss the relationship between multiple integrals, line integrals, and surface integrals. Chapters 10 on vectors and 11 on partial derivatives are prerequisites.

12.1 DOUBLE INTEGRALS

The *double integral* is the analogue of the *single integral* (definite integral) suggested by Figure 12.1.1. Figure 12.1.1(a) shows the area A bounded by the interval $[a, b]$ and the curve $y = f(x)$, and corresponds to the single integral

$$A = \int_a^b f(x)\,dx.$$

Figure 12.1.1(b) shows the volume V bounded by the plane region D and the surface $z = f(x, y)$, and corresponds to the double integral

$$V = \iint_D f(x, y)\,dx\,dy.$$

Our development of double integrals will be similar to our development of single integrals in Chapter 4. Before going into detail, we give a brief intuitive preview.

Instead of closed intervals $[u, v]$ in the line, we deal with closed regions D in the plane. A *volume function* for $f(x, y)$ is a function B, which assigns a real number $B(D)$ to each closed region D, and has the following two properties: Addition Property and Cylinder Property.

(a) Area (b) Volume

Figure 12.1.1

ADDITION PROPERTY

If D is divided into two regions D_1 and D_2 which meet only on a common boundary curve, then

$$B(D) = B(D_1) + B(D_2).$$

(Intuitively, the volume over D is the sum of the volumes over D_1 and D_2.)

This property is illustrated in Figure 12.1.2(a).

CYLINDER PROPERTY

Let m and M be the minimum and maximum values of $f(x, y)$ on D and let A be the area of D. Then

$$mA \leq B(D) \leq MA.$$

(Intuitively, the volume over D is between the volumes of the cylinders over D of height m and M. This corresponds to the Rectangle Property for single integrals.)

This property is illustrated in Figure 12.1.2(b).

We shall see at the end of this section that the double integral

$$\iint_D f(x, y)\, dx\, dy$$

is the unique volume function for a continuous function $f(x, y)$. The double integral will be constructed using double Riemann sums, just as the single integral was constructed from single Riemann sums.

We now begin the construction of the double integral, starting with a careful discussion of closed regions in the plane.

A *closed region* in the (x, y) plane is a set D of real points (x, y) given by inequalities

$$a_1 \leq x \leq a_2, \qquad b_1(x) \leq y \leq b_2(x),$$

(a) Addition Property (b) Cylinder Property

Figure 12.1.2

where $b_1(x)$ and $b_2(x)$ are continuous and $b_1(x) \leq b_2(x)$ for x in $[a_1, a_2]$ (Figure 12.1.3(a)). The *boundary* of D is the set of points in D which are on the curves

$$x = a_1, \qquad x = a_2, \qquad y = b_1(x), \qquad y = b_2(x).$$

The simplest type of closed region is a *closed rectangle*

$$a_1 \leq x \leq a_2, \qquad b_1 \leq y \leq b_2,$$

shown in Figure 12.1.3(b).

Remark In this course we are restricting our attention to a very simple type of closed region, sometimes called a basic closed region. In advanced calculus and beyond, a much wider class of closed regions is studied.

An *open region* is a set of real points defined by strict inequalities of the form

$$c_1 < x < c_2, \qquad d_1(x) < y < d_2(x).$$

(a) Closed region (b) Closed rectangle

Figure 12.1.3

We shall usually be working with closed regions. So from now on when we use the word *region* alone we mean *closed region.*

To simplify our treatment we shall consider only continuous functions.

PERMANENT ASSUMPTION FOR CHAPTER 12

Whenever we refer to a function $f(x, y)$ and a region D, we assume that $f(x, y)$ is continuous on some open region containing D.

If $f(x, y) \geq 0$ on D, the double integral is intuitively the volume of the solid over D between the surfaces $z = 0$ and $z = f(x, y)$; i.e., the solid consisting of all points (x, y, z) where (x, y) is in D and

$$0 \leq z \leq f(x, y).$$

If $f(x, y) \leq 0$ on D the double integral is intuitively the negative of the volume of the solid under D between the surfaces $z = f(x, y)$ and $z = 0$. Thus volumes above the plane $z = 0$ are counted positively and volumes below $z = 0$ are counted negatively (Figure 12.1.4).

Figure 12.1.4

We now define the double Riemann sum and use it to give a precise definition of the double integral. We first consider the case where D is a rectangle

$$a_1 \leq x \leq a_2, \qquad b_1 \leq y \leq b_2,$$

shown in Figure 12.1.5.

Let Δx and Δy be positive real numbers. We partition the interval $[a_1, a_2]$ into subintervals of length Δx and $[b_1, b_2]$ into subintervals of length Δy. The partition points are

$$x_0 = a_1, \quad x_1 = a_1 + \Delta x, \quad x_2 = a_1 + 2\,\Delta x, \ldots, x_n = a_1 + n\,\Delta x,$$
$$y_0 = b_1, \quad y_1 = b_1 + \Delta y, \quad y_2 = b_1 + 2\,\Delta y, \ldots, y_p = b_1 + p\,\Delta y$$

where $$x_n < a_2 \leq x_n + \Delta x, \qquad y_p < b_2 \leq y_p + \Delta y.$$

Figure 12.1.5

If Δx and Δy do not evenly divide $a_2 - a_1$ and $b_2 - b_1$, there will be little pieces left over at the end. We have partitioned the rectangle D into Δx by Δy subrectangles with partition points

$$(x_k, y_l), \qquad 0 \le k \le n, \qquad 0 \le l \le p,$$

as in Figure 12.1.6.

Figure 12.1.6

The *double Riemann sum* for a rectangle D is the sum

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y = \sum_{k=0}^{n}\sum_{l=0}^{p} f(x_k, y_l)\,\Delta x\,\Delta y.$$

This is the sum of the volume of the rectangular solids with base $\Delta x\,\Delta y$ and height $f(x_k, y_l)$.

As we can see from Figure 12.1.7,

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y$$

approximates the volume of the solid over D between $z = 0$ and $z = f(x, y)$.

Now let D be a general region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x).$$

The *circumscribed rectangle* of D is the rectangle

$$a_1 \le x \le a_2, \qquad B_1 \le y \le B_2,$$

Figure 12.1.7

where B_1 = minimum value of $b_1(x)$,

B_2 = maximum value of $b_2(x)$.

It is shown in Figure 12.1.8.

Figure 12.1.8

Given positive real numbers Δx and Δy, we partition the circumscribed rectangle of D into Δx by Δy subrectangles with partition points

$$(x_k, y_l), \qquad 0 \le k \le n, \qquad 0 \le l \le p.$$

DEFINITION

The ***double Riemann sum*** *over D is defined as the sum of the volumes of the rectangular solids with base $\Delta x\,\Delta y$ and height $f(x_k, y_l)$ corresponding to partition points (x_k, y_l) which belong to D. In symbols,*

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y = \sum_{(x_k, y_l) \text{ in } D}\sum f(x_k, y_l)\,\Delta x\,\Delta y.$$

Notice that in the double Riemann sum over D, we only use partition points (x_k, y_l) which belong to D (Figure 12.1.9).

Figure 12.1.9 Double Riemann Sum

EXAMPLE 1 Find the double Riemann sum

$$\sum_{D_1}\sum x^2 y\,\Delta x\,\Delta y,$$

where D_1 is the square

$$0 \le x \le 1, \qquad 0 \le y \le 1,$$

and

$$\Delta x = \tfrac{1}{4}, \qquad \Delta y = \tfrac{1}{5}.$$

The partition of D_1 is shown in Figure 12.1.10 and the values of x^2y at the partition points are shown in the table.

x^2y	$y_0 = 0$	$y_1 = \frac{1}{5}$	$y_2 = \frac{2}{5}$	$y_3 = \frac{3}{5}$	$y_4 = \frac{4}{5}$
$x_0 = 0$	0	0	0	0	0
$x_1 = \frac{1}{4}$	0	$\frac{1}{80}$	$\frac{2}{80}$	$\frac{3}{80}$	$\frac{4}{80}$
$x_2 = \frac{1}{2}$	0	$\frac{4}{80}$	$\frac{8}{80}$	$\frac{12}{80}$	$\frac{16}{80}$
$x_3 = \frac{3}{4}$	0	$\frac{9}{80}$	$\frac{18}{80}$	$\frac{27}{80}$	$\frac{36}{80}$

The double Riemann sum is

$$\sum_{D_1}\sum x^2 y\,\Delta x\,\Delta y$$
$$= (1 + 2 + 3 + 4 + 4 + 8 + 12 + 16 + 9 + 18 + 27 + 36)\tfrac{1}{80}\cdot\tfrac{1}{4}\cdot\tfrac{1}{5} = 0.0875.$$

A similar computation with $\Delta x = \frac{1}{10}$, $\Delta y = \frac{1}{10}$ gives

$$\sum_{D_1}\sum x^2 y\,\Delta x\,\Delta y = 0.12825.$$

EXAMPLE 2 Find the double Riemann sum

$$\sum\sum_{D_2} x^2 y \, \Delta x \, \Delta y,$$

where D_2 is the region

$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}$$

and

$$\Delta x = \tfrac{1}{4}, \qquad \Delta y = \tfrac{1}{5}.$$

The circumscribed rectangle of D_2 is the unit square. The partition and D_2 are shown in Figure 12.1.11 and the partition points which actually belong to D_2 are circled. The table shows the values of x^2y at the partition points which belong to D_2. It is a part of the table from Example 1.

x^2y	$y_0 = 0$	$y_1 = \frac{1}{5}$	$y_2 = \frac{2}{5}$	$y_3 = \frac{3}{5}$	$y_4 = \frac{4}{5}$
$x_0 = 0$	0				
$x_1 = \frac{1}{4}$		$\frac{1}{80}$	$\frac{2}{80}$		
$x_2 = \frac{1}{2}$			$\frac{8}{80}$	$\frac{12}{80}$	
$x_3 = \frac{3}{4}$				$\frac{27}{80}$	$\frac{36}{80}$

The double Riemann sum is

$$\sum\sum_{D_2} x^2 y \, \Delta x \, \Delta y$$

$$= \left(\frac{1}{80} + \frac{2}{80} + \frac{8}{80} + \frac{12}{80} + \frac{27}{80} + \frac{36}{80}\right) \frac{1}{4} \cdot \frac{1}{5} = \frac{86}{80 \cdot 4 \cdot 5} = 0.05375.$$

A similar computation with $\Delta x = \frac{1}{10}$, $\Delta y = \frac{1}{10}$ gives

$$\sum\sum_{D_2} x^2 y \, \Delta x \, \Delta y = 0.04881.$$

Figure 12.1.10

Figure 12.1.11

Given the function $f(x, y)$ and the region D, the double Riemann sum

$$\sum_D\sum f(x, y)\,\Delta x\,\Delta y$$

is a real function of Δx and Δy. When we replace Δx and Δy by positive infinitesimals dx and dy (Figure 12.1.12), we obtain (by the Function Axiom) the *infinite double Riemann sum*

$$\sum_D\sum f(x, y)\,dx\,dy.$$

The infinite double Riemann sum is in general a hyperreal number. Intuitively, it is equal to the sum of the volumes of infinitely many rectangular solids of infinitesimal base $dx\,dy$ and height $f(x_K, y_L)$. The double integral is defined as the standard part of the infinite double Riemann sum. The following lemma, based on our Permanent Assumption for Chapter 12, shows that this sum has a standard part.

Infinite Double Riemann Sum

Figure 12.1.12

LEMMA

For any positive infinitesimals dx and dy, the double Riemann sum

$$\sum_D\sum f(x, y)\,dx\,dy$$

is a finite hyperreal number and thus has a standard part.

We omit the proof, which is similar to the proof that single Riemann sums are finite. We are now ready to define the double integral.

DEFINITION

Given positive infinitesimals dx and dy, the ***double integral*** *of a continuous function $f(x, y)$ over D is the standard part of the double Riemann sum·*

$$\iint_D f(x, y)\,dx\,dy = st\left(\sum_D\sum f(x, y)\,dx\,dy\right).$$

Here is a list of properties of the double integral. Each property is analogous to a property of the single integral given in Chapter Four and has a similar proof.

INDEPENDENCE OF dx AND dy

The value of the double integral $\iint_D f(x, y)\, dx\, dy$ does not depend on dx and dy. That is, if dx, d_1x, dy, and d_1y are positive infinitesimals then

$$\iint_D f(x, y)\, dx\, dy = \iint_D f(x, y)\, d_1x\, d_1y.$$

This theorem shows that the value of the double integral depends only on the function f and the region D. From now on we shall usually use the simpler notation $dA = dx\, dy$ for the area of an infinitesimal dx by dy rectangle, and

$$\iint_D f(x, y)\, dA \qquad \text{for} \qquad \iint_D f(x, y)\, dx\, dy.$$

ADDITION PROPERTY

Let D be divided into two regions D_1 and D_2 which meet only on a common boundary as in Figure 12.1.13. Then

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA.$$

Figure 12.1.13

Interpreting the double integral as a volume, the Addition Property says that the volume of the solid over D is equal to the sum of the volume over D_1 and the volume over D_2, as shown in Figure 12.1.14.

A continuous function $z = f(x, y)$ always has a minimum and maximum value on a closed region D. The proof is similar to the one-variable case.

Figure 12.1.14 Addition Property

CYLINDER PROPERTY

Let m and M be the minimum and maximum values of $f(x, y)$ on D and let A be the area of D. Then

$$mA \leq \iint_D f(x, y)\, dA \leq MA.$$

This corresponds to the Rectangle Property for single integrals. The solid with base D and constant height m is called the *inscribed cylinder*, and the solid with base D and height M is called the *circumscribed cylinder*. The inscribed cylinder and the circumscribed cylinder are shown in Figure 12.1.15. Intuitively, the volume of a cylinder is equal to the area of the base A times the height. Thus the Cylinder Property states that the volume of the solid is between the volumes of the inscribed and circumscribed cylinders.

Here are two consequences of the Cylinder Property.

Figure 12.1.15 Cylinder Property

COROLLARY 1

The area of D is equal to the double integral of the constant function 1 *over D,* (Figure 12.1.16):

$$A = \iint_D dA.$$

Figure 12.1.16

PROOF Both m and M are equal to 1, so $1 \cdot A \leq \iint_D dA \leq 1 \cdot A$.

COROLLARY 2

If $f(x, y) \geq 0$ *on* D *then* $\iint_D f(x, y)\, dA \geq 0$. *If* $f(x, y) \leq 0$ *on* D *then* $\iint_D f(x, y)\, dA \leq 0$.

To really be sure that the double integral corresponds to the volume, we need to know that it is the only operation that has the Addition and Cylinder Properties. To make this precise, we introduce the notion of a volume function.

We suppose $f(x, y)$ is continuous at every point of an open region D_0, and consider subregions D of D_0. A *volume function* for f is a function B which assigns a real number $B(D)$ to each subregion D of D_0 and has the Addition Property

$$B(D) = B(D_1) + B(D_2)$$

and the Cylinder Property

$$mA \leq B(D) \leq MA,$$

where m is the minimum and M the maximum value of f on D.

UNIQUENESS THEOREM

The double integral $\iint_D f(x, y)\, dA$ *is the only volume function for* f. *That is, if* B *is a function which has the Addition and Cylinder Properties, then*

$$B(D) = \iint_D f(x, y)\, dA \qquad \textit{for every } D.$$

Given a continuous function f such that $f(x, y) \geq 0$ for all (x, y), the function

$$V(D) = \text{volume over } D$$

certainly has the Addition and Cylinder Properties. Thus we are justified in defining the volume as the double integral.

DEFINITION

Let $f(x, y) \geq 0$ for (x, y) in D. Then the volume over D between $z = 0$ and $z = f(x, y)$ is the double integral

$$V = \iint_D f(x, y)\, dA.$$

When $f(x, y)$ is the constant 1, we have

$$A = \iint_D dA = V.$$

That is, the area of D is equal to the volume of the cylinder with base D and height 1, as in Figure 12.1.17.

Given any unit of length (say meters), if the height is one meter then the area is in square meters and the volume has the same value but in cubic meters.

Figure 12.1.17

PROBLEMS FOR SECTION 12.1

Compute the following double Riemann sums.

1 $\sum\sum_D (3x + 4y)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{4}, \quad \Delta y = \frac{1}{4}, \quad D: 0 \leq x \leq 1, \quad 0 \leq y \leq 1$

2 $\sum\sum_D (4 + 2x - 5y)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{2}, \quad \Delta y = \frac{1}{5}, \quad D: -2 \leq x \leq 2, \quad -1 \leq y \leq 1$

3 $\sum\sum_D (x^2 + y^2)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{2}, \quad \Delta y = \frac{1}{2}, \quad D: -2 \leq x \leq 2, \quad -2 \leq y \leq 2$

4 $\sum\sum_D (1 + xy)\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{3}, \quad \Delta y = \frac{1}{5}, \quad D: 0 \leq x \leq 2, \quad 0 \leq y \leq 1$

5 $\sum\sum_D \frac{x}{y}\,\Delta x\,\Delta y, \quad \Delta x = \frac{1}{4}, \quad \Delta y = \frac{1}{5}, \quad D: 1 \leq x \leq 2, \quad 1 \leq y \leq 2$

6 $\sum\sum_D (\cos x + \sin y)\,\Delta x\,\Delta y,\quad \Delta x = \frac{\pi}{6},\quad \Delta y = \frac{\pi}{6},\quad D: -\frac{\pi}{2} \le x \le \frac{\pi}{2},\quad 0 \le y \le \pi$

7 $\sum\sum_D (\cos x \sin y)\,\Delta x\,\Delta y,\quad \Delta x = \frac{\pi}{6},\quad \Delta y = \frac{\pi}{6},\quad D: -\frac{\pi}{2} \le x \le \frac{\pi}{2},\quad 0 \le y \le \pi$

8 $\sum\sum_D xe^y\,\Delta x\,\Delta y,\quad \Delta x = \frac{1}{2},\quad \Delta y = 1,\quad D: 0 \le x \le 2,\quad -2 \le y \le 3$

9 $\sum\sum_D e^{2x-y}\,\Delta x\,\Delta y,\quad \Delta x = 1,\quad \Delta y = 1,\quad D: -2 \le x \le 2,\quad -2 \le y \le 2$

10 $\sum\sum_D (x + 2y)\,\Delta x\,\Delta y,\quad \Delta x = \frac{1}{4},\quad \Delta y = \frac{1}{4},\quad D: 0 \le x \le 1,\quad 0 \le y \le x$

11 $\sum\sum_D (2 + x + 3y)\,\Delta x\,\Delta y,\quad \Delta x = \frac{1}{4},\quad \Delta y = \frac{1}{4},\quad D: 0 \le x \le 1,\quad x \le y \le 1$

12 $\sum\sum_D (x^2 + \sqrt{y})\,\Delta x\,\Delta y,\quad \Delta x = \frac{1}{5},\quad \Delta y = \frac{1}{4},\quad D: -1 \le x \le 1,\quad 0 \le y \le x^2$

13 $\sum\sum_D y \sin x\,\Delta x\,\Delta y,\quad \Delta x = \frac{\pi}{4},\quad \Delta y = \frac{1}{4},\quad D: 0 \le x \le \pi,\quad \sin^2 x \le y \le 2\sin x$

14 $\sum\sum_D (e^x + e^y)\,\Delta x\,\Delta y,\quad \Delta x = 1,\quad \Delta y = 1,\quad D: -3 \le x \le 3,\quad -x \le y \le x$

15 $\sum\sum_D 4\,\Delta x\,\Delta y,\quad \Delta x = 1,\quad \Delta y = 1,\quad D: x^2 + y^2 \le 9$

16 $\sum\sum_D -10\,\Delta x\,\Delta y,\quad \Delta x = 1,\quad \Delta y = 1,\quad D: -3 \le x \le 3,\quad x^2 \le y \le 18 - x^2$

☐ 17 Show that if D is a region with area A and c is constant, then $\iint_D c\,dA = cA$.

☐ 18 Prove the Constant Rule:

$$\sum\sum_D cf(x, y)\,\Delta x\,\Delta y = c\sum\sum_D f(x, y)\,\Delta x\,\Delta y,$$

$$\iint_D cf(x, y)\,dx\,dy = c\iint_D f(x, y)\,dx\,dy.$$

☐ 19 Prove the Sum Rule:

$$\sum\sum_D f(x, y) + g(x, y)\,\Delta x\,\Delta y = \sum\sum_D f(x, y)\,\Delta x\,\Delta y + \sum\sum_D g(x, y)\,\Delta x\,\Delta y,$$

$$\iint_D f(x, y) + g(x, y)\,dx\,dy = \iint_D f(x, y)\,dx\,dy + \iint_D g(x, y)\,dx\,dy.$$

12.2 ITERATED INTEGRALS

In this section we shall learn how to evaluate double integrals. A double integral can be evaluated by two single integrations. The Iterated Integral Theorem gives the key formula.

The *iterated integral*

$$\int_{a_1}^{a_2} \left[\int_{b_1(x)}^{b_2(x)} f(x, y)\,dy \right] dx$$

is an integral of an integral of $f(x, y)$. It is evaluated in two stages. First evaluate the inside integral

$$g(x) = \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy$$

by ordinary definite integration, treating x as a constant. This gives us a function of x alone. Second, evaluate the outside integral

$$\int_{a_1}^{a_2} g(x)\,dx = \int_{a_1}^{a_2} \left[\int_{b_1(x)}^{b_2(x)} f(x, y)\,dy \right] dx$$

by a second definite integration.

We shall usually drop the brackets around the inside integral and write the iterated integral as

$$\int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy\,dx.$$

ITERATED INTEGRAL THEOREM

Let D be a region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x).$$

The double integral over D is equal to the iterated integral:

$$\iint_D f(x, y)\,dA = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy\,dx.$$

Discussion For a fixed x_0, $\int_{b_1(x_0)}^{b_2(x_0)} f(x_0, y)\,dy$ is the area of the cross section shown in Figure 12.2.1. The Iterated Integral Theorem states that the volume is equal to the integral of the areas of the cross sections.

The proof of the Iterated Integral Theorem is given at the end of this section. When using iterated integrals we must be sure that:

(1) $a_1 \le a_2$ and $b_1(x) \le b_2(x)$.
(2) The differentials dx and dy appear in the right order.
(3) The outer integral sign has constant limits.

Figure 12.2.1

While the order of the differentials, $dx\,dy$ or $dy\,dx$, does not matter in a double integral, it is important in an iterated integral. The inside integral sign goes with the inside differential, and is performed first.

$$\int_{a_1}^{a_2} \underbrace{\int_{b_1(x)}^{b_2(x)} f(x, y)\,dy}_{\text{do first}}\,dx$$

do second

When the region D is a rectangle, there are two possible orders of integration, because all the boundaries are constant. Thus there are two different iterated integrals over a rectangle. Integrating first with respect to y we have

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\,dy\,dx,$$

and integrating first with respect to x we have

$$\int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y)\,dx\,dy.$$

Using the Iterated Integral Theorem twice, we see that both iterated integrals must equal the double integral.

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\,dy\,dx = \iint_D f(x, y)\,dA,$$

$$\int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y)\,dx\,dy = \iint_D f(x, y)\,dA.$$

Therefore the two iterated integrals are equal to each other. We have proved a corollary.

COROLLARY

The two iterated integrals over a rectangle are equal:

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\,dy\,dx = \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y)\,dx\,dy.$$

Discussion This corollary is the simplest form of a result known as Fubini's Theorem. Remember that by our Permanent Assumption, $f(x, y)$ is continuous on D. For an idea of the difficulties that arise when $f(x, y)$ is not assumed to be continuous, see Problem 49 at the end of this section.

There are also other regions besides rectangles over which we can integrate in either of two orders, such as Example 5 in this section.

In the following two examples we evaluate the double integrals which were approximated by double Riemann sums in the preceding section.

EXAMPLE 1 Evaluate
$$\iint_{D_1} x^2 y \, dA$$
where D_1 is the unit square
$$0 \le x \le 1, \qquad 0 \le y \le 1.$$
The limits of the outside integral are given by $0 \le x \le 1$, and those of the inside integral are given by $0 \le y \le 1$. The iterated integral is thus
$$\iint_{D_1} x^2 y \, dA = \int_0^1 \int_0^1 x^2 y \, dy \, dx.$$
The inside integral is
$$\int_0^1 x^2 y \, dy = \tfrac{1}{2} x^2 y^2 \Big]_{y=0}^{y=1} = \tfrac{1}{2} x^2.$$
Then
$$\iint_{D_1} x^2 y \, dA = \int_0^1 \tfrac{1}{2} x^2 \, dx = \tfrac{1}{6} x^3 \Big]_0^1 = \tfrac{1}{6} \sim 0.16667.$$
Since D_1 is a rectangle we may also integrate in the other order, and should get the same answer.
$$\iint_{D_1} x^2 y \, dA = \int_0^1 \int_0^1 x^2 y \, dx \, dy.$$
$$\int_0^1 x^2 y \, dx = \tfrac{1}{3} x^3 y \Big]_0^1 = \tfrac{1}{3} y.$$
$$\iint_{D_1} x^2 y \, dA = \int_0^1 \tfrac{1}{3} y \, dy = \tfrac{1}{6} y^2 \Big]_0^1 = \tfrac{1}{6} \sim 0.16667.$$
The Riemann sums in Section 12.1 were 0.0875, 0.12825.

EXAMPLE 2 Evaluate $\iint_{D_2} x^2 y \, dA$ where D_2 is the region in Figure 12.2.2:
$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}.$$
The limits on the outside integral are given by $0 \le x \le 1$, and those on the inside integral by $x^2 \le y \le \sqrt{x}$, so the iterated integral is

Figure 12.2.2

$$\iint_{D_2} x^2 y\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y\, dy\, dx.$$

$$\int_{x^2}^{\sqrt{x}} x^2 y\, dy = \tfrac{1}{2}x^2 y^2 \Big]_{y=x^2}^{y=\sqrt{x}} = \tfrac{1}{2}x^3 - \tfrac{1}{2}x^6.$$

$$\iint_{D_2} x^2 y\, dA = \int_0^1 (\tfrac{1}{2}x^3 - \tfrac{1}{2}x^6)\, dx = \tfrac{1}{8}x^4 - \tfrac{1}{14}x^7 \Big]_0^1$$

$$= \tfrac{3}{56} \sim 0.05357.$$

The Riemann sums in Section 12.1 were 0.05375, 0.04881.

In many applications the region D is given verbally, and part of the problem is to find inequalities which describe D.

EXAMPLE 3 Let D be the region bounded by the curve $xy = 1$ and the line $y = \frac{5}{2} - x$. Find inequalities which describe D, and write down an iterated integral equal to $\iint_D f(x, y)\, dA$.

Step 1 Sketch the region D as in Figure 12.2.3.

Figure 12.2.3

Step 2 The line and curve intersect where

$$x(\tfrac{5}{2} - x) = 1,$$
$$x^2 - \tfrac{5}{2}x + 1 = 0,$$
$$(x - \tfrac{1}{2})(x - 2) = 0,$$
$$x = \tfrac{1}{2}, \qquad x = 2.$$

For $1/2 \le x \le 2$, the curve $y = 1/x$ is below the line $y = 5/2 - x$. Therefore D is the region

$$\tfrac{1}{2} \le x \le 2, \qquad 1/x \le y \le \tfrac{5}{2} - x.$$

Step 3 The inequalities for x give the limits of the outside integral, and those for y give the limits of the inside integral. Thus

$$\iint_D f(x, y)\, dA = \int_{1/2}^{2} \int_{1/x}^{(5/2)-x} f(x, y)\, dy\, dx.$$

EXAMPLE 4 Find the volume of the solid bounded by the surfaces $z = 0, z = y - x^2$, $y = 1$.

Step 1 Sketch the solid and the region D, as in Figure 12.2.4.

Figure 12.2.4

Step 2 Find the inequalities describing the region D.

This is the hardest step, and gives us the limits of integration. The surfaces $z = 0$ and $z = y - x^2$ intersect at the curve $y = x^2$. We see from the figure that D is the region between the curves $y = x^2$ and $y = 1$, so D is given by

$$-1 \le x \le 1, \qquad x^2 \le y \le 1.$$

Step 3 Set up the iterated integral and evaluate it.

$$V = \iint_D y - x^2 \, dA = \int_{-1}^{1} \int_{x^2}^{1} y - x^2 \, dy \, dx.$$

$$\begin{aligned}\int_{x^2}^{1} y - x^2 \, dy &= \tfrac{1}{2}y^2 - x^2 y \Big]_{x^2}^{1} \\ &= (\tfrac{1}{2} \cdot 1^2 - x^2 \cdot 1) - (\tfrac{1}{2}(x^2)^2 - x^2 \cdot x^2) \\ &= \tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4.\end{aligned}$$

$$V = \int_{-1}^{1} \tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4 \, dx = \tfrac{16}{30}.$$

Multiple integration problems can be solved by a three-step process as shown in Examples 3 and 4.

Step 1 Sketch the problem.

Step 2 Find the inequalities describing the region D.

Step 3 Set up the iterated integral and evaluate.

We can also integrate over a region in the (y, x) plane instead of the (x, y) plane. A region D in the (y, x) plane has the form

$$b_1 \le y \le b_2, \qquad a_1(y) \le x \le a_2(y),$$

as shown in Figure 12.2.5.

The double integral over D is equal to the iterated integral with dy on the outside and dx inside,

Figure 12.2.5

$$\iint_D f(x, y)\, dA = \int_{b_1}^{b_2} \int_{a_1(y)}^{a_2(y)} f(x, y)\, dx\, dy.$$

Some regions, such as rectangles and ellipses, may be regarded as regions in either the (x, y) plane or the (y, x) plane (Figure 12.2.6).

(a) D as an (x, y) Region (b) D as a (y, x) Region

Figure 12.2.6

EXAMPLE 5 Let D be the region bounded by the curves

$$x = y^2, \qquad x = y + 2.$$

Evaluate the double integral $\iint_D xy\, dA$.

Step 1 The region D is sketched in Figure 12.2.7.

Figure 12.2.7

Step 2 Find inequalities for D. To do this we must find the points where the curves

$$x = y^2, \qquad x = y + 2$$

intersect. Solving for y and then x, we see that they intersect at

$$(1, -1), \qquad (4, 2).$$

We see from the figure that D is a region in either the (x, y) plane or the (y, x) plane. However, the boundary curves are simpler in the (y, x) plane. D is the region

$$-1 \le y \le 2, \qquad y^2 \le x \le y + 2.$$

Step 3 Set up the iterated integral and evaluate.

$$\iint_D xy\,dA = \int_{-1}^{2}\int_{y^2}^{y+2} xy\,dx\,dy.$$

$$\begin{aligned}\int_{y^2}^{y+2} xy\,dx &= \tfrac{1}{2}x^2y\Big]_{y^2}^{y+2}\\ &= \tfrac{1}{2}(y+2)^2y - \tfrac{1}{2}(y^2)^2y\\ &= \tfrac{1}{2}y^3 + 2y^2 + 2y - \tfrac{1}{2}y^5.\end{aligned}$$

$$\iint_D xy\,dA = \int_{-1}^{2} \tfrac{1}{2}y^3 + 2y^2 + 2y - \tfrac{1}{2}y^5\,dy = \tfrac{135}{24}.$$

PROOF OF THE ITERATED INTEGRAL THEOREM For any region D, let $B(D)$ be the iterated integral over D. Our plan is to prove that B has the Addition and Cylinder Properties, so that by the Uniqueness Theorem $B(D)$ will equal the double integral.

PROOF OF ADDITION PROPERTY

Case 1 Let D be divided into D_1 and D_2 as in Figure 12.2.8(a). By the Addition Property for single integrals,

$$\begin{aligned}B(D) &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx\\ &= \int_{a_1}^{a_3}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx + \int_{a_3}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx\\ &= B(D_1) + B(D_2).\end{aligned}$$

Case 2 Let D be divided into D_1 and D_2 as in Figure 12.2.8(b). Then

$$\begin{aligned}B(D) &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx\\ &= \int_{a_1}^{a_2}\left[\int_{b_1(x)}^{b_3(x)} f\,dy + \int_{b_3(x)}^{b_2(x)} f\,dy\right]dx\\ &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_3(x)} f\,dy\,dx + \int_{a_1}^{a_2}\int_{b_3(x)}^{b_2(x)} f\,dy\,dx\\ &= B(D_1) + B(D_2).\end{aligned}$$

Figure 12.2.8

PROOF OF CYLINDER PROPERTY Let m be the minimum value and M the maximum value of $f(x, y)$ on D For each fixed value of x,

$$\int_{b_1(x)}^{b_2(x)} m\,dy \le \int_{b_1(x)}^{b_2(x)} f(x, y)\,dy.$$

Integrating from a_1 to a_2,

$$\int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} m\,dy\,dx \le \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f\,dy\,dx = B(D).$$

But
$$\int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} m\,dy\,dx = \int_{a_1}^{a_2} m(b_2(x) - b_1(x))\,dx$$
$$= m\int_{a_1}^{a_2} (b_2(x) - b_1(x))\,dx = mA.$$

Therefore $\quad mA \le B(D).$

By a similar argument, $\quad B(D) \le MA.$

Since B has both the Addition and Cylinder Properties,

$$B(D) = \iint_D f(x, y)\,dA.$$

The Constant, Sum, and Inequality Rules for double integrals follow easily from the corresponding rules for single integrals, using the Iterated Integral Theorem.

CONSTANT RULE

$$\iint_D cf(x, y)\,dA = c\iint_D f(x, y)\,dA.$$

SUM RULE

$$\iint_D f(x, y) + g(x, y)\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA.$$

INEQUALITY RULE

If $f(x, y) \leq g(x, y)$ *for all* (x, y) *in* D,

$$\iint_D f(x, y)\, dA \leq \iint_D g(x, y)\, dA.$$

PROOF As an illustration we prove the Sum Rule.

$$\begin{aligned}
\iint_D f + g\, dA &= \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} f + g\, dy\, dx \\
&= \int_{a_1}^{a_2} \left[\int_{b_1(x)}^{b_2(x)} f\, dy + \int_{b_1(x)}^{b_2(x)} g\, dy \right] dx \\
&= \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} f\, dy\, dx + \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} g\, dy\, dx \\
&= \iint_D f\, dA + \iint_D g\, dA.
\end{aligned}$$

The Iterated Integral Theorem gives another proof that the area of D is equal to the double integral of 1 over D.

By definition of area between two curves,

$$A = \int_{a_1}^{a_2} (b_2(x) - b_1(x))\, dx.$$

Using iterated integrals,

$$\begin{aligned}
\iint_D dA &= \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} dy\, dx \\
&= \int_{a_1}^{a_2} (b_2(x) - b_1(x))\, dx = A.
\end{aligned}$$

PROBLEMS FOR SECTION 12.2

In Problems 1–16, evaluate the double integrals (compare these with the problems from Section 12.1).

1 $\displaystyle\iint_D (3x + 4y)\, dA, \quad D: 0 \leq x \leq 1, 0 \leq y \leq 1$

2 $\displaystyle\iint_D (4 + 2x - 5y)\, dA, \quad D: -2 \leq x \leq 2, -1 \leq y \leq 1$

3 $\displaystyle\iint_D (x^2 + y^2)\, dA, \quad D: -2 \leq x \leq 2, -2 \leq y \leq 2$

4 $\displaystyle\iint_D (1 + xy)\, dA, \quad D: 0 \leq x \leq 2, 0 \leq y \leq 1$

5 $\displaystyle\iint_D \frac{x}{y}\,dA, \quad D: 1 \le x \le 2,\ 1 \le y \le 2$

6 $\displaystyle\iint_D (\cos x + \sin y)\,dA, \quad D: -\pi/2 \le x \le \pi/2,\ 0 \le y \le \pi$

7 $\displaystyle\iint_D (\cos x \sin y)\,dA, \quad D: -\pi/2 \le x \le \pi/2,\ 0 \le y \le \pi$

8 $\displaystyle\iint_D xe^y\,dA, \quad D: 0 \le x \le 2,\ -2 \le y \le 3$

9 $\displaystyle\iint_D e^{2x-y}\,dA, \quad D: -2 \le x \le 2,\ -2 \le y \le 2$

10 $\displaystyle\iint_D (x + 2y)\,dA, \quad D: 0 \le x \le 1,\ 0 \le y \le x$

11 $\displaystyle\iint_D (2 + x + 3y)\,dA, \quad D: 0 \le x \le 1,\ x \le y \le 1$

12 $\displaystyle\iint_D (x^2 + \sqrt{y})\,dA, \quad D: -1 \le x \le 1,\ 0 \le y \le x^2$

13 $\displaystyle\iint_D y \sin x\,dA, \quad D: 0 \le x \le \pi,\ \sin^2 x \le y \le 2\sin x$

14 $\displaystyle\iint_D (e^x + e^y)\,dA, \quad D: -3 \le x \le 3,\ -x \le y \le x$

15 $\displaystyle\iint_D 4\,dA, \quad D: x^2 + y^2 \le 9$

16 $\displaystyle\iint_D -10\,dA, \quad D: -3 \le x \le 3,\ x^2 \le y \le 18 - x^2$

In Problems 17–24, evaluate the iterated integral. Then check your answer, by evaluating in the other order.

17 $\displaystyle\int_0^1\int_0^1 (x^2y - 3xy^2 + 5)\,dy\,dx$

18 $\displaystyle\int_0^1\int_0^1 xy(2y + 1)\,dy\,dx$

19 $\displaystyle\int_3^6\int_{-2}^8 dy\,dx$

20 $\displaystyle\int_2^4\int_1^6 3x\,dy\,dx$

21 $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} \sin(x + y)\,dy\,dx$

22 $\displaystyle\int_{-1}^1\int_0^2 \frac{y}{1 + x^2}\,dy\,dx$

23 $\displaystyle\int_0^3\int_1^6 \sqrt{x + y}\,dy\,dx$

24 $\displaystyle\int_1^2\int_0^1 \frac{1}{x + y}\,dy\,dx$

In Problems 25–30 evaluate the iterated integral.

25 $\displaystyle\int_0^1\int_0^{e^x} dy\,dx$

26 $\displaystyle\int_0^{\pi}\int_{\sin x}^1 dy\,dx$

27 $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} y\,dy\,dx$

28 $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2\,dy\,dx$

29 $\displaystyle\int_0^3\int_{y^2}^{3y} x^2y\,dx\,dy$

30 $\displaystyle\int_0^1\int_0^y \frac{2}{\sqrt{1-x^2}}\,dx\,dy$

In Problems 31–38, find inequalities which describe the given region D, and write down an iterated integral equal to $\iint_D f(x, y)\,dA$.

31 The triangle with vertices (0, 0), (5, 0), (0, 5).

32 The triangle with vertices (1, −2), (1, 4), (5, 0).

33 The circle of radius 2 with center at the origin.

34 The bottom half of the circle of radius 1 with center at (2, 3).

35 The region bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$.

36 The region above the parabola $y = x^2$ and inside the circle $x^2 + y^2 = 1$.

37 The region bounded by the curves $x = \frac{1}{2}$ and $x = 1/(1 + y^2)$.

38 The region bounded by the curves $x = 12 + y^2$ and $x = y^4$.

39 Find the volume of the solid over the region $x^2 + y^2 \le 1$ and between the surfaces $z = 0$, $z = x^2$.

40 Find the volume of the solid over the region

$$D: 1 \le x \le 2,\ x \le y \le x^2$$

and between the surfaces $z = 0$, $z = y/x$.

41 Find the volume of the solid between the surfaces $z = 0$, $z = 2 + 3x - y$, over the region $0 \le x \le 2$, $0 \le y \le x$.

42 Find the volume of the solid between the surfaces $z = 0$, $z = \sqrt{y - x}$, over the region $0 \le x \le 1$, $x \le y \le 1$.

43 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

44 Find the volume of the solid bounded by the three coordinate planes and the plane $ax + by + cz = 1$, where a, b, and c are positive.

□ 45 Show that

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x)\,dy\,dx = (b_2 - b_1)\int_{a_1}^{a_2} f(x)\,dx.$$

□ 46 Show that

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x) + g(y)\,dy\,dx = (b_2 - b_1)\int_{a_1}^{a_2} f(x)\,dx + (a_2 - a_1)\int_{b_1}^{b_2} g(y)\,dy.$$

□ 47 Show that

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x)g(y)\,dy\,dx = \left(\int_{a_1}^{a_2} f(x)\,dx\right)\left(\int_{b_1}^{b_2} g(y)\,dy\right).$$

□ 48 Show that

$$\int_a^b\int_{-g(x)}^{g(x)} y\,dy\,dx = 0.$$

□ 49 Let

$$f(x, y) = \begin{cases} y & \text{if } x \text{ is rational,} \\ 1 - y & \text{if } x \text{ is irrational.} \end{cases}$$

Show that:

(a)
$$\int_0^1\int_0^1 f(x, y)\,dy\,dx = \int_0^1 \tfrac{1}{2}\,dx = \tfrac{1}{2}.$$

(b) For each constant $y_0 \ne \frac{1}{2}$, the function $g(x) = f(x, y_0)$ is everywhere discontinuous, so that the iterated integral $\int_0^1\int_0^1 f(x, y)\,dx\,dy$ is undefined.

12.3 INFINITE SUM THEOREM AND VOLUME

The double integral, like the single integral, has a number of applications to geometry and physics. The basic theorem which justifies these applications is the Infinite Sum Theorem. It shows how to get an integration formula by considering an infinitely small element of area.

An *element of area* is a rectangle ΔD whose sides are infinitesimal and parallel to the x and y axes. Given an element of area ΔD, we let

$$\begin{aligned}(x, y) &= \text{lower left corner of } \Delta D,\\ \Delta x, \Delta y &= \text{dimensions of } \Delta D,\\ \Delta A &= \Delta x\, \Delta y = \text{area of } \Delta D.\end{aligned}$$

ΔD is illustrated in Figure 12.3.1.

An element of area

Figure 12.3.1

INFINITE SUM THEOREM

Let $h(x, y)$ be continuous on an open region D_0 and let B be a function which assigns a real number $B(D)$ to each region D contained in D_0. Assume that

(i) *B has the Addition Property $B(D) = B(D_1) + B(D_2)$.*

(ii) *$B(D) \geq 0$ for every D.*

(iii) *For every element of area ΔD, $B(\Delta D) \approx h(x, y)\,\Delta A$ (compared to ΔA).*

Then
$$B(D) = \iint_D h(x, y)\, dA.$$

We shall use the notation

$$\Delta B = B(\Delta D).$$

Given (i) and (ii), the theorem shows that if we always have

$$\Delta B \approx h(x, y)\,\Delta A \qquad \text{(compared to } \Delta A)$$

then
$$B(D) \approx \sum\sum_D h(x, y)\,\Delta A.$$

The proof is simplest in the case that D is a rectangle.

PROOF WHEN D IS A RECTANGLE Choose positive infinitesimal Δx and Δy and partition D into elements of area ΔD (Figure 12.3.2). Since B has the Addition Property, $B(D)$ is the sum of the ΔB's. Let c be any positive real number. For each ΔD we have

$$\Delta B \approx h(x, y)\,\Delta A \qquad \text{(compared to } \Delta A\text{)},$$

$$\frac{\Delta B}{\Delta A} \approx h(x, y),$$

$$\frac{\Delta B}{\Delta A} - c < h(x, y) < \frac{\Delta B}{\Delta A} + c,$$

$$\Delta B - c\,\Delta A < h(x, y)\,\Delta A < \Delta B + c\,\Delta A.$$

Letting A be the area of D and adding up,

$$B(D) - cA < \sum_D\sum h(x, y)\,\Delta A < B(D) + cA.$$

Taking standard parts,

$$B(D) - cA \le \iint_D h(x, y)\,dA \le B(D) + cA,$$

so

$$B(D) = \iint_D h(x, y)\,dA.$$

Figure 12.3.2

The proof in the general case is similar except that some of the elements of area ΔD will overlap the boundary of D and thus be only partly within D. (See Figure 12.3.3.) The method of proof is to change D to include all instead of part of each ΔD, use hypothesis (ii) to show that the new $B(D)$ is infinitely close to the old one, and then show as above that the new $B(D)$ is infinitely close to the double integral $\iint_D h(x, y)\,dA$.

In most applications of the Infinite Sum Theorem, hypotheses (i) and (ii) are automatic. To get a formula for $B(D)$ in practice, we take an element of area ΔD and

Figure 12.3.3

find an $h(x, y)$ such that

$$\Delta B \approx h(x, y)\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

Our first application is to the volume between two surfaces.

DEFINITION

Let $f(x, y) \leq g(x, y)$ for (x, y) in D and let E be the set of all points in space such that

$$(x, y) \text{ is in } D, \qquad f(x, y) \leq z \leq g(x, y).$$

The ***volume*** *of E is*

$$V = \iint_D g(x, y) - f(x, y)\, dA.$$

V is called the volume over D between the surfaces $z = f(x, y)$ and $z = g(x, y)$ (Figure 12.3.4).

Figure 12.3.4 Volume between two surfaces

JUSTIFICATION The part ΔE of the solid E over an element of area ΔD is a rectangular solid with base ΔA and height $g(x, y) - f(x, y)$, except that the top and bottom surfaces are curved (Figure 12.3.5). Therefore the volume of ΔE is

$$\Delta V \approx (g(x, y) - f(x, y))\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

By the Infinite Sum Theorem,

$$V = \iint_D g(x, y) - f(x, y)\,dA.$$

Figure 12.3.5

EXAMPLE 1 Find the volume of the solid

$$0 \le x \le 1, \quad 0 \le y \le x, \qquad x + y \le z \le e^{x+y}.$$

Step 1 D is the triangle shown in Figure 12.3.6.

Step 2 D is the region $0 \le x \le 1, 0 \le y \le x$.

Figure 12.3.6

Step 3

$$V = \iint_D e^{x+y} - (x + y)\, dA$$

$$= \int_0^1 \int_0^x e^{x+y} - (x + y)\, dy\, dx.$$

$$\int_0^x e^{x+y} - (x + y)\, dy = e^{x+y} - xy - \tfrac{1}{2}y^2 \Big]_0^x = e^{2x} - e^x - \tfrac{3}{2}x^2.$$

$$V = \int_0^1 e^{2x} - e^x - \tfrac{3}{2}x^2\, dx = \tfrac{1}{2}e^2 - e.$$

EXAMPLE 2 Find the volume of the solid bounded by the four planes

$$x = 0, \qquad y = 0, \qquad z = x + y, \qquad z = 1 - x - y.$$

Step 1 Sketch the planes. We see from Figure 12.3.7 that $z = x + y$ is below $z = 1 - x - y$.

Figure 12.3.7

Step 2 Find inequalities for the region D. Since the two planes

$$z = x + y, \qquad z = 1 - x - y$$

meet at the line $\quad 2x + 2y = 1, \qquad y = \tfrac{1}{2} - x,$

D is the region $\quad 0 \le x \le \tfrac{1}{2}, \qquad 0 \le y \le \tfrac{1}{2} - x.$

Step 3

$$V = \iint_D (1 - x - y) - (x + y)\, dA = \iint_D 1 - 2x - 2y\, dA$$

$$= \int_0^{1/2} \int_0^{1/2 - x} 1 - 2x - 2y\, dy\, dx.$$

$$\int_0^{1/2-x} 1 - 2x - 2y\,dy = y - 2xy - y^2\Big]_0^{1/2-x}$$
$$= \tfrac{1}{2} - x - 2x(\tfrac{1}{2} - x) - (\tfrac{1}{2} - x)^2 = \tfrac{1}{4} - x + x^2.$$
$$V = \int_0^{1/2} \tfrac{1}{4} - x + x^2\,dx = \tfrac{1}{24}.$$

EXAMPLE 3 Find the volume of the solid bounded by the plane $z = 2y$ and the paraboloid $z = 1 - 2x^2 - y^2$.

Step 1 The surfaces and the region D are sketched in Figure 12.3.8.

Figure 12.3.8

Step 2 The two surfaces intersect on the curve

$$2y = 1 - 2x^2 - y^2,$$

or solving for y, $\qquad y = -1 \pm \sqrt{2 - 2x^2}.$

Therefore D is the region

$$-1 \le x \le 1, \qquad -1 - \sqrt{2 - 2x^2} \le y \le -1 + \sqrt{2 - 2x^2}.$$

Step 3 We see from the figure that the plane is the lower surface and the paraboloid is the upper surface.

$$V = \iint_D (1 - 2x^2 - y^2) - 2y\,dA$$
$$= \int_{-1}^{1}\int_{-1-\sqrt{2-2x^2}}^{-1+\sqrt{2-2x^2}} (1 - 2x^2 - y^2 - 2y)\,dy\,dx.$$

$$\int_{-1-\sqrt{2-2x^2}}^{-1+\sqrt{2-2x^2}} (1 - 2x^2 - y^2 - 2y)\,dy = \int_{-\sqrt{2-2x^2}}^{\sqrt{2-2x^2}} (2 - 2x^2 - u^2)\,du$$

$$= \frac{8\sqrt{2}}{3}(1 - x^2)^{3/2}.$$

$$V = \int_{-1}^{1} \frac{8\sqrt{2}}{3}(1 - x^2)^{3/2}\,dx.$$

Put $x = \sin\theta$, $\sqrt{1 - x^2} = \cos\theta$, $dx = \cos\theta\,d\theta$ (Figure 12.3.9).

Figure 12.3.9

$$V = \frac{8\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta$$

$$= \frac{8\sqrt{2}}{3}\left(\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\right)\Bigg]_{-\pi/2}^{\pi/2}$$

$$= \frac{8\sqrt{2}}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi = \sqrt{2}\pi.$$

Answer $V = \sqrt{2}\pi$.

PROBLEMS FOR SECTION 12.3

Find the volumes of the following solids in Problems 1–8.

1 $0 \le x \le 1,\quad 0 \le y \le 1,\quad xy \le z \le 1$

2 $0 \le x \le 2,\quad 0 \le y \le 2,\quad x^2 + y^2 \le z \le 8$

3 $0 \le x \le 1,\quad 1 \le y \le 2,\quad x \le z \le y$

4 $0 \le x \le 4,\quad 0 \le y \le 1,\quad x \le z \le xe^y$

5 $0 \le x \le 2,\quad 0 \le y \le x,\quad y \le z \le x$

6 $1 \le x \le 4,\quad x \le y \le 4,\quad y \le x \le xy$

7 $-1 \le x \le 1,\quad x^2 \le y \le 1,\quad x\sqrt{y} \le z \le y$

8 $0 \le x \le \pi,\quad -\sin x \le y \le \sin x,\quad -\sin x \le z \le \sin x$

In Problems 9–16, find the volume of the solid bounded by the given surfaces.

9 The planes $y = 0,\quad x + y = 2,\quad z = -x,\quad z = x$

10 The planes $x = 0,\quad y = 0,\quad 2x + 3y + z = 4,\quad 6x + y - z = 8$

11 $z = x^2 + y^2, \quad z = 4$

12 $z = x^2 + y^2 + 1, \quad z = 2x + 2y$

13 $y = 0, \quad z = x^2 + y, \quad z = 1$

14 $x = 0, \quad x = y, \quad z^2 = 1 - y$

15 $x^2 + y^2 = 9, \quad x^2 + z^2 = 9$

16 $z = x^2 + y^2, \quad z = 2 - x^2 - y^2$

17 Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

18 Find the volume of the solid bounded by the paraboloid $z = x^2/a^2 + y^2/b^2$ and the plane $z = c$, where c is positive.

12.4 APPLICATIONS TO PHYSICS

In this section we obtain double integrals for mass, center of mass, and moment of inertia.

DEFINITION

If a plane object fills a region D and has continuous density $\rho(x, y)$, its mass is

$$m = \iint_D \rho(x, y)\, dA.$$

On an element of area ΔD, the density is infinitely close to $\rho(x, y)$ (Figure 12.4.1). Therefore the mass is

$$\Delta m \approx \rho(x, y)\, \Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

By the Infinite Sum Theorem $m = \iint_D \rho(x, y)\, dx\, dy$.

Figure 12.4.1

In Chapter 6 we were able to find the mass of a plane object whose density $\rho(x)$ depends only on x by a single integral,

$$m = \int_{a_1}^{a_2} \rho(x)(b_2(x) - b_1(x))\,dx.$$

Our new formula for mass reduces to the old formula in this case, for by the Iterated Integral Theorem,

$$\begin{aligned} m &= \iint_D \rho(x)\,dA \\ &= \int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} \rho(x)\,dy\,dx \\ &= \int_{a_1}^{a_2} \rho(x)(b_2(x) - b_1(x))\,dx. \end{aligned}$$

Now we can find the mass of a plane object whose density $\rho(x, y)$ depends on both x and y instead of on x alone.

EXAMPLE 1 Find the mass of an object in the shape of a unit square whose density is the sum of the distance from one edge and twice the distance from a second perpendicular edge.

Step 1 The region D is shown in Figure 12.4.2.

Figure 12.4.2

Step 2 Place the object so the first two edges are on the x and y axes. Then D is the region

$$0 \le x \le 1, \qquad 0 \le y \le 1.$$

Step 3 The density is $\rho(x, y) = y + 2x$.

$$m = \iint_D y + 2x\,dA = \int_0^1\int_0^1 y + 2x\,dy\,dx.$$

$$\int_0^1 y + 2x\,dy = \tfrac{1}{2}y^2 + 2xy\Big]_0^1 = \tfrac{1}{2} + 2x.$$

$$m = \int_0^1 \tfrac{1}{2} + 2x\,dx = \tfrac{3}{2}.$$

DEFINITION

A plane object which fills a region D and has continuous density $\rho(x, y)$ has ***moments*** *about the x and y axes given by*

$$M_x = \iint_D y\rho(x, y)\, dA.$$

$$M_y = \iint_D x\rho(x, y)\, dA.$$

M_x and M_y are sometimes called first moments to distinguish them from moments of inertia (which are called second moments).

*The **center of mass** of the object is the point* $(\bar{x}, \bar{y})$ *with coordinates*

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\rho(x, y)\, dA}{\iint_D \rho(x, y)\, dA},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_D y\rho(x, y)\, dA}{\iint_D \rho(x, y)\, dA}.$$

JUSTIFICATION The piece of the object on an element of area ΔD has mass

$$\Delta m \approx \rho(x, y)\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

A point mass $\bar{m}$ at (x, y) has moments

$$\bar{M}_x = y\bar{m}, \qquad \bar{M}_y = x\bar{m}.$$

Therefore the piece of the object at ΔD has moments

$$\Delta M_x \approx y\,\Delta m \approx y\rho(x, y)\,\Delta A \qquad \text{(compared to } \Delta A\text{)},$$
$$\Delta M_y \approx x\,\Delta m \approx x\rho(x, y)\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

The double integrals for M_x and M_y now follow from the Infinite Sum Theorem.

An object will balance on a pin at its center of mass (Figure 12.4.3). The center of mass is useful in finding the work done against gravity when moving the

Figure 12.4.3

object. The work is the same as if the mass were all concentrated at the center of mass, and is given by

$$W = mgs$$

where s is the distance the center of mass is raised and g is constant.

EXAMPLE 2 A triangular plate bounded by the lines $x = 0$, $x = y$, $y = 1$ has density $\rho(x, y) = x + y$. Find the moments and center of mass.

Step 1 Sketch the region D, as in Figure 12.4.4.

Figure 12.4.4

Step 2 We see from the figure that D is the region

$$0 \le x \le 1, \qquad x \le y \le 1.$$

Step 3 Set up and evaluate the iterated integrals for the mass m and moments M_x and M_y.

$$m = \iint_D x + y\, dA = \int_0^1 \int_x^1 x + y\, dy\, dx.$$

$$\int_x^1 x + y\, dy = x + \tfrac{1}{2} - \tfrac{3}{2}x^2.$$

$$m = \int_0^1 x + \tfrac{1}{2} - \tfrac{3}{2}x^2\, dx = \tfrac{1}{2}.$$

$$M_x = \iint_D y(x + y)\, dA = \int_0^1 \int_x^1 yx + y^2\, dy\, dx.$$

$$\int_x^1 yx + y^2\, dy = \tfrac{1}{2}x + \tfrac{1}{3} - \tfrac{5}{6}x^3.$$

$$M_x = \int_0^1 \tfrac{1}{2}x + \tfrac{1}{3} - \tfrac{5}{6}x^3\, dx = \tfrac{9}{24}.$$

$$M_y = \iint_D x(x + y)\, dA = \int_0^1 \int_x^1 x^2 + xy\, dy\, dx$$

$$\int_x^1 x^2 + xy\, dy = x^2 + \tfrac{1}{2}x - \tfrac{3}{2}x^3.$$

$$M_y = \int_0^1 x^2 + \tfrac{1}{2}x - \tfrac{3}{2}x^3\, dx = \tfrac{5}{24}.$$

The answers are $$M_x = \frac{9}{24}, \qquad M_y = \frac{5}{24}.$$

$$\bar{x} = \frac{M_y}{m} = \frac{5/24}{1/2} = \frac{5}{12}.$$

$$\bar{y} = \frac{M_x}{m} = \frac{9/24}{1/2} = \frac{9}{12}.$$

The point $(\bar{x}, \bar{y})$ is shown in Figure 12.4.5.

EXAMPLE 3 The object in Example 2 is lying horizontally on the ground. Find the work required to stand the object up with the hypotenuse of the triangle on the ground (Figure 12.4.6).

We use the formula $$W = mgs.$$

From Example 2, $m = \frac{1}{2}$. We must find s.

$$s = \text{minimum distance from } \left(\frac{5}{12}, \frac{9}{12}\right) \text{ to the line } x = y.$$

$$s = \text{minimum value of } z = \sqrt{\left(x - \frac{5}{12}\right)^2 + \left(x - \frac{9}{12}\right)^2}.$$

$$z = \sqrt{2x^2 - \frac{28}{12}x + \frac{106}{144}}.$$

$$\frac{dz}{dx} = \left(4x - \frac{28}{12}\right)\frac{1}{2}z^{-1/2}.$$

$$\frac{dz}{dx} = 0 \quad \text{at} \quad 4x = \frac{28}{12}, \qquad x = \frac{7}{12}.$$

$$s = \sqrt{2\left(\frac{7}{12}\right)^2 - \frac{28}{12}\cdot\frac{7}{12} + \frac{106}{144}} = \frac{\sqrt{2}}{6}.$$

$$W = mgs = \frac{1}{2}\cdot g \cdot \frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{12}g.$$

The *second moment*, or *moment of inertia*, of a point mass m about the origin is the mass times the square of the distance to the origin,

Figure 12.4.5

Figure 12.4.6

$$I = m(x^2 + y^2).$$

The moment of inertia is related to the kinetic energy of rotation. A mass m moving at speed v has kinetic energy

$$KE = \tfrac{1}{2} mv^2.$$

Hence if m is rotating about the origin with angular velocity ω radians per second, its speed is $v = \omega\sqrt{x^2 + y^2}$ and

$$KE = \tfrac{1}{2}m(\omega\sqrt{x^2 + y^2})^2 = \tfrac{1}{2}I\omega^2.$$

Thus moment of inertia is the rotational analogue of mass.

DEFINITION

Given a plane object on the region D with continuous density $\rho(x, y)$, the ***moment of inertia*** *about the origin is*

$$I = \iint_D \rho(x, y)(x^2 + y^2)\, dA.$$

JUSTIFICATION On an element of volume ΔD, the moment of inertia is

$$\Delta I \approx (x^2 + y^2)\,\Delta m \approx \rho(x, y)(x^2 + y^2)\,\Delta A \qquad \text{(compared to } \Delta A\text{)}.$$

The integral for I follows by the Infinite Sum Theorem.

EXAMPLE 4 Find the moment of inertia about the origin of an object with constant density $\rho = 1$ which covers the square shown in Figure 12.4.7:

$$-\tfrac{1}{2} \le x \le \tfrac{1}{2}, \qquad -\tfrac{1}{2} \le y \le \tfrac{1}{2}.$$

$$I = \iint_D (x^2 + y^2)\, dA = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} x^2 + y^2\, dy\, dx.$$

$$\int_{-1/2}^{1/2} x^2 + y^2\, dy = x^2y + \tfrac{1}{3}y^3\Big]_{-1/2}^{1/2} = x^2 + \tfrac{1}{12}.$$

$$I = \int_{-1/2}^{1/2} x^2 + \tfrac{1}{12}\, dx = \tfrac{1}{6}.$$

Figure 12.4.7

PROBLEMS FOR SECTION 12.4

In Problems 1–10, find (a) the mass, (b) the center of mass, (c) the moment of inertia about the origin, of the given plane object.

1 $-a \le x \le a, \quad -b \le y \le b, \quad \rho(x, y) = k$

2 $0 \le x \le a, \quad 0 \le y \le b, \quad \rho(x, y) = k$

3 $0 \le x \le 1, \quad x^2 \le y \le 1, \quad \rho(x, y) = k$

4 $0 \le x \le a, \quad 0 \le y \le bx, \quad \rho(x, y) = k$

5 $0 \le x \le 2, \quad x \le y \le 2x, \quad \rho(x, y) = x + y + 1$

6 $0 \le x \le 1, \quad 0 \le y \le x, \quad \rho(x, y) = x - y$

7 $0 \le x \le 1, \quad 0 \le y \le x^2, \quad \rho(x, y) = \sqrt{x} + \sqrt{y}$

8 $1 \le x \le 2, \quad x \le y \le x^2, \quad \rho(x, y) = 1/\sqrt{xy}$

9 $0 \le x \le 2, \quad e^{-x} \le y \le e^x, \quad \rho(x, y) = 1$

10 $-1 \le x \le 1, \quad 0 \le y \le 1/\sqrt{1 + x^2}, \quad \rho(x, y) = y$

11 Find the mass of an object in the shape of a unit square whose density is the sum of the four distances from the sides.

12 Find the mass of an object in the shape of a unit square whose density is the product of the distances from the four sides.

13 An object on the triangle $0 \le x \le 1, 0 \le y \le x$ has density equal to the distance from the hypotenuse $y = x$. Find the amount of work required to stand the object up (a) on one of the short sides, (b) on the hypotenuse.

14 An object in the shape of a unit square has density equal to the distance to the nearest side. Find the mass and the amount of work needed to stand the object up on a side.

15 An object on the plane region $-1 \le x \le 1, x^2 \le y \le 1$ has density $\rho(x, y) = 1 + x + \sqrt{y}$. Find the mass and the work needed to stand the object up on the flat side.

16 An object on the unit square $0 \le x \le 1, 0 \le y \le 1$ has density $\rho(x, y) = ax + by + c$. Find the mass and center of mass.

□ 17 The moment of an object of density $\rho(x, y)$ in the region D about the vertical line $x = a$ is defined as

$$M_{y,x=a} = \iint_D (x - a)\rho(x, y)\, dA.$$

Show that

$$M_{y,x=a} = M_y - a \cdot m$$

where M_y is the moment about the y-axis and m is the mass.

□ 18 The moment of inertia of an object in the region D of density $\rho(x, y)$ about the point $P(a, b)$ is defined as

$$I_P = \iint_D \rho(x, y)((x - a)^2 + (y - b)^2)\, dA.$$

Show that

$$I_P = I - 2aM_x - 2bM_y + m(a^2 + b^2)$$

where I is the moment of inertia about the origin, M_x and M_y are the first moments, and m is the mass.

12.5 DOUBLE INTEGRALS IN POLAR COORDINATES

A point with polar coordinates (θ, r) has rectangular coordinates

$$(x, y) = (r \cos \theta, r \sin \theta).$$

DEFINITION

A ***polar region*** *is a region D in the (x, y) plane given by polar coordinate inequalities*

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta),$$

where $a(\theta)$ and $b(\theta)$ are continuous. To avoid overlaps, we also require that for all (θ, r) in D,

$$0 \le \theta \le 2\pi \quad \text{and} \quad 0 \le r.$$

The last requirement means that the limits α and β are between 0 and 2π, while the limits $a(\theta)$ and $b(\theta)$ are ≥ 0. Figure 12.5.1 shows a polar region.

The simplest polar regions are the *polar rectangles*

$$\alpha \le \theta \le \beta, \qquad a \le r \le b.$$

We see in Figure 12.5.2 that the θ boundaries are radii and the r boundaries are circular arcs.

A polar region

Figure 12.5.1

A polar rectangle

Figure 12.5.2

The polar rectangle

$$\alpha \le \theta \le \beta, \qquad 0 \le r \le b$$

is a sector of a circle of radius b (Figure 12.5.3(a)).

The polar rectangle

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b$$

is a whole circle of radius b (Figure 12.5.3(b)).

Less trivial examples of polar regions are the *circle with diameter from* $(0, 0)$ *to* $(0, b)$,

(a) sector: $\alpha \le \theta \le \beta,\ 0 \le r \le b$

(b) circle: $0 \le \theta \le 2\pi,\ 0 \le r \le b$

Figure 12.5.3

$$0 \le \theta \le \pi, \qquad 0 \le r \le b \sin \theta,$$

and the *cardioid* $\qquad 0 \le \theta \le 2\pi, \qquad 0 \le r \le 1 + \cos \theta.$

Both of these regions are shown in Figure 12.5.4.

We shall use the Infinite Sum Theorem to get a formula for the double integral over a polar region. In the proof we take for ΔD an infinitely small polar rectangle.

POLAR INTEGRATION FORMULA

Let D be the polar region

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta).$$

The double integral of $f(x, y)$ over D is

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{a(\theta)}^{b(\theta)} f(x, y) r\, dr\, d\theta$$

$$= \int_\alpha^\beta \int_{a(\theta)}^{b(\theta)} f(r \cos \theta, r \sin \theta) r\, dr\, d\theta.$$

Notice that in the iterated integral for a polar region we do not integrate $f(x, y)$ but the product of $f(x, y)$ and r. Intuitively, the extra r comes from the fact that a polar element of area is almost a rectangle of area $r\,\Delta\theta\,\Delta r$ (see Figure 12.5.6(b)).

PROOF We shall work with the rectangular (θ, r) plane. Let C be the region in the (θ, r) plane given by the inequalities

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta).$$

Thus C has the same inequalities as D but they refer to the (θ, r) plane instead of the (x, y) plane. D and C are shown in Figure 12.5.5.

We must prove that

$$\iint_D f(x, y)\, dx\, dy = \iint_C f(x, y) r\, d\theta\, dr.$$

(a) circle:
$0 \le \theta \le \pi,\ 0 \le r \le b \sin\theta$

(b) cardioid:
$0 \le \theta \le 2\pi,\ 0 \le r \le 1 + \cos\theta$

Figure 12.5.4

Figure 12.5.5

Our plan is to use the Infinite Sum Theorem in the (θ, r) plane. Assume first that $f(x, y) > 0$ for all (x, y) in D.

For any (θ, r) region C_1 corresponding to a polar region D_1 in the (x, y) plane, let

$$B(C_1) = \iint_{D_1} f(x, y)\, dx\, dy.$$

Then B has the Addition Property and is always ≥ 0. Consider an element of area ΔC in the (θ, r) plane with area $\Delta\theta\, \Delta r$. ΔC corresponds to a polar rectangle ΔD in the (x, y) plane. As we can see from Figure 12.5.6, ΔD is almost a rectangle with sides $r\,\Delta\theta$ and Δr and area $r\,\Delta\theta\,\Delta r$.

The volume over ΔD is almost a rectangular solid with base of area $r\,\Delta\theta\,\Delta r$ and height

$$f(x, y) = f(r\cos\theta, r\sin\theta).$$

Therefore $\quad B(\Delta C) \approx f(x, y) r\,\Delta\theta\,\Delta r \qquad$ (compared to $\Delta\theta\,\Delta r$).

By the Infinite Sum Theorem

$$B(C) = \iint_C f(x, y) r\, d\theta\, dr,$$

and by definition $\qquad B(C) = \iint_D f(x, y)\, dx\, dy.$

Figure 12.5.6

Finally we consider the case where $f(x, y)$ is not always positive. Pick a real constant $k > 0$ such that $f(x, y) + k$ is always positive for (x, y) in D. By the above proof,

$$\iint_D (f(x, y) + k)\, dx\, dy = \iint_C (f(x, y) + k) r\, d\theta\, dr,$$

$$\iint_D k\, dx\, dy = \iint_C kr\, d\theta\, dr.$$

When we use the Sum Rule and subtract the second equation from the first, we get

$$\iint_D f(x, y)\, dx\, dy = \iint_C f(x, y) r\, d\theta\, dr.$$

In a double integration problem where the region D is a circle or a sector of a circle, it is usually best to take the center as the origin and represent D as a polar rectangle.

EXAMPLE 1 Find the volume over the unit circle $x^2 + y^2 \leq 1$ between the surfaces $z = 0$ and $z = x^2$.

Step 1 Sketch D and the solid, as in Figure 12.5.7.

Figure 12.5.7

Step 2 D is the polar region $0 \le \theta \le 2\pi, 0 \le r \le 1$.

Step 3 $$V = \iint_D x^2\,dx\,dy = \int_0^{2\pi}\int_0^1 x^2 r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r\cos\theta)^2 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4}\cos^2\theta\,d\theta = \tfrac{1}{4}(\tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta)\Big]_0^{2\pi} = \pi/4.$$

For comparison let us also work this problem in rectangular coordinates. We can see that it is easier to use polar coordinates.

D is the region $-1 \le x \le 1, \qquad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$

$$V = \iint_D x^2\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2\,dy\,dx = \int_{-1}^{1} 2x^2\sqrt{1-x^2}\,dx.$$

We make the trigonometric substitution shown in Figure 12.5.8:

$$x = \sin\phi, \qquad \sqrt{1-x^2} = \cos\phi, \qquad dx = \cos\phi\,d\phi.$$

Figure 12.5.8

Then $\phi = -\pi/2$ at $x = -1$ and $\phi = \pi/2$ at $x = 1$, so

$$V = \int_{-\pi/2}^{\pi/2} 2\sin^2\phi\cos^2\phi\,d\phi$$

$$= \int_{-\pi/2}^{\pi/2} 2\left(\frac{1-\cos 2\phi}{2}\right)\left(\frac{1+\cos 2\phi}{2}\right)d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(1 - \cos^2 2\phi)\, d\phi$$
$$= \int_{-\pi}^{\pi} \tfrac{1}{4}(1 - \cos^2 u)\, du$$
$$= \tfrac{1}{4}(u - \tfrac{1}{2}\cos u \sin u - \tfrac{1}{2}u)\Big]_{-\pi}^{\pi} = \pi/4.$$

EXAMPLE 2 Find the mass and center of mass of a flat plate in the shape of a semicircle of radius one whose density is equal to the distance from the center of the circle.

Step 1 The region D is sketched in Figure 12.5.9.

Step 2 Take the origin at the center of the circle and the x-axis as the base of the semicircle. D is the polar region $0 \le \theta \le \pi$, $0 \le r \le 1$.

Step 3 The density is

$$\rho(x, y) = \sqrt{x^2 + y^2} = r.$$

$$m = \iint_D \sqrt{x^2 + y^2}\, dA = \int_0^{\pi}\int_0^1 r \cdot r\, dr\, d\theta$$
$$= \int_0^{\pi}\int_0^1 r^2\, dr\, d\theta = \int_0^{\pi} \frac{1}{3}\, d\theta = \frac{\pi}{3}.$$
$$M_x = \iint_D y\sqrt{x^2 + y^2}\, dA = \int_0^{\pi}\int_0^1 r \sin\theta \cdot r \cdot r\, dr\, d\theta$$
$$= \int_0^{\pi}\int_0^1 r^3 \sin\theta\, dr\, d\theta = \int_0^{\pi} \tfrac{1}{4}\sin\theta\, d\theta = \tfrac{1}{2}.$$
$$M_y = \iint_D x\sqrt{x^2 + y^2}\, dA = \int_0^{\pi}\int_0^1 r\cos\theta \cdot r \cdot r\, dr\, d\theta$$
$$= \int_0^{\pi}\int_0^1 r^3 \cos\theta\, dr\, d\theta = \int_0^{\pi} \tfrac{1}{4}\cos\theta\, d\theta = 0.$$

Answer $m = \dfrac{\pi}{3}$, $\quad \bar{x} = \dfrac{M_y}{m} = 0$, $\quad \bar{y} = \dfrac{M_x}{m} = \dfrac{3}{2\pi} \sim 0.477.$

The point $(\bar{x}, \bar{y})$ is shown in Figure 12.5.10.

Figure 12.5.9

Figure 12.5.10

EXAMPLE 3 Find the moment of inertia of a circle of radius b and constant density ρ about the center of the circle.

Step 1 Draw the region D (Figure 12.5.11).

Figure 12.5.11

Step 2 Put the origin at the center, so D is the polar region

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b.$$

Step 3 $x^2 + y^2 = r^2$, so

$$\begin{aligned} I &= \iint_D \rho \cdot (x^2 + y^2)\, dA = \int_0^{2\pi} \int_0^b \rho r^2 \cdot r\, dr\, d\theta \\ &= \rho \int_0^{2\pi} \int_0^b r^3\, dr\, d\theta = \rho \int_0^{2\pi} \tfrac{1}{4} b^4\, d\theta \\ &= \frac{\rho b^4 \pi}{2}. \end{aligned}$$

PROBLEMS FOR SECTION 12.5

In Problems 1–16, find the volume using polar coordinates.

1 $x^2 + y^2 \le 1, \quad 0 \le z \le 6$

2 $x^2 + y^2 \le 1, \quad 0 \le z \le x^2 + y^2$

3 $x^2 + y^2 \le 4, \quad 0 \le z \le x + 2$

4 $x^2 + y^2 \le 4, \quad 0 \le z \le \sqrt{x^2 + y^2}$

5 $x^2 + y^2 \le 9, \quad x^2 + y^2 \le z \le 9$

6 $x^2 + y^2 \le 25, \quad 0 \le z \le e^{-x^2-y^2}$

7 $1 \le x^2 + y^2 \le 4, \quad (x^2 + y^2)^{-1} \le z \le (x^2 + y^2)^{-1/2}$

8 $1 \le x^2 + y^2 \le 9, \quad 1/\sqrt{x^2 + y^2} \le z \le 1$

9 $0 \le x \le 1, \quad 0 \le y \le \sqrt{1 - x^2}, \quad 0 \le z \le x\sqrt{y}$

10 $-2 \le x \le 2, \quad 0 \le y \le \sqrt{4 - x^2}, \quad x \le z \le y + 2$

11 $\pi/4 \le \theta \le \pi/3, \quad 0 \le r \le 1, \quad 0 \le z \le r^2$

12 $0 \le \theta \le \pi/6, \quad 1 \le r \le 2, \quad 0 \le z \le \sqrt{9 - r^2}$

13 $0 \le \theta \le \pi, \quad 0 \le r \le 2\sin\theta, \quad 0 \le z \le r$

14 $0 \le \theta \le \pi/2, \quad 0 \le r \le \cos\theta, \quad r^3 \le z \le r^2$

15 $0 \le \theta \le 2\pi, \quad 0 \le r \le \theta, \quad 0 \le z \le r^2\theta^3 + 2r\theta$

16 $0 \le \theta \le 2\pi, \quad 0 \le r \le e^\theta, \quad 0 \le z \le \sqrt{r}$

17 Find the volume of the solid over the cardioid $r = 1 + \cos\theta$ between the plane $z = 0$ and the cone $z = r$.

18 Find the volume of the solid over the cardioid $r = 1 + \cos\theta$ between the paraboloids $z = r^2$ and $z = 8 - r^2$.

19 Find the volume of the solid over the circle $r = \sin\theta$ between the plane $z = 0$ and the hemisphere $z = \sqrt{1 - r^2}$.

20 Find the volume of the solid over the circle $r = 2\cos\theta$ between the plane $z = 0$ and the cone $z = 2 - r$.

21 Find the volume of the solid over the polar rectangle $\alpha \le \theta \le \beta$, $a \le r \le b$, between the plane $z = 0$ and the cone $z = r$.

22 Find the volume of the portion of the hemisphere $0 \le z \le \sqrt{1 - r^2}$ over the polar rectangle $\alpha \le \theta \le \beta$, $a \le r \le b$ (assuming $b \le 1$).

23 A circular object of radius b has density equal to the distance from the outside of the circle. Find (a) the mass, (b) the moment of inertia about the origin.

24 A circular object of radius b has density equal to the cube of the distance from the center. Find (a) the mass, (b) the moment of inertia about the origin.

25 Find the moment of inertia about the origin of a circular ring $a \le r \le b$, $0 \le \theta \le 2\pi$, of constant density k.

26 Find the moment of inertia of a circular object of radius b and constant density k about a point on its circumference. (The center can be put at $(0, b)$, so the object is on the polar region $0 \le r \le 2b\sin\theta$, $0 \le \theta \le \pi$.)

27 An object has constant density k on the circular sector $0 \le x \le 1$, $0 \le y \le \sqrt{1 - x^2}$. Find (a) the center of mass, (b) the moment of inertia about the origin.

28 An object of constant density k covers the cardioid $r \le 1 + \cos\theta$, $0 \le \theta \le 2\pi$. Find (a) the center of mass, (b) the moment of inertia about the origin.

29 An object of constant density k covers the region inside the circle $r = 2b\sin\theta$ and outside the circle $r = b$. Find (a) the center of mass, (b) the moment of inertia about the origin.

30 An object of constant density k covers the polar region

$$0 \le \theta \le \pi/2, \qquad 0 \le r \le b\sin 2\theta.$$

Find (a) the center of mass, (b) the moment of inertia about the origin.

□ 31 (a) Use polar coordinates to evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx$.

(b) Show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

(c) Now evaluate the single integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

12.6 TRIPLE INTEGRALS

A *closed region in space*, or *solid region*, is a set E of points given by inequalities

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x, y) \le z \le c_2(x, y)$$

where the functions $b_1(x)$, $b_2(x)$ and $c_1(x, y)$, $c_2(x, y)$ are continuous.

The *boundary* of E is the part of E on the following surfaces:

The planes $x = a_1$, $x = a_2$.

The cylinders $y = b_1(x)$, $y = b_2(x)$.

The surfaces $z = c_1(x, y)$, $z = c_2(x, y)$.

The simplest type of closed region is a *rectangular solid*, or *rectangular box*,

$$a_1 \le x \le a_2, \qquad b_1 \le y \le b_2, \qquad c_1 \le z \le c_2.$$

Figure 12.6.1 shows a solid region and a rectangular box.

Region in space Rectangular box

Figure 12.6.1

An *open region* in space is defined in a similar way but with strict inequalities. As in the two-dimensional case, the word *region* alone will mean *closed region.*

PERMANENT ASSUMPTION

Whenever we refer to a function $f(x, y, z)$ and a solid region E, we assume that $f(x, y, z)$ is continuous on some open region containing E.

The triple integral $$\iiint_E f(x, y, z)\, dx\, dy\, dz$$

is analogous to the double integral.

The first step in defining the triple integral is to form the *circumscribed rectangular box* of E (Figure 12.6.2). This is the rectangular box

$$a_1 \le x \le a_2, \qquad B_1 \le y \le B_2, \qquad C_1 \le z \le C_2,$$

where

$$\begin{aligned} B_1 &= \text{minimum value of } b_1(x), \\ B_2 &= \text{maximum value of } b_2(x), \\ C_1 &= \text{minimum value of } c_1(x, y), \\ C_2 &= \text{maximum value of } c_2(x, y). \end{aligned}$$

Figure 12.6.2 The circumscribed rectangular box

Our next step is to define the triple Riemann sum. Given positive real numbers Δx, Δy, and Δz, we partition the circumscribed rectangular box of E into rectangular boxes with sides Δx, Δy, and Δz (Figure 12.6.3). The partition points of this three-dimensional partition have the form

$$(x_k, y_l, z_m), \qquad 0 \le k \le n, \qquad 0 \le l \le p, \qquad 0 \le m \le q.$$

The *triple Riemann sum* of $f(x, y, z)\,\Delta x\,\Delta y\,\Delta z$ over E is defined as the sum

$$\sum_E\sum\sum f(x, y, z)\,\Delta x\,\Delta y\,\Delta z = \sum_{(x_k, y_l, z_m) \text{ in } E}\sum\sum f(x_k, y_l, z_m)\,\Delta x\,\Delta y\,\Delta z.$$

When we replace Δx, Δy, Δz by positive infinitesimals dx, dy, dz we obtain an *infinite triple Riemann sum*

$$\sum_E\sum\sum f(x, y, z)\,dx\,dy\,dz.$$

A partition

Figure 12.6.3

LEMMA

For all positive infinitesimals dx, dy, and dz, the triple Riemann sum

$$\sum_E\sum\sum f(x, y, z)\,dx\,dy\,dz$$

is a finite hyperreal number and therefore has a standard part.

We are now ready to define the triple integral (see Figure 12.6.4).

Figure 12.6.4

DEFINITION

Given positive infinitesimal dx, dy, and dz, the ***triple integral*** *of a continuous function f(x, y, z) over E is*

$$\iiint_E f(x, y, z)\,dx\,dy\,dz = st\left(\sum_E\sum\sum f(x, y, z)\,dx\,dy\,dz\right).$$

We shall now briefly state some basic theorems on triple integrals, which are exactly like the corresponding theorems for double integrals.

INDEPENDENCE OF *dx, dy,* AND *dz*

The value of $\iiint_E f(x, y, z)\,dx\,dy\,dz$ *does not depend on dx, dy, or dz.*

We shall usually use the notation $dV = dx\,dy\,dz$ for the volume of an infinitesimal dx by dy by dz rectangular box, and write

$$\iiint_E f(x, y, z)\,dV \qquad \text{for} \qquad \iiint_E f(x, y, z)\,dx\,dy\,dz.$$

ADDITION PROPERTY

If E is divided into two regions E_1 *and* E_2 *which meet only on a common boundary then*

$$\iiint_E f(x, y, z)\,dV = \iiint_{E_1} f(x, y, z)\,dV + \iiint_{E_2} f(x, y, z)\,dV.$$

ITERATED INTEGRAL THEOREM

If E is the region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x, y) \le z \le c_2(x, y),$$

then

$$\iiint_E f(x, y, z)\, dV = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

If the region E is a rectangular box

$$a_1 \le x \le a_2, \qquad b_1 \le y \le b_2, \qquad c_1 \le z \le c_2,$$

there are six different iterated integrals over E, corresponding to six different orders of integration. Here they are (in "alphabetical" order).

(1) $\displaystyle\int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dy\, dx$ (2) $\displaystyle\int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dz\, dx$

(3) $\displaystyle\int_{b_1}^{b_2} \int_{a_1}^{a_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dx\, dy$ (4) $\displaystyle\int_{b_1}^{b_2} \int_{c_1}^{c_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dz\, dy$

(5) $\displaystyle\int_{c_1}^{c_2} \int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dx\, dz$ (6) $\displaystyle\int_{c_1}^{c_2} \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dy\, dz.$

The Iterated Integral Theorem shows that each of these iterated integrals is equal to the triple integral

$$\iiint_E f(x, y, z)\, dV.$$

EXAMPLE 1 Evaluate $\displaystyle\iiint_E xy^2z^3\, dV$ where E is the rectangular box

$$0 \le x \le 2, \qquad 0 \le y \le 1, \qquad 0 \le z \le 4.$$

There are six iterated integrals which all have the same value. We compute one of them, and then another to check our answer.

FIRST SOLUTION $\displaystyle\iiint_E xy^2z^3\, dV = \int_0^2 \int_0^1 \int_0^4 xy^2z^3\, dz\, dy\, dx.$

The inside integral is

$$\int_0^4 xy^2z^3\, dz = \left.\frac{xy^2z^4}{4}\right]_0^4 = 64xy^2.$$

The second integral is

$$\int_0^1 64xy^2\, dy = \left.\tfrac{64}{3}xy^3\right]_0^1 = \tfrac{64}{3}x.$$

The final answer is

$$\int_0^2 \tfrac{64}{3}x\, dx = \left.\tfrac{64}{6}x^2\right]_0^2 = \tfrac{256}{6} = \tfrac{128}{3}.$$

SECOND SOLUTION $\displaystyle\iiint_E xy^2z^3\,dV = \int_0^4\int_0^2\int_0^1 xy^2z^3\,dy\,dx\,dz.$

The inside integral is

$$\int_0^1 xy^2z^3\,dy = \tfrac{1}{3}xy^3z^3\Big]_0^1 = \tfrac{1}{3}xz^3.$$

The second integral is

$$\int_0^2 \tfrac{1}{3}xz^3\,dx = \tfrac{1}{6}x^2z^3\Big]_0^2 = \tfrac{4}{6}z^3.$$

The final answer is

$$\int_0^4 \tfrac{4}{6}z^3\,dz = \tfrac{1}{6}z^4\Big]_0^4 = \tfrac{256}{6} = \tfrac{128}{3}.$$

Triple integrals can be evaluated by iterated integrals.

EXAMPLE 2 Evaluate $\displaystyle\iiint_E y + z\,dV$ where E is the region shown in Figure 12.6.5,

$$0 \le x \le \pi/2, \qquad 0 \le y \le \sin x, \qquad 0 \le z \le y\cos x.$$

Figure 12.6.5

SOLUTION $\displaystyle\iiint_E y + z\,dV = \int_0^{\pi/2}\int_0^{\sin x}\int_0^{y\cos x} y + z\,dz\,dy\,dx.$

We first evaluate the inside integral.

$$\int_0^{y\cos x} y + z\,dz = yz + \tfrac{1}{2}z^2\Big]_0^{y\cos x} = y^2\cos x + \tfrac{1}{2}y^2\cos^2 x.$$

Now we evaluate the second integral.

$$\int_0^{\sin x} y^2\cos x + \tfrac{1}{2}y^2\cos^2 x\,dy = \tfrac{1}{3}y^3(\cos x + \tfrac{1}{2}\cos^2 x)\Big]_0^{\sin x}$$
$$= \tfrac{1}{3}\sin^3 x(\cos x + \tfrac{1}{2}\cos^2 x).$$

Finally we evaluate the outside integral.

$$\iiint_E y + z\,dV = \int_0^{\pi/2} \tfrac{1}{3}\sin^3 x(\cos x + \tfrac{1}{2}\cos^2 x)\,dx$$
$$= \int_0^{\pi/2} \tfrac{1}{3}(1 - \cos^2 x)(\cos x + \tfrac{1}{2}\cos^2 x)\sin x\,dx$$
$$= \int_1^0 -\tfrac{1}{3}(1 - u^2)(u + \tfrac{1}{2}u^2)\,du$$
$$= \int_0^1 \tfrac{1}{3}(u + \tfrac{1}{2}u^2 - u^3 - \tfrac{1}{2}u^4)\,du = \tfrac{19}{180}.$$

COROLLARY

The volume of a region E in space is equal to the triple integral of the constant 1 *over E* as illustrated in Figure 12.6.6,

$$V = \iiint_E dV.$$

Figure 12.6.6

PROOF E is the solid over the plane region D given by

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x)$$

between the surfaces $z = c_1(x, y)$ and $z = c_2(x, y)$. By definition of the volume between two surfaces,

$$V = \iint_D c_2(x, y) - c_1(x, y)\,dA.$$

Using the Iterated Integral Theorem,

$$\iiint_E dV = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} dz\,dy\,dx$$
$$= \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} [c_2(x, y) - c_1(x, y)]\,dy\,dx = V.$$

We now come to the Infinite Sum Theorem for triple integrals, which is, again, the key result for applications.

We shall use Δx, Δy, and Δz for positive infinitesimals. By an *element of volume* we mean a rectangular box ΔE with sides Δx, Δy, and Δz (Figure 12.6.7). The volume of ΔE is

$$\Delta V = \Delta x\, \Delta y\, \Delta z.$$

Figure 12.6.7 An element of volume

INFINITE SUM THEOREM

Let $h(x, y, z)$ be continuous on an open region E_0 and let B be a function which assigns a real number $B(E)$ to each region E contained in E_0. Assume that:

(i) *B has the Addition Property.*

(ii) *$B(E) \geq 0$ for every E.*

(iii) *For every element of volume ΔE,*

$$B(\Delta E) \approx h(x, y, z)\, \Delta V \qquad \text{(compared to } \Delta V\text{)}.$$

Then

$$B(E) = \iiint_E h(x, y, z)\, dV.$$

Here are some applications of the triple Infinite Sum Theorem. Perhaps the simplest physical interpretation of the triple integral is mass as the triple integral of density.

DEFINITION

*The **mass** of an object filling a solid region E with continuous density $\rho(x, y, z)$ is*

$$m = \iiint_E \rho(x, y, z)\, dV.$$

JUSTIFICATION At every point of an element of volume ΔE the density is infinitely close to $\rho(x, y, z)$, so the element of mass is

$$\Delta m \approx \rho(x, y, z)\, \Delta V \qquad \text{(compared to } \Delta V\text{)}.$$

(See Figure 12.6.8.) By the Infinite Sum Theorem,

$$m = \iiint_E \rho(x, y, z)\, \Delta V.$$

Figure 12.6.8

EXAMPLE 3 Find the mass of an object in the unit cube

$$0 \le x \le 1, \qquad 0 \le y \le 1, \qquad 0 \le z \le 1$$

with density $\qquad \rho(x, y, z) = x + y + z.$

$$\begin{aligned} m &= \iiint_E x + y + z\, dV \\ &= \int_0^1 \int_0^1 \int_0^1 x + y + z\, dz\, dy\, dx \\ &= \int_0^1 \int_0^1 x + y + \tfrac{1}{2}\, dy\, dx = \int_0^1 x + \tfrac{1}{2} + \tfrac{1}{2}\, dx = \tfrac{3}{2}. \end{aligned}$$

An object in space has a moment about each coordinate plane.

DEFINITION

If an object in space fills a region E and has continuous density $\rho(x, y, z)$, its ***moments*** *about the coordinate planes are*

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV.$$

$$M_{xz} = \iiint_E y\rho(x, y, z)\, dV.$$

$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV.$$

The ***center of mass*** *of the object is the point $(\bar{x}, \bar{y}, \bar{z})$, where m is mass and*

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}.$$

JUSTIFICATION A point mass m has moment $M_{xy} = mz$ about the (x, y) plane (Figure 12.6.9). In an element of volume ΔE, the object has moment

$$\Delta M_{xy} \approx z\,\Delta m \approx z\rho(x, y, z)\,\Delta V \qquad \text{(compared to } \Delta V\text{)}.$$

Figure 12.6.9

By the Infinite Sum Theorem,

$$M_{xy} = \iiint_E z\rho(x, y, z)\, dV.$$

EXAMPLE 4 An object has constant density and the shape of a tetrahedron with vertices at the four points

$$(0, 0, 0), \qquad (1, 0, 0), \qquad (0, 1, 0), \qquad (0, 0, 1).$$

Find the center of mass.

Step 1 The region is sketched in Figure 12.6.10.

Figure 12.6.10

Step 2 The region E is the solid bounded by the coordinate planes and the plane $x + y + z = 1$ which passes through $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Solving for z, the plane is

$$z = 1 - x - y.$$

This plane meets the plane $z = 0$ at the line $1 - x - y = 0$, or $y = 1 - x$. Therefore E is the region

$$0 \le x \le 1, \qquad 0 \le y \le 1 - x, \qquad 0 \le z \le 1 - x - y.$$

Step 3 Let the density be $\rho = 1$.

$$m = \iiint_E dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} 1 - x - y\, dy\, dx$$
$$= \int_0^1 \tfrac{1}{2}(1 - x)^2\, dx = \tfrac{1}{6}.$$
$$M_{yz} = \iiint_E x\, dV$$
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} x(1 - x - y)\, dy\, dx$$
$$= \int_0^1 \tfrac{1}{2}x(1 - x)^2\, dx = \tfrac{1}{24}.$$
$$\bar{x} = \frac{M_{yz}}{m} = \frac{1/24}{1/6} = \frac{1}{4}.$$

Similarly
$$\bar{y} = \frac{1}{4}, \qquad \bar{z} = \frac{1}{4}.$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

An object in space has a moment of inertia about each coordinate axis. Intuitively, the moment of inertia about an axis is the analogue of mass for rotations about the axis.

DEFINITION

If an object in space fills a region E and has continuous density $\rho(x, y, z)$, its ***moments of inertia*** *about the coordinate axes are*

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV,$$
$$I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV,$$
$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV.$$

JUSTIFICATION A point mass m has a moment of inertia about the x-axis of

$$I_x = (y^2 + z^2)m.$$

On an element of volume ΔE, the object has moment of inertia

$$\Delta I_x \approx (y^2 + z^2)\,\Delta m \approx (y^2 + z^2)\rho(x, y, z)\,\Delta V \qquad \text{(compared to } \Delta V\text{)}.$$

The triple integral for I_x follows by the Infinite Sum Theorem.

EXAMPLE 5 Find the moments of inertia about the three axes of an object with constant density 1 filling the cube shown in Figure 12.6.11,

$$0 \le x \le a, \qquad 0 \le y \le a, \qquad 0 \le z \le a.$$

$$I_x = \iiint_E y^2 + z^2\,dV = \int_0^a \int_0^a \int_0^a y^2 + z^2\,dz\,dy\,dx$$

$$= \int_0^a \int_0^a ay^2 + \tfrac{1}{3}a^3\,dy\,dx = \int_0^a \tfrac{2}{3}a^4\,dx = \tfrac{2}{3}a^5.$$

Similarly, $\qquad I_y = \tfrac{2}{3}a^5, \qquad I_z = \tfrac{2}{3}a^5.$

Figure 12.6.11

PROBLEMS FOR SECTION 12.6

In Problems 1–8, evaluate the iterated integral.

1 $\displaystyle\int_0^1 \int_2^4 \int_1^3 xyz\,dz\,dy\,dx$

2 $\displaystyle\int_1^2 \int_0^2 \int_{-1}^4 (x - 2y + 4z)\,dz\,dy\,dx$

3 $\displaystyle\int_0^4 \int_0^1 \int_0^1 (3y^2 + 6z^2)\,dz\,dy\,dx$

4 $\displaystyle\int_0^2 \int_1^2 \int_1^4 e^{x+z}\,dz\,dy\,dx$

5 $\displaystyle\int_0^1 \int_0^x \int_{xy}^1 (x^2 + yz)\,dz\,dy\,dx$

6 $\displaystyle\int_0^1 \int_x^1 \int_y^1 2x^2 z\,dz\,dy\,dx$

7 $\displaystyle\int_0^\pi \int_0^{\sin x} \int_0^{\sin x} \sqrt{yz}\,dz\,dy\,dx$

8 $\displaystyle\int_0^{\pi/2} \int_0^{\cos x} \int_0^{y \sin x} (x + 2z)\,dz\,dy\,dx$

In Problems 9–16, evaluate the triple integral.

9 $\iiint_E (x + 2y)\,dV, \quad E: 0 \le x \le 2, 1 \le y \le 3, 2 \le z \le 4$

10 $\iiint_E x^2yz^3\,dV, \quad E: 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$

11 $\iiint_E (4xy + yz)\,dV, \quad E: 0 \le x \le 10, 0 \le y \le x^2, 0 \le z \le xy$

12 $\iiint_E \left(\frac{1}{x} + \frac{2}{y} + \frac{3}{z}\right) dV, \quad E: 1 \le x \le e, 1 \le y \le x, 1 \le z \le x$

13 $\iiint_E e^{x+2y+3z}\,dV, \quad E: -1 \le x \le 1, x \le y \le 1, x \le z \le y$

14 $\iiint_E xe^{y+z}\,dV, \quad E: 1 \le x \le 2, 0 \le y \le \ln x, 0 \le z \le y$

15 $\iiint_E \sqrt{x + y + z}\,dV, \quad E: 0 \le x \le 1, 0 \le y \le x, y \le z \le 2y$

16 $\iiint_E dV, \quad E: 0 \le x \le 1, x^2 \le y \le x, x^2y \le z \le x\sqrt{y}$

In Problems 17–26, find (a) the mass, (b) the center of mass, (c) the moments of inertia about the three coordinate axes, of an object with density $\rho(x, y, z)$ filling the region E.

17 $E: 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \qquad \rho(x, y, z) = x + 2y + 3z$

18 $E: 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1 \qquad \rho(x, y, z) = x^2 + y^2 + z^2$

19 $E: 0 \le x \le 1, 0 \le y \le x, 0 \le z \le x + y \qquad \rho(x, y, z) = 2$

20 $E: -1 \le x \le 1, x^2 \le y \le 1, x^2 \le z \le 1 \qquad \rho(x, y, z) = z$

21 $E: 0 \le x \le 1, 0 \le y \le 1, \sqrt{xy} \le z \le 1 \qquad \rho(x, y, z) = xyz$

22 $E: 0 \le x \le 1, x \le y \le 1, x \le z \le y \qquad \rho(x, y, z) = 10$

23 E is the tetrahedron with vertices at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $\rho(x, y, z) = k$.

24 E is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, $\rho(x, y, z) = x + y + z$.

25 E is the rectangular box $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, $\rho(x, y, z) = k$.

26 E is the rectangular box $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$, $\rho(x, y, z) = k$.

12.7 CYLINDRICAL AND SPHERICAL COORDINATES

In evaluating triple integrals it is sometimes easier to use cylindrical or spherical coordinates instead of rectangular coordinates.

A point (x, y, z) has *cylindrical coordinates* (θ, r, z) if

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

That is, as we see in Figure 12.7.1, (θ, r) is a polar coordinate representation of (x, y), and z is the height above the (x, y) plane.

The name cylindrical coordinates is used because the graph of the cylindrical coordinate equation $r =$ constant is a circular cylinder as shown in Figure 12.7.2.

Cylindrical coordinates

Figure 12.7.1

Figure 12.7.2

DEFINITION

A ***cylindrical region*** *is a region E in (x, y, z) space given by cylindrical coordinate inequalities*

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta), \qquad c_1(\theta, r) \le z \le c_2(\theta, r),$$

where all the functions are continuous. To avoid overlaps we also require that for (θ, r, z) in E,

$$0 \le \theta \le 2\pi \quad \text{and} \quad 0 \le r.$$

A cylindrical region is shown in Figure 12.7.3.

The simplest kind of cylindrical region is the *cylindrical box*

$$\alpha \le \theta \le \beta, \qquad a \le r \le b, \qquad c_1 \le z \le c_2.$$

This is a cylinder whose base is a polar rectangle and whose upper and lower faces are horizontal, as in Figure 12.7.4.

A cylindrical region

Figure 12.7.3

A cylindrical box

Figure 12.7.4

The cylinder box

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad c_1 \le z \le c_2$$

is a cylinder whose base is a circle of radius b and whose top and bottom faces are horizontal (Figure 12.7.5).

The cylindrical box

$$0 \le \theta \le 2\pi, \qquad a \le r \le b, \qquad c_1 \le z \le c_2$$

is a circular pipe with inner radius a and outer radius b (Figure 12.7.6).

Figure 12.7.5

Figure 12.7.6

To get a formula for the triple integral over a cylindrical region E, we use the Infinite Sum Theorem but take for ΔE an infinitely small cylindrical box instead of rectangular box.

CYLINDRICAL INTEGRATION FORMULA

Let E be the cylindrical region

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta), \qquad c_1(\theta, r) \le z \le c_2(\theta, r).$$

The triple integral of $f(x, y, z)$ over E is

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{a(\theta)}^{b(\theta)} \int_{c_1(\theta,r)}^{c_2(\theta,r)} f(x, y, z) r\, dz\, dr\, d\theta.$$

To evaluate the triple integral we substitute

$$f(x, y, z) = f(r\cos\theta, r\sin\theta, z).$$

This is like the Polar Integration Formula but has an extra variable z. In the iterated integral we do not integrate $f(x, y, z)$ but the product of $f(x, y, z)$ and r.

PROOF Let C be the region in the rectangular (θ, r, z) space given by

$$\alpha \le \theta \le \beta, \qquad a(\theta) \le r \le b(\theta), \qquad c_1(\theta, r) \le z \le c_2(\theta, r).$$

The region C is shown in Figure 12.7.7.

Figure 12.7.7

We must prove that

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \iiint_C f(x, y, z) r\, d\theta\, dr\, dz.$$

Assume first that $f(x, y, z) > 0$ on E. For any (θ, r, z) region C_1 corresponding to a cylindrical region E_1, define

$$B(C_1) = \iiint_{E_1} f(x, y, z)\, dx\, dy\, dz.$$

B has the Addition Property and is ≥ 0. An element of volume ΔC in the (θ, r, z) space has volume $\Delta\theta\, \Delta r\, \Delta z$. As we can see from Figure 12.7.8, ΔC corresponds to a cylindrical box ΔE. ΔE is almost a rectangular box with sides $r\, \Delta\theta$, Δr, and Δz, and volume $r\, \Delta\theta\, \Delta r\, \Delta z$.

At any point of ΔE, f has value infinitely close to

$$f(x, y, z) = f(r \cos\theta, r \sin\theta, z).$$

Therefore $\qquad B(\Delta C) \approx f(x, y, z) r\, \Delta\theta\, \Delta r\, \Delta z \qquad$ (compared to $\Delta\theta\, \Delta r\, \Delta z$).

Figure 12.7.8

By the Infinite Sum Theorem

$$B(C) = \iiint_C f(x, y, z) r \, d\theta \, dr \, dr,$$

and by definition

$$B(C) = \iiint_E f(x, y, z) \, dx \, dy \, dz.$$

The general case where $f(x, y, z)$ is not always positive is dealt with as in the Polar Integration Formula proof.

When integrating over a solid region E whose base is a circle or polar rectangle, it is often easier to use cylindrical instead of rectangular coordinates.

EXAMPLE 1 Find the moment of inertia of a cylinder of height h, base a circle of radius b, and constant density 1, about its axis.

Step 1 Draw the region as in Figure 12.7.9.

Step 2 The problem is greatly simplified by a wise choice of coordinate axes. Let the z-axis be the axis of the cylinder and put the origin at the center of the base. Then the region E in rectangular coordinates is

$$-b \le x \le b, \qquad -\sqrt{b^2 - x^2} \le y \le \sqrt{b^2 - x^2}, \qquad 0 \le z \le h,$$

and in cylindrical coordinates is

$$0 \le \theta \le 2\pi, \quad 0 \le r \le b, \quad 0 \le z \le h.$$

Step 3 The problem looks easier in cylindrical coordinates.

$$x^2 + y^2 = r^2.$$

$$\begin{aligned} I_z &= \iiint_E (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^b \int_0^h r^2 r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^b \int_0^h r^3 \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^b r^3 h \, dr \, d\theta = \int_0^{2\pi} \frac{1}{4} b^4 h \, d\theta = \frac{\pi b^4 h}{2}. \end{aligned}$$

Figure 12.7.9

EXAMPLE 2 Find the center of mass of a cone of constant density with height h and base a circle of radius b.

Step 1 The region is sketched in Figure 12.7.10.

Step 2 Put the origin at the center of the base and let the z-axis be the axis of the cone. E is the cylindrical region

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad 0 \le z \le h - \frac{h}{b}r.$$

Step 3 Let the density be 1.

$$\begin{aligned}
m &= \iiint_E dV = \int_0^{2\pi}\int_0^b\int_0^{h-hr/b} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^b r\left(h - \frac{h}{b}r\right) dr\,d\theta \\
&= \int_0^{2\pi}\left(\frac{1}{2}b^2h - \frac{1}{3}b^3\frac{h}{b}\right) d\theta = \int_0^{2\pi}\frac{1}{6}b^2h\,d\theta = \frac{\pi b^2 h}{3}. \\
M_{xy} &= \iiint_E z\,dV = \int_0^{2\pi}\int_0^b\int_0^{h-hr/b} zr\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^b \frac{1}{2}r\left(h - \frac{hr}{b}\right)^2 dr\,d\theta \\
&= \frac{1}{2}h^2\int_0^{2\pi}\int_0^b r - \frac{2r^2}{b} + \frac{r^3}{b^2}\,dr\,d\theta \\
&= \frac{1}{2}h^2\int_0^{2\pi}\frac{1}{2}b^2 - \frac{2}{3}\cdot\frac{b^3}{b} + \frac{1}{4}\frac{b^4}{b^2}\,d\theta \\
&= \frac{1}{2}h^2\int_0^{2\pi}\frac{1}{12}b^2\,d\theta = \frac{\pi b^2 h^2}{12}.
\end{aligned}$$

Since the cone is symmetric about the z-axis, $\bar{x} = 0$ and $\bar{y} = 0$.

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{4}h, \qquad (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{1}{4}h\right).$$

The point $(\bar{x}, \bar{y}, \bar{z})$ is shown in Figure 12.7.11.

Figure 12.7.10

Figure 12.7.11

To express a point $P(x, y, z)$ in *spherical coordinates* we let ρ (rho) be the distance from the origin to P, let θ be the same angle as in cylindrical coordinates, and let ϕ be the angle between the positive z-axis and the line OP. Note that ϕ can always be chosen between 0 and π.

Figure 12.7.12

We see from Figure 12.7.12 that a point (x, y, z) has spherical coordinates (θ, ϕ, ρ) if

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi.$$

The graph of the equation $\rho =$ constant is a sphere with center at the origin (hence the name spherical coordinates). The graph of $\phi =$ constant is a vertical cone with vertex at the origin. The graph of $\theta =$ constant is a half-plane through the z-axis. These surfaces are shown in Figure 12.7.13.

Figure 12.7.13

DEFINITION

A ***spherical region*** *E is a region in (x, y, z) space given by spherical coordinate inequalities*

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1(\theta) \le \phi \le \beta_2(\theta), \qquad c_1(\theta, \phi) \le \rho \le c_2(\theta, \phi),$$

where all the functions are continuous. To avoid overlaps we also require that for (θ, ϕ, ρ) *in* E,

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho.$$

A *spherical box* is a spherical region of the simple form

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1 \le \phi \le \beta_2, \qquad c_1 \le \rho \le c_2.$$

The θ-boundaries are planes, the ϕ-boundaries are portions of cone surfaces, and the ρ-boundaries are portions of spherical surfaces. Figure 12.7.14 shows a spherical box.

Figure 12.7.14 A spherical box

The spherical box

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le c$$

is a sphere of radius c with center at the origin.

The spherical box

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \beta, \qquad 0 \le \rho \le c$$

is a cone whose vertex is at the origin and whose top is spherical instead of flat. (See Figure 12.7.15.)

Figure 12.7.15

Another important example is the spherical region

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi/2, \qquad 0 \le \rho \le c \cos \phi,$$

which is a sphere of radius $\frac{1}{2}c$ whose center is on the z-axis at $z = \frac{1}{2}c$ (Figure 12.7.16).

Figure 12.7.16

When integrating over a solid region E made up of spheres or cones, it is often easiest to use spherical coordinates.

SPHERICAL INTEGRATION FORMULA

Let E be a spherical region

$$\alpha_1 \le \theta \le \alpha_2, \qquad \beta_1(\theta) \le \phi \le \beta_2(\theta), \qquad c_1(\theta, \phi) \le \rho \le c_2(\theta, \phi).$$

The triple integral of $f(x, y, z)$ over E is

$$\iiint_E f(x, y, z)\, dV = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1(\theta)}^{\beta_2(\theta)} \int_{c_1(\phi,\theta)}^{c_2(\phi,\theta)} f(x, y, z)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

In practice we make the substitution

$$f(x, y, z) = f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

before integrating.

PROOF Let C be the region in the rectangular (θ, ϕ, ρ) space which has the same inequalities as E. We prove

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \iiint_C f(x, y, z)\rho^2 \sin \phi \, d\theta \, d\phi \, d\rho.$$

As usual we let $f(x, y, z) > 0$ on E and put

$$B(C_1) = \iiint_{E_1} f(x, y, z)\, dx\, dy\, dz.$$

Consider an element of volume ΔC. As we see from Figure 12.7.17, ΔC corresponds to a spherical box ΔE. ΔE is almost a rectangular box with sides

Figure 12.7.17 Spherical Element of Volume

$$\Delta\rho, \quad \rho\,\Delta\phi, \quad \rho \sin\phi\,\Delta\theta$$

and volume $\qquad \rho^2 \sin\phi\,\Delta\theta\,\Delta\phi\,\Delta\rho.$

Thus $\qquad B(\Delta C) \approx f(x, y, z)\rho^2 \sin\phi\,\Delta\theta\,\Delta\phi\,\Delta\rho \qquad$ (compared to $\Delta\theta\,\Delta\phi\,\Delta\rho$).

By the Infinite Sum Theorem

$$B(C) = \iiint_C f(x, y, z)\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho,$$

and by definition

$$B(C) = \iiint_E f(x, y, z)\, dx\, dy\, dz.$$

The triple integral for volume,

$$V = \iiint_E dV,$$

gives us iterated integral formulas for volume in rectangular, cylindrical, and spherical coordinates.

Rectangular $\qquad V = \displaystyle\int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} dz\, dy\, dx.$

Cylindrical $\qquad V = \displaystyle\int_{\alpha}^{\beta} \int_{a(\theta)}^{b(\theta)} \int_{c_1(\theta, r)}^{c_2(\theta, r)} r\, dz\, dr\, d\theta.$

Spherical $\qquad V = \displaystyle\int_{\alpha_1}^{\alpha_2} \int_{\beta_1(\theta)}^{\beta_2(\theta)} \int_{c_1(\theta,\phi)}^{c_2(\theta,\phi)} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$

The rectangular formula is really equivalent to the double integral for the volume between two surfaces. Similarly, the cylindrical formula is equivalent to the double integral in polar coordinates for the volume between two surfaces.

On the other hand, the volume formula in spherical coordinates is something new which is useful for finding volumes of spherical regions.

EXAMPLE 3 Find the volume of the region above the cone $\phi = \beta$ and inside the sphere $\rho = c$.

The region, shown in Figure 12.7.18, is given by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \beta, \qquad 0 \le \rho \le c.$$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\beta} \int_0^{c} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\beta} \frac{c^3}{3} \sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} (1 - \cos\beta)\frac{c^3}{3} \, d\theta = \frac{2\pi}{3}(1 - \cos\beta)c^3. \end{aligned}$$

Figure 12.7.18

EXAMPLE 4 A sphere of diameter a passes through the center of a sphere of radius b, and $a > b$. Find the volume of the region inside the sphere of diameter a and outside the sphere of radius b.

Step 1 The region is sketched in Figure 12.7.19.

Figure 12.7.19

Step 2 We let the z-axis be the line through the two centers and put the origin at the center of the sphere of radius b. The two spheres have the spherical equations

$$\rho = a\cos\phi, \qquad \rho = b.$$

They intersect at
$$\cos\phi = \frac{b}{a}.$$

Thus E is the region

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \arccos\frac{b}{a}, \qquad b \le \rho \le a\cos\phi.$$

Step 3
$$V = \int_0^{2\pi}\int_0^{\arccos(b/a)}\int_b^{a\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \int_0^{2\pi}\int_0^{\arccos(b/a)} \frac{1}{3}(a^3\cos^3\phi - b^3)\sin\phi \, d\phi \, d\theta.$$

Put $u = \cos\phi$, $du = -\sin\phi \, d\phi$. Then

$$V = \int_0^{2\pi}\int_1^{b/a} -\frac{1}{3}(a^3u^3 - b^3)\, du \, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi}\int_{b/a}^{1} a^3u^3 - b^3 \, du \, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi} \frac{a^3}{4} - b^3 + \frac{3}{4}\frac{b^4}{a} \, d\theta = \frac{\pi}{6}\left(a^3 - 4b^3 + 3\frac{b^4}{a}\right).$$

EXAMPLE 5 Find the mass of a sphere of radius c whose density is equal to the distance from the surface. The sphere is shown in Figure 12.7.20.

Figure 12.7.20

Put the center at the origin. The sphere is then given by

$$0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi, \qquad 0 \le \rho \le c.$$

The density at (θ, ϕ, ρ) is

$$\text{density} = c - \rho.$$

The mass is

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^{\pi}\int_0^{c}(c-\rho)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{c}{3}c^3-\frac{1}{4}c^4\right)\sin\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\frac{1}{12}c^4\sin\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi}\frac{1}{6}c^4\,d\theta = \frac{\pi}{3}c^4. \end{aligned}$$

PROBLEMS FOR SECTION 12.7

In Problems 1–6, evaluate the integral using cylindrical coordinates.

1 $\iiint_E \sqrt{x^2+y^2}z\,dV$, E is the cylinder $x^2+y^2 \le 1, 0 \le z \le 2$

2 $\iiint_E x^2+z\,dV$, E is the cylinder $x^2+y^2 \le 9, 0 \le z \le 6$

3 $\iiint_E x^2+y^2\,dV$, E is the cone $x^2+y^2 \le 1, 0 \le z \le 1-\sqrt{x^2+y^2}$

4 $\iiint_E 4+\sqrt{z}\,dV$, E is the cone $x^2+y^2 \le 1, \sqrt{x^2+y^2} \le z \le 1$

5 $\iiint_E (x+y)z\,dV$, E is the region $0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}, 0 \le z \le x^2+y^2$

6 $\iiint_E \frac{z}{\sqrt{x^2+y^2}}\,dV$, E is the region $1 \le x^2+y^2 \le 4, 0 \le z \le |x|$

7 Find the mass of an object in the shape of a cylinder of radius b and height h whose density is equal to the distance from the axis.

8 Find the mass of an object in the shape of a cylinder of radius b and height h whose density is equal to the distance from the base.

9 Find the mass of an object in the shape of a cone of radius b and height h whose density is equal to the square of the distance from the axis.

10 Find the mass of an object in the shape of a cone of radius b and height h whose density is equal to the sum of the distance from the base and the distance from the axis.

11 Find the center of mass of an object of constant density filling the region above the paraboloid $z = x^2+y^2$ and below the plane $z = 1$.

12 Find the center of mass of an object of constant density filling the region

$$x^2+y^2 \le b, \qquad 0 \le z \le \sqrt{x^2+y^2}.$$

13 Find the moment of inertia of an object of constant density k in the cylinder $0 \le r \le b$, $-c \le z \le c$, about the x-axis.

14 Find the moment of inertia of an object of constant density k in the cylindrical shell $a \le r \le b$, $-c \le z \le c$, about the z-axis.

15 Find the moment of inertia of an object of constant density k in a cone of radius b and height h about its axis.

16 Find the moment of inertia of an object of constant density k in a cone of radius b and height h about a line through its apex and perpendicular to its axis.

In Problems 17–24, evaluate the integral using spherical coordinates.

17 $\iiint_E x^2 + y^2 + z^2\, dV$, E is the sphere $x^2 + y^2 + z^2 \le b^2$

18 $\iiint_E \sqrt{x^2 + y^2 + z^2}\, dV$, E is the sphere $x^2 + y^2 + z^2 \le b^2$

19 $\iiint_E x^2\, dV$, E is the sphere $x^2 + y^2 + z^2 \le 1$

20 $\iiint_E z^2\, dV$, E is the sphere $x^2 + y^2 + z^2 \le 1$

21 $\iiint_E z\, dV$, E is the sphere $\rho \le 2b \cos \phi$

22 $\iiint_E (x^2 + y^2 + z^2)^{3/2}\, dV$, E is the intersection of the spheres $\rho \le 2b \cos \phi$, $\rho \le b$

23 $\iiint_E z\sqrt{x^2 + y^2 + z^2}\, dV$, E is the region above the cone $\phi = \alpha$ and inside the sphere $\rho = b$

24 $\iiint_E \dfrac{1}{x^2 + y^2 + z^2}\, dV$, E is the spherical shell $a \le \rho \le b$

25 Find the volume of the spherical shell $a \le \rho \le b$.

26 Find the volume of the spherical box $\alpha_1 \le \theta \le \alpha_2$, $\beta_1 \le \phi \le \beta_2$, $c_1 \le \rho \le c_2$.

27 Find the volume of the region above the cone $\phi = \beta$ and inside the sphere $\rho = b \cos \phi$.

28 Find the volume of the spherical region $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, $0 \le \rho \le \sin \phi$.

29 Find the mass of an object in the shape of a sphere of radius c whose density is equal to the distance from the center.

30 Find the mass of a spherical shell $a \le \rho \le b$ whose density is equal to the reciprocal of the distance from the center.

31 Find the moment of inertia of a spherical object of radius b and constant density k about a diameter of the sphere.

32 Find the moment of inertia of a spherical shell $a \le \rho \le b$ of constant density k about any diameter.

33 A hole of radius a is bored through a sphere of radius b, and the surface of the hole passes through the center of the sphere, $a = \frac{1}{2}b$. Find the volume removed.

34 A hole of radius a is bored through a cone of height h and base of radius b, and the axis of the cone is on the surface of the hole ($a \le \frac{1}{2}b$). Find the volume removed.

35 Find the center of mass of a hemisphere of constant density and radius b.

36 Find the moment of inertia of an object of constant density k in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

about the z-axis. *Hint*: Change variables $x_1 = x/a$, $y_1 = y/b$, $z_1 = z/c$ and use spherical coordinates.

EXTRA PROBLEMS FOR CHAPTER 12

1 Compute the Riemann sum

$$\sum_D\sum x^2 - y^2\,\Delta x\,\Delta y, \Delta x = \tfrac{1}{2}, \Delta y = \tfrac{1}{2}, \quad D: -2 \le x \le 2, -2 \le y \le 2.$$

2 Compute the Riemann sum

$$\sum_D\sum x^2 - \sqrt{y}\,\Delta x\,\Delta y, \Delta x = \tfrac{1}{5}, \Delta y = \tfrac{1}{4}, \quad D: -1 \le x \le 1, 0 \le y \le 1 - x^2.$$

3 Evaluate $\displaystyle\iint_D x^2 - y^2\,dA, \quad D: -2 \le x \le 2, -2 \le y \le 2.$

4 Evaluate $\displaystyle\iint_D x^2 - \sqrt{y}\,dA, \quad D: -1 \le x \le 1, 0 \le y \le 1 - x^2.$

5 Evaluate $\displaystyle\int_0^{\sqrt{2}/2}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} x\,dy\,dx.$

6 Evaluate $\displaystyle\int_0^1\int_{\sqrt{x}}^1 ye^x\,dy\,dx.$

7 Find the volume of the solid over the region $-1 \le x \le 1, 0 \le y \le 1 - x^2$ and between the surfaces $z = 0$, $z = 1 - y$.

8 Find the volume of the solid over the region $x^2 + y^2 = 4$ and between the surfaces $z = 0$ and $z = y^2 + x + 2$.

9 Find the volume of the solid $1 \le x \le 2, 0 \le y \le \ln x, y/x \le z \le 1/x$.

10 Find the volume of the solid $x^2 + y^2 \le 1, x^2y^3 \le z \le 1$.

11 Find the volume of the solid bounded by the planes

$$x = 1, \qquad x = y, \qquad z = x + y, \qquad z = x + 2.$$

12 Find the volume of the solid bounded by the cylinders

$$x^2 + y^2 = 1, \quad x^2 + z^2 = 1.$$

13 Find the mass, center of mass, and moment of inertia about the origin of the plane object

$$0 \le x \le \pi, \qquad 0 \le y \le \sin x, \qquad \rho(x, y) = k.$$

14 Find the mass, center of mass, and moment of inertia about the origin of the plane object

$$0 \le x \le 1, \qquad x \le y \le 1, \qquad \rho(x, y) = x^2y.$$

15 A circular disc filling the region $x^2 + y^2 \le r^2$ has density $\rho(x, y) = y^2$. Find the mass, center of mass, and moment of inertia about the origin.

16 A semicircular object on the region

$$-r \le x \le r, \qquad 0 \le y \le \sqrt{r^2 - x^2}$$

has density $\rho(x, y) = y$. Find the work required to stand the object up on its flat side.

17 Using polar coordinates, find the volume of the solid

$$x^2 + y^2 \le 9, \qquad y \le z \le x + 5.$$

18 Find the volume of the solid over the region $0 \le r \le 3 + \cos\theta$ between the plane $z = 0$ and the cone $z = r$.

19 Find the volume of the solid over the circle $0 \le r \le a$ between the plane $z = 0$ and the surface $z = 1/r$.

20 Find the mass and the moment of inertia about the origin of a semicircular object $0 \le r \le 1, 0 \le \theta \le \pi$ whose density is $\rho(r, \theta) = r\theta$.

21 A plane object covers the circle $0 \le r \le a$ and its density depends only on the distance r from the center, $\rho(r, \theta) = f(r)$. Show that the center of mass is at the origin.

22 Evaluate $\displaystyle\int_0^{\pi}\int_0^{\pi/2}\int_0^1 z\sin x + z\cos y\,dz\,dy\,dx.$

23 Evaluate $\int_0^1 \int_0^{x^2} \int_0^{y^2} x + y + z \, dz \, dy \, dx$.

24 Evaluate the triple integral

$$\iiint_E \frac{y+z}{x} dV, \quad E: 1 \le x \le 4, 1 \le y \le x, 1 \le z \le y.$$

25 An object has constant density k in the region

$$E: 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le xy.$$

Find its center of mass and its moments of inertia about the coordinate axes.

26 Use cylindrical coordinates to evaluate $\iiint_E z \, dV$, where E is the region inside the cylinder $x^2 + y^2 = 1$ which is above the plane $z = 0$ and within the sphere $x^2 + y^2 + z^2 = 9$.

27 An object of constant density k has the shape of a parabolic bowl

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le b, \qquad r^2 \le z \le r^2 + c.$$

Find its center of mass and its moment of inertia about the z-axis.

28 Use spherical coordinates to evaluate the integral

$$\iiint_E x + y + z \, dV,$$

E is the spherical octant

$$x^2 + y^2 + z^2 \le 1, \qquad 0 \le x, \qquad 0 \le y, \qquad 0 \le z.$$

29 A spherical shell $a \le \rho \le b$ has density equal to the distance from the center. Find its mass and its moment of inertia about a diameter.

☐ 30 Prove that the double Riemann sum $\sum\sum_D f(x, y) \, dx \, dy$ is finite whenever $f(x, y)$ is continuous, D is a closed region, and dx, dy are positive infinitesimals.

☐ 31 Suppose a plane object is symmetric about the x-axis, that is, it covers a region D of the form

$$D: a \le x \le b, -g(x) \le y \le g(x)$$

and has density $\rho(x, y) = \rho(x, -y)$. Prove that the center of mass is on the x-axis.

☐ 32 The moment of inertia about the x-axis of a point in the plane of mass m is $I_x = my^2$. Use the Infinite Sum Theorem to show that the moment of inertia about the x-axis of a plane object with density $\rho(x, y)$ in the region D is $I_x = \iint_D \rho(x, y) y^2 \, dA$.

☐ 33 The kinetic energy of a point of mass m moving at speed v is $KE = \frac{1}{2}mv^2$. A rigid object of density $\rho(x, y)$ in the plane region D is rotating about the origin with angular velocity ω (so a point at distance d from the origin has speed ωd). Use the Infinite Sum Theorem to show that the kinetic energy of the object is

$$KE = \iint_D \tfrac{1}{2}\omega^2(x^2 + y^2)\rho(x, y) \, dA = \tfrac{1}{2}\omega^2 I.$$

☐ 34 Suppose a plane object is symmetric about the origin; that is, it fills a polar region $0 \le r \le g(\theta)$, $-\pi \le \theta \le \pi$, such that $g(\theta \pm \pi) = g(\theta)$, and its density has the property $\rho(r, \theta) = \rho(r, \theta \pm \pi)$. Show that the center of mass is at the origin.

☐ 35 Use the Infinite Sum Theorem to show that if D is a polar region of the form $a \le r \le b$, $\alpha(r) \le \theta \le \beta(r)$, then

$$\iint_D f(x, y) \, dA = \int_a^b \int_{\alpha(r)}^{\beta(r)} f(r, \theta) r \, d\theta \, dr.$$

Contents 29

Integral Vector Calculus

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Learning outcomes

In this Workbook you will learn how to integrate functions involving vectors. You will learn how to evaluate line integrals i.e. where a scalar or a vector is summed along a line or contour. You will be able to evaluate surface and volume integrals where a function involving vectors is summed over a surface or volume. You will learn about some theorems relating to line, surface or volume integrals viz Stokes' theorem, Gauss' divergence theorem and Green's theorem.

Line Integrals

29.1

Introduction

HELM workbook 28 considered the differentiation of scalar and vector fields. Here we consider how to integrate such fields along a line. Firstly, integrals involving scalars along a line will be considered. Subsequently, line integrals involving vectors will be considered. These can give scalar or vector answers depending on the form of integral involved. Of particular interest are the integrals of conservative vector fields.

Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the basic techniques of integration
- be familiar with the operators div, grad and curl

Learning Outcomes

On completion you should be able to ...

- integrate a scalar or vector quantity along a line

1. Line integrals

HELM 28 was concerned with evaluating an integral over **all** points within a rectangle or other shape (or over a cuboid or other volume). In a related manner, an integral can take place over a line or curve running through a two-dimensional (or three-dimensional) region. Line integrals may involve scalar or vector fields. Those involving scalar fields are dealt with first.

Line integrals in two dimensions

A line integral in two dimensions may be written as

$$\int_C F(x,y)dw$$

There are three main features determining this integral:

$F(x,y)$: This is the scalar function to be integrated e.g. $F(x,y) = x^2 + 4y^2$.

C: This is the curve along which integration takes place. e.g. $y = x^2$ or $x = \sin y$ or $x = t - 1;\ \ y = t^2$ (where x and y are expressed in terms of a parameter t).

dw: This gives the variable of the integration. Three main cases are dx, dy and ds. Here 's' is arc length and so indicates position along the curve C.

ds may be written as $ds = \sqrt{(dx)^2 + (dy)^2}$ or $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx$.

A fourth case is when $F(x,y)\ dw$ has the form: $F_1dx + F_2dy$. This is a combination of the cases dx and dy.

The integral $\displaystyle\int_C F(x,y)\ ds$ represents the area beneath the surface $z = F(x,y)$ but above the curve C.

The integrals $\displaystyle\int_C F(x,y)\ dx$ and $\displaystyle\int_C F(x,y)\ dy$ represent the projections of this area onto the xz and yz planes respectively.

A particular case of the integral $\displaystyle\int_C F(x,y)\ ds$ is the integral $\displaystyle\int_C 1\ ds$. This is a means of calculating the length along a curve i.e. an arc length.

Figure 1: Representation of a line integral and its projections onto the xz and yz planes

The technique for evaluating a line integral is to express all quantities in the integral in terms of a single variable. If the integral is with respect to 'x' or 'y', then the curve 'C' and the function 'F' may be expressed in terms of the relevant variable. If the integral is with respect to ds, normally all quantities are expressed in terms of x. If x and y are given in terms of a parameter t, then t is used as the variable.

Example 1

Find $\int_c x\,(1+4y)\,dx$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$.

Solution

As this integral concerns only points along C and the integration is carried out with respect to x, y may be replaced by x^2. The limits on x will be 0 to 1. So the integral becomes

$$\begin{aligned}\int_C x(1+4y)\,dx &= \int_{x=0}^{1} x\left(1+4x^2\right)dx = \int_{x=0}^{1}\left(x+4x^3\right)dx \\ &= \left[\frac{x^2}{2}+x^4\right]_0^1 = \left(\frac{1}{2}+1\right)-(0) = \frac{3}{2}\end{aligned}$$

Example 2

Find $\int_c x\,(1+4y)\;dy$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same as Example 1 other than dx being replaced by dy.

Solution

As this integral concerns only points along C and the integration is carried out with respect to y, everything may be expressed in terms of y, i.e. x may be replaced by $y^{1/2}$. The limits on y will be 0 to 1. So the integral becomes

$$\begin{aligned}\int_C x(1+4y)\,dy &= \int_{y=0}^{1} y^{1/2}\,(1+4y)\,dx = \int_{y=0}^{1}\left(y^{1/2}+4y^{3/2}\right)dx \\ &= \left[\frac{2}{3}y^{3/2}+\frac{8}{5}x^{5/2}\right]_0^1 = \left(\frac{2}{3}+\frac{8}{5}\right)-(0) = \frac{34}{15}\end{aligned}$$

Example 3

Find $\int_c x\,(1+4y)\,ds$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same integral and curve as the previous two examples but the integration is now carried out with respect to s, the arc length parameter.

Solution

As this integral is with respect to x, all parts of the integral can be expressed in terms of x, Along $y = x^2$, $\quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (2x)^2}dx = \sqrt{1+4x^2}dx$

So, the integral is

$$\int_c x\,(1+4y)\,ds = \int_{x=0}^{1} x\left(1+4x^2\right)\sqrt{1+4x^2}\;dx = \int_{x=0}^{1} x\left(1+4x^2\right)^{3/2}dx$$

This can be evaluated using the transformation $u = 1 + 4x^2$ so $du = 8xdx$ i.e. $x\;dx = \frac{du}{8}$.
When $x = 0,\ u = 1$ and when $x = 1,\ u = 5$.
Hence,

$$\begin{aligned}\int_{x=0}^{1} x\left(1+4x^2\right)^{3/2}dx &= \frac{1}{8}\int_{u=1}^{5} u^{3/2}du \\ &= \frac{1}{8}\times\frac{2}{5}\left[\;u^{5/2}\right]_1^5 = \frac{1}{20}\left[5^{5/2}-1\right] \approx 2.745\end{aligned}$$

Note that the results for Examples 1,2 and 3 are all different: Example 3 is the area between a curve and a surface above; Examples 1 and 2 give projections of this area onto other planes.

Example 4

Find $\int_C xy\;dx$ where, on C, x and y are given in terms of a parameter t by

$x = 3t^2$, $y = t^3 - 1$ for t varying from 0 to 1.

Solution

Everything can be expressed in terms of t, the parameter. Here $x = 3t^2$ so $dx = 6t\;dt$. The limits on t are $t = 0$ and $t = 1$. The integral becomes

$$\begin{aligned}\int_C xy\;dx &= \int_{t=0}^{1} 3t^2\left(t^3-1\right)6t\;dt = \int_{t=0}^{1}\left(18t^6 - 18t^3\right)dt \\ &= \left[\frac{18}{7}t^7 - \frac{18}{4}t^4\right]_0^1 = \frac{18}{7} - \frac{9}{2} - 0 = -\frac{27}{14}\end{aligned}$$

Key Point 1

A line integral is normally evaluated by expressing all variables in terms of one variable.

In general

$$\int_C f(x,y)\ ds \neq \int_C f(x,y)\ dy \neq \int_C f(x,y)\ dx$$

Task

For $F(x,y) = 2x + y^2$, find (i) $\displaystyle\int_C F(x,y)\ dx$, (ii) $\displaystyle\int_C F(x,y)\ dy$, (iii) $\displaystyle\int_C F(x,y)\ ds$ where C is the line $y = 2x$ from $(0,0)$ to $(1,2)$.

Express each integral as a simple integral with respect to a single variable and hence evaluate each integral:

Your solution

Answer

(i) $\displaystyle\int_{x=0}^{1} (2x + 4x^2)\ dx = \frac{7}{3}$, (ii) $\displaystyle\int_{y=0}^{2} (y + y^2)\ dy = \frac{14}{3}$, (iii) $\displaystyle\int_{x=0}^{1} (2x + 4x^2)\sqrt{5}\ dx = \frac{7}{3}\sqrt{5}$

Task

Find (i) $\int_C F(x,y)\,dx$, (ii) $\int_C F(x,y)\,dy$, (iii) $\int_C F(x,y)\,ds$ where $F(x,y)=1$ and C is the curve $y=\frac{1}{2}x^2-\frac{1}{4}\ln x$ from $(1,\frac{1}{2})$ to $(2,2-\frac{1}{4}\ln 2)$.

Your solution

Answer

(i) $\int_1^2 1\,dx = 1$, (ii) $\int_{1/2}^{2-(1/4)\ln 2} 1\,dy = \frac{3}{2}-\frac{1}{4}\ln 2$, (iii) $y=\frac{1}{2}x^2-\frac{1}{4}\ln x \Rightarrow \frac{dy}{dx}=x-\frac{1}{4x}$

$$\therefore \int 1\,ds = \int_1^2 \sqrt{1+(x-\frac{1}{4x})^2}\,dx = \int_1^2 \sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}\,dx = \int_1^2 (x+\frac{1}{4x})\,dx = \frac{3}{2}+\frac{1}{4}\ln 2.$$

Task

Find (i) $\int_C F(x,y)\,dx$, (ii) $\int_C F(x,y)\,dy$, (iii) $\int_C F(x,y)\,ds$ where $F(x,y)=\sin 2x$ and C is the curve $y=\sin x$ from $(0,0)$ to $(\frac{\pi}{2},1)$.

Your solution

Answer

(i) $\int_0^{\pi/2} \sin 2x\,dx = 1$, (ii) $\int_0^{\pi/2} 2\sin x\cos^2 x\,dx = \frac{2}{3}$

(iii) $\int_0^{\pi/2} \sin 2x\sqrt{1+\cos^2 x}\,dx = \frac{2}{3}(2\sqrt{2}-1)$, using the substitution $u=1+\cos^2 x$.

2. Line integrals of scalar products

Integrals of the form $\int_C \underline{F} \cdot \underline{dr}$ occur in applications such as the following.

Figure 2: Schematic for cyclist travelling from A to B into a head wind

Consider a cyclist riding along the road from A to B (Figure 2). Suppose it is necessary to find the total work the cyclist has to do in overcoming a wind of velocity $\underline{v}$.

On moving from S to T, along an element $\underline{\delta r}$ of road, the work done is given by 'Force $\times$ distance' $= |\underline{F}| \times |\underline{\delta r}| \cos\theta$ where $\underline{F}$, the force, is directly proportional to $\underline{v}$, but in the opposite direction, and $|\underline{\delta r}| \cos\theta$ is the component of the distance travelled in the direction of the wind.
So, the work done travelling $\underline{\delta r}$ is $-k\underline{v} \cdot \underline{\delta r}$. Letting $\underline{\delta r}$ become infinitesimally small, the work done becomes $-k\underline{v} \cdot \underline{dr}$ and the total work is $-k \int_A^B \underline{v} \cdot \underline{dr}$.
This is an example of the integral along a line, of the scalar product of a vector field, with a vector element of the line. The term **scalar line integral** is often used for integrals of this form. The vector $\underline{dr}$ may be considered to be $dx\ \underline{i} + dy\ \underline{j} + dz\ \underline{k}$.

Multiplying out the scalar product, the 'scalar line integral' of the vector $\underline{F}$ along contour C, is given by $\int_C \underline{F} \cdot \underline{dr}$ and equals $\int_C \{F_x\ dx + F_y\ dy + F_z\ dz\}$ in three dimensions, and $\int_C \{F_x\ dx + F_y\ dy\}$ in two dimensions, where F_x, F_y, F_z are the components of $\underline{F}$.

If the contour C has its start and end points in the same positions i.e. it represents a closed contour, the symbol $\oint_C$ rather than $\int_C$ is used, i.e. $\oint_C \underline{F} \cdot \underline{dr}$.
As before, to evaluate the line integral, express the path and the function $\underline{F}$ in terms of either x, y and z, or in terms of a parameter t. Note that t often represents time.

Example 5

Find $\int_C \{2xy\ dx - 5x\ dy\}$ where C is the curve $y = x^3 \quad 0 \le x \le 1$.

[This is the integral $\int_C \underline{F} \cdot \underline{dr}$ where $\underline{F} = 2xy\underline{i} - 5x\underline{j}$ and $\underline{dr} = dx\ \underline{i} + dy\ \underline{j}$.]

Solution

It is possible to split this integral into two different integrals and express the first term as a function of x and the second term as a function of y. However, it is also possible to express everything in terms of x. Note that on C, $y = x^3$ so $dy = 3x^2\,dx$ and the integral becomes

$$\begin{aligned}\int_C \{2xy\,dx - 5x\,dy\} &= \int_{x=0}^{1} \left(2x\,x^3\,dx - 5x\,3x^2\,dx\right) = \int_0^1 (2x^4 - 15x^3)\,dx \\ &= \left[\frac{2}{5}x^5 - \frac{15}{4}x^4\right]_0^1 = \frac{2}{5} - \frac{15}{4} - 0 = -\frac{67}{20}\end{aligned}$$

Key Point 2

An integral of the form $\int_C \underline{F} \cdot \underline{dr}$ may be expressed as $\int_C \{F_x\,dx + F_y\,dy + F_z\,dz\}$. Knowing the expression for the path C, every term in the integral can be further expressed in terms of one of the variables x, y or z or in terms of a parameter t and hence integrated.

If an integral is two-dimensional there are no terms involving z.

The integral $\int_C \underline{F} \cdot \underline{dr}$ evaluates to a scalar.

Example 6

Three paths from $(0,0)$ to $(1,2)$ are defined by

(a) $C_1: \quad y = 2x$
(b) $C_2: \quad y = 2x^2$
(c) $C_3: \quad y = 0$ from $(0,0)$ to $(1,0)$ and $x = 1$ from $(1,0)$ to $(1,2)$

Sketch each path, and along each path find $\int \underline{F} \cdot \underline{dr}$, where $\underline{F} = y^2\underline{i} + xy\underline{j}$.

Solution

(a) $\int \underline{F} \cdot \underline{dr} = \int \left\{y^2 dx + xydy\right\}$. Along $y = 2x$, $\dfrac{dy}{dx} = 2$ so $dy = 2dx$. Then

$$\begin{aligned}\int_{C_1} \underline{F} \cdot \underline{dr} &= \int_{x=0}^{1} \left\{(2x)^2\, dx + x\,(2x)\,(2dx)\right\} \\ &= \int_0^1 \left(4x^2 + 4x^2\right) dx = \int_0^1 8x^2 dx = \left[\frac{8}{3}x^2\right]_0^1 = \frac{8}{3}\end{aligned}$$

Figure 3(a): Integration along path C_1

(b) $\int \underline{F} \cdot \underline{dr} = \int \left\{y^2 dx + xydy\right\}$. Along $y = 2x^2$, $\dfrac{dy}{dx} = 4x$ so $dy = 4xdx$. Then

$$\int_{C_2} \underline{F} \cdot \underline{dr} = \int_{x=0}^{1} \left\{\left(2x^2\right)^2 dx + x\left(2x^2\right)(4xdx)\right\} = \int_0^1 12x^4 dx = \left[\frac{12}{5}x^5\right]_0^1 = \frac{12}{5}$$

Figure 3(b): Integration along path C_2

Note that the answer is different to part (a), i.e., the line integral depends upon the path taken.

(c) As the contour C_3, has two distinct parts with different equations, it is necessary to break the full contour OA into the two parts, namely OB and BA where B is the point $(1, 0)$. Hence

$$\int_{C_3} \underline{F} \cdot \underline{dr} = \int_O^B \underline{F} \cdot \underline{dr} + \int_B^A \underline{F} \cdot \underline{dr}$$

Solution (contd.)

Along OB, $y = 0$ so $dy = 0$. Then

$$\int_O^B \underline{F} \cdot \underline{dr} = \int_{x=0}^{1} \left(0^2 dx + x \times 0 \times 0\right) = \int_0^1 0 dx = 0$$

Along AB, $x = 1$ so $dx = 0$. Then

$$\int_A^B \underline{F} \cdot \underline{dr} = \int_{y=0}^{2} \left(y^2 \times 0 + 1 \times y \times dy\right) = \int_0^2 y dy = \left[\frac{1}{2}y^2\right]_0^2 = 2.$$

Hence $\displaystyle\int_{C_3} \underline{F} \cdot \underline{dr} = 0 + 2 = 2$

Figure 3(c): Integration along path C_3

Once again, the result is path dependent.

Key Point 3

In general, the value of a line integral depends on the path of integration as well as upon the end points.

Example 7

Find $\displaystyle\int_A^O \underline{F}\cdot\underline{dr}$, where $\underline{F} = y^2\underline{i} + xy\underline{j}$ (as in Example 6) and the path C_4 from A to O is the straight line from $(1,2)$ to $(0,0)$, that is the reverse of C_1 in Example 6(a).

Deduce $\displaystyle\oint_C \underline{F}\cdot\underline{dr}$, the integral around the closed path C formed by the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$ and the line $y = 2x$ from $(1,2)$ to $(0,0)$.

Solution

Reversing the path interchanges the limits of integration, which results in a change of sign for the value of the integral.

$$\int_A^O \underline{F}\cdot\underline{dr} = -\int_O^A \underline{F}\cdot\underline{dr} = -\frac{8}{3}$$

The integral along the parabola (calculated in Example 6(b)) evaluates to $\dfrac{12}{5}$, then

$$\oint_C \underline{F}\cdot\underline{dr} = \int_{C_2} \underline{F}\cdot\underline{dr} + \int_{C_4} \underline{F}\cdot\underline{dr} = \frac{12}{5} - \frac{8}{3} = -\frac{4}{15} \approx -0.267$$

Example 8

Consider the vector field

$$\underline{F} = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$$

Let C_1 and C_2 be the curves from $O = (0,0,0)$ to $A = (1,1,1)$, given by

$$\begin{array}{lllll} C_1 & : & x = t, & y = t, & z = t \quad (0 \le t \le 1) \\ C_2 & : & x = t^2, & y = t, & z = t^2 \quad (0 \le t \le 1) \end{array}$$

(a) Evaluate the scalar integral of the vector field along each path.

(b) Find the value of $\displaystyle\oint_C \underline{F}\cdot\underline{dr}$ where C is the closed path along C_1 from O to A and back along C_2 from A to O.

Solution

(a) The path C_1 is given in terms of the parameter t by $x = t$, $y = t$ and $z = t$. Hence

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1 \text{ and } \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

Now by substituting for $x = y = z = t$ in $\underline{F}$ we have

$$\underline{F} = t^5\underline{i} + 2t^5\underline{j} + 3t^5\underline{k}$$

Hence $\underline{F} \cdot \dfrac{d\underline{r}}{dt} = t^5 + 2t^5 + 3t^5 = 6t^5$. The values of $t = 0$ and $t = 1$ correspond to the start and end point of C_1 and so these are the required limits of integration. Now

$$\int_{C_1} \underline{F} \cdot \underline{dr} = \int_0^1 \underline{F} \cdot \frac{d\underline{r}}{dt} dt = \int_0^1 6t^5 dt = \Big[t^6\Big]_0^1 = 1$$

For the path C_2 the parameterisation is $x = t^2$, $y = t$ and $z = t^2$ so $\dfrac{d\underline{r}}{dt} = 2t\underline{i} + \underline{j} + 2t\underline{k}$. Substituting $x = t^2$, $y = t$ and $z = t^2$ in $\underline{F}$ we have

$$\underline{F} = t^8\underline{i} + 2t^9\underline{j} + 3t^8\underline{k} \text{ and } \underline{F} \cdot \frac{d\underline{r}}{dt} = 2t^9 + 2t^9 + 6t^9 = 10t^9$$

$$\int_{C_2} \underline{F} \cdot \underline{dr} = \int_0^1 10t^9 dt = \Big[t^{10}\Big]_0^1 = 1$$

(b) For the closed path C

$$\oint_C \underline{F} \cdot \underline{dr} = \int_{C_1} \underline{F} \cdot \underline{dr} - \int_{C_2} \underline{F} \cdot \underline{dr} = 1 - 1 = 0$$

(Note: A line integral round a closed path is not necessarily zero - see Example 7.)

Further points on Example 8

Vector Field	Path	Line Integral
$\underline{F}$	C_1	1
$\underline{F}$	C_2	1
$\underline{F}$	closed	0

Note that the value of the line integral of $\underline{F}$ is 1 for both paths C_1 and C_2. In fact, this result would hold for any path from $(0, 0, 0)$ to $(1, 1, 1)$.

The field $\underline{F}$ is an example of a **conservative vector field**; these are discussed in detail in the next subsection.

In $\displaystyle\int_C \underline{F} \cdot \underline{dr}$, the vector field $\underline{F}$ may be the gradient of a scalar field or the curl of a vector field.

Task

Consider the vector field

$$\underline{G} = x\underline{i} + (4x - y)\underline{j}$$

Let C_1 and C_2 be the curves from $O = (0, 0, 0)$ to $A = (1, 1, 1)$, given by

$$\begin{array}{lllll} C_1 & : & x = t, & y = t, & z = t \quad (0 \le t \le 1) \\ C_2 & : & x = t^2, & y = t, & z = t^2 \quad (0 \le t \le 1) \end{array}$$

(a) Evaluate the scalar integral $\displaystyle\int_C \underline{G} \cdot \underline{dr}$ of each vector field along each path.

(b) Find the value of $\displaystyle\oint_C \underline{G} \cdot \underline{dr}$ where C is the closed path along C_1 from O to A and back along C_2 from A to O.

Your solution

Answer

(a) The path C_1 is given in terms of the parameter t by $x = t$, $y = t$ and $z = t$. Hence

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1 \text{ and } \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

Substituting for $x = y = z = t$ in $\underline{G}$ we have

$$\underline{G} = t\underline{i} + 3t\underline{j} \text{ and } \underline{G} \cdot \frac{d\underline{r}}{dt} = t + 3t = 4t$$

The limits of integration are $t = 0$ and $t = 1$, then

$$\int_{C_1} \underline{G} \cdot \underline{dr} = \int_0^1 \underline{G} \cdot \frac{d\underline{r}}{dt} dt = \int_0^1 4t dt = \Big[2t^2\Big]_0^1 = 2$$

For the path C_2 the parameterisation is $x = t^2$, $y = t$ and $z = t^2$ so $\dfrac{d\underline{r}}{dt} = 2t\underline{i} + \underline{j} + 2t\underline{k}$.

Substituting $x = t^2$, $y = t$ and $z = t^2$ in $\underline{G}$ we have

$$\underline{G} = t^2\underline{i} + \left(4t^2 - t\right)\underline{j} \text{ and } \underline{G} \cdot \frac{d\underline{r}}{dt} = 2t^3 + 4t^2 - t$$

$$\int_{C_2} \underline{G} \cdot \underline{dr} = \int_0^1 \left(2t^3 + 4t^2 - t\right) dt = \left[\frac{1}{2}t^4 + \frac{4}{3}t^3 - \frac{1}{2}t^2\right]_0^1 = \frac{4}{3}$$

(b) For the closed path C $\quad \displaystyle\oint_C \underline{G} \cdot \underline{dr} = \int_{C_1} \underline{G} \cdot \underline{dr} - \int_{C_2} \underline{G} \cdot \underline{dr} = 2 - \frac{4}{3} = \frac{2}{3}$

(Note: The value of the integral around the closed path is non-zero, unlike Example 8.)

Example 9

Find $\displaystyle\int_C \left\{\underline{\nabla}(x^2y)\right\} \cdot \underline{dr}$ where C is the contour $y = 2x - x^2$ from $(0, 0)$ to $(2, 0)$.

Here, $\underline{\nabla}$ refers to the gradient operator, i.e. $\underline{\nabla}\phi \equiv$ grad ϕ

Solution

Note that $\underline{\nabla}(x^2y) = 2xy\underline{i} + x^2\underline{j}$ so the integral is $\displaystyle\int_C \left\{2xy\, dx + x^2\, dy\right\}$.

On $y = 2x - x^2$, $dy = (2 - 2x)\, dx$ so the integral becomes

$$\begin{aligned}\int_C \left\{2xy\, dx + x^2\, dy\right\} &= \int_{x=0}^2 \left\{2x(2x - x^2)\, dx + x^2(2 - 2x)\, dx\right\} \\ &= \int_0^2 (6x^2 - 4x^3)\, dx = \Big[2x^3 - x^4\Big]_0^2 = 0\end{aligned}$$

Example 10

Two paths from $(0,0)$ to $(4,2)$ are defined by

(a) $C_1: \quad y = \frac{1}{2}x \qquad 0 \le x \le 4$

(b) $C_2:$ The straight line $y = 0$ from $(0,0)$ to $(4,0)$ followed by
$C_3:$ The straight line $x = 4$ from $(4,0)$ to $(4,2)$

For each path find $\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = 2x\underline{i} + 2y\underline{j}$.

Solution

(a) For the straight line $y = \frac{1}{2}x$ we have $dy = \frac{1}{2}dx$

Then, $$\int_{C_1} \underline{F} \cdot \underline{dr} = \int_{C_1} 2x\,dx + 2y\,dy = \int_0^4 \left(2x + \frac{x}{2}\right)\,dx = \int_0^4 \frac{5x}{2}\,dx = 20$$

(b) For the straight line from $(0,0)$ to $(4,0)$ we have $$\int_{C_2} \underline{F} \cdot \underline{dr} = \int_0^4 2x\,dx = 16$$

For the straight line from $(4,0)$ to $(4,2)$ we have $$\int_{C_3} \underline{F} \cdot \underline{dr} = \int_0^2 2y\,dy = 4$$

Adding these two results gives $\int_C \underline{F} \cdot \underline{dr} = 16 + 4 = 20$

Task

Evaluate $\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = (x-y)\underline{i} + (x+y)\underline{j}$ along each of the following paths

(a) C_1 : from $(1,1)$ to $(2,4)$ along the straight line $y = 3x - 2$:

(b) C_2 : from $(1,1)$ to $(2,4)$ along the parabola $y = x^2$:

(c) C_3 : along the straight line $x = 1$ from $(1,1)$ to $(1,4)$ then along the straight line $y = 4$ from $(1,4)$ to $(2,4)$.

Your solution

Answer

(a) $\displaystyle\int_1^2 (10x - 4)\, dx = 11,$

(b) $\displaystyle\int_1^2 (x + x^2 + 2x^3)\, dx = \frac{34}{3}$, (this differs from (a) showing path dependence)

(c) $\displaystyle\int_1^4 (1 + y)\, dy + \int_1^2 (x - 4)\, dx = 8$

Task

For the function $\underline{F}$ and paths in the last Task, deduce $\oint \underline{F} \cdot \underline{dr}$ for the closed paths

(a) C_1 followed by the reverse of C_2.

(b) C_2 followed by the reverse of C_3.

(c) C_3 followed by the reverse of C_1.

Your solution

Answer

(a) $-\dfrac{1}{3}$, (b) $\dfrac{10}{3}$, (c) -3. (note that all these are non-zero.)

Exercises

1. Consider $\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$. Find the value of the line integral along each of the paths from $(0,0)$ to $(1,4)$.

   (a) $y = 4x$ (b) $y = 4x^2$ (c) $y = 4x^{1/2}$ (d) $y = 4x^3$

2. Consider the vector field $\underline{F} = 2x\underline{i} + (xz - 2)\underline{j} + xy\underline{k}$ and the two curves between $(0,0,0)$ and $(1,-1,2)$ defined by

   $C_1 : x = t^2,\ y = -t,\ z = 2t$ for $0 \le t \le 1$.
   $C_2 : x = t - 1,\ y = 1 - t,\ z = 2t - 2$ for $1 \le t \le 2$.

   (a) Find $\int_{C_1} \underline{F} \cdot \underline{dr}$, $\int_{C_2} \underline{F} \cdot \underline{dr}$

   (b) Find $\oint_C \underline{F} \cdot \underline{dr}$ where C is the closed path from $(0,0,0)$ to $(1,-1,2)$ along C_1 and back to $(0,0,0)$ along C_2.

3. Consider the vector field $\underline{G} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3 + y^3)\underline{k}$ and the two curves between $(0,0,0)$ and $(1,-1,2)$ defined by

   $C_1 : x = t^2,\ y = -t,\ z = 2t$ for $0 \le t \le 1$.
   $C_2 : x = t - 1,\ y = 1 - t,\ z = 2t - 2$ for $1 \le t \le 2$.

   (a) Find $\int_{C_1} \underline{G} \cdot \underline{dr}$, $\int_{C_2} \underline{G} \cdot \underline{dr}$

   (b) Find $\oint_C \underline{G} \cdot \underline{dr}$ where C is the closed path from $(0,0,0)$ to $(1,-1,2)$ along C_1 and back to $(0,0,0)$ along C_2.

4. Find $\int_C \underline{F} \cdot \underline{dr})$ along $y = 2x$ from $(0,0)$ to $(2,4)$ for

   (a) $\underline{F} = \underline{\nabla}(x^2y)$
   (b) $\underline{F} = \underline{\nabla} \times (\frac{1}{2}x^2y^2\underline{k})$ [Here $\underline{\nabla} \times \underline{f}$ represents the curl of $\underline{f}$]

Answers

1. All are 12, and in fact the integral would be 12 for **any** path from (0,0) to (1,4).

2 (a) 2, $\frac{5}{3}$ (b) $\frac{1}{3}$.

3 (a) 0, 0 (b) 0.

4. (a) $\int_C 2xy\ dx + x^2\ dy = 16$, (b) $\int_C x^2y\ dx - xy^2\ dy = -24$.

3. Conservative vector fields

For some line integrals in the previous section, the value of the integral depended only on the vector field $\underline{F}$ and the start and end points of the line but not on the actual path between the start and end points. However, for other line integrals, the result depended on the actual details of the path of the line.

Vector fields are classified according to whether the line integrals are path dependent or path independent. Those vector fields for which *all* line integrals between *all* pairs of points are path independent are called **conservative vector fields**.

There are five properties of a conservative vector field (P1 to P5 below). It is impossible to check the value of every line integral over every path, but it *is* possible to use any one of these five properties (particularly property P3 below) to determine whether or not a vector field is conservative. These properties are also used to simplify calculations with conservative vector fields over non-closed paths.

P1 The line integral $\displaystyle\int_A^B \underline{F} \cdot \underline{dr}$ depends only on the end points A and B and is independent of the actual path taken.

P2 The line integral around any closed curve is zero. That is $\displaystyle\oint_C \underline{F} \cdot \underline{dr} = 0$ for all C.

P3 The curl of a conservative vector field $\underline{F}$ is zero i.e. $\underline{\nabla} \times \underline{F} = \underline{0}$.

P4 For any conservative vector field $\underline{F}$, it is possible to find a scalar field ϕ such that $\underline{\nabla}\phi = \underline{F}$. Then, $\displaystyle\oint_C \underline{F} \cdot \underline{dr} = \phi(B) - \phi(A)$ where A and B are the start and end points of contour C. [This is sometimes called the Fundamental Theorem of Line Integrals and is comparable with the Fundamental Theorem of Calculus.]

P5 All gradient fields are conservative. That is, $\underline{F} = \underline{\nabla}\phi$ is a conservative vector field for any scalar field ϕ.

Example 11

Consider the following vector fields.

1. $\underline{F}_1 = y^2\underline{i} + xy\underline{j}$ (Example 6)
2. $\underline{F}_2 = 2x\underline{i} + 2y\underline{j}$ (Example 10)
3. $\underline{F}_3 = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$ (Example 8)
4. $\underline{F}_4 = x\underline{i} + (4x - y)\,\underline{j}$ (Task on page 14)

Determine which of these vector fields are conservative where possible by referring to the answers given in the solution. For those that are conservative find a scalar field ϕ such that $\underline{F} = \underline{\nabla}\phi$ and use property P4 to verify the values of the line integrals.

Solution

1. Two different values were obtained for line integrals over the paths C_1 and C_2. Hence, by P1, $\underline{F}_1$ is not conservative. [It is also possible to reach this conclusion from P3 by finding that $\underline{\nabla} \times \underline{F} = -y\underline{k} \neq \underline{0}$.]

Solution (contd.)

2. For the closed path consisting of C_2 and C_3 from $(0,0)$ to $(4,2)$ and back to $(0,0)$ along C_1 we obtain the value $20+(-20)=0$. This alone does not mean that $\underline{F}_2$ is conservative as there could be other paths giving different values. So by using P3

$$\underline{\nabla}\times\underline{F}_2=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x & 2y & 0\end{vmatrix}=\underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(0-0)=\underline{0}$$

As $\underline{\nabla}\times\underline{F}_2=\underline{0}$, P3 gives that $\underline{F}_2$ is a conservative vector field.

Now, find a ϕ such that $\underline{F}_2=\underline{\nabla}\phi$. Then $\dfrac{\partial\phi}{\partial x}\underline{i}+\dfrac{\partial\phi}{\partial y}\underline{j}=2x\underline{i}+2y\underline{j}$.

Thus

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=2x \quad\Rightarrow\quad \phi=x^2+f(y)\\ \frac{\partial\phi}{\partial y}&=2y \quad\Rightarrow\quad \phi=y^2+g(x)\end{aligned}\right\}\Rightarrow\phi=x^2+y^2(+\text{ constant})$$

Using P4: $\displaystyle\int_{(0,0)}^{(4,2)}\underline{F}_2\cdot\underline{dr}=\int_{(0,0)}^{(4,2)}(\underline{\nabla}\phi)\cdot\underline{dr}=\phi(4,2)-\phi(0,0)=(4^2+2^2)-(0^2+0^2)=20.$

3. The fact that line integrals along two different paths between the same start and end points have the same value is consistent with $\underline{F}_3$ being a conservative field according to P1. So too is the fact that the integral around a closed path is zero according to P2. However, neither fact can be used to *conclude* that $\underline{F}_3$ is a conservative field. This can be done by showing that $\underline{\nabla}\times\underline{F}_3=\underline{0}$.

Now, $$\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2z^3 & 2xyz^3 & 3xy^2z^2\end{vmatrix}=(6xyz^2-6xyz^2)\underline{i}-(3y^2z^2-3y^2z^2)\underline{j}+(2yz^3-2yz^3)\underline{k}=\underline{0}.$$

As $\underline{\nabla}\times\underline{F}_3=\underline{0}$, P3 gives that $\underline{F}_3$ is a conservative field.

To find ϕ that satisfies $\underline{\nabla}\phi=\underline{F}_3$, it is necessary to satisfy

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=y^2z^3 \quad\rightarrow\quad \phi=xy^2z^3+f(y,z)\\ \frac{\partial\phi}{\partial y}&=2xyz^3 \quad\rightarrow\quad \phi=xy^2z^3+g(x,z)\\ \frac{\partial\phi}{\partial z}&=3xy^2z^2 \quad\rightarrow\quad \phi=xy^2z^3+h(x,y)\end{aligned}\right\}\rightarrow\phi=xy^2z^3$$

Using P4: $\displaystyle\int_{(0,0,0)}^{(1,1,1)}\underline{F}_3\cdot\underline{dr}=\phi(1,1,1)-\phi(0,0,0)=1-0=1$ in agreement with Example 8(a).

Solution (contd.)

4. As the integral along C_1 is 2 and the integral along C_2 (same start and end points but different intermediate points) is $\frac{4}{3}$, F_4 is **not** a conservative field using P1.
Note that $\underline{\nabla} \times \underline{F}_4 = 4\underline{k} \neq \underline{0}$ so, using P3, this is an independent conclusion that $\underline{F}_4$ is **not** conservative.

Engineering Example 1

Work done moving a charge in an electric field

Introduction

If a charge, q, is moved through an electric field, $\underline{E}$, from A to B, then the work required is given by the line integral

$$W_{AB} = -q \int_A^B \underline{E} \cdot \underline{dr}$$

Problem in words

Compare the work done in moving a charge through the electric field around a point charge in a vacuum via two different paths.

Mathematical statement of problem

An electric field $\underline{E}$ is given by

$$\begin{aligned} \underline{E} &= \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\underline{r}} \\ &= \frac{Q}{4\pi\varepsilon_0(x^2+y^2+z^2)} \times \frac{x\underline{i}+y\underline{j}+z\underline{k}}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{Q(x\underline{i}+y\underline{j}+z\underline{k})}{4\pi\varepsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}} \end{aligned}$$

where $\underline{r}$ is the position vector with magnitude r and unit vector $\hat{\underline{r}}$, and $\dfrac{1}{4\pi\epsilon_0}$ is a combination of constants of proportionality, where $\epsilon_0 = 10^{-9}/36\pi \text{ F m}^{-1}$.

Given that $Q = 10^{-8}$C, find the work done in bringing a charge of $q = 10^{-10}$C from the point $A = (10, 10, 0)$ to the point $B = (1, 1, 0)$ (where the dimensions are in metres)

(a) by the direct straight line $y = x$, $z = 0$

(b) by the straight line pair via $C = (10, 1, 0)$

Figure 4: Two routes (a and b) along which a charge can move through an electric field

The path comprises two straight lines from $A = (10, 10, 0)$ to $B = (1, 1, 0)$ via $C = (10, 1, 0)$ (see Figure 4).

Mathematical analysis

(a) Here $Q/(4\pi\varepsilon_0) = 90$ so

$$\underline{E} = \frac{90[x\underline{i} + y\underline{j}]}{(x^2 + y^2)^{\frac{3}{2}}}$$

as $z = 0$ over the region of interest. The work done

$$\begin{aligned} W_{AB} &= -q \int_A^B \underline{E} \cdot \underline{dr} \\ &= -10^{-10} \int_A^B \frac{90}{(x^2 + y^2)^{\frac{3}{2}}} \, [x\underline{i} + y\underline{j}] \cdot [dx\underline{i} + dy\underline{j}] \end{aligned}$$

Using $y = x, \;\; dy = dx$

$$\begin{aligned} W_{AB} &= -10^{-10} \int_{x=10}^{1} \frac{90}{(2x^2)^{\frac{3}{2}}} \; \{x \, dx + x \, dx\} \\ &= -10^{-10} \int_{10}^{1} \frac{90}{(2\sqrt{2})} \; x^{-3} \, 2x \, dx \\ &= \frac{90 \times -10^{-10}}{\sqrt{2}} \int_{10}^{1} x^{-2} \, dx \\ &= \frac{9 \times -10^{-9}}{\sqrt{2}} \Big[-x^{-1} \Big]_{10}^{1} \\ &= \frac{9 \times 10^{-9}}{\sqrt{2}} \Big[x^{-1} \Big]_{10}^{1} \\ &= \frac{9 \times 10^{-9}}{\sqrt{2}} [1 - 0.1] \\ &= 5.73 \times 10^{-9} \text{ J} \end{aligned}$$

(b) The first part of the path is A to C where $x = 10$, $dx = 0$ and y goes from 10 to 1.

$$\begin{aligned}
W_{AC} &= -q\int_A^C \underline{E}\cdot\underline{dr} \\
&= -10^{-10}\int_{y=10}^{1} \frac{90}{(100+y^2)^{\frac{3}{2}}}\ [x\underline{i}+y\underline{j}]\cdot[0\underline{i}+dy\underline{j}] \\
&= -10^{-10}\int_{10}^{1} \frac{90y\ dy}{(100+y^2)^{\frac{3}{2}}} \\
&= -10^{-10}\int_{u=200}^{101} \frac{45\ du}{u^{\frac{3}{2}}} \quad \text{(substituting} \quad u = 100+y^2,\ du = 2y\ dy) \\
&= -45\times 10^{-10}\int_{200}^{101} u^{-\frac{3}{2}}\ du \\
&= -45\times 10^{-10}\left[-2u^{-\frac{1}{2}}\right]_{200}^{101} \\
&= 45\times 10^{-10}\left(\frac{2}{\sqrt{101}}-\frac{2}{\sqrt{200}}\right) = 2.59\times 10^{-10}\,\text{J}
\end{aligned}$$

The second part is C to B, where $y = 1$, $dy = 0$ and x goes from 10 to 1.

$$\begin{aligned}
W_{CB} &= -10^{-10}\int_{x=10}^{1} \frac{90}{(x^2+1)^{\frac{3}{2}}}\ [x\underline{i}+y\underline{j}]\cdot[dx\underline{i}+0\underline{j}] \\
&= -10^{-10}\int_{10}^{1} \frac{90x\ dx}{(x^2+1)^{\frac{3}{2}}} \\
&= -10^{-10}\int_{u=101}^{2} \frac{45\ du}{u^{\frac{3}{2}}} \quad \text{(substituting} \quad u = x^2+1,\ du = 2x\ dx) \\
&= -45\times 10^{-10}\int_{101}^{2} u^{-\frac{3}{2}}\ du \\
&= -45\times 10^{-10}\left[-2u^{-\frac{1}{2}}\right]_{101}^{2} \\
&= 45\times 10^{-10}\left(\frac{2}{\sqrt{2}}-\frac{2}{\sqrt{101}}\right) = 5.468\times 10^{-9}\,\text{J}
\end{aligned}$$

The sum of the two components W_{AC} and W_{CB} is 5.73×10^{-9}J.

Therefore the work done over the two paths (a) and (b) is identical.

Interpretation

In fact, the work done is independent of the route taken as the electric field $\underline{E}$ around a point charge in a vacuum is a **conservative** field.

Example 12

1. Show that $I = \int_{(0,0)}^{(2,1)} \left\{(2xy+1)dx + (x^2-2y)dy\right\}$ is independent of the path taken.

2. Find I using property P1. (Page 19)

3. Find I using property P4. (Page 19)

4. Find $I = \oint_C \left\{(2xy+1)dx + (x^2-2y)dy\right\}$ where C is

   (a) the circle $x^2+y^2=1$

   (b) the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Solution

1. The integral $I = \int_{(0,0)}^{(2,1)} \left\{(2xy+1)dx + (x^2-2y)dy\right\}$ may be re-written $\int_C \underline{F} \cdot \underline{dr}$ where $\underline{F} = (2xy+1)\underline{i} + (x^2-2y)\underline{j}$.

   Now $\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+1 & x^2-2y & 0 \end{vmatrix} = 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$

   As $\underline{\nabla} \times \underline{F} = \underline{0}$, $\underline{F}$ is a conservative field and I is independent of the path taken between $(0,0)$ and $(2,1)$.

2. As I is independent of the path taken from $(0,0)$ to $(2,1)$, it can be evaluated along **any** such path. One possibility is the straight line $y = \frac{1}{2}x$. On this line, $dy = \frac{1}{2}dx$. The integral I becomes

$$\begin{aligned} I &= \int_{(0,0)}^{(2,1)} \left\{(2xy+1)dx + (x^2-2y)dy\right\} \\ &= \int_{x=0}^{2} \left\{(2x \times \frac{1}{2}x + 1)dx + (x^2-x)\frac{1}{2}dx\right\} \\ &= \int_0^2 (\frac{3}{2}x^2 - \frac{1}{2}x + 1)dx \\ &= \left[\frac{1}{2}x^3 - \frac{1}{4}x^2 + x\right]_0^2 = 4 - 1 + 2 - 0 = 5 \end{aligned}$$

Solution (contd.)

3. If $\underline{F} = \underline{\nabla}\phi$ then

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= 2xy + 1 \quad \rightarrow \quad \phi = x^2y + x + f(y) \\ \frac{\partial\phi}{\partial y} &= x^2 - 2y \quad \rightarrow \quad \phi = x^2y - y^2 + g(x) \end{aligned}\right\} \rightarrow \phi = x^2y + x - y^2 + C.$$

These are consistent if $\phi = x^2y + x - y^2$ (plus a constant which may be omitted since it cancels).
So $I = \phi(2,1) - \phi(0,0) = (4 + 2 - 1) - 0 = 5$

4. As $\underline{F}$ is a conservative field, all integrals around a closed contour are zero.

Exercises

1. Determine whether the following vector fields are conservative

   (a) $\underline{F} = (x - y)\underline{i} + (x + y)\underline{j}$
   (b) $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$
   (c) $\underline{F} = 2x\underline{i} + (xz - 2)\underline{j} + xy\underline{k}$
   (d) $\underline{F} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3 + y^3)\underline{k}$

2. Consider the integral $\displaystyle\int_C \underline{F} \cdot \underline{dr}$ with $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$. From Exercise 1(b) $\underline{F}$ is a conservative vector field. Find a scalar field ϕ so that $\underline{\nabla}\phi = \underline{F}$. Use property P4 to evaluate the integral $\displaystyle\int_C \underline{F} \cdot \underline{dr}$ where C is an integral with start-point $(0,0)$ and end point $(1,4)$.

3. For the following conservative vector fields $\underline{F}$, find a scalar field ϕ such that $\underline{\nabla}\phi = \underline{F}$ and hence evaluate the $I = \displaystyle\int_C \underline{F} \cdot \underline{dr}$ for the contours C indicated.

   (a) $\underline{F} = (4x^3y - 2x)\underline{i} + (x^4 - 2y)\underline{j}$; any path from $(0,0)$ to $(2,1)$.
   (b) $\underline{F} = (e^x + y^3)\underline{i} + (3xy^2)\underline{j}$; closed path starting from any point on the circle $x^2 + y^2 = 1$.
   (c) $\underline{F} = (y^2 + \sin z)\underline{i} + 2xy\underline{j} + x\cos z\underline{k}$; any path from $(1,1,0)$ to $(2,0,\pi)$.
   (d) $\underline{F} = \dfrac{1}{x}\underline{i} + 4y^3z^2\underline{j} + 2y^4z\underline{k}$; any path from $(1,1,1)$ to $(1,2,3)$.

Answers

1. (a) No, (b) Yes, (c) No, (d) Yes

2. $x^3y^2 - y + C$, 12

3. (a) $x^4y - x^2 - y^2$, 11; (b) $e^x + xy^3$, 0; (c) $xy^2 + x\sin z$, -1; (d) $\ln x + y^4z^2$,143

4. Vector line integrals

It is also possible to form less commonly used integrals of the types:

$$\int_C f(x,y,z)\ \underline{dr} \quad \text{and} \quad \int_C \underline{F}(x,y,z) \times \underline{dr}.$$

Each of these integrals evaluates to a vector.

Remembering that $\underline{dr} = dx\ \underline{i} + dy\ \underline{j} + dz\ \underline{k}$, an integral of the form $\int_C f(x,y,z)\ \underline{dr}$ becomes $\int_C f(x,y,z)dx\ \ \underline{i} + \int_C f(x,y,z)\ \ dy\ \ \underline{j} + \int_C f(x,y,z)dz\ \ \underline{k}$. The first term can be evaluated by expressing y and z in terms of x. Similarly the second and third terms can be evaluated by expressing all terms as functions of y and z respectively. Alternatively, all variables can be expressed in terms of a parameter t. If an integral is two-dimensional, the term in z will be absent.

Example 13

Evaluate the integral $\int_C xy^2\underline{dr}$ where C represents the contour $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution

This is a two-dimensional integral so the term in z will be absent.

$$\begin{aligned}
I &= \int_C xy^2\underline{dr} \\
&= \int_C xy^2(dx\underline{i} + dy\underline{j}) \\
&= \int_C xy^2 dx\ \underline{i} + \int_C xy^2\ dy\ \underline{j} \\
&= \int_{x=0}^{1} x(x^2)^2 dx\ \underline{i} + \int_{y=0}^{1} y^{1/2}y^2\ dy\ \underline{j} \\
&= \int_0^1 x^5 dx\ \underline{i} + \int_0^1 y^{5/2}\ dy\ \underline{j} \\
&= \left[\frac{1}{6}x^6\right]_0^1\ \underline{i} + \left[\frac{2}{7}x^{7/2}\right]_0^1\ \underline{j} \\
&= \frac{1}{6}\underline{i} + \frac{2}{7}\underline{j}
\end{aligned}$$

Example 14

Find $I = \int_C x\underline{dr}$ for the contour C given parametrically by $x = \cos t$, $y = \sin t$, $z = t - \pi$ starting at $t = 0$ and going to $t = 2\pi$, i.e. the contour starts at $(1, 0, -\pi)$ and finishes at $(1, 0, \pi)$.

Solution

The integral becomes $\int_C x(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k})$.

Now, $x = \cos t$, $y = \sin t$, $z = t - \pi$ so $dx = -\sin t\,dt$, $dy = \cos t\,dt$ and $dz = dt$. So

$$\begin{aligned}
I &= \int_0^{2\pi} \cos t(-\sin t\,dt\,\underline{i} + \cos t\,dt\,\underline{j} + dt\,\underline{k}) \\
&= -\int_0^{2\pi} \cos t \sin t\,dt\,\underline{i} + \int_0^{2\pi} \cos^2 t\,dt\,\underline{j} + \int_0^{2\pi} \cos t\,dt\,\underline{k} \\
&= -\frac{1}{2}\int_0^{2\pi} \sin 2t\,dt\,\underline{i} + \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt\,\underline{j} + \Big[\sin t\Big]_0^{2\pi}\,\underline{k} \\
&= \frac{1}{4}\Big[\cos 2t\Big]_0^{2\pi}\,\underline{i} + \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi}\,\underline{j} + 0\underline{k} \\
&= 0\underline{i} + \pi\,\underline{j} = \pi\underline{j}
\end{aligned}$$

Integrals of the form $\int_C \underline{F} \times \underline{dr}$ can be evaluated as follows. If the vector field $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$ and $\underline{dr} = dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}$ then:

$$\begin{aligned}
\underline{F} \times \underline{dr} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix} &= (F_2\,dz - F_3\,dy)\underline{i} + (F_3\,dx - F_1\,dz)\underline{j} + (F_1\,dy - F_2\,dx)\underline{k} \\
&= (F_3\underline{j} - F_2\underline{k})dx + (F_1\underline{k} - F_3\underline{i})dy + (F_2\underline{i} - F_1\underline{j})dz
\end{aligned}$$

There are thus a maximum of six terms involved in one such integral; the exact details may dictate which method to use.

Example 15

Evaluate the integral $\int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr}$ where C represents the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution

Note that the z components of both $\underline{F}$ and $\underline{dr}$ are zero.

$$\underline{F} \times \underline{dr} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x^2 & 3xy & 0 \\ dx & dy & 0 \end{vmatrix} = (x^2dy - 3xydx)\underline{k} \quad \text{and}$$

$$\int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr} = \int_C (x^2dy - 3xydx)\underline{k}$$

Now, on C, $\quad y = 2x^2 \quad dy = 4xdx \quad$ and

$$\begin{aligned} \int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr} &= \int_C \{x^2dy - 3xydx\}\underline{k} \\ &= \int_{x=0}^{1} \left\{x^2 \times 4xdx - 3x \times 2x^2dx\right\} \underline{k} \\ &= \int_0^1 -2x^3dx\underline{k} \\ &= -\left[\frac{1}{2}x^4\right]_0^1 \underline{k} \\ &= -\frac{1}{2}\underline{k} \end{aligned}$$

Engineering Example 2

Force on a loop due to a magnetic field

Introduction

A current $\underline{I}$ in a magnetic field $\underline{B}$ is subject to a force $\underline{F}$ given by

$$\underline{F} = I\ \underline{dr} \times \underline{B}$$

where the current can be regarded as having magnitude I and flowing (positive charge) in the direction given by the vector $\underline{dr}$. The force is known as the Lorentz force and is responsible for the workings of an electric motor. If current flows around a loop, the total force on the loop is given by the integral of $\underline{F}$ around the loop, i.e.

$$\underline{F} = \oint (I\ \underline{dr} \times \underline{B}) = -I \oint (\underline{B} \times \underline{dr})$$

where the closed path of the integral represents one circuit of the loop.

Figure 5: The magnetic field through a loop of current

Problem in words

A current of 1 amp flows around a circuit in the shape of the unit circle in the Oxy plane. A magnetic field of 1 tesla (T) in the positive z-direction is present. Find the total force on the circuit loop.

Mathematical statement of problem

Choose an origin at the centre of the circuit and use polar coordinates to describe the position of any point on the circuit and the length of a small element.

Calculate the line integral around the circuit to give the force required using the given values of current and magnetic field.

Mathematical analysis

The circuit is described parametrically by

$$x = \cos\theta \quad y = \sin\theta \quad z = 0$$

with

$$\begin{aligned} \underline{dr} &= -\sin\theta\ d\theta\ \underline{i} + \cos\theta\ d\theta\ \underline{j} \\ \underline{B} &= B\ \underline{k} \end{aligned}$$

since $\underline{B}$ is constant. Therefore, the force on the circuit is given by

$$\underline{F} = -IB \oint \underline{k} \times \underline{dr} = -\oint \underline{k} \times \underline{dr} \qquad \text{(since } I = 1 \text{ A and } B = 1 \text{ T)}$$

where

$$\begin{aligned} \underline{k} \times \underline{dr} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ -\sin\theta\, d\theta & \cos\theta\, d\theta & 0 \end{vmatrix} \\ &= \left(-\cos\theta\, \underline{i} - \sin\theta\, \underline{j}\,\right) d\theta \end{aligned}$$

So

$$\begin{aligned} \underline{F} &= -\int_{\theta=0}^{2\pi} \left(-\cos\theta\, \underline{i} - \sin\theta\, \underline{j}\,\right) d\theta \\ &= \Big[\sin\theta\, \underline{i} - \cos\theta\, \underline{j}\,\Big]_{\theta=0}^{2\pi} \\ &= (0-0)\, \underline{i} - (1-1)\, \underline{j} = \underline{0} \end{aligned}$$

Hence there is no net force on the loop.

Interpretation

At any given point of the circle, the force on the point opposite is of the same magnitude but opposite direction, and so cancels, leaving a zero net force.

Tip: Use symmetry arguments to avoid detailed calculations whenever possible!

A scalar or vector involved in a vector line integral may itself be a vector derivative as this next Example illustrates.

Example 16

Find the vector line integral $\int_C (\underline{\nabla} \cdot \underline{F})\ \underline{dr}$ where $\underline{F}$ is the vector $x^2\underline{i} + 2xy\underline{j} + 2xz\underline{k}$ and C is the curve $y = x^2$, $z = x^3$ from $x = 0$ to $x = 1$ i.e. from $(0, 0, 0)$ to $(1, 1, 1)$. Here $\underline{\nabla} \cdot \underline{F}$ is the (scalar) divergence of the vector $\underline{F}$.

Solution

As $\underline{F} = x^2\underline{i} + 2xy\underline{j} + 2xz\underline{k}$, $\underline{\nabla} \cdot \underline{F} = 2x + 2x + 2x = 6x$.
The integral

$$\begin{aligned}\int_C (\underline{\nabla} \cdot \underline{F})\ \underline{dr} &= \int_C 6x(dx\ \underline{i} + dy\ \underline{j} + dz\ \underline{k}) \\ &= \int_C 6x\ dx\ \underline{i} + \int_C 6x\ dy\ \underline{j} + \int_C 6x\ dz\ \underline{k}\end{aligned}$$

The first term is

$$\int_C 6x\ dx\ \underline{i} = \int_{x=0}^{1} 6x\ dx\ \underline{i} = \Big[3x^2\Big]_0^1 \underline{i} = 3\underline{i}$$

In the second term, as $y = x^2$ on C, dy may be replaced by $2x\ dx$ so

$$\int_C 6x\ dy\ \underline{j} = \int_{x=0}^{1} 6x \times 2x\ dx\ \underline{j} = \int_0^1 12x^2\ dx\ \underline{j} = \Big[4x^3\Big]_0^1 \underline{j} = 4\underline{j}$$

In the third term, as $z = x^3$ on C, dz may be replaced by $3x^2\ dx$ so

$$\int_C 6x\ dz\ \underline{k} = \int_{x=0}^{1} 6x \times 3x^2\ dx\ \underline{k} = \int_0^1 18x^3\ dx\ \underline{k} = \Big[\frac{9}{2}x^4\Big]_0^1 \underline{k} = \frac{9}{2}\underline{k}$$

On summing, $\int_C (\underline{\nabla} \cdot \underline{F})\ \underline{dr} = 3\underline{i} + 4\underline{j} + \frac{9}{2}\underline{k}$.

Task

Find the vector line integral $\int_C f\underline{dr}$ where $f = x^2$ and C is

(a) the curve $y = x^{1/2}$ from $(0, 0)$ to $(9, 3)$.

(b) the line $y = x/3$ from $(0, 0)$ to $(9, 3)$.

Your solution

Answer

(a) $\displaystyle\int_0^9 (x^2\underline{i} + \frac{1}{2}x^{3/2}\underline{j})dx = 243\underline{i} + \frac{243}{5}\underline{j}$, (b) $\displaystyle\int_0^9 (x^2\underline{i} + \frac{1}{3}x^2\underline{j})dx = 243\underline{i} + 81\underline{j}$.

Task

Evaluate the vector line integral $\displaystyle\int_C \underline{F} \times \underline{dr}$ when C represents the contour $y = 4-4x$, $z = 2-2x$ from $(0, 4, 2)$ to $(1, 0, 0)$ and $\underline{F}$ is the vector field $(x-z)\underline{j}$.

Your solution

Answer

$$\int_0^1 \{(4-6x)\underline{i} + (2-3x)\underline{k}\} = \underline{i} + \frac{1}{2}\underline{k}$$

Exercises

1. Evaluate the vector line integral $\int_C (\underline{\nabla} \cdot \underline{F})\ \underline{dr}$ in the case where $\underline{F} = x\underline{i} + xy\underline{j} + xy^2\underline{k}$ and C is the contour described by $x = 2t$, $y = t^2$, $z = 1 - t$ for t starting at $t = 0$ and going to $t = 1$.

2. When C is the contour $y = x^3$, $z = 0$, from $(0, 0, 0)$ to $(1, 1, 0)$, evaluate the vector line integrals

   (a) $\int_C \{\underline{\nabla}(xy)\} \times \underline{dr}$

   (b) $\int_C \left\{\underline{\nabla} \times (x^2\underline{i} + y^2\underline{k})\right\} \times \underline{dr}$

Answers

1. $\int_C (1 + x)(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}) = 4\underline{i} + \frac{7}{3}\underline{j} - 2\underline{k}$,

2. (a) $\underline{k}\int_C y\ dy - x\ dx = 0\underline{k} = \underline{0}$, (b) $\underline{k}\int_C 2y\ dy = 1\underline{k} = \underline{k}$

Chapter 7

Improper integrals

7.1 Introduction

The goal of this chapter is to meaningfully extend our theory of integrals to *improper* integrals. There are two types of so-called improper integrals: the first involves integrating a function over an infinite domain and the second involves integrands that are undefined at points within the domain of integration. In order to integrate over an infinite domain, we consider limits of the form $\lim_{b\to\infty}\int_a^b f(x)dx$. If the integrand is not defined at c ($a < c < b$) then we split the integral and consider the limits $\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon\to 0}\int_{c+\epsilon}^b f(x)dx$. The latter is sometimes also referred to as improper integrals of the second kind. Such situations occur, for example, for rational functions $f(x) = p(x)/q(x)$ whenever $q(x)$ has zeroes in the domain of integration.

The notions of *convergence* and *divergence* as discussed in Chapter 10 & 11 in the context of sequences and series will be very important to determine these limits. Improper integrals (of both types) arise frequently in applications and in probability. By relating improper integrals to infinite series we derive the last convergence test: the *Integral Comparison test*. As an application we finally prove that the p-series $\sum_{k=1}^{\infty} k^{-p}$ converges for $p > 1$ and diverges otherwise.

7.2 Integration over an infinite domain

We will see that there is a close connection between certain infinite series and improper integrals, which involve integrals over an infinite domain. We have already encountered examples of improper integrals in Section 3.8 and in the context of radioactive decay in Section 8.4. Recall the following definition:

Definition 1 (Improper integral (first kind)). *An improper integral of the first kind is an integral performed over an infinite domain, e.g.*

$$\int_a^{\infty} f(x)\, dx.$$

The value of such an integral is understood to be a limit, as given in the following definition:

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.}$$

We evaluate an improper integral by first computing a definite integral over a finite domain $a \le x \le b$, and then taking a limit as the endpoint b moves off to larger and larger values. The definite integral can be interpreted as an area under the graph of the function. The essential question being addressed here is whether that area remains bounded when we include the "infinite tail" of the function (i.e. as the endpoint b moves to larger values.) For some functions (whose values get small enough, fast enough) the answer is "yes".

Definition 2 (Convergence). *If the limit*

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

exists, we say that the improper integral converges*. Otherwise we say that the improper integral* diverges*.*

With these definitions in mind, we can compute a number of classic integrals.

7.2.1 Example: Decaying exponential

Recall that the improper integral of a decaying exponential converges – we have seen this earlier, in Section 3.8.5, and again in applications in Sections 7.3 and 8.4.1. Here we recapitulate this important result in the context of improper integrals. Suppose that $r > 0$ and let

$$I = \int_0^\infty e^{-rt}\,dt \equiv \lim_{b\to\infty}\int_0^b e^{-rt}\,dt.$$

Then

$$I = \lim_{b\to\infty} -\frac{1}{r}e^{-rt}\Big|_0^b = -\frac{1}{r}\lim_{b\to\infty}(e^{-rb} - e^0) = -\frac{1}{r}(\underbrace{\lim_{b\to\infty} e^{-rb}}_{0} - 1) = \frac{1}{r},$$

where we have used the fact that $\lim_{b\to\infty} e^{-rb} = 0$ for $r > 0$. Thus the limit exists (is finite) and hence the integral *converges*. More precisely, it converges to the value $I = 1/r$.

7.2.2 Example: The improper integral of $1/\mathrm{x}$ diverges

We now consider a classic and counter-intuitive result, and *one of the most important results in this chapter*. Consider the function

$$y = f(x) = \frac{1}{x}.$$

Examining the graph of this function for positive x, e.g. for the interval $(1, \infty)$, we know that values decrease to zero as x increases[26]. The function is not only bounded, but also falls to arbitrarily small values as x increases (see Figure 7.1). Nevertheless, this is *insufficient*

Figure 7.1. *In Sections 7.2.2 and 7.2.3, we consider two functions whose values decrease along the x axis, $f(x) = 1/x$ and $f(x) = 1/x^2$. We show that one, but not the other encloses a finite (bounded) area over the interval $(1, \infty)$. To do so, we compute an improper integral for each one. The heavy arrow is meant to remind us that we are considering areas over an unbounded domain.*

to guarantee that the enclosed area remains finite! We made a similar observation in the context of series in Section 11.3.1. We show this in the following calculation.

$$\begin{aligned} I &= \int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big|_1^b = \lim_{b\to\infty} (\ln(b) - \ln(1)) \\ &= \lim_{b\to\infty} \ln(b) = \infty \end{aligned}$$

The fact that we get an infinite value for this integral follows from the observation that $\ln(b)$ increases without bound as b increases, that is *the limit does not exist (is not finite)*. Thus, the area under the curve $f(x) = 1/x$ over the interval $1 \le x \le \infty$ is infinite. We say that the improper integral of $1/x$ *diverges* (or does not converge). We will use this result again in Section 11.4.2.

7.2.3 Example: The improper integral of $1/x^2$ converges

Now consider the related function

$$y = f(x) = \frac{1}{x^2}$$

and the corresponding integral

$$I = \int_1^\infty \frac{1}{x^2}\,dx.$$

[26] We do not chose the interval $(0, \infty)$ because this function is undefined at $x = 0$. Here we want to emphasize the behaviour at infinity, not the blow up that occurs close to $x = 0$.

Then

$$I = \lim_{b\to\infty} \int_1^b x^{-2}\, dx = \lim_{b\to\infty} \left(-x^{-1}\right)\Big|_1^b = -\lim_{b\to\infty} \left(\frac{1}{b} - 1\right) = 1.$$

Thus, the limit *exists*, and is $I = 1$. In contrast to the example in Section 7.2.2, this integral *converges*.

We observe that the behaviours of the improper integrals of the functions $1/x$ and $1/x^2$ are very different. The former diverges, while the latter converges. The only difference between these functions is the power of x. As shown in Figure 7.1, that power affects how rapidly the graph "falls off" to zero as x increases. The function $1/x^2$ decreases much faster than $1/x$. Consequently $1/x^2$ has a sufficiently "slim" infinite "tail", such that the area under its graph does not become infinite - not an easy concept to digest! This observations leads us to wonder what power p is needed to make the improper integral of a function $1/x^p$ converge. We answer this question below.

7.2.4 When does the integral of $1/x^p$ converge?

Here we consider an arbitrary power, p, that can be any real number. We ask when the corresponding improper integral converges or diverges. Let

$$I = \int_1^\infty \frac{1}{x^p}\, dx.$$

For $p = 1$ we have already established that this integral diverges (see Section 7.2.2), and for $p = 2$ we have seen that it is convergent (see Section 7.2.3). By a similar calculation, we find that

$$I = \lim_{b\to\infty} \frac{x^{1-p}}{(1-p)}\Big|_1^b = \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left(b^{1-p} - 1\right).$$

Thus, this integral converges provided that the term b^{1-p} does not "blow up" as b increases. For this to be true, we require that the exponent $(1-p)$ should be negative, i.e. $1 - p < 0$ or $p > 1$. In this case, we have

$$I = \frac{1}{p-1}.$$

To summarize our result,

$$\boxed{\int_1^\infty \frac{1}{x^p}\, dx \qquad \text{converges if } p > 1, \ \text{ and diverges if } p \le 1.}$$

Examples:

(i) The integral

$$\int_1^\infty \frac{1}{\sqrt{x}}\, dx, \quad \text{diverges.}$$

We see this from the following argument: $\sqrt{x} = x^{\frac{1}{2}}$, so $p = \frac{1}{2} < 1$. Thus, by the general result, this integral diverges.

(ii) The integral

$$\int_1^\infty x^{-1.01}\,dx, \quad \text{converges.}$$

Here $p = 1.01 > 1$, so the result implies convergence of the integral.

7.3 Application: Present value of a continuous income stream

Here we discuss the value of an annuity, which is a kind of savings account that guarantees a continuous stream of income. You would like to pay P dollars to purchase an annuity that will pay you an income $f(t)$ every year from now on, for $t > 0$. In some cases, we might want a constant income every year, in which case $f(t)$ would be constant. More generally, we can consider the case that at each future year t, we ask for income $f(t)$ that could vary from year to year. If the bank interest rate is r, how much should you pay now?

Solution

If we invest P dollars (the "principal" i.e., the amount deposited) in the bank with interest r then the amount $A(t)$ in the account at time t (in years), will grow as follows:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$$

where r is the annual interest rate (e.g. 5%) and n is the number of times per year that interest is compound (e.g. $n = 2$ means interest compounded twice per year, $n = 12$ means monthly compounded interest, etc.). Define $h = \frac{r}{n}$. Then at time t, we have that

$$\begin{aligned} A(t) &= P(1+h)^{\frac{1}{h}rt} \\ &= P\left[(1+h)^{\frac{1}{h}}\right]^{rt} \\ &\approx Pe^{rt} \quad \text{for large } n \text{ or small } h. \end{aligned}$$

Here we have used the fact that when h is small (i.e. frequent intervals of compounding) the expression in square brackets above can be approximated by e, the base of the natural logarithms. Recall that

$$e = \lim_{h\to 0}\left[(1+h)^{\frac{1}{h}}\right].$$

This result was obtained in a first semester calculus course by selecting the base of exponentials such that the derivative of e^x is just e^x itself. Thus, we have found that the amount in the bank at time t will grow as

$$\boxed{A(t) = Pe^{rt}, \quad \text{continually compounded interest.} \qquad (7.1)}$$

Having established the exponential growth of an investment, we return to the question of how to set up an annuity for a continuous stream of income in the future. Rewriting

Eqn. (7.1), the principle amount that we should invest in order to have $A(t)$ to spend at time t is

$$P = A(t)e^{-rt}.$$

Suppose we want to have $f(t)$ spending money for each year t. We refer to the *present value* of year t as the quantity

$$P = f(t)e^{-rt},$$

i.e. we must pay P now, in the present, to get $f(t)$ in a future year t. Summing over all the years, we find that the present value of the continuous income stream is

$$P = \sum_{t=1}^{L} f(t)e^{-rt} \cdot \underbrace{1}_{\text{“}\Delta t\text{”}} \approx \int_0^L f(t)e^{-rt}\, dt,$$

where L is the expected number of years left in the lifespan of the individual to whom this annuity will be paid, and where we have approximated a sum of payments by an integral (of a continuous income stream). One problem is that we do not know in advance how long the lifespan L will be. As a crude approximation, we could assume that this income stream continues forever, i.e. that $L \approx \infty$. In such an approximation, we have to compute the integral:

$$P = \int_0^\infty f(t)e^{-rt}\, dt. \tag{7.2}$$

The integral in Eqn. (7.2) is an **improper integral** (i.e. integral over an unbounded domain), as we have already encountered in Section 3.8.5. We shall have more to say about the properties of such integrals, and about their technical definition, existence, and properties in Chapter 7. We refer to the quantity

$$P = \int_0^\infty f(t)e^{-rt}\, dt, \tag{7.3}$$

as the *present value of a continuous income stream* $f(t)$.

Example: Setting up an annuity

Suppose we want an annuity that provides us with an annual payment of $10,000$ from the bank, i.e. in this case $f(t) = \$10,000$ is a function that has a constant value for every year. Then from Eqn (7.3),

$$P = \int_0^\infty 10000e^{-rt}\, dt = 10000 \int_0^\infty e^{-rt}\, dt.$$

By a previous calculation in Section 3.8.5, we find that

$$P = 10000 \cdot \frac{1}{r},$$

e.g. if interest rate is 5% (and assumed constant over future years), then

$$P = \frac{10000}{0.05} = \$200,000.$$

Therefore, we need to pay \$200,000 today to get $10,000$ annually for every future year.

7.4 Integral comparison test

The integrals discussed above can be used to make comparisons that help us to identify when other improper integrals converge or diverge[27]. The following important result establishes how these comparisons work:

Suppose we are given two functions, $f(x)$ and $g(x)$, both continuous on some infinite interval $[a, \infty)$. Suppose, moreover, that at all points on this interval the first function is smaller than the second, i.e.

$$0 \leq f(x) \leq g(x).$$

Then the following conclusions can be made:[a]

(i) $\displaystyle\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$
The area under $f(x)$ is smaller than the area under $g(x)$.

(ii) If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
If the larger area is finite, so is the smaller one.

(iii) If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.
If the smaller area is infinite, so is the larger one.

[a] These statements have to be carefully noted. What is assumed and what is concluded works "one way". That is the order "if . . . then" is important. Reversing that order leads to a common error.

Example: Determine whether the following integral converges:

$$\int_1^\infty \frac{x}{1+x^3}\,dx.$$

Solution: by noting that for all $x > 0$

$$0 \leq \frac{x}{1+x^3} \leq \frac{x}{x^3} = \frac{1}{x^2}.$$

Note that for $x > 0$

$$0 \leq \frac{x}{1+x^3} \leq \frac{x}{x^3} = \frac{1}{x^2}.$$

Thus,

$$\int_1^\infty \frac{x}{1+x^3}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx.$$

Since the larger integral on the right is known to converge, so does the smaller integral on the left. ♢

[27] Similar ideas will be employed for the comparison of infinite series in Chapter 11. A recurring theme in this course is the close connection between series and integrals, for example, recall the Riemann sums in Chapter 2.

7.5 Integration of an unbounded integrand

The second kind of improper integrals refers to integrands that are undefined at one (or more) points of the domain of integration $[a, b]$. Suppose $f(x)$ is continuous on the open interval (a, b) but becomes infinite at the lower bound, $x = a$. Then the integral of $f(x)$ over the domain $[a+\epsilon, b]$ for $\epsilon > 0$ has a definite value regardless of how small ϵ is chosen. Therefore, we can consider the limit

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)dx,$$

where $\epsilon \to 0^+$ means that ϵ approaches 0 from 'above', i.e. $\epsilon > 0$ always holds. If this limit exists and is equal to L, then we define

$$\int_a^b f(x)dx = L.$$

If an anti-derivative of $f(x)$, say $F(x)$ is known, then the fundamental theorem of calculus permits us to compute

$$\int_{a+\epsilon}^{b} f(x)dx = F(b) - F(a+\epsilon).$$

We are thus led to determine the existence (or nonexistence) of the limit

$$\lim_{\epsilon \to 0^+} F(a+\epsilon).$$

Example 1: Calculate the following integral for $p \neq 1$:

$$I = \int_a^b \frac{dx}{(x-a)^p}.$$

Solution: We interpret the integral as the following limit:

$$I = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} \frac{dx}{(x-a)^p} = -\lim_{\epsilon \to 0^+} \frac{1}{p-1} \left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\epsilon^{p-1}} \right].$$

Thus, for $p > 1$ the term ϵ^{1-p} becomes arbitrarily large as $\epsilon \to 0^+$ and hence the area diverges and the integral does not exist. Conversely, for $p < 1$ the term ϵ^{1-p} converges to 0 as $\epsilon \to 0^+$ and hence the improper integral exists and is

$$I = \frac{(b-a)^{1-p}}{1-p}.$$

Finally, note that for $p = 1$ the anti-derivative is undefined and the integral does not exist. Alternatively, for $p = 1$ we directly see that

$$\int_{a+\epsilon}^{b} \frac{dx}{x-a} = \ln\left(\frac{b-a}{\epsilon}\right)$$

Figure 7.2. *Consider the improper integral* $\int_{-e^2}^{e^2} \ln\left(x^2\right) dx$. *The integrand* $\ln\left(x^2\right)$ *does not exist for* $x = 0$. *Nevertheless, the definite integral exists and equals* $4e^2$.

diverges as $\epsilon \to 0^+$. ♢

Note that for $a = 0$ the above example recovers the integrand $1/x^p$ that was discussed in Section 7.2.4. In particular, we find that the improper integral of the second kind

$$\int_0^1 \frac{1}{x^p} dx$$

exists for $p < 1$ and equals $1/(1-p)$ but does not exist for $p \geq 1$. Conversely, in Section 7.2.4 we observed that the improper integral of the first kind

$$\int_1^\infty \frac{1}{x^p} dx$$

exists for $p > 1$ and equals $1/(p-1)$ but does not exist for $p \leq 1$. Note that for $p = 1$ neither of the integrals exists.

Example 2: Calculate the following integral:

$$I = \int_{-e^2}^{e^2} \ln\left(x^2\right) dx.$$

Solution: A graph of the integrand is shown in Figure 7.2. First, we note that the integrand is not defined at $x = 0$. Therefore, we split the integral into two parts such that

the undefined point marks once the upper and once the lower bound and we write the two integrals as a limit:

$$I = \int_{-e^2}^{e^2} \ln\left(x^2\right) dx = \lim_{\epsilon \to 0^-} \int_{-e^2}^{\epsilon} \ln\left(x^2\right) dx + \lim_{\epsilon \to 0^+} \int_{\epsilon}^{e^2} \ln\left(x^2\right) dx,$$

where $\epsilon \to 0^-$ means ϵ approaches 0 from below, i.e. $\epsilon < 0$ always holds. Second, the integrand is an even function and the integral runs over a symmetric domain. Hence we get

$$I = \lim_{\epsilon \to 0^+} 2 \int_{\epsilon}^{e^2} \ln\left(x^2\right) dx.$$

Check it!

The integral can be solved using the substitution $u = x^2$ followed by an integration by parts. This yields

$$I = \lim_{\epsilon \to 0^+} 2x\left(\ln\left(x^2\right) - 2\right)\Big|_{\epsilon}^{e^2} = 4 \lim_{\epsilon \to 0^+} x\left(\ln(x) - 1\right)\Big|_{\epsilon}^{e^2} = 4e^2 - \lim_{\epsilon \to 0^+} \epsilon(\ln(\epsilon) - 1).$$

The fact that the limit exists and converges to 0 can be seen by setting $\epsilon = 1/k$ and considering the limit $k \to \infty$:

$$\lim_{\epsilon \to 0^+} \epsilon(\ln(\epsilon) - 1) = \lim_{k \to \infty} \frac{1}{k}(\ln(\frac{1}{k}) - 1) = -\lim_{k \to \infty} \frac{1}{k}(\ln(k) + 1).$$

Now we can use either de l'Hôpital's rule or simply recognize that k grows much faster than $\ln k$ and hence the limit converges to 0. Thus, we find that the improper integral exists and is

$$I = 4e^2.$$

$\diamondsuit$

7.6 L'Hôpital's rule

This section introduces a powerful method to evaluate tricky limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$. The rule is named after the French mathematician Guillaume de l'Hôpital, who published it in the 17th century. For our purposes, the rule is often particularly useful to evaluate the limits that arise in improper integrals of the first (unbounded domain) and second kind (unbounded integrand).

Consider two functions, $f(x)$ and $g(x)$, and suppose that the following four prerequisites are satisfied:

(a) $f(x)$ and $g(x)$ are differentiable near $x = a$, but not necessarily at $x = a$.

(b) $g'(x) \neq 0$ for x near a but $x \neq a$.

(c) $\displaystyle\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

(d) and either

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or

(ii) $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$.

Then, l'Hôpital's rule states that

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \qquad (7.4)}$$

Note: values of the limit a can include $\pm\infty$.

In *loose* terms, l'Hôpital's rule can be used if

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{is of type} \quad \frac{0}{0} \quad \text{or} \quad \pm\frac{\infty}{\infty}.$$

Let us now explore the power of l'Hôpital's rule through several examples.

Example 1: Calculate the following limit:

$$\lim_{x\to 0} \frac{\sin x}{e^x - 1}.$$

Solution: In this example l'Hôpital's rule can be used because $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0}(e^x - 1) = 0$. Thus,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{e^x - 1} &= \lim_{x\to 0} \frac{(\sin x)'}{(e^x - 1)'} = \lim_{x\to 0} \frac{\cos x}{e^x} \\
&= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} e^x} \qquad \text{(because of non-zero denominator)} \\
&= \frac{\cos 0}{1} = 1.
\end{aligned}$$

$\diamondsuit$

Example 2: Calculate the following limit:

$$\lim_{t\to\infty} \frac{t \ln t}{t^2 + 1}.$$

Solution: In this example, we can use l'Hôpital's rule because both the numerator and the denominator diverge: $\lim_{t\to\infty}(t\ln t) = \infty$ and $\lim_{t\to\infty}(t^2+1) = \infty$. Thus,

$$\lim_{t\to\infty}\frac{t\ln t}{t^2+1} = \lim_{t\to\infty}\frac{(t\ln t)'}{(t^2+1)'} = \lim_{t\to\infty}\frac{\ln t+1}{2t}.$$

Unfortunately, the new limit that results from applying l'Hôpital's rule remains tricky. However, we can simply apply l'Hôpital's rule again because the prerequisites are still satisfied. In this case, the numerator and denominator still diverge: $\lim_{t\to\infty}(\ln t+1) = \infty$ and $\lim_{t\to\infty} 2t = \infty$. Using l'Hôpital's rule again, we obtain

$$\lim_{t\to\infty}\frac{\ln t+1}{2t} = \lim_{t\to\infty}\frac{(\ln t+1)'}{(2t)'} = \lim_{t\to\infty}\frac{\frac{1}{t}}{2} = \frac{\lim\limits_{t\to\infty}\frac{1}{t}}{2} = 0.$$

Note: As long as *all* prerequisites of l'Hôpital's rule remain satisfied, the rule can be applied repeatedly. ♢

Example 3: Calculate the following limit:

$$\lim_{x\to 0+}\frac{\sin x}{x^2}.$$

Note: a plus-sign (+) added to the limit means that x *approaches the limit from above* (or *from the right*). In the present case, $x > 0$ always holds as x approaches zero. Later in this example we will see why this subtle point is important. In analogy, a minus-sign (−) added to the limit means that *the limit is approached from below* (or *from the left*). If no sign is added, it does not matter whether the limit is approached from above, below or even in an alternating manner.

Solution: Again, l'Hôpital's rule can be used because $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0} x^2 = 0$. Thus,

$$\lim_{x\to 0+}\frac{\sin x}{x^2} = \lim_{x\to 0+}\frac{(\sin x)'}{(x^2)'} = \lim_{x\to 0+}\frac{\cos x}{2x}.$$

Check it!

In order to evaluate this new limit, we might be tempted to apply l'Hôpital's rule again. Is this permissible? Stop for a moment and think about why or why not.

Of course, it is *not* permissible because the numerator $\lim_{x\to 0+}\cos x = 1$ and hence violates the prerequisites for applying l'Hôpital's rule. Ignoring the prerequisites and blindly applying l'Hôpital's rule again would yield an incorrect limit of zero. Instead, we get

$$\lim_{x\to 0+}\frac{\cos x}{2x} = \lim_{x\to 0+}\cos x\cdot\lim_{x\to 0+}\frac{1}{2x} = +\infty.$$

Hence, the limit does not exist, it diverges to $+\infty$.

Check it!

In order to see why it was important to approach the limit from above, $x \to 0+$, calculate the limit as x approaches zero from below:

$$\lim_{x\to 0-} \frac{\sin x}{x^2}.$$

In contrast to the above result, the limit is now $-\infty$ because $x < 0$ always holds as x approaches zero. ♢

Example 4: Calculate the following limit:

$$\lim_{a\to 0+} a \ln a.$$

Note: The limit of zero can only be approached from above, $a \to 0+$, because $\ln a$ is undefined for $a < 0$.

Solution: At first, this limit does not seem to have the correct form to apply l'Hôpital's rule. However, we can use a neat little trick and rewrite the limit as a quotient of two functions:

$$\lim_{a\to 0+} a \ln a = \lim_{a\to 0+} \frac{\ln a}{\frac{1}{a}}.$$

Once rewritten, it becomes apparent that l'Hôpital's rule can indeed be applied because again both the numerator and the denominator diverge: $\lim_{a\to 0+} \ln a = -\infty$ and $\lim_{a\to 0+} 1/a = \infty$. Thus,

$$\begin{aligned}\lim_{a\to 0+} \frac{\ln a}{\frac{1}{a}} &= \lim_{a\to 0+} \frac{(\ln a)'}{(\frac{1}{a})'} = \lim_{a\to 0+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} \\ &= \lim_{a\to 0+} (-a) \quad = 0.\end{aligned}$$

♢

7.7 Summary

The main points of this chapter can be summarized as follows:

1. We reviewed the definition of an improper integral (type one) over an infinite domain:

$$\int_a^\infty f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

2. We computed some examples of improper integrals and discussed their convergence or divergence. We recalled (from earlier chapters) that

$$I = \int_0^\infty e^{-rt}\, dt \quad \text{converges,}$$

whereas

$$I = \int_1^\infty \frac{1}{x}\, dx \quad \text{diverges.}$$

3. More generally, we showed that

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \text{converges if } p > 1, \qquad \text{diverges if } p \leq 1.$$

4. We reviewed the definition of improper integrals (type two) for integrands that are unbounded at either end of the domain of integration, say $x = a$:

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)\,dx.$$

5. If the integrand is not defined at one (or more) point(s) in the interior of the domain of integration, then the integral is split into two (or more) parts and we proceed as above.

6. In particular, we showed that

$$\int_0^1 \frac{1}{x^p}\,dx \qquad \text{converges if } p < 1, \qquad \text{diverges if } p \geq 1.$$

7. L'Hôpital's rule is a powerful tool to evaluate tricky limits that may arise for improper integrals of both kinds. It states that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if $f(a) = g(a) = 0$ or $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$ as well as some, more subtle prerequisites.

7.8 Exercises

Exercise 7.1 Consider the integral

$$A = \int_1^D \frac{1}{x^p}\,dx$$

(a) Sketch a region in the plane whose area represents this if (i) $p > 1$ and (ii) $p < 1$.

(b) Evaluate the integral for $p \neq 1$.

(c) How does the area A depend on the value of D in each of the cases (i) and (ii). Does the area increase without bound as D increases? Or does the area approach some constant?

(d) With this in mind, how might we try to understand an integral of the form

$$\int_1^\infty \frac{1}{x^p}\,dx$$

Exercise 7.2 Which of the following improper integrals converge? Give a reason in each case.

(a) $\displaystyle\int_1^\infty \frac{1}{x^{1.001}}\,dx$ (b) $\displaystyle\int_1^\infty x\,dx$ (c) $\displaystyle\int_1^\infty x^{-3}\,dx$

(d) $\displaystyle\int_0^\infty e^x\,dx$ (e) $\displaystyle\int_0^\infty e^{-2x}\,dx$ (f) $\displaystyle\int_0^\infty x\,e^{-x}\,dx$

Exercise 7.3 The gravitational force between two objects of mass m_1 and m_2 is $F = Gm_1m_2/r^2$ where r is the distance of separation. Initially the objects are a distance D apart. The work done in moving an object from position D to position x against a force F is defined as

$$W = \int_D^x F(r)\,dr.$$

Find the total work needed to move one of these objects infinitely far away.

Exercise 7.4 "Gabriel's Horn" is the surface of revolution formed by rotating the graph of the function $y = f(x) = 1/x$ about the x axis for $1 \leq x < \infty$.

(a) Find the volume of air inside this shape and show that it is finite.

(b) When we cut a cross-section of this horn along the $x-y$-plane, we see a flat area which is wedged between the curves $y = 1/x$ and $y = -1/x$. Show that this "cross-sectional area" is infinite.

(c) The surface area of a surface of revolution generated by revolving the function $y = f(x)$ for $a \leq x \leq b$ about the x-axis is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Write down an integral that would represent the surface area of "Gabriel's Horn".

(d) The integral in part (c) is not easy to evaluate explicitly – i.e. we cannot find an anti-derivative. However, we can show that it diverges. Set up a comparison that shows that the surface area of Gabriel's horn is infinite.

Exercise 7.5 Does the integral $\int_0^\infty \frac{\sin(x)}{x^4+x^2+1}\,dx$ converge or diverge?

Exercise 7.6 Suppose that an airborne disease is introduced into a large population at time $t=0$. At time $t \geqslant 0$, the rate at which this disease is spreading is $r(t) = 4000te^{-4t}$ new infections per day.

(a) After how long is this disease most infectious?

(b) How many people in total acquire the disease?

(c) What eventually happens to the rate of new infections?

Exercise 7.7 Does the integral $\int_0^5 \frac{x-1}{x^2+x-2}\,dx$ converge or diverge?

Exercise 7.8 Suppose that you place \$1,000,000 in an account that earns 5% annual interest, and you wish to withdraw z dollars from this account one year from now, $2z$ dollars two years from now, $3z$ dollars three years from now, and so on. What is the maximum value of z so that you never run out of money?

Exercise 7.9 For which values of p does the integral $\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$ converge?

Exercise 7.10 Evaluate the following limits:

(a) $\lim\limits_{x\to-\infty} xe^{4x}$ (b) $\lim\limits_{x\to\infty} x\sin\left(\frac{1}{x}\right)$ (c) $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\cos(\theta)}{\frac{\pi}{2}-\theta}$

Exercise 7.11 Determine the convergence of the following integrals. If they converge, find their values:

(a) $\int_0^\infty xe^{-x}\,dx$ (b) $\int_0^1 \frac{2x}{\sqrt{1-x^2}}\,dx$ (c) $\int_0^1 \frac{1}{1-x^3}\,dx$

Exercise 7.12 Convergence of an integral of the form $\int_{-\infty}^\infty f(x)\,dx$ is determined by splitting the integral up into two parts:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx.$$

The integral $\int_{-\infty}^{\infty} f(x)\, dx$ is said to converge if **both** of these two integrals converge. Can you come up with an example of a function $f(x)$ for which

$$\lim_{b\to\infty} \int_{-b}^{b} f(x)\, dx < \infty$$

but the integral $\int_{-\infty}^{\infty} f(x)\, dx$ diverges?

Exercise 7.13 Evaluate the integral $\int_0^{\pi} \frac{\cos\theta}{\sqrt{1-\sin^3\theta}}\, d\theta$.

Exercise 7.14 Evaluate the integral $\int_0^{\infty} e^{-x} \sin x\, dx$.

7.9 Solutions

Solution to 7.1

(a) See Figure 7.3

(b) $A = \dfrac{D^{1-p} - 1}{1-p}$

(c) $p > 1$: A approaches a constant as D increases
$p < 1$: A increases with D.

(d) $p > 1$: $A = \dfrac{1}{p-1}$
$p < 1$: A DNE.

DNE: does not exist.

Figure 7.3. *Solution for problem* **??**

Solution to 7.2

(a) convergent, $p > 1$ (b) divergent, $p < 1$ (c) convergent, $p > 1$

(d) divergent, $\to \infty$ (e) convergent, $\dfrac{1}{2}$ (f) convergent, 1

Solution to 7.3 $W = G\, m_1\, m_2 \dfrac{1}{D}$

Solution to 7.4

(a) $V = \pi$ (b) $A \to \infty$

(c) $S = 2\pi \displaystyle\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$ (d) $\dfrac{\sqrt{x^4+1}}{x^3} > \dfrac{1}{x}$ for $x > 1 \Rightarrow S \to \infty$

Solution to 7.5 The integral converges.

Solution to 7.6

(a) after 6 hours (b) 250 people (c) the rate tends to 0

Solution to 7.7 It converges: $\int_0^5 \frac{x-1}{x^2+x-2}\,dx = \int_0^5 \frac{1}{x+2}\,dx = \ln(7) - \ln(2)$.

Solution to 7.8 $z = 2500$ dollars

Solution to 7.9 It converges if $p > 1$ and diverges otherwise.

Solution to 7.10

(a) 0 (b) 1 (c) 1

Solution to 7.11

(a) it converges to 1 (b) it converges to 2 (c) it diverges

Solution to 7.12 One possible example is $f(x) = xe^{x^2}$; any odd function will do.

Solution to 7.13 $$\int_0^{\pi} \frac{\cos\theta}{\sqrt{1-\sin^3\theta}}\,d\theta = 0.$$

Solution to 7.14 $$\int_0^{\infty} e^{-x}\sin x\,dx = \frac{1}{2}.$$

Section 8.8: Improper Integrals

Switching up the Limits of Integration: Up until now, we have required two properties of *definite* integral:

1. the domain of integration, $[a, b]$, is finite
2. the range of the integrand is finite on this domain.

We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.

First, we examine what the area looks like over finite intervals. That is, we integrate over $[0, b]$.

$$A(b) := \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2} - \left[-2e^{-0/2}\right] = 2\left(1 - e^{-b/2}\right).$$

Now we have an expression for the area over a finite integral, we can let $b \longrightarrow \infty$ by calculating the limit of this expression.

$$A = \lim_{b\to\infty} A(b) = \lim_{b\to\infty} 2\left(1 - e^{-b/2}\right) = 2\,(1 - 0) = 2.$$

So,

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$

So this is how we deal with infinite limits of integration - with a limit! Remember those?

Definition: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty} \int_{-a}^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we sat that the improper integral _____converges_____ and that the limit is the _____value_____ of the improper integral. If the limit fails to exist, the improper integral _____diverges_____

Any of the integrals in the above definition can be interpreted as an area if $f(x) \geq 0$ on the interval of integration. If $f(x) \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1: Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

$$u = \ln(x) \qquad dv = \frac{1}{x^2}\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$$

$$\begin{aligned}
\int_1^b \frac{\ln(x)}{x^2}\,dx &= -\frac{\ln(x)}{x}\bigg|_1^b - \int_1^b -\frac{1}{x^2}\,dx \\
&= -\frac{\ln(x)}{x} - \frac{1}{x}\bigg|_1^b \\
&= -\frac{\ln(b)}{b} - \frac{1}{b} - \left[-\frac{\ln(1)}{1} - \frac{1}{1}\right] \\
&= -\frac{\ln(b)}{b} - \frac{1}{b} + 1
\end{aligned}$$

Now we take a limit,

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b} - \frac{1}{b} + 1\right] = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b}\right] - 0 + 1 \overset{\text{L'H}}{=} \lim_{b\to\infty}\left[-\frac{1/b}{1}\right] + 1 = 0 + 1 = \boxed{1}$$

L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that $f(x)$ and $g(x)$ are differentiable on an open interval I containing a and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the left and right both exist.

Example 2: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

According to part 3 of our definition, we can choose any real number c and split this integral into two integrals and then apply parts 1 and 2 to each piece. Let's choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Now we will evaluate each piece separately.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to-\infty} \tan^{-1}(x)\Big|_{1}^{0} \\
&= \lim_{a\to-\infty} \tan^{-1}(0) - \tan^{-1}(a) \\
&= \lim_{a\to-\infty} -\tan^{-1}(a) \\
&= \frac{\pi}{2},
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx \\
&= \lim_{b\to\infty} \tan^{-1}(x)\Big|_{0}^{b} \\
&= \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(0) \\
&= \lim_{b\to\infty} \tan^{-1}(b) \\
&= \frac{\pi}{2}.
\end{aligned}$$

So,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Since $1/(1+x^2) > 0$ on $\mathbb{R}$, the improper integral can be interpreted as the (finite) area between the curve and the x-axis.

A Special Example: For what values of p does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

We split this investigation into two cases; when $p \neq 1$ and when $p = 1$.

If $p \neq 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^b \\ &= \lim_{b\to\infty} \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\bigg|_1^b \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p<1.\end{cases}\end{aligned}$$

If $p = 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln(x)\bigg|_1^b \\ &= \lim_{b\to\infty}[\ln(b)-\ln(1)] \\ &= \lim_{b\to\infty}\ln(b) = \infty\end{aligned}$$

Combining these two results we have

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p\le 1\end{cases}}$$

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration. We apply a similar technique as in the previous examples of integrating over an altered interval before obtaining the integral we want by taking limits.

Example 4: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

First we find the integral over the region $[a, 1]$ where $0 < a \le 1$.

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \int_a^1 x^{-1/2}\,dx = 2x^{1/2}\bigg|_a^1 = 2\sqrt{x}\bigg|_a^1 = 2 - 2\sqrt{a} = 2(1-\sqrt{a}).$$

Then we find the limit as $a \to 0^+$:

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\left(1-\sqrt{a}\right) = 2.$$

Therefore,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \boxed{2}$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^a f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we sat that the improper integral _____converges_____ and that the limit is the _____value_____ of the improper integral. If the limit fails to exist, the improper integral _____diverges_____

Example 5: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

$$\begin{aligned}
\int_0^1 \frac{1}{1-x}\,dx &= \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx \\
&= \lim_{b\to 1^-} -\int_0^b \frac{1}{x-1}\,dx \\
&= \lim_{b\to 1^-} -\ln|x-1|\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(x-1)\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(1-b) \\
&= -(-\infty) \\
&= \boxed{\infty}
\end{aligned}$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Why does this make sense?

1. If the area under the curve of $g(x)$ is *finite* and $f(x)$ is bounded above by $g(x)$ (and below by 0), then the area under the curve of $f(x)$ must be *less than or equal to* the area under the curve of $g(x)$. A positive number less that a *finite* number is also *finite*.
2. If the area under the curve of $f(x)$ is *infinite* and $g(x)$ is bounded below by $f(x)$, then the area under the curve of $g(x)$ must be *"less than or equal to"* the area under the curve of $g(x)$. Since there is no finite number "greater than" infinity, the area under $g(x)$ must also be *infinite*.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

We want to find a function $g(x)$ such that for some $a \in \mathbb{R}$, $f(x) = \dfrac{\cos^2(x)}{x^2} \leq g(x)$ or $f(x) = \dfrac{\cos^2(x)}{x^2} \geq g(x)$ for all $x \geq a$. One way we can do this is by finding *bounds* for $f(x)$. Since $0 \leq \cos^2(x) \leq 1$ for all x,

$$\frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}.$$

So then we can use $g(x) := \dfrac{1}{x^2}$. So,

$$0 \leq \int_2^\infty \frac{\cos^2(x)}{x^2}\,dx \leq \int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{b} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$$

So $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges.

Example 7: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Since $x \geq x - e^{-x}$, $f(x) := \dfrac{1}{x} \leq \dfrac{1}{x - e^{-1}} =: g(x)$ for all $x \geq 3$. So,

$$0 \leq \int_3^\infty f(x)\,dx \leq \int_3^\infty g(x)\,dx.$$

By the Direct Comparison Test then, $\displaystyle\int_3^\infty \frac{1}{x - e^{-x}}\,dx$ *diverges* since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges.

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Why does this make sense? The convergence is really only dependent on the "tail" of the integral. That is, the convergence is dictated by what happens "at infinity." If for sufficiently large values of x, $f(x) \approx Lg(x)$ and one of the two integrals converges, then the other one should also converge, since it is only off by "about a scalar multiple." The same goes for diverging, if one diverges, then multiplying it by a positive number won't suddenly make it converge, so the other one should also diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Let $f(x) := \dfrac{1}{1+x^2}$ and $g(x) := \dfrac{1}{x^2}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1+x^2-1}{1+x^2} = \lim_{x\to\infty} \left(1 - \frac{1}{1+x^2}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ *converges.*

Example 9: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.

Let $f(x) := \dfrac{1-e^{-x}}{x}$ and $g(x) := \dfrac{1}{x}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(1 - e^{-x}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ *diverges.*

Integral Vector Theorems

29.3

Introduction

Various theorems exist relating integrals involving vectors. Those involving line, surface and volume integrals are introduced here.

They are the multivariable calculus equivalent of the fundamental theorem of calculus for single variables ("integration and differentiation are the reverse of each other").

Use of these theorems can often make evaluation of certain vector integrals easier. This Section introduces the main theorems which are Gauss' divergence theorem, Stokes' theorem and Green's theorem.

Prerequisites

Before starting this Section you should ...

- be able to find the gradient of a scalar field and the divergence and curl of a vector field
- be familiar with the integration of vector functions

Learning Outcomes

On completion you should be able to ...

- use vector integral theorems to facilitate vector integration

1. Stokes' theorem

This is a theorem that equates a line integral to a surface integral. For any vector field $\underline{F}$ and a contour C which bounds an area S,

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Figure 16: A surface for Stokes' theorem

Notes

(a) $\underline{dS}$ is a vector perpendicular to the surface S and $\underline{dr}$ is a line element along the contour C. The sense of $\underline{dS}$ is linked to the direction of travel along C by a right hand screw rule.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a line integral along a contour composed of several distinct parts (e.g. a square or other figure).

(d) $\underline{\nabla} \times \underline{F}$ is a vector field representing the curl of the vector field $\underline{F}$ and may, alternatively, be written as curl $\underline{F}$.

Justification of Stokes' theorem

Imagine that the surface S is divided into a set of infinitesimally small rectangles $ABCD$ where the axes are adjusted so that AB and CD lie parallel to the new x-axis i.e. $AB = \delta x$ and BC and AD lie parallel to the new y-axis i.e. $BC = \delta y$.

Now, $\oint_C \underline{F} \cdot \underline{dr}$ is calculated, where C is the boundary of a typical such rectangle.

The contributions along AB, BC, CD and DA are

$$\underline{F}(x, y, 0) \cdot \underline{\delta x} = F_x(x, y, z)\delta x,$$

$$\underline{F}(x + \delta x, y, 0) \cdot \underline{\delta y} = F_y(x + \delta x, y, z)\delta y,$$

$$\underline{F}(x, y + \delta y, 0) \cdot (-\underline{\delta x}) = -F_x(x, y + \delta y, z)\delta x$$

$$\underline{F}(x, y, 0) \cdot (-\underline{\delta x}) = -F_y(x, y, z)\delta y.$$

Thus,

$$\begin{aligned}
\oint_C \underline{F} \cdot \underline{dr} &\approx (F_x(x, y, z) - F_x(x, y + \delta y, z))\delta x + (F_y(x + \delta x, y, z) - F_y(x, y, z))\delta y \\
&\approx \frac{\partial F_y}{\partial x}\delta x \delta y - \frac{\partial F_x}{\partial y}\delta x \delta y \\
&\approx (\underline{\nabla} \times \underline{F})_z \delta S \\
&= (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}
\end{aligned}$$

as $\underline{dS}$ is perpendicular to the x- and y- axes.

Thus, for each small rectangle, $\oint_C \underline{F} \cdot \underline{dr} \approx (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$

When the contributions over all the small rectangles are summed, the line integrals along the inner parts of the rectangles cancel and all that remains is the line integral around the outside of the surface S. The surface integrals sum. Hence, the theorem applies for the area S bounded by the contour C. While the above does not constitute a formal proof of Stokes' theorem, it does give an appreciation of the origin of the theorem.

Figure 17: Line integral cancellation and non-cancellation

Key Point 8

Stokes' Theorem

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$$

The closed contour integral of the scalar product of a vector function with the vector along the contour is equal to the integral of the scalar product of the curl of that vector function and the unit normal, over the corresponding surface.

Example 32

Verify Stokes' theorem for the vector function $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ and the unit square $0 \le x \le 1$, $0 \le y \le 1$, $z = 0$.

Solution

If $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ then $\underline{\nabla} \times \underline{F} = (z+1)\underline{i} + (-1-2y)\underline{k} = \underline{i} + (-1-2y)\underline{k}$ (as $z = 0$). Note that $\underline{dS} = dxdy\underline{k}$ so that $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (-1-2y)dydx$

Thus
$$\begin{aligned}\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{1} (-1-2y)dydx \\ &= \int_{x=0}^{1} \Big[(-y-y^2)\Big]_{y=0}^{1} dx = \int_{x=0}^{1} (-2)dx \\ &= \Big[-2x\Big]_0^1 = -2+0 = -2\end{aligned}$$

To evaluate $\oint_C \underline{F} \cdot \underline{dr}$, we must consider the four sides separately.

When $y = 0$, $\underline{F} = -x\underline{j}$ and $\underline{dr} = dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = 0$ i.e. the contribution of this side to the integral is zero.

When $x = 1$, $\underline{F} = y^2\underline{i} - \underline{j}$ and $\underline{dr} = dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = -dy$ so the contribution to the integral is

$$\int_{y=0}^{1} (-dy) = \Big[-y\Big]_0^1 = -1.$$

When $y = 1$, $\underline{F} = \underline{i} - x\underline{j}$ and $\underline{dr} = -dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = -dx$ so the contribution to the integral is

$$\int_{x=0}^{1} (-dx) = \Big[-x\Big]_0^1 = -1.$$

When $x = 0$, $\underline{F} = y^2\underline{i}$ and $\underline{dr} = -dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = 0$ so the contribution to the integral is zero.

The integral $\oint_C \underline{F} \cdot \underline{dr}$ is the sum of the contributions i.e. $0 - 1 - 1 + 0 = -2$.

Thus $\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr} = -2$ i.e. Stokes' theorem has been verified.

Example 33

Using cylindrical polar coordinates verify Stokes' theorem for the function $\underline{F} = \rho^2\underline{\hat{\phi}}$ the circle $\rho = a, z = 0$ and the surface $\rho \leq a, z = 0$.

Solution

Firstly, find $\oint_C \underline{F} \cdot \underline{dr}$. This can be done by integrating along the contour $\rho = a$ from $\phi = 0$ to $\phi = 2\pi$. Here $\underline{F} = a^2\underline{\hat{\phi}}$ (as $\rho = a$) and $\underline{dr} = a\ d\phi\ \underline{\hat{\phi}}$ (remembering the scale factor) so $\underline{F}\cdot\underline{dr} = a^3 d\phi$ and hence

$$\oint_C \underline{F} \cdot \underline{dr} = \int_0^{2\pi} a^3 d\phi = 2\pi a^3$$

As $\underline{F} = \rho^2\underline{\hat{\phi}}$, $\underline{\nabla} \times \underline{F} = 3\rho\underline{\hat{z}}$ and $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 3\rho$ as $\underline{dS} = \underline{\hat{z}}$.
Thus

$$\begin{aligned}\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} &= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{1} 3\rho \times \rho d\rho d\phi = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} 3\rho^2 d\rho d\phi \\ &= \int_{\phi=0}^{2\pi} \left[\rho^3\right]_{\rho=0}^{a} d\phi = \int_0^{2\pi} a^3 d\phi = 2\pi a^3\end{aligned}$$

Hence

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 2\pi a^3$$

Example 34

Find the closed line integral $\oint_C \underline{F}\cdot\underline{dr}$ for the vector field $\underline{F} = y^2\underline{i} + (x^2 - z)\underline{j} + 2xy\underline{k}$ and for the contour $ABCDEFGHA$ in Figure 18.

Figure 18: Closed contour $ABCDEFGHA$

Solution

To find the line integral directly would require eight line integrals i.e. along AB, BC, CD, DE, EF, FG, GH and HA. It is easier to carry out a surface integral to find $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ which is equal to the required line integral $\displaystyle\oint_C \underline{F} \cdot \underline{dr}$ by Stokes' theorem.

As $\underline{F} = y^2\underline{i} + (x^2 - z)\underline{j} + 2xy\underline{k}$, $\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 - z & 2xy \end{vmatrix} = (2x+1)\underline{i} - 2y\underline{j} + (2x - 2y)\underline{k}$

As the contour lies in the x-y plane, the unit normal is $\underline{k}$ and $\underline{dS} = dxdy\underline{k}$

Hence $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (2x - 2y)dxdy$.

To work out $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$, it is necessary to divide the area inside the contour into two smaller areas i.e. the rectangle $ABCDGH$ and the trapezium $DEFG$. On $ABCDGH$, the integral is

$$\begin{aligned} \int_{y=0}^{4}\int_{x=0}^{6} (2x-2y)dxdy &= \int_{y=0}^{4} \Big[x^2 - 2xy\Big]_{x=0}^{6} dy = \int_{y=0}^{4} (36 - 12y)dy \\ &= \Big[36y - 6y^2\Big]_0^4 = 36 \times 4 - 6 \times 16 - 0 = 48 \end{aligned}$$

On $DEFG$, the integral is

$$\begin{aligned} \int_{y=4}^{7}\int_{x=1}^{y-2} (2x-2y)dxdy &= \int_{y=4}^{7} \Big[x^2 - 2xy\Big]_{x=1}^{y-2} dy = \int_{y=4}^{7} (-y^2 + 2y + 3)dy \\ &= \Big[-\frac{1}{3}y^3 + y^2 + 3y\Big]_4^7 = -\frac{343}{3} + 49 + 21 + \frac{64}{3} - 16 - 12 = -51 \end{aligned}$$

So the full integral is, $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 48 - 51 = -3$.

$\therefore$ By Stokes' theorem, $\displaystyle\oint_C \underline{F} \cdot \underline{dr} = -3$

From Stokes' theorem, it can be seen that surface integrals of the form $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ depend only on the contour bounding the surface and not on the internal part of the surface.

Task

Verify Stokes' theorem for the vector field $\underline{F} = x^2\underline{i} + 2xy\underline{j} + z\underline{k}$ and the triangle with vertices at $(0, 0, 0)$, $(3, 0, 0)$ and $(3, 1, 0)$.

First find the normal vector $\underline{dS}$:

Your solution

Answer

$dxdy\underline{k}$

Then find the vector $\underline{\nabla} \times \underline{F}$:

Your solution

Answer

$2y\underline{k}$

Now evaluate the double integral $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ over the triangle:

Your solution

Answer

1

Finally find the integral $\int \underline{F} \cdot \underline{dr}$ along the 3 sides of the triangle and so verify that the two sides of the Stokes' theorem are equal:

Your solution

Answer

$9 + 3 - 11 = 1$, Both sides of Stokes' theorem have value 1.

Exercises

1. Using plane-polar coordinates (or cylindrical polar coordinates with $z = 0$), verify Stokes' theorem for the vector field $\underline{F} = \rho\underline{\hat{\rho}} + \rho\cos\left(\frac{\pi\rho}{2}\right)\underline{\hat{\phi}}$ and the semi-circle $\rho \leq 1$, $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$.

2. Verify Stokes' theorem for the vector field $\underline{F} = 2x\underline{i} + (y^2 - z)\underline{j} + xz\underline{k}$ and the contour around the rectangle with vertices at $(0, -2, 0)$,$(2, -2, 0)$, $(2, 0, 1)$ and $(0, 0, 1)$.

3. Verify Stokes' theorem for the vector field $\underline{F} = -y\underline{i} + x\underline{j} + z\underline{k}$

   (a) Over the triangle $(0, 0, 0), (1, 0, 0), (1, 1, 0)$.

   (b) Over the triangle $(1, 0, 0), (1, 1, 0), (1, 1, 1)$.

4. Use Stokes' theorem to evaluate the integral

   $$\oint_C \underline{F} \cdot \underline{dr} \quad \text{where} \quad \underline{F} = \left(\sin(\frac{1}{x} + 1) + 5y\right)\underline{i} + (2x - e^{y^2})\underline{j}$$

   and C is the contour starting at $(0, 0)$ and going to $(5, 0)$, $(5, 2)$, $(6, 2)$, $(6, 5)$, $(3, 5)$, $(3, 2)$, $(0, 2)$ and returning to $(0, 0)$.

Answers

1. Both integrals give 0,

2. Both integrals give 1

3. (a) Both integrals give 1 (b) Both integrals give 0 (as $\underline{\nabla} \times \underline{F}$ is perpendicular to $\underline{dS}$)

4. -57, $[\underline{\nabla} \times \underline{F} = -3\,\underline{k}]$.

2. Gauss' theorem

This is sometimes known as the **divergence theorem** and is similar in form to Stokes' theorem but equates a surface integral to a volume integral. Gauss' theorem states that for a volume V, bounded by a closed surface S, any 'well-behaved' vector field $\underline{F}$ satisfies

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F}\, dV$$

Notes:

(a) $\underline{dS}$ is a unit normal pointing outwards from the interior of the volume V.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a surface integral over a surface composed of several distinct parts (e.g. a cube).

(d) $\underline{\nabla} \cdot \underline{F}$ is a scalar field representing the divergence of the vector field $\underline{F}$ and may, alternatively, be written as div $\underline{F}$.

(e) Gauss' theorem can be justified in a manner similar to that used for Stokes' theorem (i.e. by proving it for a small volume element, then summing up the volume elements and allowing the internal surface contributions to cancel.)

Key Point 9

Gauss' Theorem

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} dV$$

The closed surface integral of the scalar product of a vector function with the unit normal (or flux of a vector function through a surface) is equal to the integral of the divergence of that vector function over the corresponding volume.

Example 35

Verify Gauss' theorem for the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and the function $\underline{F} = x\underline{i} + z\underline{j}$

Solution

To find $\displaystyle\iint_S \underline{F} \cdot \underline{dS}$, the integral must be evaluated for all six faces of the cube and the results summed.

On the face $x = 0$, $\underline{F} = z\underline{j}$ and $\underline{dS} = -\underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\, dydz = 0$$

On the face $x = 1$, $\underline{F} = \underline{i} + z\underline{j}$ and $\underline{dS} = \underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 1\, dydz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 1\, dydz = 1$$

On the face $y = 0$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = -\underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = -z\, dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = -\int_0^1 \int_0^1 z\, dxdz = -\frac{1}{2}$$

On the face $y = 1$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = \underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = z\, dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 z\, dxdz = \frac{1}{2}$$

On the face $z = 0$, $\underline{F} = x\underline{i}$ and $\underline{dS} = -\underline{k}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0\, dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\, dxdy = 0$$

On the face $z = 1$, $\underline{F} = x\underline{i} + \underline{j}$ and $\underline{dS} = \underline{k}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0\, dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\, dxdy = 0$$

Thus, summing over all six faces, $\displaystyle\iint_S \underline{F} \cdot \underline{dS} = 0 + 1 - \frac{1}{2} + \frac{1}{2} + 0 + 0 = 1.$

To find $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\, dV$ note that $\underline{\nabla} \cdot \underline{F} = \dfrac{\partial}{\partial x}x + \dfrac{\partial}{\partial y}z = 1 + 0 = 1.$

So $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\, dV = \int_0^1 \int_0^1 \int_0^1 1\, dxdydz = 1.$

So $\displaystyle\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F}\, dV = 1$ hence verifying Gauss' theorem.

Note: The volume integral needed just one triple integral, but the surface integral required six double integrals. Reducing the number of integrals is often the motivation for using Gauss' theorem.

Example 36

Use Gauss' theorem to evaluate the surface integral $\int\int_S \underline{F} \cdot \underline{dS}$ where $\underline{F}$ is the vector field $x^2y\underline{i} + 2xy\underline{j} + z^3\underline{k}$ and S is the surface of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

Solution

Note that to carry out the surface integral directly will involve, as in Example 35, the evaluation of six double integrals. However, by Gauss' theorem, the same result comes from the volume integral $\int\int\int_V \underline{\nabla} \cdot \underline{F}\, dV$. As $\underline{\nabla} \cdot \underline{F} = 2xy + 2x + 3z^2$, we have the triple integral

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_0^1 (2xy + 2x + 3z^2)\, dxdydz \\
= & \int_0^1 \int_0^1 \left[x^2y + x^2 + 3xz^2 \right]_{x=0}^1 dydz = \int_0^1 \int_0^1 (y + 1 + 3z^2) dydz \\
= & \int_0^1 \left[\frac{1}{2}y^2 + y + 3yz^2 \right]_{y=0}^1 dz = \int_0^1 (\frac{1}{2} + 1 + 3z^2) dz = \int_0^1 (\frac{3}{2} + 3z^2) dz \\
= & \left[\frac{3}{2}z + z^3 \right]_0^1 = \frac{5}{2}
\end{aligned}$$

The six double integrals would also sum to $\frac{5}{2}$ but this approach would require much more effort.

Engineering Example 5

Gauss' law

Introduction

From Gauss' theorem, it is possible to derive a result which can be used to gain insight into situations arising in Electrical Engineering. Knowing the electric field on a closed surface, it is possible to find the electric charge within this surface. Alternatively, in a sufficiently symmetrical situation, it is possible to find the electric field produced by a given charge distribution.
Gauss' theorem states

$$\int\int_S \underline{F} \cdot \underline{dS} = \int\int\int_V \underline{\nabla} \cdot \underline{F}\, dV$$

If $\underline{F} = \underline{E}$, the electric field, it can be shown that,

$$\underline{\nabla} \cdot \underline{F} = \underline{\nabla} \cdot \underline{E} = \frac{q}{\varepsilon_0}$$

where q is the amount of charge per unit volume, or charge density, and ε_0 is the permittivity of free space: $\varepsilon_0 = 10^{-9}/36\pi$ F m^{-1} $\approx$ 8.84$\times 10^{-12}$ F m^{-1}. Gauss' theorem becomes in this case

$$\int\int_S \underline{E}\cdot\underline{dS} = \iiint_V \underline{\nabla}\cdot\underline{E}\,dV = \iiint_V \frac{q}{\varepsilon_0}\,dV = \frac{1}{\varepsilon_0}\iiint_V q\,dV = \frac{Q}{\varepsilon_0}$$

i.e.

$$\int\int_S \underline{E}\cdot\underline{dS} = \frac{Q}{\varepsilon_0}$$

which is known as Gauss' law. Here Q is the total charge inside the surface S.

Note: this is one of the important Maxwell's Laws.

Problem in words

A point charge lies at the centre of a cube. Given the electric field, find the magnitude of the charge, using Gauss' law .

Mathematical statement of problem

Consider the cube $-\frac{1}{2} \le x \le \frac{1}{2}$, $-\frac{1}{2} \le y \le \frac{1}{2}$, $-\frac{1}{2} \le z \le \frac{1}{2}$ where the dimensions are in metres. A point charge Q lies at the centre of the cube. If the electric field on the top face $(z = \frac{1}{2})$ is given by

$$\underline{E} = 10\frac{x\underline{i} + y\underline{j} + z\underline{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

find the charge Q from Gauss' law .

$$\left[\text{Hint : } \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}}\left(x^2+y^2+\frac{1}{4}\right)^{-\frac{3}{2}}\,dy\,dx = \frac{4\pi}{3}\right]$$

Mathematical analysis

From Gauss' law

$$\int\int_S \underline{E}\cdot\underline{dS} = \frac{Q}{\varepsilon_0}$$

so

$$Q = \varepsilon_0\int\int_S \underline{E}\cdot\underline{dS} = 6\varepsilon_0\int\int_{S(top)} \underline{E}\cdot\underline{dS}$$

since, using the symmetry of the six faces of the cube, it is possible to integrate over just one of them (here the top face is chosen) and multiply by 6. On the top face

$$\underline{E} = 10\frac{x\underline{i} + y\underline{j} + \frac{1}{2}\underline{k}}{\left(x^2+y^2+\frac{1}{4}\right)^{\frac{3}{2}}}$$

and

$$\begin{aligned}\underline{dS} &= \text{(element of surface area)} \times \text{(unit normal)}\\ &= dx\,dy\,\underline{k}\end{aligned}$$

So

$$\begin{aligned}\underline{E}\cdot\underline{dS} &= 10\frac{\frac{1}{2}}{\left(x^2+y^2+\frac{1}{4}\right)^{\frac{3}{2}}}\,dy\,dx\\ &= 5\left(x^2+y^2+\frac{1}{4}\right)^{-\frac{3}{2}}\,dy\,dx\end{aligned}$$

Now

$$\begin{aligned}\int\int_{S(top)}\underline{E}\cdot\underline{dS} &= \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}} 5\left(x^2+y^2+\frac{1}{4}\right)^{-\frac{3}{2}}\,dy\,dx\\ &= 5\times\frac{4\pi}{3}\quad\text{(using the hint)}\\ &= \frac{20\pi}{3}\end{aligned}$$

So, from Gauss' law,

$$Q = 6\varepsilon_0\times\frac{20\pi}{3} = 40\pi\varepsilon_0 \approx 10^{-9}\text{C}$$

Interpretation

Gauss' law can be used to find a charge from its effects elsewhere.

The form of $\underline{E} = 10\dfrac{x\underline{i}+y\underline{j}+\frac{1}{2}\underline{k}}{\left(x^2+y^2+\frac{1}{4}\right)^{\frac{3}{2}}}$ comes from the fact that $\underline{E}$ is radial and equals $10\dfrac{\underline{r}}{r^3} = 10\dfrac{\hat{\underline{r}}}{r^2}$

Example 37

Verify Gauss' theorem for the vector field $\underline{F} = y^2\underline{j} - xz\underline{k}$ and the triangular prism with vertices at $(0,0,0)$, $(2,0,0)$, $(0,0,1)$, $(0,4,0)$, $(2,4,0)$ and $(0,4,1)$ (see Figure 19).

Figure 19: The triangular prism defined by six vertices

Solution

As $\underline{F} = y^2\underline{j} - xz\underline{k}$, $\underline{\nabla} \cdot \underline{F} = 0 + 2y - x = 2y - x$.
Thus

$$\begin{aligned}
\iiint_V \underline{\nabla} \cdot \underline{F} dV &= \int_{x=0}^{2}\int_{y=0}^{4}\int_{z=0}^{1-x/2}(2y-x)dzdydx \\
&= \int_{x=0}^{2}\int_{y=0}^{4}\Big[2yz - xz\Big]_{z=0}^{1-x/2} dydx = \int_{x=0}^{2}\int_{y=0}^{4}(2y - xy - x + \frac{1}{2}x^2)dydx \\
&= \int_{x=0}^{2}\Big[y^2 - \frac{1}{2}xy^2 - xy + \frac{1}{2}x^2y\Big]_{y=0}^{4} dx \quad = \int_{x=0}^{2}(16 - 12x + 2x^2)dx \\
&= \Big[16x - 6x^2 + \frac{2}{3}x^3\Big]_0^2 = \frac{40}{3}
\end{aligned}$$

To work out $\iint_S \underline{F} \cdot \underline{dS}$, it is necessary to consider the contributions from the five faces separately.
On the front face, $y = 0$, $\underline{F} = -xz\underline{k}$ and $\underline{dS} = -\underline{j}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the back face, $y = 4$, $\underline{F} = 16\underline{j} - xz\underline{k}$ and $\underline{dS} = \underline{j}$ thus $\underline{F} \cdot \underline{dS} = 16$ and the contribution to the integral is

$$\int_{x=0}^{2}\int_{z=0}^{1-x/2} 16dzdx = \int_{x=0}^{2}\Big[\ 16z\Big]_{z=0}^{1-x/2} dx = \int_{x=0}^{2} 16(1 - x/2)dx = \Big[\ 16x - 4x^2\Big]_0^2 = 16.$$

On the left face, $x = 0$, $\underline{F} = y^2\underline{j}$ and $\underline{dS} = -\underline{i}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the bottom face, $z = 0$, $\underline{F} = y^2\underline{j}$ and $\underline{dS} = -\underline{k}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the top right (sloping) face, $z = 1 - x/2$, $\underline{F} = y^2\underline{j} + (\frac{1}{2}x^2 - x)\underline{k}$ and the unit normal $\underline{\hat{n}} = \frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k}$
Thus $\underline{dS} = \left[\frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k}\right] dydw$ where dw measures the distance along the slope for a constant y.
As $dw = \frac{\sqrt{5}}{2}dx$, $\underline{dS} = \left[\frac{1}{2}\underline{i} + \underline{k}\right] dydx$ thus $\underline{F} \cdot \underline{dS} = 16$ and the contribution to the integral is

$$\int_{x=0}^{2}\int_{y=0}^{4}(\frac{1}{2}x^2 - x)dydx = \int_{x=0}^{2}(2x^2 - 4x)dx = \Big[\frac{2}{3}x^3 - 2x^2\Big]_0^2 = -\frac{8}{3}.$$

Adding the contributions, $\iint_S \underline{F} \cdot \underline{dS} = 0 + 16 + 0 + 0 - \dfrac{8}{3} = \dfrac{40}{3}$.

Thus $\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} dV = \dfrac{40}{3}$ hence verifying Gauss' divergence theorem.

Engineering Example 6

Field strength around a charged line

Problem in words

Find the electric field strength at a given distance from a uniformly charged line.

Mathematical statement of problem

Determine the electric field at a distance r from a uniformly charged line (charge per unit length ρ_L). You may assume from symmetry that the field points directly away from the line.

Figure 20: Field strength around a line charge

Mathematical analysis

Imagine a cylinder a distance r from the line and of length l (see Figure 20). From Gauss' law

$$\int\int_S \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

As the charge per unit length is ρ_L, then the right-hand side equals $\rho_L l/\varepsilon_0$. On the left-hand side, the integral can be expressed as the sum

$$\int\int_S \underline{E} \cdot \underline{dS} = \int\int_{S(ends)} \underline{E} \cdot \underline{dS} \; + \; \int\int_{S(curved)} \underline{E} \cdot \underline{dS}$$

Looking first at the circular ends of the cylinder, the fact that the field lines point radially away from the charged line implies that the electric field is in the plane of these circles and has no normal component. Therefore $\underline{E} \cdot \underline{dS}$ will be zero for these ends.
Next, over the curved surface of the cylinder, the electric field is normal to it, and the symmetry of the problem implies that the strength of the electric field will be constant (here denoted by E). Therefore the integral = Total curved surface area $\times$ Field strength $= 2\pi r l E$.

So, by Gauss' law

$$\int\int_{S(ends)} \underline{E} \cdot \underline{dS} \; + \; \int\int_{S(curved)} \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

or

$$0 \; + \; 2\pi r l E = \frac{\rho_L l}{\varepsilon_0}$$

Interpretation

Hence, the field strength E is given by $\quad E = \dfrac{\rho_L}{2\pi\varepsilon_0 r}$

Engineering Example 7

Field strength on a cylinder

Problem in words

Given the electric field $\underline{E}$ on the surface of a cylinder, use Gauss' law to find the charge per unit length.

Mathematical statement of problem

On the surface of a long cylinder of radius a and length l, the electric field is given by

$$\underline{E} = \frac{\rho_L}{2\pi\varepsilon_0}\,\frac{(a + b\cos\theta)\,\underline{\hat{r}} - b\sin\theta\,\underline{\hat{\theta}}}{(a^2 + 2ab\cos\theta + b^2)}$$

(using cylindrical polar co-ordinates) due to a line of charge a distance b ($< a$) from the centre of the cylinder. Using Gauss' law , find the charge per unit length.

Hint:- $$\int_0^{2\pi} \frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\,d\theta = \frac{2\pi}{a}$$

Mathematical analysis

Consider a cylindrical section - as in the previous example, there are no contributions from the ends of the cylinder since the electric field has no normal component here. However, on the curved surface

$$\underline{dS} = a\;d\theta\;dz\;\underline{\hat{r}}$$

so

$$\underline{E}\cdot\underline{dS} = \frac{\rho_L}{2\pi\varepsilon_0}\,\frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\;a\;d\theta\;dz$$

Integrating over the curved surface of the cylinder

$$\begin{aligned}
\int\int_S \underline{E}\cdot\underline{dS} &= \int_{z=0}^{l}\int_{\theta=0}^{\theta=2\pi} \frac{a\rho_L}{2\pi\varepsilon_0}\,\frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\;d\theta\;dz \\
&= \frac{a\rho_L l}{2\pi\varepsilon_0}\int_0^{2\pi} \frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\;d\theta \\
&= \frac{\rho_L l}{\varepsilon_0} \quad \text{using the given result for the integral.}
\end{aligned}$$

Then, if Q is the total charge inside the cylinder, from Gauss' law

$$\frac{\rho_L l}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \quad \text{so} \quad \rho_L = \frac{Q}{l} \quad \text{as one would expect.}$$

Interpretation

Therefore the charge per unit length on the line of charge is given by ρ_L (i.e. the charge per unit length is constant).

Task

Verify Gauss' theorem for the vector field $\underline{F} = x\underline{i} - y\underline{j} + z\underline{k}$ and the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

(a) Find the vector $\underline{\nabla} \cdot \underline{F}$.

(b) Evaluate the integral $\displaystyle\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} \underline{\nabla} \cdot \underline{F} dxdydz$.

(c) For each side, evaluate the normal vector $\underline{dS}$ and the surface integral $\displaystyle\iint_S \underline{F} \cdot \underline{dS}$.

(d) Show that the two sides of the statement of Gauss' theorem are equal.

Your solution

Answer

(a) $1 - 1 + 1 = 1$

(b) 1

(c) $-dxdy\underline{k},\ 0;\quad dxdy\underline{k},\ \ 1;\quad -dxdz\underline{j},\ 0;\quad dxdz\underline{j},\ \ -1;\quad -dydz\underline{i},\ 0;\quad dydz\underline{i},\ \ 1$

(d) Both sides are equal to 1.

Exercises

1. Verify Gauss' theorem for the vector field $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and the cuboid $0 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 4$.

2. Verify Gauss' theorem, using cylindrical polar coordinates, for the vector field $\underline{F} = \rho^{-2}\underline{\hat{\rho}}$ over the cylinder $0 \le \rho \le r_0$, $-1 \le z \le 1$ for

   (a) $r_0 = 1$

   (b) $r_0 = 2$

3. If S is the surface of the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, find the surface integral
   $$\iint_S (x\underline{i} + yz\underline{j}) \cdot \underline{dS}$$

   (a) directly

   (b) by using Gauss' theorem

   Hint :- When evaluating directly, show that the unit normal on the sloping face is $\frac{1}{\sqrt{3}}(\underline{i}+\underline{j}+\underline{k})$ and that $\underline{dS} = (\underline{i} + \underline{j} + \underline{k})dxdy$

Answers

1. Both sides are 156,

2. Both sides equal (a) 4π, (b) 2π,

3. (a) $\dfrac{5}{24}$ [only contribution is from the sloping face] (b) $\dfrac{5}{24}$ [by volume integral of $(1+z)$].

3. Green's Identities (3D)

Like Gauss' theorem, Green's identities relate surface integrals to volume integrals. However, Green's identities are concerned with two scalar fields $u(x, y, z)$ and $w(x, y, z)$. Two statements of Green's identities are as follows

$$\iint_S (u\underline{\nabla}w)\cdot\underline{dS} = \iiint_V \left\{\underline{\nabla}u\cdot\underline{\nabla}w + u\underline{\nabla}^2 w\right\} dV \qquad [1]$$

and

$$\iint_S \{u\underline{\nabla}w - v\underline{\nabla}u\}\cdot\underline{dS} = \iiint_V \left\{u\underline{\nabla}^2 w - w\underline{\nabla}^2 u\right\} dV \qquad [2]$$

Proof of Green's identities

Green's identities can be derived from Gauss' theorem and a vector derivative identity.

Vector identity (1) from subsection 6 of 28.2 states that $\underline{\nabla}\cdot(\phi\underline{A}) = (\underline{\nabla}\phi)\cdot\underline{A} + \phi(\underline{\nabla}\cdot\underline{A})$.
Letting $\phi = u$ and $\underline{A} = \underline{\nabla}w$ in this identity,

$$\underline{\nabla}\cdot(u\underline{\nabla}w) = (\underline{\nabla}u)\cdot(\underline{\nabla}w) + u(\underline{\nabla}\cdot(\underline{\nabla}w)) = (\underline{\nabla}u)\cdot(\underline{\nabla}w) + u\underline{\nabla}^2 w$$

Gauss' theorem states

$$\iint_S \underline{F}\cdot\underline{dS} = \iiint_V \underline{\nabla}\cdot\underline{F}dV$$

Now, letting $\underline{F} = u\underline{\nabla}w$,

$$\begin{aligned}\iint_S (u\underline{\nabla}w)\cdot\underline{dS} &= \iiint_V \underline{\nabla}\cdot(u\underline{\nabla}w)dV \\ &= \iiint_V \left\{(\underline{\nabla}u)\cdot(\underline{\nabla}w) + u\underline{\nabla}^2 w\right\} dV\end{aligned}$$

This is Green's identity [1].
Reversing the roles of u and w,

$$\iint_S (w\underline{\nabla}u)\cdot\underline{dS} = \iiint_V \left\{(\underline{\nabla}w)\cdot(\underline{\nabla}u) + w\underline{\nabla}^2 u\right\} dV$$

Subtracting the last two equations yields Green's identity [2].

Key Point 10

Green's Identities

[1] $$\iint_S (u\underline{\nabla}w)\cdot\underline{dS} = \iiint_V \left\{\underline{\nabla}u\cdot\underline{\nabla}w + u\underline{\nabla}^2 w\right\} dV$$

[2] $$\iint_S \{u\underline{\nabla}w - v\underline{\nabla}u\}\cdot\underline{dS} = \iiint_V \left\{u\underline{\nabla}^2 w - w\underline{\nabla}^2 u\right\} dV$$

Example 38

Verify Green's first identity for $u = (x - x^2)y$, $w = xy + z^2$ and the unit cube, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

Solution

As $w = xy + z^2$, $\underline{\nabla} w = y\underline{i} + x\underline{j} + 2z\underline{k}$. Thus $u\underline{\nabla} w = (xy - x^2y)(y\underline{i} + x\underline{j} + 2z\underline{k})$ and the surface integral is of this quantity (scalar product with $\underline{dS}$) integrated over the surface of the unit cube.

On the three faces $x = 0$, $x = 1$, $y = 0$, the vector $u\underline{\nabla} w = \underline{0}$ and so the contribution to the surface integral is zero.

On the face $y = 1$, $u\underline{\nabla} w = (x - x^2)(\underline{i} + x\underline{j} + 2z\underline{k})$ and $\underline{dS} = dxdz\underline{j}$ so $(u\underline{\nabla} w) \cdot \underline{dS} = (x^2 - x^3)dxdz$ and the contribution to the integral is

$$\int_{x=0}^{1}\int_{z=0}^{1}(x^2 - x^3)dzdx = \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$

On the face $z = 0$, $u\underline{\nabla} w = (x - x^2)y(y\underline{i} + x\underline{j})$ and $\underline{dS} = -dxdz\underline{k}$ so $(u\underline{\nabla} w) \cdot \underline{dS} = 0$ and the contribution to the integral is zero.

On the face $z = 1$, $u\underline{\nabla} w = (x - x^2)y(y\underline{i} + x\underline{j} + 2\underline{k})$ and $\underline{dS} = dxdy\underline{k}$ so $(u\underline{\nabla} w) \cdot \underline{dS} = 2y(x - x^2)dxdy$ and the contribution to the integral is

$$\int_{x=0}^{1}\int_{y=0}^{1} 2y(x - x^2)dydx = \int_{x=0}^{1}\left[y^2(x - x^2)\right]_{y=0}^{1} dx = \int_0^1 (x - x^2)dx = \frac{1}{6}.$$

Thus, $\displaystyle\iint_S (u\underline{\nabla} w) \cdot \underline{dS} = 0 + 0 + 0 + \frac{1}{12} + 0 + \frac{1}{6} = \frac{1}{4}.$

Now evaluate $\displaystyle\iiint_V \left\{\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV.$

Note that $\underline{\nabla} u = (1 - 2x)y\underline{i} + (x - x^2)\underline{j}$ and $\underline{\nabla}^2 w = 2$ so

$$\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w = (1 - 2x)y^2 + (x - x^2)x + 2(x - x^2)y = x^2 - x^3 + 2xy - 2x^2y + y^2 - 2xy^2$$

and the integral

$$\begin{aligned}
\iiint_V \left\{\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV &= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(x^2 - x^3 + 2xy - 2x^2y + y^2 - 2xy^2)dxdydz \\
&= \int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^3}{3} - \frac{x^4}{4} + x^2y - \frac{2}{3}x^3y + xy^2 - x^2y^2\right]_{x=0}^{1} dydz \\
&= \int_{z=0}^{1}\int_{y=0}^{1}(\frac{1}{12} + \frac{y}{3})dydz = \int_{z=0}^{1}\left[\frac{y}{12} + \frac{y^2}{6}\right]_{y=0}^{1} dz \\
&= \int_{z=0}^{1}(\frac{1}{4})dz = \left[\frac{z}{4}\right]_{z=0}^{1} = \frac{1}{4}
\end{aligned}$$

Hence $\displaystyle\iint_S (u\underline{\nabla} w) \cdot \underline{dS} = \iiint_V \left[\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right] dV = \frac{1}{4}$ and Green's first identity is verified.

Green's theorem in the plane

This states that

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

S is a 2-D surface with perimeter C; $P(x, y)$ and $Q(x, y)$ are scalar functions.

This should not be confused with Green's identities.

Justification of Green's theorem in the plane

Green's theorem in the plane can be derived from Stokes' theorem.

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Now let $\underline{F}$ be the vector field $P(x, y)\underline{i} + Q(x, y)\underline{j}$ i.e. there is no dependence on z and there are no components in the $z-$ direction. Now

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \underline{k}$$

and $\underline{dS} = dxdy\underline{k}$ giving $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dxdy.$

Thus Stokes' theorem becomes

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C \underline{F} \cdot \underline{dr}$$

and Green's theorem in the plane follows.

Key Point 11

Green's Theorem in the Plane

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

This relates a line integral around a closed path C with a double integral over the region S enclosed by C. It is effectively a two-dimensional form of Stokes' theorem.

Example 39

Evaluate the line integral $\oint_C \left[(4x^2+y-3)dx+(3x^2+4y^2-2)dy\right]$ around the rectangle $0 \le x \le 3$, $0 \le y \le 1$.

Solution

The integral could be obtained by evaluating four line integrals but it is easier to note that $[(4x^2+y-3)dx+(3x^2+4y^2-2)dy]$ is of the form $Pdx+Qdy$ with $P=4x^2+y-3$ and $Q=3x^2+4y^2-2$. It is thus of a suitable form for Green's theorem in the plane.

Note that $\dfrac{\partial Q}{\partial x}=6x$ and $\dfrac{\partial P}{\partial y}=1$.

Green's theorem in the plane becomes

$$\begin{aligned}\oint_C \{(4x^2+y-3)dx+(3x^2+4y^2-2)dy\} &= \int_{y=0}^{1}\int_{x=0}^{3}(6x-1)\,dxdy \\ &= \int_{y=0}^{1}\Big[3x^2-x\Big]_{x=0}^{3}dy = \int_{y=0}^{1}24\;dy = 24\end{aligned}$$

Example 40

Verify Green's theorem in the plane for the integral $\oint_C \left[4zdy + (y^2-2)dz\right]$ and the triangular contour starting at the origin $O = (0,0,0)$ and going to $A = (0,2,0)$ and $B = (0,0,1)$ before returning to the origin.

Solution

The whole of the contour is in the plane $x = 0$ and Green's theorem in the plane becomes

$$\oint_C (Pdy + Qdz) = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz$$

(a) Firstly evaluate $\oint_C \left\{4zdy + (y^2-2)dz\right\}$.

On OA, $z = 0$ and $dz = 0$. As the integrand is zero, the integral will also be zero.

On AB, $z = (1 - \frac{y}{2})$ and $dz = -\frac{1}{2}dy$. The integral is

$$\int_{y=2}^{0} \left((4-2y)dy - \frac{1}{2}(y^2-2)dy\right) = \int_2^0 (5 - 2y - \frac{1}{2}y^2)dy = \left[5y - y^2 - \frac{1}{6}y^3\right]_2^0 = -\frac{14}{3}$$

On BO, $y = 0$ and $dy = 0$. The integral is $\int_1^0 (-2)dz = \Big[\ -2z\Big]_1^0 = 2.$

Summing, $\oint_C \left(4zdy + (y^2-2)dz\right) = -\dfrac{8}{3}$

(b) Secondly evaluate $\iint_S \left(\dfrac{\partial Q}{\partial y} - \dfrac{\partial P}{\partial z}\right) dydz$

In this example, $P = 4z$ and $Q = y^2 - 2$. Thus $\dfrac{\partial P}{\partial z} = 4$ and $\dfrac{\partial Q}{\partial y} = 2y$. Hence,

$$\begin{aligned}
\iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz &= \int_{y=0}^{2}\int_{z=0}^{1-y/2} (2y-4)\, dzdy \\
&= \int_{y=0}^{2} \Big[2yz - 4z\Big]_{z=0}^{1-y/2} dy = \int_{y=0}^{2} \left(-y^2 + 4y - 4\right) dy \\
&= \left[-\frac{1}{3}y^3 + 2y^2 - 4y\right]_0^2 = -\frac{8}{3}
\end{aligned}$$

Hence:

$\oint_C (Pdy + Qdz) = \iint_S \left(\dfrac{\partial Q}{\partial y} - \dfrac{\partial P}{\partial z}\right) dydz = -\dfrac{8}{3}$ and Green's theorem in the plane is verified.

One very useful, special case of Green's theorem in the plane is when $Q = x$ and $P = -y$. The theorem becomes

$$\oint_C \{-ydx + xdy\} = \iint_S (1 - (-1))\, dxdy$$

The right-hand side becomes $\iint_S 2\, dxdy$ i.e. $2A$ where A is the area inside the contour C. Hence

$$A = \frac{1}{2}\oint_C \{xdy - ydx\}$$

This result is known as the **area theorem**. It gives us the area bounded by a curve C in terms of a line integral around C.

Example 41

Verify the area theorem for the segment of the circle $x^2 + y^2 = 4$ lying above the line $y = 1$.

Solution

Firstly, the area of the segment $ADBC$ can be found by subtracting the area of the triangle $OADB$ from the area of the sector $OACB$. The triangle has area $\frac{1}{2} \times 2\sqrt{3} \times 1 = \sqrt{3}$. The sector has area $\frac{\pi}{3} \times 2^2 = \frac{4}{3}\pi$. Thus segment $ADBC$ has area $\frac{4}{3}\pi - \sqrt{3}$.

Now, evaluate the integral $\oint_C \{xdy - ydx\}$ around the segment.

Along the line, $y = 1$, $dy = 0$ so the integral $\int_C \{xdy - ydx\}$ becomes $\int_{-\sqrt{3}}^{\sqrt{3}} (x \times 0 - 1 \times dx) = \int_{-\sqrt{3}}^{\sqrt{3}} (-dx) = -2\sqrt{3}$.

Along the arc of the circle, $y = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$ so $dy = -x(4 - x^2)^{-1/2}dx$. The integral $\int_C \{xdy - ydx\}$ becomes

$$\begin{aligned}
\int_{\sqrt{3}}^{-\sqrt{3}} \{-x^2(4 - x^2)^{-1/2} - (4 - x^2)^{1/2}\}dx &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{4}{\sqrt{4 - x^2}}dx \\
&= \int_{-\pi/3}^{\pi/3} 4\frac{1}{2\cos\theta} 2\cos\theta\, d\theta \text{ (letting } x = 2\sin\theta) \\
&= \int_{-\pi/3}^{\pi/3} 4d\theta = \frac{8}{3}\pi
\end{aligned}$$

So, $\frac{1}{2}\oint_C \{xdy - ydx\} = \frac{1}{2}\left[\frac{8}{3}\pi - 2\sqrt{3}\right] = \frac{4}{3}\pi - \sqrt{3}$.

Hence both sides of the area theorem equal $\frac{4}{3}\pi - \sqrt{3}$ thus verifying the theorem.

Task

Verify Green's theorem in the plane when applied to the integral

$$\oint_C \{(5x+2y-7)dx + (3x-4y+5)dy\}$$

where C represents the perimeter of the trapezium with vertices at $(0,0)$, $(3,0)$, $(6,1)$ and $(1,1)$.

First let $P = 5x+2y-7$ and $Q = 3x-4y+5$ and find $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$:

Your solution

Answer

1

Now find $\displaystyle\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ over the trapezium:

Your solution

Answer

4 (by elementary geometry)

Now find $\displaystyle\int (Pdx + Qdy)$ along the four sides of the trapezium, beginning with the line from $(0,0)$ to $(3,0)$, and then proceeding anti-clockwise.

Your solution

Answers $1.5,\ 66,\ -62.5,\ -1$ whose sum is 4.

Finally show that the two sides of the statement of Green's theorem are equal:

Your solution

Answer

Both sides are 4.

Exercises

1. Verify Green's identity [1] (page 73) for the functions $u = xyz$, $w = y^2$ and the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

2. Verify the area theorem for

   (a) The area above $y = 0$, but below $y = 1 - x^2$.

   (b) The segment of the circle $x^2 + y^2 = 1$, to the upper left of the line $y = 1 - x$.

Answers

1. Both integrals in [1] equal $\dfrac{1}{2}$

2. (a) both sides give a value of $\dfrac{4}{3}$, (b) both sides give a value of $\dfrac{\pi}{4} - \dfrac{1}{2}$.

Chapter 3

Integral Theorems

[Anton, pp. 1124–1130, pp. 1145–1160] & [Bourne, pp. 195–224]

First of all some definitions which we will need in the following:

Definition 3.1. (a) A domain (region) Ω is an open connected subset of $\mathbb{R}^n$.

(b) A domain $\Omega \subset \mathbb{R}^3$ is **bounded**, if there exists an $R > 0$ such that $\Omega \subset \mathcal{B}_R$, where $\mathcal{B}_R$ is the ball with radius R and centre $\mathbf{0}$.

(c) A surface $\mathcal{S} \subset \mathbb{R}^3$ is **open**, if for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \notin S$ there exists a continuous curve from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ which does not cross $\mathcal{S}$. A surface $\mathcal{S} \subset \mathbb{R}^3$ is **closed**, if it is not open.

(d) A closed surface $\mathcal{S} \subset \mathbb{R}^3$ is **convex**, if every straight line intersects (meets) $\mathcal{S}$ at two points at most. **Examples.**

(e) A closed surface $\mathcal{S} \subset \mathbb{R}^3$ is **semi–convex**, if we can choose a coordinate system $0xyz$ so that every straight line *parallel to the coordinate axes* intersects $\mathcal{S}$ at two points at most. **Examples.**

Note. Recall also (Remark 1.24) that a surface $\mathcal{S}$ is smooth, if its parametrisation is continuously differentiable. $\mathcal{S}$ is piecewise smooth, if $\mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i$ and $\mathcal{S}_i$ smooth.

3.1 The Divergence Theorem of Gauss

Theorem 3.2 (Divergence Theorem). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with piecewise smooth, closed boundary (surface) $\mathcal{S}$. Suppose also that $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is a continuously differentiable vector field. Then*

$$\iiint_\Omega \nabla \cdot \boldsymbol{F}\, dV \;=\; \iint_{\mathcal{S}} \boldsymbol{F} \cdot \boldsymbol{dS}\ . \tag{3.1}$$

Proof. (Only for $\mathcal{S}$ smooth and semi–convex).

Let D be the projection of Ω onto the (x, y)–plane. Consider the line L through the point $(x, y, 0)$ parallel to the z–axis. Since $\mathcal{S}$ is semi-convex, L intersects $\mathcal{S}$ at two points $(x, y, f(x, y))^T$ and $(x, y, g(x, y))^T$, where $f(x, y) \le g(x, y)$ for all $(x, y) \in D$ (otherwise change the coordinate system).

Hence,

(i) Let us first show that

$$\iint_{\mathcal{S}} F_3 \boldsymbol{k} \cdot \boldsymbol{dS} \;=\; \iint_D \Big\{ F_3(x, y, g(x, y)) - F_3(x, y, f(x, y)) \Big\}\, dx\, dy\ . \tag{3.2}$$

(ii) Now we show that

$$\iiint_\Omega \frac{\partial F_3}{\partial z}\, dV \;=\; \iint_{\mathcal{S}} F_3 \boldsymbol{k} \cdot \boldsymbol{dS} \;. \tag{3.3}$$

(iii) Similarly, by projecting onto the (x, z)-plane and onto the (y, z)-plane we can establish

$$\iiint_\Omega \frac{\partial F_2}{\partial y}\, dV \;=\; \iint_{\mathcal{S}} F_2 \boldsymbol{j} \cdot \boldsymbol{dS} \;, \tag{3.4}$$

$$\iiint_\Omega \frac{\partial F_1}{\partial x}\, dV \;=\; \iint_{\mathcal{S}} F_1 \boldsymbol{i} \cdot \boldsymbol{dS} \;, \tag{3.5}$$

and

□

Remark 3.3. This proof can be extended in a straightforward way to domains Ω with piecewise smooth and non-semi-convex boundary $\mathcal{S}$, if $\Omega = \bigcup_{i=1}^n \Omega_i$, where each of the Ω_i has a smooth, semi-convex boundary $\mathcal{S}_i$, e.g. torus.

Example 3.4. Find $\iint_{\mathcal{S}} \boldsymbol{F} \cdot \boldsymbol{dS}$ where $\mathcal{S}$ is the surface of the unit cube and $\boldsymbol{F} := (x^2, y^2, z^2)^T$.

Corollary 3.5. *Let Ω and $\mathcal{S}$ be as in Theorem 3.2. Suppose $f : \Omega \to \mathbb{R}$ and $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ are continuously differentiable. Then*

$$\iiint_\Omega \nabla f\, dV \;=\; \iint_{\mathcal{S}} f\, \boldsymbol{dS} \tag{3.6}$$

$$\iiint_\Omega \nabla \wedge \boldsymbol{F}\, dV \;=\; -\iint_{\mathcal{S}} \boldsymbol{F} \wedge \boldsymbol{dS} \tag{3.7}$$

Proof. Let $\boldsymbol{a} \in \mathbb{R}^3$ be constant.

(i) Apply the Divergence Theorem to $\boldsymbol{G} := f\,\boldsymbol{a}$:

(ii) Apply the Divergence Theorem to $\boldsymbol{G} := \boldsymbol{a} \wedge \boldsymbol{F}$:

$\square$

3.2 Green's Theorem in the Plane

Note. In this section we work in $\mathbb{R}^2$ not in $\mathbb{R}^3$!

Definition 3.6. (a) A closed curve $\mathcal{C} \subset \mathbb{R}^2$, is **simple**, if it does not intersect itself, e.g.

(b) A closed curve $\mathcal{C} \subset \mathbb{R}^2$ is **convex**, if every straight line intersects $\mathcal{C}$ at 2 points at most.

(c) A closed curve $\mathcal{C} \subset \mathbb{R}^2$ is **semi–convex**, if we can choose a coordinate system $0xy$ so that every straight line *parallel to the coordinate axes* intersects $\mathcal{C}$ at 2 points at most.

Theorem 3.7 (Green's Theorem in the Plane). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with simple, piecewise smooth boundary (curve) $\mathcal{C} \subset \mathbb{R}^2$ described in the anticlockwise sense. Suppose that $\boldsymbol{\Phi} : \Omega \to \mathbb{R}^2$ is a continuously differentiable vector field in $\mathbb{R}^2$, i.e. $\boldsymbol{\Phi} = \Phi_1 \boldsymbol{i} + \Phi_2 \boldsymbol{j}$. Then*

$$\iint_\Omega \left(\frac{\partial \Phi_2}{\partial x} - \frac{\partial \Phi_1}{\partial y} \right) dx\, dy \;=\; \oint_{\mathcal{C}} \boldsymbol{\Phi} \cdot d\boldsymbol{r} \;. \tag{3.8}$$

Proof. See **Handout** or [Bourne, pp.210–213]. □

Remark 3.8. Green's Theorem in the plane is sometimes also referred to as **<u>Stokes' Theorem in the plane</u>** (e.g. in [Bourne, pp. 210–213]).

Corollary 3.9. *The area bounded by a simple, closed, piecewise smooth curve $\mathcal{C} \subset \mathbb{R}^2$ is given by*

$$\frac{1}{2} \left| \oint_{\mathcal{C}} (-y\boldsymbol{i} + x\boldsymbol{j}) \cdot d\boldsymbol{r} \right| .$$

Proof. Apply Green's Theorem in the plane with $\Phi_1(x, y) = -y$ and $\Phi_2(x, y) = x$.

□

3.3 Stokes' Theorem

Definition 3.10. (a) A closed curve $\mathcal{C} \subset \mathbb{R}^3$, is **<u>simple</u>**, if it does not intersect itself.

(b) A surface $\mathcal{S} \subset \mathbb{R}^3$ is **<u>orientable</u>**, if a unique normal can be assigned at each point $\boldsymbol{x} \in \mathcal{S}$.

<u>Example.</u> A Möbius strip for example is **not** orientable:

(c) Let $\mathcal{S}$ be an open, orientable surface with simple boundary (curve) $\mathcal{C}$. Let $\hat{\boldsymbol{n}}$ be the unit normal on $\mathcal{S}$. Imagine a person walking along the curve $\mathcal{C}$ (in the positive direction) with its head pointing in the direction of $\hat{\boldsymbol{n}}$.

Then $\mathcal{S}$ and $\mathcal{C}$ are said to be **correspondingly orientated**, if the surface is to the left of the person. [Anton, p. 1154], [Bourne, p. 210].

Theorem 3.11 (Stokes' Theorem). *Let $\mathcal{S} \subset \mathbb{R}^3$ be an open, orientable, piecewise smooth surface with correspondingly orientated, simple, piecewise smooth boundary (curve) $\mathcal{C} \subset \mathbb{R}^3$. Suppose that the vector field $\boldsymbol{F}$ is continuously differentiable (in a neighbourhood of $\mathcal{S}$). Then*

$$\iint_{\mathcal{S}} (\nabla \wedge \boldsymbol{F}) \cdot \boldsymbol{dS} \;=\; \oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} \;. \tag{3.9}$$

Proof. See **Handout** or [Bourne, pp.213–216]. □

Remark 3.12. (a) Stokes' Theorem implies that the flux of $\nabla \wedge \boldsymbol{F}$ through a surface $\mathcal{S}$ depends only on the boundary $\mathcal{C}$ of $\mathcal{S}$ and is therefore independent of its shape. In other words,

$\iint_{\mathcal{S}} (\nabla \wedge \boldsymbol{F}) \cdot \boldsymbol{dS}$ is the same for

and for

(b) Note that Theorem 3.7 is a special case of Theorem 3.11. To see this, assume that $\mathcal{S}$ in Theorem 3.11 is flat, i.e. $\mathcal{S} \subset \mathbb{R}^2 \times \{0\}$. Then

MATHEMATICAL METHOD 1

MTS 241

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Topics to be covered

- ✔ Multiple and Line Integrals
- ✔ Improper Integrals
- ✔ Relationship between Vector Field Functions
- ✔ Integral Theorems

Multiple and Line Integrals

5.1 INTRODUCTION TO DEFINITE INTEGRALS AND DOUBLE INTEGRALS

Definite Integrals

The concept of definite integral

$$\int_a^b f(x)\,dx \qquad \ldots(1)$$

is physically the area under a curve $y = f(x)$, (say), the x-axis and the two ordinates $x = a$ and $x = b$. It is defined as the limit of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots + f(x_n)\delta x_n$$

when $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots, \delta x_n$ tends to zero.

Fig. 5.1

Here $\delta x_1, \delta x_2, \ldots, \delta x_n$ are n subdivisions into which the range of integration has been divided and $x_1, x_2, \ldots, x_n$ are the values of x lying respectively in the Ist, 2nd, ..., nth subintervals.

Double Integrals

A double integral is the counter part of the above definition in two dimensions.

Let $f(x, y)$ be a single valued and bounded function of two independent variables x and y defined in a closed region A in xy plane. Let A be divided into n elementary areas $\delta A_1, \delta A_2, \ldots, \delta A_n$.

Let (x_r, y_r) be any point inside the rth elementary area δA_r.

Fig. 5.2

Consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots + f(x_n, y_n)\delta A_n = \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \ldots(2)$$

Then the limit of the sum (2), if exists, as $n \to \infty$ and each sub-elementary area approaches to zero, is termed as '*double integral*' of $f(x, y)$ over the region A and expressed as $\iint_A f(x, y)\,dA$.

Thus $$\iint_A f(x,y)\,dA = \underset{\substack{n\to\infty \\ \delta A_r \to 0}}{Lt} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots(3)$$

Observations: Double integrals are of limited use if they are evaluated as the limit of the sum. However, they are very useful for physical problems when they are evaluated by treating as successive single integrals.

Further just as the definite integral (1) can be interpreted as an area, similarly the double integrals (3) can be interpreted as a volume (see Figs. 5.1 and 5.2).

5.2 EVALUATION OF DOUBLE INTEGRAL

Evaluation of double integral $\iint_R f(x,y)\,dx\,dy$ is discussed under following three possible cases:

Case I: *When the region R is bounded by two continuous curves $y = \psi(x)$ and $y = \phi(x)$ and the two lines (ordinates) $x = a$ and $x = b$.*

In such a case, integration is first performed with respect to y keeping x as a constant and then the resulting integral is integrated within the limits $x = a$ and $x = b$.

Mathematically expressed as:

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \left(\int_{y=\phi(x)}^{y=\Psi(x)} f(x,y)\,dy \right) dx$$

Fig. 5.3

Geometrically the process is shown in Fig. 5.3, where integration is carried out from inner rectangle (i.e., along the one edge of the 'vertical strip PQ' from P to Q) to the outer rectangle.

Case 2: *When the region R is bounded by two continuous curves $x = \phi(y)$ and $x = \Psi(y)$ and the two lines (abscissa) $y = a$ and $y = b$.*

In such a case, integration is first performed with respect to x. keeping y as a constant and then the resulting integral is integrated between the two limits $y = a$ and $y = b$.

Mathematically expressed as:

$$\iint_R f(x,y)\,dx\,dy = \int_{y=a}^{y=b}\left(\int_{x=\theta(y)}^{x=\Psi(y)} f(x,y)\,dx\right)dy$$

Geometrically the process is shown in Fig. 5.4, where integration is carried out from inner rectangle (*i.e.*, along the one edge of the horizontal strip PQ from P to Q) to the outer rectangle.

Case 3: *When both pairs of limits are constants, the region of integration is the rectangle ABCD (say).*

Fig. 5.4

Fig. 5.5

In this case, it is immaterial whether $f(x, y)$ is integrated first with respect to x or y, the result is unaltered in both the cases (Fig. 5.5).

Observations: While calculating double integral, in either case, we proceed outwards from the innermost integration and this concept can be generalized to repeated integrals with three or more variable also.

EXAMPLE Evaluate $$\iint_{D_1} x^2 y \, dA$$

where D_1 is the unit square

$$0 \le x \le 1, \qquad 0 \le y \le 1.$$

The limits of the outside integral are given by $0 \le x \le 1$, and those of the inside integral are given by $0 \le y \le 1$. The iterated integral is thus

$$\iint_{D_1} x^2 y \, dA = \int_0^1 \int_0^1 x^2 y \, dy \, dx.$$

The inside integral is

$$\int_0^1 x^2 y \, d \quad = \tfrac{1}{2} x^2 y^2 \Big]_{y=0}^{y=1} = \tfrac{1}{2} x^2.$$

Then $$\iint_{D_1} x^2 y \, dA = \int_0^1 \tfrac{1}{2} x^2 \, dx = \tfrac{1}{6} x^3 \Big]_0^1 = \tfrac{1}{6} \sim 0.16667.$$

Since D_1 is a rectangle we may also integrate in the other order, and should get the same answer.

$$\iint_{D_1} x^2 y \, dA = \int_0^1 \int_0^1 x^2 y \, dx \, dy.$$

$$\int_0^1 x^2 y \, dx = \tfrac{1}{3} x^3 y \Big]_0^1 = \tfrac{1}{3} y.$$

$$\iint_{D_1} x^2 y \, dA = \int_0^1 \tfrac{1}{3} y \, dy = \tfrac{1}{6} y^2 \Big]_0^1 = \tfrac{1}{6} \sim 0.16667.$$

EXAMPLE Evaluate $\iint_{D_2} x^2 y\, dA$ where D_2 is the region in Figure 12.2.2:

$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}.$$

The limits on the outside integral are given by $0 \le x \le 1$, and those on the inside integral by $x^2 \le y \le \sqrt{x}$, so the iterated integral is

Figure 12.2.2

$$\iint_{D_2} x^2 y\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y\, dy\, dx.$$

$$\int_{x^2}^{\sqrt{x}} x^2 y\, dy = \tfrac{1}{2} x^2 y^2 \Big]_{y=x^2}^{y=\sqrt{x}} = \tfrac{1}{2}x^3 - \tfrac{1}{2}x^6.$$

$$\iint_{D_2} x^2 y\, dA = \int_0^1 (\tfrac{1}{2}x^3 - \tfrac{1}{2}x^6)\, dx = \tfrac{1}{8}x^4 - \tfrac{1}{14}x^7 \Big]_0^1$$

$$= \tfrac{3}{56} \sim 0.05357.$$

The Riemann sums in Section 12.1 were 0.05375, 0.04881.

In many applications the region D is given verbally, and part of the problem is to find inequalities which describe D.

EXAMPLE Let D be the region bounded by the curve $xy = 1$ and the line $y = \frac{5}{2} - x$. Find inequalities which describe D, and write down an iterated integral equal to $\iint_D f(x, y)\, dA$.

Step 1 Sketch the region D as in Figure 12.2.3.

Figure 12.2.3

Step 2 The line and curve intersect where

$$\begin{aligned} x(\tfrac{5}{2} - x) &= 1, \\ x^2 - \tfrac{5}{2}x + 1 &= 0, \\ (x - \tfrac{1}{2})(x - 2) &= 0, \\ x = \tfrac{1}{2}, \qquad x &= 2. \end{aligned}$$

For $1/2 \le x \le 2$, the curve $y = 1/x$ is below the line $y = 5/2 - x$. Therefore D is the region

$$\tfrac{1}{2} \le x \le 2, \qquad 1/x \le y \le \tfrac{5}{2} - x.$$

EXAMPLE 4 Find the volume of the solid bounded by the surfaces $z = 0$, $z = y - x^2$, $y = 1$.

Step 1 Sketch the solid and the region D, as in Figure 12.2.4.

Figure 12.2.4

Step 2 Find the inequalities describing the region D.

This is the hardest step, and gives us the limits of integration. The surfaces $z = 0$ and $z = y - x^2$ intersect at the curve $y = x^2$. We see from the figure that D is the region between the curves $y = x^2$ and $y = 1$, so D is given by

$$-1 \le x \le 1, \qquad x^2 \le y \le 1.$$

Step 3 Set up the iterated integral and evaluate it.

$$V = \iint_D y - x^2 \, dA = \int_{-1}^{1} \int_{x^2}^{1} y - x^2 \, dy \, dx.$$

$$\int_{x^2}^{1} y - x^2 \, dy = \tfrac{1}{2}y^2 - x^2 y \Big]_{x^2}^{1}$$

$$= (\tfrac{1}{2} \cdot 1^2 - x^2 \cdot 1) - (\tfrac{1}{2}(x^2)^2 - x^2 \cdot x^2)$$

$$= \tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4.$$

$$V = \int_{-1}^{1} \tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4 \, dx = \tfrac{16}{30}.$$

Multiple integration problems can be solved by a three-step process as shown in Examples 3 and 4.

Step 1 Sketch the problem.

Step 2 Find the inequalities describing the region D.

Step 3 Set up the iterated integral and evaluate.

We can also integrate over a region in the (y, x) plane instead of the (x, y) plane. A region D in the (y, x) plane has the form

$$b_1 \le y \le b_2, \qquad a_1(y) \le x \le a_2(y),$$

as shown in Figure 12.2.5.

The double integral over D is equal to the iterated integral with dy on the outside and dx inside,

Example Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2+y^2)} dydx$ [Madras 2000; Rajasthan 2005].

Solution: Clearly, here $y = f(x)$ varies from 0 to $\sqrt{1+x^2}$ and finally x (as an independent variable) goes between 0 to 1.

Fig. 5.6

$$I = \int_0^1 \left(\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2} dy \right) dx$$

$$= \int_0^1 \left(\int_0^{\sqrt{1+x^2}} \frac{1}{a^2+y^2} dy \right) dx, \; a^2 = (1+x^2)$$

$$= \int_0^1 \left(\frac{1}{a} \tan^{-1} \frac{y}{a} \right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1} 0 \right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4} - 0 \right) dx = \frac{\pi}{4} \left[\log \left\{ x + \sqrt{1+x^2} \right\} \right]_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Example Evaluate the integral $\iint_R xy(x+y)\,dxdy$ over the area between the curves $y = x^2$ and $y = x$.

Solution: We have $y = x^2$ and $y = x$ which implies $x^2 - x = 0$ i.e. either $x = 0$ or $x = 1$

Further, if $x = 0$ then $y = 0$; if $x = 1$ then $y = 1$. Means the two curves intersect at points (0, 0), (1, 1).

$\therefore$ The region R of integration is doted and can be expressed as: $0 \le x \le 1$, $x^2 \le y \le x$.

Fig. 5.8

$$\therefore \quad \iint_R xy(x+y)\,dxdy = \int_0^1 \left(\int_{x^2}^{x} xy(x+y)\,dy \right) dx$$

$$= \int_0^1 \left\{ \left(x^2 \frac{y^2}{2} + x\frac{y^3}{3} \right)_{x^2}^{x} \right\} dx$$

$$= \int_0^1 \left\{ \left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) \right\} dx$$

$$= \int_0^1 \left(\frac{5}{6}x^4 - \frac{1}{2}x^6 - \frac{1}{3}x^7 \right) dx$$

$$= \left[\frac{5}{6} \times \frac{x^5}{5} - \frac{1}{2}\frac{x^7}{7} - \frac{1}{3}\frac{x^8}{8} \right]_0^1 = \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}$$

ASSIGNMENT 1

1. Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

2. Evaluate $\iint_R xy\,dx\,dy$, where A is the domain bounded by the x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$. [M.D.U., 2000]

3. Evaluate $\iint e^{ax+by}\,dy\,dx$, where R is the area of the triangle $x = 0$, $y = 0$, $ax + by = 1$ $(a > 0, b > 0)$. [Hint: See example 2]

4. Prove that $\int_1^2 \int_3^1 (xy + e^y)\,dy\,dx = \int_3^1 \int_1^2 (xy + e^y)\,dx\,dy$.

5. Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx$.

6. Evaluate $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)}\,x\,dx\,dy$ [Hint: Put $x^2(1 + y^2) = t$, taking y as const.]

Application to Volume

DEFINITION

Let $f(x, y) \leq g(x, y)$ for (x, y) in D and let E be the set of all points in space such that

$$(x, y) \text{ is in } D, \qquad f(x, y) \leq z \leq g(x, y).$$

The ***volume*** *of E is*

$$V = \iint_D g(x, y) - f(x, y)\, dA.$$

V is called the volume over D between the surfaces $z = f(x, y)$ and $z = g(x, y)$ (Figure 12.3.4).

Figure 12.3.4 Volume between two surfaces

EXAMPLE 1 Find the volume of the solid

$$0 \le x \le 1, \quad 0 \le y \le x, \qquad x + y \le z \le e^{x+y}.$$

Step 1 D is the triangle shown in Figure 12.3.6.

Step 2 D is the region $0 \le x \le 1, 0 \le y \le x$.

Figure 12.3.6

Step 3

$$V = \iint_D e^{x+y} - (x + y)\, dA$$

$$= \int_0^1 \int_0^x e^{x+y} - (x + y)\, dy\, dx.$$

$$\int_0^x e^{x+y} - (x + y)\, dy = e^{x+y} - xy - \tfrac{1}{2}y^2 \Big]_0^x = e^{2x} - e^x - \tfrac{3}{2}x^2.$$

$$V = \int_0^1 e^{2x} - e^x - \tfrac{3}{2}x^2\, dx = \tfrac{1}{2}e^2 - e.$$

EXAMPLE 2 Find the volume of the solid bounded by the four planes

$$x = 0, \qquad y = 0, \qquad z = x + y, \qquad z = 1 - x - y.$$

Step 1 Sketch the planes. We see from Figure 12.3.7 that $z = x + y$ is below $z = 1 - x - y$.

Figure 12.3.7

Step 2 Find inequalities for the region D. Since the two planes

$$z = x + y, \qquad z = 1 - x - y$$

meet at the line $\quad 2x + 2y = 1, \qquad y = \frac{1}{2} - x,$

D is the region $\quad 0 \le x \le \frac{1}{2}, \qquad 0 \le y \le \frac{1}{2} - x.$

Step 3

$$V = \iint_D (1 - x - y) - (x + y)\, dA = \iint_D 1 - 2x - 2y\, dA$$

$$= \int_0^{1/2} \int_0^{1/2 - x} 1 - 2x - 2y\, dy\, dx.$$

$$\int_0^{1/2 - x} 1 - 2x - 2y\, dy = y - 2xy - y^2 \Big]_0^{1/2 - x}$$

$$= \tfrac{1}{2} - x - 2x(\tfrac{1}{2} - x) - (\tfrac{1}{2} - x)^2 = \tfrac{1}{4} - x + x^2.$$

$$V = \int_0^{1/2} \tfrac{1}{4} - x + x^2\, dx = \tfrac{1}{24}.$$

12.6 TRIPLE INTEGRALS

A *closed region in space*, or *solid region*, is a set E of points given by inequalities

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x, y) \le z \le c_2(x, y)$$

where the functions $b_1(x)$, $b_2(x)$ and $c_1(x, y)$, $c_2(x, y)$ are continuous.

The *boundary* of E is the part of E on the following surfaces:

The planes $x = a_1$, $x = a_2$.

The cylinders $y = b_1(x)$, $y = b_2(x)$.

The surfaces $z = c_1(x, y)$, $z = c_2(x, y)$.

The simplest type of closed region is a *rectangular solid*, or *rectangular box*,

$$a_1 \le x \le a_2, \qquad b_1 \le y \le b_2, \qquad c_1 \le z \le c_2.$$

Figure 12.6.1 shows a solid region and a rectangular box.

Figure 12.6.1

An *open region* in space is defined in a similar way but with strict inequalities. As in the two-dimensional case, the word *region* alone will mean *closed region*.

PERMANENT ASSUMPTION

Whenever we refer to a function $f(x, y, z)$ and a solid region E, we assume that $f(x, y, z)$ is continuous on some open region containing E.

The triple integral
$$\iiint_E f(x, y, z)\, dx\, dy\, dz$$
is analogous to the double integral.

The first step in defining the triple integral is to form the *circumscribed rectangular box* of E (Figure 12.6.2). This is the rectangular box

$$a_1 \le x \le a_2, \qquad B_1 \le y \le B_2, \qquad C_1 \le z \le C_2,$$

where
$$\begin{aligned} B_1 &= \text{minimum value of } b_1(x), \\ B_2 &= \text{maximum value of } b_2(x), \\ C_1 &= \text{minimum value of } c_1(x, y), \\ C_2 &= \text{maximum value of } c_2(x, y). \end{aligned}$$

DEFINITION

Given positive infinitesimal dx, dy, and dz, the ***triple integral*** *of a continuous function $f(x, y, z)$ over E is*

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = st\left(\sum\sum_E\sum f(x, y, z)\, dx\, dy\, dz\right).$$

We shall now briefly state some basic theorems on triple integrals, which are exactly like the corresponding theorems for double integrals.

INDEPENDENCE OF *dx, dy,* AND *dz*

The value of $\iiint_E f(x, y, z)\, dx\, dy\, dz$ does not depend on dx, dy, or dz.

We shall usually use the notation $dV = dx\, dy\, dz$ for the volume of an infinitesimal dx by dy by dz rectangular box, and write

$$\iiint_E f(x, y, z)\, dV \qquad \text{for} \qquad \iiint_E f(x, y, z)\, dx\, dy\, dz.$$

ADDITION PROPERTY

If E is divided into two regions E_1 and E_2 which meet only on a common boundary then

$$\iiint_E f(x, y, z)\, dV = \iiint_{E_1} f(x, y, z)\, dV + \iiint_{E_2} f(x, y, z)\, dV.$$

ITERATED INTEGRAL THEOREM

If E is the region

$$a_1 \le x \le a_2, \qquad b_1(x) \le y \le b_2(x), \qquad c_1(x, y) \le z \le c_2(x, y),$$

then

$$\iiint_E f(x, y, z)\, dV = \int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{c_1(x,y)}^{c_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

If the region E is a rectangular box

$$a_1 \le x \le a_2, \qquad b_1 \le y \le b_2, \qquad c_1 \le z \le c_2,$$

there are six different iterated integrals over E, corresponding to six different orders of integration. Here they are (in "alphabetical" order).

(1) $\displaystyle\int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dy\, dx$ (2) $\displaystyle\int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dz\, dx$

(3) $\displaystyle\int_{b_1}^{b_2} \int_{a_1}^{a_2} \int_{c_1}^{c_2} f(x, y, z)\, dz\, dx\, dy$ (4) $\displaystyle\int_{b_1}^{b_2} \int_{c_1}^{c_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dz\, dy$

(5) $\displaystyle\int_{c_1}^{c_2} \int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y, z)\, dy\, dx\, dz$ (6) $\displaystyle\int_{c_1}^{c_2} \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x, y, z)\, dx\, dy\, dz.$

The Iterated Integral Theorem shows that each of these iterated integrals is equal to the triple integral

$$\iiint_E f(x, y, z)\, dV.$$

EXAMPLE 1 Evaluate $\iiint_E xy^2z^3\,dV$ where E is the rectangular box

$$0 \le x \le 2, \qquad 0 \le y \le 1, \qquad 0 \le z \le 4.$$

There are six iterated integrals which all have the same value. We compute one of them, and then another to check our answer.

FIRST SOLUTION $\iiint_E xy^2z^3\,dV = \int_0^2 \int_0^1 \int_0^4 xy^2z^3\,dz\,dy\,dx.$

The inside integral is

$$\int_0^4 xy^2z^3\,dz = \left.\frac{xy^2z^4}{4}\right]_0^4 = 64xy^2.$$

The second integral is

$$\int_0^1 64xy^2\,dy = \left.\tfrac{64}{3}xy^3\right]_0^1 = \tfrac{64}{3}x.$$

The final answer is

$$\int_0^2 \tfrac{64}{3}x\,dx = \left.\tfrac{64}{6}x^2\right]_0^2 = \tfrac{256}{6} = \tfrac{128}{3}.$$

SECOND SOLUTION $\iiint_E xy^2z^3\,dV = \int_0^4 \int_0^2 \int_0^1 xy^2z^3\,dy\,dx\,dz.$

The inside integral is

$$\int_0^1 xy^2z^3\,dy = \tfrac{1}{3}xy^3z^3\Big]_0^1 = \tfrac{1}{3}xz^3.$$

The second integral is

$$\int_0^2 \tfrac{1}{3}xz^3\,dx = \tfrac{1}{6}x^2z^3\Big]_0^2 = \tfrac{4}{6}z^3.$$

The final answer is

$$\int_0^4 \tfrac{4}{6}z^3\,dz = \tfrac{1}{6}z^4\Big]_0^4 = \tfrac{256}{6} = \tfrac{128}{3}.$$

Triple integrals can be evaluated by iterated integrals.

EXAMPLE 2 Evaluate $\iiint_E y + z\,dV$ where E is the region shown in Figure 12.6.5,

$$0 \le x \le \pi/2, \qquad 0 \le y \le \sin x, \qquad 0 \le z \le y \cos x.$$

Figure 12.6.5

SOLUTION

$$\iiint_E y + z\,dV = \int_0^{\pi/2} \int_0^{\sin x} \int_0^{y \cos x} y + z\,dz\,dy\,dx.$$

We first evaluate the inside integral.

$$\int_0^{y \cos x} y + z\,dz = yz + \tfrac{1}{2}z^2\Big]_0^{y \cos x} = y^2 \cos x + \tfrac{1}{2}y^2 \cos^2 x.$$

Now we evaluate the second integral.

$$\int_0^{\sin x} y^2 \cos x + \tfrac{1}{2}y^2 \cos^2 x\,dy = \tfrac{1}{3}y^3(\cos x + \tfrac{1}{2}\cos^2 x)\Big]_0^{\sin x}$$

$$= \tfrac{1}{3}\sin^3 x(\cos x + \tfrac{1}{2}\cos^2 x).$$

Finally we evaluate the outside integral.

$$\iiint_E y + z\,dV = \int_0^{\pi/2} \tfrac{1}{3}\sin^3 x(\cos x + \tfrac{1}{2}\cos^2 x)\,dx$$

$$= \int_0^{\pi/2} \tfrac{1}{3}(1 - \cos^2 x)(\cos x + \tfrac{1}{2}\cos^2 x)\sin x\,dx$$

$$= \int_1^0 -\tfrac{1}{3}(1 - u^2)(u + \tfrac{1}{2}u^2)\,du$$

$$= \int_0^1 \tfrac{1}{3}(u + \tfrac{1}{2}u^2 - u^3 - \tfrac{1}{2}u^4)\,du = \tfrac{19}{180}.$$

1. Line integrals

HELM 28 was concerned with evaluating an integral over **all** points within a rectangle or other shape (or over a cuboid or other volume). In a related manner, an integral can take place over a line or curve running through a two-dimensional (or three-dimensional) region. Line integrals may involve scalar or vector fields. Those involving scalar fields are dealt with first.

Line integrals in two dimensions

A line integral in two dimensions may be written as

$$\int_C F(x,y)dw$$

There are three main features determining this integral:

$F(x,y)$: This is the scalar function to be integrated e.g. $F(x,y) = x^2 + 4y^2$.

C: This is the curve along which integration takes place. e.g. $y = x^2$ or $x = \sin y$ or $x = t - 1;\ y = t^2$ (where x and y are expressed in terms of a parameter t).

dw: This gives the variable of the integration. Three main cases are dx, dy and ds. Here 's' is arc length and so indicates position along the curve C.

ds may be written as $ds = \sqrt{(dx)^2 + (dy)^2}$ or $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$.

A fourth case is when $F(x,y)\,dw$ has the form: $F_1dx + F_2dy$. This is a combination of the cases dx and dy.

The integral $\int_C F(x,y)\,ds$ represents the area beneath the surface $z = F(x,y)$ but above the curve C.

The integrals $\int_C F(x,y)\,dx$ and $\int_C F(x,y)\,dy$ represent the projections of this area onto the xz and yz planes respectively.

A particular case of the integral $\int_C F(x,y)\,ds$ is the integral $\int_C 1\,ds$. This is a means of calculating the length along a curve i.e. an arc length.

Example 1

Find $\int_c x\,(1+4y)\,dx$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$.

Solution

As this integral concerns only points along C and the integration is carried out with respect to x, y may be replaced by x^2. The limits on x will be 0 to 1. So the integral becomes

$$\begin{aligned}\int_C x(1+4y)\,dx &= \int_{x=0}^{1} x\left(1+4x^2\right)dx = \int_{x=0}^{1}\left(x+4x^3\right)dx \\ &= \left[\frac{x^2}{2}+x^4\right]_0^1 = \left(\frac{1}{2}+1\right)-(0) = \frac{3}{2}\end{aligned}$$

Example 2

Find $\int_c x(1+4y)\ dy$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same as Example 1 other than dx being replaced by dy.

Solution

As this integral concerns only points along C and the integration is carried out with respect to y, everything may be expressed in terms of y, i.e. x may be replaced by $y^{1/2}$. The limits on y will be 0 to 1. So the integral becomes

$$\begin{aligned}\int_C x(1+4y)\,dy &= \int_{y=0}^{1} y^{1/2}(1+4y)\,dx = \int_{y=0}^{1}\left(y^{1/2}+4y^{3/2}\right)dx \\ &= \left[\frac{2}{3}y^{3/2}+\frac{8}{5}x^{5/2}\right]_0^1 = \left(\frac{2}{3}+\frac{8}{5}\right)-(0) = \frac{34}{15}\end{aligned}$$

Example 3

Find $\int_c x\,(1+4y)\,ds$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same integral and curve as the previous two examples but the integration is now carried out with respect to s, the arc length parameter.

Solution

As this integral is with respect to x, all parts of the integral can be expressed in terms of x, Along $y = x^2$, $\quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (2x)^2}dx = \sqrt{1 + 4x^2}dx$

So, the integral is

$$\int_c x\,(1+4y)\,ds = \int_{x=0}^{1} x\left(1+4x^2\right)\sqrt{1+4x^2}\;dx = \int_{x=0}^{1} x\left(1+4x^2\right)^{3/2}dx$$

This can be evaluated using the transformation $u = 1 + 4x^2$ so $du = 8xdx$ i.e. $x\,dx = \frac{du}{8}$.

When $x = 0$, $u = 1$ and when $x = 1$, $u = 5$.

Hence,

$$\begin{aligned}\int_{x=0}^{1} x\left(1+4x^2\right)^{3/2}dx &= \frac{1}{8}\int_{u=1}^{5} u^{3/2}du \\ &= \frac{1}{8}\times\frac{2}{5}\left[\;u^{5/2}\right]_1^5 = \frac{1}{20}\left[5^{5/2}-1\right] \approx 2.745\end{aligned}$$

Example 4

Find $\int_C xy\,dx$ where, on C, x and y are given in terms of a parameter t by $x = 3t^2$, $y = t^3 - 1$ for t varying from 0 to 1.

Solution

Everything can be expressed in terms of t, the parameter. Here $x = 3t^2$ so $dx = 6t\,dt$. The limits on t are $t = 0$ and $t = 1$. The integral becomes

$$\begin{aligned}\int_C xy\,dx &= \int_{t=0}^{1} 3t^2\,(t^3 - 1)\,6t\,dt = \int_{t=0}^{1} (18t^6 - 18t^3)\,dt \\ &= \left[\frac{18}{7}t^7 - \frac{18}{4}t^4\right]_0^1 = \frac{18}{7} - \frac{9}{2} - 0 = -\frac{27}{14}\end{aligned}$$

Improper integrals

The goal of this chapter is to meaningfully extend our theory of integrals to *improper* integrals. There are two types of so-called improper integrals: the first involves integrating a function over an infinite domain and the second involves integrands that are undefined at points within the domain of integration. In order to integrate over an infinite domain, we consider limits of the form $\lim_{b\to\infty}\int_a^b f(x)dx$. If the integrand is not defined at c ($a < c < b$) then we split the integral and consider the limits $\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon\to 0}\int_{c+\epsilon}^b f(x)dx$. The latter is sometimes also referred to as improper integrals of the second kind. Such situations occur, for example, for rational functions $f(x) = p(x)/q(x)$ whenever $q(x)$ has zeroes in the domain of integration.

Definition 1 (Improper integral (first kind)). *An improper integral of the first kind is an integral performed over an infinite domain, e.g.*

$$\int_a^\infty f(x)\,dx.$$

The value of such an integral is understood to be a limit, as given in the following definition:

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.}$$

We evaluate an improper integral by first computing a definite integral over a finite domain $a \le x \le b$, and then taking a limit as the endpoint b moves off to larger and larger values. The definite integral can be interpreted as an area under the graph of the function. The essential question being addressed here is whether that area remains bounded when we include the "infinite tail" of the function (i.e. as the endpoint b moves to larger values.) For some functions (whose values get small enough, fast enough) the answer is "yes".

Definition 2 (Convergence). *If the limit*

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

exists, we say that the improper integral converges. *Otherwise we say that the improper integral* diverges.

Example

Suppose that r > 0, evaluate

$$I=\int_0^\infty e^{-rt}\,dt\equiv\lim_{b\to\infty}\int_0^b e^{-rt}\,dt.$$

Solution

$$I=\lim_{b\to\infty}-\frac{1}{r}e^{-rt}\Big|_0^b=-\frac{1}{r}\lim_{b\to\infty}(e^{-rb}-e^0)=-\frac{1}{r}(\underbrace{\lim_{b\to\infty}e^{-rb}}_{0}-1)=\frac{1}{r},$$

where we have used the fact that $\lim_{b\to\infty}e^{-rb}=0$ for $r>0$. Thus the limit exists (is finite) and hence the integral *converges*. More precisely, it converges to the value $I=1/r$.

Example: The improper integral of $1/x$ diverges

We now consider a classic and counter-intuitive result, and *one of the most important results in this chapter*. Consider the function

$$y = f(x) = \frac{1}{x}.$$

Examining the graph of this function for positive x, e.g. for the interval $(1, \infty)$, we know that values decrease to zero as x increases[26]. The function is not only bounded, but also falls to arbitrarily small values as x increases (see Figure 7.1). Nevertheless, this is *insufficient*

The graph above shows *two functions whose values decrease along the x axis, $f(x) = 1/x$ and $f(x) = 1/x^2$. We show that one, but not the other encloses a finite (bounded) area over the interval $(1, \infty)$. To do so, we compute an improper integral for each one. The heavy arrow is meant to remind us that we are considering areas over an unbounded domain.*

to guarantee that the enclosed area remains finite! We show this in the following calculation.

$$I = \int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big|_1^b = \lim_{b\to\infty} (\ln(b) - \ln(1))$$
$$= \lim \ln(b) = \infty$$

The fact that we get an infinite value for this integral follows from the observation that $\ln(b)$ increases without bound as b increases, that is *the limit does not exist (is not finite)*. Thus, the area under the curve $f(x) = 1/x$ over the interval $1 \le x \le \infty$ is infinite. We say that the improper integral of $1/x$ *diverges* (or does not converge). We will use this result again later

7.2.3 Example: The improper integral of $1/x^2$ converges

Now consider the related function

$$y = f(x) = \frac{1}{x^2}$$

and the corresponding integral

$$I = \int_1^\infty \frac{1}{x^2}\, dx.$$

Then

$$I = \lim_{b\to\infty} \int_1^b x^{-2}\, dx = \lim_{b\to\infty} (-x^{-1})\Big|_1^b = -\lim_{b\to\infty}\left(\frac{1}{b} - 1\right) = 1.$$

Thus, the limit *exists*, and is $I = 1$. In contrast to the example in Section 7.2.2, this integral *converges*.

We observe that the behaviours of the improper integrals of the functions $1/x$ and $1/x^2$ are very different. The former diverges, while the latter converges. The only difference between these functions is the power of x. As shown in Figure 7.1, that power affects how rapidly the graph "falls off" to zero as x increases. The function $1/x^2$ decreases much faster than $1/x$. Consequently $1/x^2$ has a sufficiently "slim" infinite "tail", such that the area under its graph does not become infinite - not an easy concept to digest! This observations leads us to wonder what power p is needed to make the improper integral of a function $1/x^p$ converge. We answer this question below.

7.2.4 When does the integral of $1/x^p$ converge?

Here we consider an arbitrary power, p, that can be any real number. We ask when the corresponding improper integral converges or diverges. Let

$$I = \int_1^\infty \frac{1}{x^p}\, dx.$$

For $p = 1$ we have already established that this integral diverges (see Section 7.2.2), and for $p = 2$ we have seen that it is convergent (see Section 7.2.3). By a similar calculation, we find that

$$I = \lim_{b\to\infty} \frac{x^{1-p}}{(1-p)}\bigg|_1^b = \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left(b^{1-p} - 1\right).$$

Thus, this integral converges provided that the term b^{1-p} does not "blow up" as b increases. For this to be true, we require that the exponent $(1-p)$ should be negative, i.e. $1 - p < 0$ or $p > 1$. In this case, we have

$$I = \frac{1}{p-1}.$$

To summarize our result,

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \qquad \text{converges if } p > 1, \text{ and diverges if } p \le 1.}$$

Examples:

(i) The integral

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx, \quad \text{diverges.}$$

We see this from the following argument: $\sqrt{x} = x^{\frac{1}{2}}$, so $p = \frac{1}{2} < 1$. Thus, by the general result, this integral diverges.

(ii) The integral

$$\int_1^\infty x^{-1.01}\,dx, \quad \text{converges.}$$

Here $p = 1.01 > 1$, so the result implies convergence of the integral.

7.4 Integral comparison test

The integrals discussed above can be used to make comparisons that help us to identify when other improper integrals converge or diverge[27]. The following important result establishes how these comparisons work:

Suppose we are given two functions, $f(x)$ and $g(x)$, both continuous on some infinite interval $[a, \infty)$. Suppose, moreover, that at all points on this interval the first function is smaller than the second, i.e.

$$0 \leq f(x) \leq g(x).$$

Then the following conclusions can be made:[a]

(i) $\displaystyle\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$
The area under $f(x)$ is smaller than the area under $g(x)$.

(ii) If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
If the larger area is finite, so is the smaller one.

(iii) If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.
If the smaller area is infinite, so is the larger one.

Example: Determine whether the following integral converges:

$$\int_1^\infty \frac{x}{1+x^3}\,dx.$$

Solution: by noting that for all $x > 0$

$$0 \le \frac{x}{1+x^3} \le \frac{x}{x^3} = \frac{1}{x^2}.$$

Note that for $x > 0$

$$0 \le \frac{x}{1+x^3} \le \frac{x}{x^3} = \frac{1}{x^2}.$$

Thus,

$$\int_1^\infty \frac{x}{1+x^3}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx.$$

Since the larger integral on the right is known to converge, so does the smaller integral on the left. $\diamond$

7.5 Integration of an unbounded integrand

The second kind of improper integrals refers to integrands that are undefined at one (or more) points of the domain of integration $[a, b]$. Suppose $f(x)$ is continuous on the open interval (a, b) but becomes infinite at the lower bound, $x = a$. Then the integral of $f(x)$ over the domain $[a+\epsilon, b]$ for $\epsilon > 0$ has a definite value regardless of how small ϵ is chosen. Therefore, we can consider the limit

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)dx,$$

where $\epsilon \to 0^+$ means that ϵ approaches 0 from 'above', i.e. $\epsilon > 0$ always holds. If this limit exists and is equal to L, then we define

$$\int_{a}^{b} f(x)dx = L.$$

If an anti-derivative of $f(x)$, say $F(x)$ is known, then the fundamental theorem of calculus permits us to compute

$$\int_{a+\epsilon}^{b} f(x)dx = F(b) - F(a+\epsilon).$$

We are thus led to determine the existence (or nonexistence) of the limit

$$\lim_{\epsilon \to 0^+} F(a+\epsilon).$$

Example 1: Calculate the following integral for $p \neq 1$:

$$I = \int_a^b \frac{dx}{(x-a)^p}.$$

Solution: We interpret the integral as the following limit:

$$I = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b \frac{dx}{(x-a)^p} = -\lim_{\epsilon \to 0^+} \frac{1}{p-1}\left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\epsilon^{p-1}}\right].$$

Thus, for $p > 1$ the term ϵ^{1-p} becomes arbitrarily large as $\epsilon \to 0^+$ and hence the area diverges and the integral does not exist. Conversely, for $p < 1$ the term ϵ^{1-p} converges to 0 as $\epsilon \to 0^+$ and hence the improper integral exists and is

$$I = \frac{(b-a)^{1-p}}{1-p}.$$

Finally, note that for $p = 1$ the anti-derivative is undefined and the integral does not exist. Alternatively, for $p = 1$ we directly see that

$$\int_{a+\epsilon}^{b} \frac{dx}{x-a} = \ln\left(\frac{b-a}{\epsilon}\right)$$

7.6 L'Hôpital's rule

This section introduces a powerful method to evaluate tricky limits of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$. The rule is named after the French mathematician Guillaume de l'Hôpital, who published it in the 17th century. For our purposes, the rule is often particularly useful to evaluate the limits that arise in improper integrals of the first (unbounded domain) and second kind (unbounded integrand).

Consider two functions, $f(x)$ and $g(x)$, and suppose that the following four prerequisites are satisfied:

(a) $f(x)$ and $g(x)$ are differentiable near $x = a$, but not necessarily at $x = a$.

(b) $g'(x) \neq 0$ for x near a but $x \neq a$.

(c) $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists.

(d) and either

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, or

(ii) $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} g(x) = \pm\infty$.

Then, l'Hôpital's rule states that

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \qquad (7.4)}$$

Note: values of the limit a can include $\pm\infty$.

In *loose* terms, l'Hôpital's rule can be used if

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{is of type} \quad \frac{0}{0} \quad \text{or} \quad \pm\frac{\infty}{\infty}.$$

Let us now explore the power of l'Hôpital's rule through several examples.

Example 1: Calculate the following limit:

$$\lim_{x\to 0} \frac{\sin x}{e^x - 1}.$$

Solution: In this example l'Hôpital's rule can be used because $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0}(e^x - 1) = 0$. Thus,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{e^x - 1} &= \lim_{x\to 0} \frac{(\sin x)'}{(e^x - 1)'} = \lim_{x\to 0} \frac{\cos x}{e^x} \\
&= \frac{\lim\limits_{x\to 0} \cos x}{\lim\limits_{x\to 0} e^x} \qquad \text{(because of non-zero denominator)} \\
&= \frac{\cos 0}{1} = 1.
\end{aligned}$$

Example 5: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

$$\begin{aligned}
\int_0^1 \frac{1}{1-x}\,dx &= \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx \\
&= \lim_{b\to 1^-} -\int_0^b \frac{1}{x-1}\,dx \\
&= \lim_{b\to 1^-} -\ln|x-1|\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(x-1)\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(1-b) \\
&= -(-\infty) \\
&= \boxed{\infty}
\end{aligned}$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\int_a^\infty g(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$.
2. If $\int_a^\infty f(x)\,dx$ diverges, then so does $\int_a^\infty g(x)\,dx$.

Why does this make sense?

1. If the area under the curve of $g(x)$ is *finite* and $f(x)$ is bounded above by $g(x)$ (and below by 0), then the area under the curve of $f(x)$ must be *less than or equal to* the area under the curve of $g(x)$. A positive number less that a *finite* number is also *finite*.
2. If the area under the curve of $f(x)$ is *infinite* and $g(x)$ is bounded below by $f(x)$, then the area under the curve of $g(x)$ must be *"less than or equal to"* the area under the curve of $g(x)$. Since there is no finite number "greater than" infinity, the area under $g(x)$ must also be *infinite*.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

We want to find a function $g(x)$ such that for some $a \in \mathbb{R}$, $f(x) = \dfrac{\cos^2(x)}{x^2} \leq g(x)$ or $f(x) = \dfrac{\cos^2(x)}{x^2} \geq g(x)$ for all $x \geq a$. One way we can do this is by finding *bounds* for $f(x)$. Since $0 \leq \cos^2(x) \leq 1$ for all x,

$$\frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}.$$

So then we can use $g(x) := \dfrac{1}{x^2}$. So,

$$0 \leq \int_2^\infty \frac{\cos^2(x)}{x^2}\,dx \leq \int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{1}{b} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$$

So $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges.

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Why does this make sense? The convergence is really only dependent on the "tail" of the integral. That is, the convergence is dictated by what happens "at infinity." If for sufficiently large values of x, $f(x) \approx Lg(x)$ and one of the two integrals converges, then the other one should also converge, since it is only off by "about a scalar multiple." The same goes for diverging, if one diverges, then multiplying it by a positive number won't suddenly make it converge, so the other one should also diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Let $f(x) := \dfrac{1}{1+x^2}$ and $g(x) := \dfrac{1}{x^2}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1+x^2-1}{1+x^2} = \lim_{x\to\infty} \left(1 - \frac{1}{1+x^2}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ *converges.*

Vector fields

Definition: A vector field in $\mathbb{R}^n$ is a function $F : \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each $\mathbf{x} \in \mathbb{R}^n$ a vector $F(\mathbf{x})$. A vector field in $\mathbb{R}^n$ with domain $U \subset \mathbb{R}^n$ is called a vector field on U.

Geometrically, a vector field F on U is interpreted as attaching a vector to each point of U. Thus, there is a subtle difference between a vector field in $\mathbb{R}^n$ and a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

When a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is viewed as a vector field, for each x the vector $F(\mathbf{x})$ is identified with the vector that starts at the point $\mathbf{x}$ and points to $F(\mathbf{x})$, i.e., $F(\mathbf{x})$ is identified with the vector that is obtained by translating $F(\mathbf{x})$ to the point $\mathbf{x}$.

Thus every vector field on $U \subset \mathbb{R}^n$ is uniquely determined by a function from $U \to \mathbb{R}^n$.

Examples of vector fields

- The gravitational force field describes the force of attraction of the earth on a mass m and is given by

$$\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r},$$

where $\mathbf{r} := (x, y, z)$ and $r := \|\mathbf{r}\|$. The vector field F points to the centre of the earth.

- The vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) := (-y, x)$ is a rotational vector field in $\mathbb{R}^2$ which rotates a vector in the anti-clockwise direction by an angle $\pi/2$.

- Let $\mathbf{x} : [0, 1] \to \mathbb{R}^n$ be C^1 and $\Gamma := \mathbf{x}([0, 1])$. Then $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\mathbf{x}(t)) = \mathbf{x}'(t)$ is a tangent vector field on Γ.

Scalar fields

A function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is called a scalar field on U as it assigns a number to each point in U.

The temperature of a metal rod that is heated at one end and cooled on another is described by a scalar field $T(x, y, z)$. The flow of heat is described by a vector field. The energy or heat flux vector field is given by $\mathbf{J} := -\kappa \nabla T$, where $\kappa > 0$.

- Let F be a vector field in $\mathbb{R}^n$. Then $F := (f_1, \ldots, f_n)$ for some scalar fields $f_1, \ldots, f_n$ on $\mathbb{R}^n$. We say that F is a C^k vector field if $f_1, \ldots, f_n$ are C^k functions.

All vector fields are assumed to be C^1 unless otherwise noted.

Gradient vector fields

If $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 scalar field then $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field in $\mathbb{R}^n$.

- A vector field F in $\mathbb{R}^n$ is said to be a gradient vector field or a conservative vector field if there is a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla f$. In such a case, f is called a scalar potential of the vector field F.

Obviously not every vector field is a gradient vector field.

Example: The vector field $F(x, y) := (y, -x)$ is not a gradient vector field. Indeed, if $F = \nabla f$ then $f_x = y$ and $f_y = -x$.

Consequently, $f_{xy} = 1$ and $f_{yx} = -1$. Hence F is NOT a C^1 vector field, which is a contradiction.

Integral curves for vector fields

Definition: Let F be a vector field in $\mathbb{R}^n$. Then a C^1 curve $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is said to be an integral curve for the vector field F if $F(\mathbf{x}(t)) = \mathbf{x}'(t)$ for $t \in [a, b]$.

Obviously, F is a tangent (velocity) vector field on the integral curve. Thus integral curves provide a geometric picture of a vector field.

Integral curves are also called flow lines or streamlines.

An integral curve may be viewed as a solution of a system of ODE. Indeed, for $n = 3$ and $F = (P, Q, R)$, we have

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)), \\ y'(t) &= Q(x(t), y(t), z(t)), \\ z'(t) &= R(x(t), y(t), z(t)). \end{aligned}$$

Example

Find integral curves for the vector field $F(x, y) = (-y, x)$.

Note that $\mathbf{r}(t) = (\cos t, \sin t)$ is an integral curve for F. The other integral curves are also circles and are of the form

$$\gamma(t) := (r\cos(t - t_0),\ r\sin(t - t_0)).$$

This follows from the system of ODE $x' = -y$ and $y' = x$ which gives $x'' + x = 0$.

Divergence of vector fields

Definition: Let $F = (f_1, \ldots, f_n)$ be a vector field in $\mathbb{R}^n$. Then the divergence of F is a scalar field on $\mathbb{R}^n$ given by

$$\mathrm{div} F = \partial_1 f_1 + \cdots + \partial_n f_n = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}.$$

Define the del operator

$$\nabla := (\partial_1, \ldots, \partial_n) = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}) = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n}.$$

Then applying ∇ to a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ we obtain the gradient vector field

$$\nabla f = (\partial_1 f, \ldots, \partial_n f).$$

Divergence of vector fields

Taking dot product of ∇ with a vector field $F = (f_1, \ldots, f_n)$ we obtain the divergence

$$\nabla \bullet F = \partial_1 f_1 + \cdots + \partial_n f_n = \mathrm{div} F.$$

Physical interpretation: If F represents velocity field of a gas (or fluid) then $\mathrm{div} F$ represents the rate of expansion per unit volume under the flow of the gas (or fluid).

The divergence of $F = (x^2 y, z, xyz)$ is given by

$$\mathrm{div} F = 2xy + 0 + xy = 3xy.$$

Examples

- The streamlines of the vector field $F(x, y) := (x, y)$ are straight lines directed away from the origin.

  For fluid flow, this means the fluid is expanding as it moves out from the origin, so $\mathrm{div} F$ should be positive. Indeed, we have

$$\mathrm{div} F = 2 > 0.$$

- Next, consider the vector field $F(x, y) := (x, -y)$. Then

$$\mathrm{div} F = 0$$

  so that neither expansion nor compression takes place.

Cross product in $\mathbb{R}^3$

The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ denoted by $\mathbf{u} \times \mathbf{v}$ is the unique vector satisfying the following conditions:

- $\mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$.
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}) \geq 0$, where $[\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}]$ is the matrix whose columns are $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$.

Cross product in $\mathbb{R}^3$

It is customary to denote the standard basis in $\mathbb{R}^3$ by

$$\mathbf{i} := (1, 0, 0), \;\; \mathbf{j} := (0, 1, 0) \;\text{ and }\; \mathbf{k} := (0, 0, 1).$$

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2v_3 - v_2u_3)\mathbf{i} + (u_1v_3 - v_1u_3)\mathbf{j} + (u_1v_2 - v_2u_1)\mathbf{k} \\ &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

the last equality is only symbolic.

Curl of vector fields in $\mathbb{R}^3$

The curl of a vector field $F = (f_1, f_2, f_3)$ in $\mathbb{R}^3$ is a vector field in $\mathbb{R}^3$ and is given by

$$\text{curl}F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

Thus curl of F is obtained by taking cross product of the del operator $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ with $F = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$.

Curl of a vector field represents rotational motion when, for example, the vector field represents flow of a fluid.

Irrotational vector field

A vector field F in $\mathbb{R}^3$ is called irrotational if $\text{curl}F = 0$. This means, in the case of a fluid flow, that the flow is free from rotational motion, i.e, no whirlpool.

Fact: If f be a C^2 scalar field in $\mathbb{R}^3$. Then ∇f is an irrotational vector field, i.e., $\text{curl}(\nabla f) = 0$.

Proof: We have $\text{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = 0$ because of the equality of mixed partial derivatives.

Observation: If $\text{curl}F \neq 0$ then F is not a conservative (gradient) vector field.

Examples

- Let $F(x, y, z) := (xy, -\sin z, 1)$. Then
$$\mathrm{curl}F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix} = (\cos z, 0, -x).$$
- Let $F(x, y, z) := (y, -x, 0)$. Then $\mathrm{curl}F = (0, 0, -2)$ and so F is not a conservative vector field.
- Let $F(x, y, z) := (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$. Then $\mathrm{curl}F = 0$ and hence F is irrotational. However, F is NOT a conservative vector field, that is, $F \neq \nabla f$ for some scalar potential f.

Scalar curl

Let $F = (P, Q)$ be a vector field in $\mathbb{R}^2$. Then identifying $\mathbb{R}^2$ with the x-y plane in $\mathbb{R}^3$, F can be identified with the vector field $F = P\mathbf{i} + Q\mathbf{j}$ in $\mathbb{R}^3$. Then we have

$$\mathrm{curl}F = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}.$$

The scalar field $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is called the scalar curl of the vector field $F = (P, Q)$.

The scalar curl of $F(x, y) := (-y^2, x)$ is given by

$$\partial_x(x) - \partial_y(-y^2) = 1 + 2y.$$

Laplace operator

Example: If $F(x, y, z) := (x, y, z)$ then $\mathrm{div} F = 3 \neq 0$ so F does not have a vector potential.

Laplace operator ∇^2: Let f be a C^2 scalar field in $\mathbb{R}^n$. Then

$$\nabla^2 f := \nabla \bullet \nabla f = \mathrm{div}(\nabla f) = \partial_1^2 f + \cdots + \partial_n^2 f$$

defines the Laplace operator ∇^2 on f.

For a C^2 function $u : \mathbb{R}^3 \to \mathbb{R}$ the partial differential equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is called Laplace equation.

Integral Vector Theorems

1. Stokes' theorem

This is a theorem that equates a line integral to a surface integral. For any vector field $\underline{F}$ and a contour C which bounds an area S,

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Figure 16: A surface for Stokes' theorem

Notes

(a) $\underline{dS}$ is a vector perpendicular to the surface S and $\underline{dr}$ is a line element along the contour C. The sense of $\underline{dS}$ is linked to the direction of travel along C by a right hand screw rule.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a line integral along a contour composed of several distinct parts (e.g. a square or other figure).

(d) $\underline{\nabla} \times \underline{F}$ is a vector field representing the curl of the vector field $\underline{F}$ and may, alternatively, be written as curl $\underline{F}$.

Example 34

Find the closed line integral $\oint_C \underline{F}\cdot\underline{dr}$ for the vector field $\underline{F} = y^2\underline{i}+(x^2-z)\underline{j}+2xy\underline{k}$ and for the contour $ABCDEFGHA$ in Figure 18.

Figure 18: Closed contour $ABCDEFGHA$

Solution

To find the line integral directly would require eight line integrals i.e. along AB, BC, CD, DE, EF, FG, GH and HA. It is easier to carry out a surface integral to find $\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ which is equal to the required line integral $\oint_C \underline{F} \cdot \underline{dr}$ by Stokes' theorem.

As $\underline{F} = y^2\underline{i} + (x^2 - z)\underline{j} + 2xy\underline{k}$, $\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 - z & 2xy \end{vmatrix} = (2x+1)\underline{i} - 2y\underline{j} + (2x - 2y)\underline{k}$

As the contour lies in the x-y plane, the unit normal is $\underline{k}$ and $\underline{dS} = dxdy\underline{k}$

Hence $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (2x - 2y)dxdy$.

To work out $\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$, it is necessary to divide the area inside the contour into two smaller areas i.e. the rectangle $ABCDGH$ and the trapezium $DEFG$. On $ABCDGH$, the integral is

$$\begin{aligned} \int_{y=0}^{4} \int_{x=0}^{6} (2x - 2y)dxdy &= \int_{y=0}^{4} \Big[x^2 - 2xy \Big]_{x=0}^{6} dy = \int_{y=0}^{4} (36 - 12y)dy \\ &= \Big[36y - 6y^2 \Big]_0^4 = 36 \times 4 - 6 \times 16 - 0 = 48 \end{aligned}$$

On $DEFG$, the integral is

$$\int_{y=4}^{7}\int_{x=1}^{y-2}(2x-2y)dxdy = \int_{y=4}^{7}\left[x^2-2xy\right]_{x=1}^{y-2}dy = \int_{y=4}^{7}(-y^2+2y+3)dy$$

$$= \left[-\frac{1}{3}y^3+y^2+3y\right]_4^7 = -\frac{343}{3}+49+21+\frac{64}{3}-16-12=-51$$

So the full integral is, $\iint_S(\underline{\nabla}\times\underline{F})\cdot\underline{dS}=48-51=-3.$

$\therefore$ By Stokes' theorem, $\oint_C \underline{F}\cdot\underline{dr}=-3$

Example 32

Verify Stokes' theorem for the vector function $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ and the unit square $0 \le x \le 1$, $0 \le y \le 1$, $z = 0$.

Solution

If $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ then $\underline{\nabla} \times \underline{F} = (z+1)\underline{i} + (-1-2y)\underline{k} = \underline{i} + (-1-2y)\underline{k}$ (as $z = 0$).
Note that $\underline{dS} = dxdy\underline{k}$ so that $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (-1-2y)dydx$

Thus
$$\begin{aligned}
\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{1} (-1-2y)dydx \\
&= \int_{x=0}^{1} \Big[(-y-y^2)\Big]_{y=0}^{1} dx = \int_{x=0}^{1} (-2)dx \\
&= \Big[-2x\Big]_0^1 = -2+0 = -2
\end{aligned}$$

To evaluate $\oint_C \underline{F} \cdot \underline{dr}$, we must consider the four sides separately.

When $y = 0$, $\underline{F} = -x\underline{j}$ and $\underline{dr} = dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = 0$ i.e. the contribution of this side to the integral is zero.

When $x = 1$, $\underline{F} = y^2\underline{i} - \underline{j}$ and $\underline{dr} = dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = -dy$ so the contribution to the integral is

$$\int_{y=0}^{1} (-dy) = \Big[-y \Big]_0^1 = -1.$$

When $y = 1$, $\underline{F} = \underline{i} - x\underline{j}$ and $\underline{dr} = -dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = -dx$ so the contribution to the integral is

$$\int_{x=0}^{1} (-dx) = \Big[-x \Big]_0^1 = -1.$$

When $x = 0$, $\underline{F} = y^2\underline{i}$ and $\underline{dr} = -dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = 0$ so the contribution to the integral is zero.

The integral $\oint_C \underline{F} \cdot \underline{dr}$ is the sum of the contributions i.e. $0 - 1 - 1 + 0 = -2$.

Thus $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr} = -2$ i.e. Stokes' theorem has been verified.

2. Gauss' theorem

This is sometimes known as the **divergence theorem** and is similar in form to Stokes' theorem but equates a surface integral to a volume integral. Gauss' theorem states that for a volume V, bounded by a closed surface S, any 'well-behaved' vector field $\underline{F}$ satisfies

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} \, dV$$

Notes:

(a) $\underline{dS}$ is a unit normal pointing outwards from the interior of the volume V.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a surface integral over a surface composed of several distinct parts (e.g. a cube).

(d) $\underline{\nabla} \cdot \underline{F}$ is a scalar field representing the divergence of the vector field $\underline{F}$ and may, alternatively, be written as div $\underline{F}$.

(e) Gauss' theorem can be justified in a manner similar to that used for Stokes' theorem (i.e. by proving it for a small volume element, then summing up the volume elements and allowing the internal surface contributions to cancel.)

Example 35

Verify Gauss' theorem for the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ and the function $\underline{F} = x\underline{i} + z\underline{j}$

Solution

To find $\iint_S \underline{F} \cdot \underline{dS}$, the integral must be evaluated for all six faces of the cube and the results summed.

On the face $x = 0$, $\underline{F} = z\underline{j}$ and $\underline{dS} = -\underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\, dydz = 0$$

On the face $x = 1$, $\underline{F} = \underline{i} + z\underline{j}$ and $\underline{dS} = \underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 1\, dydz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 1\, dydz = 1$$

On the face $y = 0$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = -\underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = -z\, dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = -\int_0^1 \int_0^1 z\, dxdz = -\frac{1}{2}$$

On the face $y = 1$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = \underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = z\, dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 z\, dxdz = \frac{1}{2}$$

On the face $z = 0$, $\underline{F} = x\underline{i}$ and $\underline{dS} = -\underline{k}\ dydz$ so $\underline{F} \cdot \underline{dS} = 0\ dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\ dxdy = 0$$

On the face $z = 1$, $\underline{F} = x\underline{i} + \underline{j}$ and $\underline{dS} = \underline{k}\ dydz$ so $\underline{F} \cdot \underline{dS} = 0\ dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\ dxdy = 0$$

Thus, summing over all six faces, $\iint_S \underline{F} \cdot \underline{dS} = 0 + 1 - \frac{1}{2} + \frac{1}{2} + 0 + 0 = 1.$

To find $\iiint_V \underline{\nabla} \cdot \underline{F}\ dV$ note that $\underline{\nabla} \cdot \underline{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} z = 1 + 0 = 1.$

So $\iiint_V \underline{\nabla} \cdot \underline{F}\ dV = \int_0^1 \int_0^1 \int_0^1 1\ dxdydz = 1.$

So $\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F}\ dV = 1$ hence verifying Gauss' theorem.

Example 36

Use Gauss' theorem to evaluate the surface integral $\iint_S \underline{F} \cdot \underline{dS}$ where $\underline{F}$ is the vector field $x^2y\underline{i} + 2xy\underline{j} + z^3\underline{k}$ and S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

Note that to carry out the surface integral directly will involve, as in Example 35, the evaluation of six double integrals. However, by Gauss' theorem, the same result comes from the volume integral $\iiint_V \underline{\nabla} \cdot \underline{F}\, dV$. As $\underline{\nabla} \cdot \underline{F} = 2xy + 2x + 3z^2$, we have the triple integral

$$\int_0^1 \int_0^1 \int_0^1 (2xy + 2x + 3z^2)\, dxdydz$$

$$= \int_0^1 \int_0^1 \left[x^2y + x^2 + 3xz^2 \right]_{x=0}^1 dydz = \int_0^1 \int_0^1 (y + 1 + 3z^2) dydz$$

$$= \int_0^1 \left[\frac{1}{2}y^2 + y + 3yz^2 \right]_{y=0}^1 dz = \int_0^1 (\frac{1}{2} + 1 + 3z^2) dz = \int_0^1 (\frac{3}{2} + 3z^2) dz$$

$$= \left[\frac{3}{2}z + z^3 \right]_0^1 = \frac{5}{2}$$

Green's theorem in the plane

This states that

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

S is a 2-D surface with perimeter C; $P(x, y)$ and $Q(x, y)$ are scalar functions.

This should not be confused with Green's identities.

Justification of Green's theorem in the plane

Green's theorem in the plane can be derived from Stokes' theorem.

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Justification of Green's theorem in the plane

Green's theorem in the plane can be derived from Stokes' theorem.

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Now let $\underline{F}$ be the vector field $P(x,y)\underline{i} + Q(x,y)\underline{j}$ i.e. there is no dependence on z and there are no components in the $z-$ direction. Now

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underline{k}$$

and $\underline{dS} = dxdy\underline{k}$ giving $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dxdy.$

Thus Stokes' theorem becomes

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C \underline{F} \cdot \underline{dr}$$

and Green's theorem in the plane follows.

Example 39

Evaluate the line integral $\oint_C \left[(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy\right]$ around the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 1$.

Solution

The integral could be obtained by evaluating four line integrals but it is easier to note that $[(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy]$ is of the form $Pdx + Qdy$ with $P = 4x^2 + y - 3$ and $Q = 3x^2 + 4y^2 - 2$. It is thus of a suitable form for Green's theorem in the plane.

Note that $\dfrac{\partial Q}{\partial x} = 6x$ and $\dfrac{\partial P}{\partial y} = 1$.

Green's theorem in the plane becomes

$$\begin{aligned}
\oint_C \{(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy\} &= \int_{y=0}^{1}\int_{x=0}^{3} (6x - 1)\, dxdy \\
&= \int_{y=0}^{1} \left[3x^2 - x\right]_{x=0}^{3} dy = \int_{y=0}^{1} 24\ dy = 24
\end{aligned}$$

Example 40

Verify Green's theorem in the plane for the integral $\oint_C [4zdy + (y^2 - 2)dz]$ and the triangular contour starting at the origin $O = (0, 0, 0)$ and going to $A = (0, 2, 0)$ and $B = (0, 0, 1)$ before returning to the origin.

Solution

The whole of the contour is in the plane $x = 0$ and Green's theorem in the plane becomes

$$\oint_C (Pdy + Qdz) = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz$$

(a) Firstly evaluate $\oint_C \{4zdy + (y^2 - 2)dz\}$.

On OA, $z = 0$ and $dz = 0$. As the integrand is zero, the integral will also be zero.

On AB, $z = (1 - \frac{y}{2})$ and $dz = -\frac{1}{2}dy$. The integral is

$$\int_{y=2}^{0} \left((4 - 2y)dy - \frac{1}{2}(y^2 - 2)dy\right) = \int_2^0 (5 - 2y - \frac{1}{2}y^2)dy = \left[5y - y^2 - \frac{1}{6}y^3\right]_2^0 = -\frac{14}{3}$$

On BO, $y = 0$ and $dy = 0$. The integral is $\int_1^0 (-2)dz = \Big[-2z\Big]_1^0 = 2.$

Summing, $\oint_C (4zdy + (y^2 - 2)dz) = -\frac{8}{3}$

(b) Secondly evaluate $\iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz$

In this example, $P = 4z$ and $Q = y^2 - 2$. Thus $\frac{\partial P}{\partial z} = 4$ and $\frac{\partial Q}{\partial y} = 2y$. Hence,

$$\begin{aligned}
\iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz &= \int_{y=0}^{2} \int_{z=0}^{1-y/2} (2y-4)\, dzdy \\
&= \int_{y=0}^{2} \Big[2yz - 4z\Big]_{z=0}^{1-y/2} dy = \int_{y=0}^{2} \left(-y^2 + 4y - 4\right) dy \\
&= \left[-\frac{1}{3}y^3 + 2y^2 - 4y\right]_0^2 = -\frac{8}{3}
\end{aligned}$$

Hence:

$\oint_C (Pdy + Qdz) = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz = -\frac{8}{3}$ and Green's theorem in the plane is verified.

MTS 241

1.0 Green's Theorem

The Green's theorem is use to relate the line integral around a closed curve with a double Integral over a region inside the curve by evaluating the line Integral of a smooth vector field around a simple closed curve.

Definition 1.1 A vector field $f(x,y) = p(x,y)i + Q(x,y)j$ is smooth if its component functions $p(x,y)$ and $Q(x,y)$ are smooth.

Theorem 1.2 (Green's Theorem)

Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C which is piecewise smooth. Let $f(x,y) = p(x,y)i + Q(x,y)j$ be a smooth vector field defined on both R and C. Then

$$\oint_C f \cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y} \right) dA. \quad (1)$$

Proof Since R is a region with boundary curve C ~~which~~ can be written as $C = C_1 \cup C_2$ in two distinct ways: ~~the~~

C_1 = the curve $y = y_1(x)$ from the point X_1 to the point X_2. (2)

C_2 = the curve $y = y_2(x)$ from the point X_2 ~~to~~ the point X_1. (3)

where X_1 and X_2 are the points on C farthest ~~to~~ the left and right respectively.

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C_1 = the curve $x = x_1(y)$ from the point T_2 to the point T_1. ——— (4)

C_2 = the curve $x = x_2(y)$ from the point T_1 to the point T_2. ——— (5)

as shown below.

Integrating $p(x,y)$ around $C = C_1 \cup C_2$ given by (2) and (3). Since $y = y_1(x)$ along C_1 (as x goes from a to b) and $y = y_2(x)$ along C_2 (x goes from b to a) we have that

$$\oint_C p(x,y)\,dx = \int_{C_1} p(x,y)\,dx + \int_{C_2} p(x,y)\,dx$$

$$= \int_a^b p(x, y_1(x))\,dx + \int_b^a (x, y_2(x))\,dx$$

$$= \int_a^b p(x, y_1(x))\,dx - \int_a^b p(x, y_2(x))\,dx$$

$$= -\int_a^b \left[p(x, y_2(x)) - p(x, y_1(x)) \right] dx$$

$$= -\int_a^b \left[p(x,y) \Big|_{y = y_1(x)}^{y = y_2(x)} \right] dx.$$

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$$= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y}\, dy\, dx.$$

$$= -\iint_R \frac{\partial P}{\partial y}\, dA.$$

Also, integrating $Q(x,y)$ around C using the representation $C = C_1 \cup C_2$ given by (4) and (5). Since $x = x_1(y)$ along C_1 and $x = x_2(y)$ along C_2 Then

$$\oint_C Q(x,y)\, dy = \int_{C_1} Q(x,y)\, dy + \int_{C_2} Q(x,y)\, dy$$

$$= \int_d^c Q(x_1(y), y)\, dy + \int_c^d Q(x_2(y)\, y)\, dy$$

$$= -\int_c^d Q(x_1(y), y)\, dy + \int_c^d Q(x_2(y), y)\, dy$$

$$= \int_c^d \left[Q(x_2(y), y) - Q(x_1(y), y) \right] dy$$

$$= \int_c^d \left[Q(x, y) \Big|_{x = x_1(y)}^{x = x_2(y)} \right] dy$$

$$= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q(x,y)}{\partial x}\, dx\, dy.$$

$$= \iint_R \frac{\partial Q}{\partial x}\, dA$$

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therefore $\oint_C f \cdot dr = \oint_C P(x,y)\,dx + \oint_C Q(x,y)\,dy$

$$= -\iint_R \frac{\partial P}{\partial y}\,dA + \iint_R \frac{\partial Q}{\partial x}\,dA.$$

$$= \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

<u>Example 1.3</u> Evaluate $\oint_C (x^2+y^2)\,dx + 2xy\,dy$ where C is the boundary (traversed counterclockwise) of the region $R = \{(x,y) : 0 \le x \le 1,\ 2x^2 \le y \le 2x\}$.

<u>Solution</u> Let $f(x,y) = P(x,y)i + Q(x,y)j = (x^2+y^2)i + 2xyj$ be the vector field. By Green's theorem

$$\oint_C (x^2+y^2)\,dx + 2xy\,dx = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R (2y - 2y)\,dA = \iint_R 0\,dA = 0.$$

<u>Example 1.4</u> Using Green's theorem, evaluate the line integral

$$\oint_C xy\,dx + (x+y)\,dy.$$

where C is the curve bounding the unit disk R.

<u>Solution</u> the component of the vector field are $P(x,y) = xy$, $Q(x,y) = x+y$

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By Green's formula

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint P\,dx + Q\,dy$$

we transform the line Integral into double Integral.

$$I = \oint_C xy\,dx + (x+y)\,dy$$

$$= \iint_R \left(\frac{\partial (x+y)}{\partial x} - \frac{\partial (xy)}{\partial y}\right) dx\,dy$$

$$= \iint_R (1-x)\,dx\,dy$$

Converting the double integral into polar coordinates, we have

$$I = \int_R (1-x)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1-r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^1 (r - r^2\cos\theta)\,dr\right] d\theta = \int_0^{2\pi}\left[\left(\frac{r^2}{2} - \frac{r^3\cos\theta}{3}\right)\right]_0^1$$

$$\Big] d\theta = \int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos\theta}{2}\right) d\theta = \left(\frac{\theta}{2} - \frac{\sin\theta}{3}\right)\Bigg|_0^{2\pi} = \pi$$

<u>Example 1.5</u> Evaluate the integral $\oint_C x^2y\,dx - xy^2\,dy$ the Curve C is the Circle. $x^2+y^2=a^2$ (traversed in the counterclockwise direction)

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Solution

We have $P(x,y) = x^2 y$, $Q(x,y) = -xy^2$

$$\frac{\partial Q}{\partial x} = \frac{\partial(-xy^2)}{\partial x} = -y^2, \quad \frac{\partial P}{\partial y} = \frac{\partial(x^2 y)}{\partial y} = x^2$$

Then $$I = \oint_C x^2 y\,dx - xy^2\,dy = \iint_R (-y^2 - x^2)\,dx\,dy$$

$$= -\iint_R (x^2+y^2)\,dx\,dy$$

where R is the circle with radius a centered at the origin. Transforming to polar coordinates we obtain

$$I = -\iint_R (x^2+y^2)\,dx\,dy = -\int_0^{2\pi}\int_0^a (r^2\cos^2\theta + r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= -\int_0^{2\pi} d\theta \int_0^a r^3\,dr = -2\pi\left[\left(\frac{r^4}{4}\right)\Big|_0^a\right] = \frac{-\pi a^4}{2}.$$

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Example 1.6 Calculate the line integral

$$\oint_C \sqrt{x^2+y^2}\,dx + y\left[xy + \ln\left(x+\sqrt{x^2+y^2}\right)\right]dy$$

Using Green's theorem. the contour of integration C is the circle. $x^2+y^2=a^2$

Solution

Let $P(x,y) = \sqrt{x^2+y^2}$, $Q = y\left[xy + \ln\left(x+\sqrt{x^2+y^2}\right)\right]$

$$\frac{\partial P}{\partial y} = \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial\left(y\left[xy + \ln\left(x+\sqrt{x^2+y^2}\right)\right]\right)}{\partial x}$$

$$= y\left(y + \frac{1+\frac{2x}{2\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}\right) = y\left(y + \frac{x+\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}\right)$$

$$= y\left(y + \frac{1}{\sqrt{x^2+y^2}}\right) = y^2 + \frac{y}{\sqrt{x^2+y^2}}$$

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by Green's formula.

$$I = \oint_C \sqrt{x^2+y^2}\,dx + y\left[xy + \ln\left(x+\sqrt{x^2+y^2}\right)\right]dy$$

$$= \iint_R \left[y^2 + \frac{y}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\right]dx\,dy = \iint_R y^2\,dx\,dy$$

It is convenient to transform the double integral to polar coordinate.

$$I = \iint_R y^2\,dx\,dy = \int_0^{2\pi}\left[\int_0^a r^2\sin^2\theta \cdot r\,dr\,d\theta\right]$$

$$= \int_0^{2\pi}\sin^2\theta\,d\theta \int_0^a r^3\,dr$$

$$\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\frac{(1-\cos 2\theta)}{2}\,d\theta = \frac{1}{2}\left[\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi}\right]$$

$$= \frac{1}{2}\cdot 2\pi = \pi$$

$$\int_0^a r^3\,dr = \frac{r^4}{4}\Big|_0^a = \frac{a^4}{4}$$

Thus $$I = \frac{\pi a^2}{4}.$$

Assignment I

1 Using Green's theorem, evaluate the integral

$$\oint_C (x+y)\,dx - (x-y)\,dy$$

where the curve C is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

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2. Evaluate the line integral

$$\oint_C (y - x^2)\,dx - (x + y^2)\,dy$$

where the contour C encloses the sector of the circle with radius a lying in the first quadrant.

The concept of Green's theorem can be extended to multiply connected regions. as shown below

In the diagrams above, we have one or more regions cut out from the interior, as opposed to discrete points being cut out. For such regions, the outer boundary and the inner boundaries are traversed so that R is always on the left. the same as in the second diagram.

For subregions, By Green's theorem holds so that

$$\oint_{\substack{bdy \\ of\, R_1}} f \cdot dr = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad \text{and}$$

$$\oint_{\substack{bdy \\ of\, R_2}} f \cdot dr = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Since the line integrals along the slits cancel out, we have

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$$\oint_{C_1 \cup C_2} f \cdot dr = \oint_{\substack{bdy \\ of R_1}} f \cdot dr + \int_{\substack{bdy \\ of R_2}} f \cdot dr.$$

and so

$$\oint_{C_1 \cup C_2} f \cdot dr = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA +$$

$$\iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

which shows that Green's theorem holds in the region R.

<u>Example 1.7</u> Evaluate $\oint_C y^3 dx - x^3 dy$ where C are the two circles of radius 2 and radius 1.

<u>Solution</u>

$$\oint_C y^3 dx - x^3 dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

$$= -3 \iint_D (x^2 + y^2)\, dA = -3 \int_0^{2\pi} \int_0^2 r^3 dr d\theta$$

by polar transformation

$$= -3 \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^2 d\theta = -3 \int_0^{2\pi} \frac{15}{4} d\theta$$

$$= -\frac{45}{2} \pi.$$

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Application of Green's Theorem

Recall that we can determine the area of a region D with the following double integral

$$A = \iint_D dA.$$

Let assume $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$

and $P = 0$, $P = -y$ $\quad P = -\frac{y}{2}$

$Q = x$ $\quad Q = 0$ $\quad Q = x/2$

Considering the Green's theorem in reverse case, we see that the area of the region D can also be computed by evaluating any of the ffg line integrals

$$A = \oint_C x\,dy = -\oint_C y\,dx$$

$$= \frac{1}{2}\oint_C x\,dy - y\,dx.$$

where C is the boundary of the region D.

<u>Example 1.8</u> Use Green's theorem to find the area of a disk of radius a.

<u>Solution</u>

Let $A = \frac{1}{2}\oint_C x\,dy - y\,dx.$

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where c is the circle of radius a. So ...

$$x = a\cos t \quad , \quad y = a\sin t \qquad 0 \le t \le 2\pi$$

then

$$A = \frac{1}{2}\oint_c x\,dy - y\,dx$$

$$= \frac{1}{2}\left(\int_0^{2\pi} a\cos t\,(a\cos t)\,dt - \int_0^{2\pi} a\sin t\,(-a\sin t)\,dt\right)$$

$$= \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t\,dt = \frac{1}{2}\int_0^{2\pi} a^2\,dt$$

$$= \pi a^2.$$

2.0 CURL AND DIVERGENCE

Given the vector field $\vec{F} = \vec{P}i + \vec{Q}j + \vec{R}k$.

the curl is defined to be.

$$\text{Curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k$$

Another easier definition of the Curl of a vector field. is by introducing an ∇ operator define as follows.

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}\vec{k}$$

So that

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k.$$

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Using the ∇, we can define the Curl as the cross product

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

<u>Remarks 2.1.1</u>

(i) If $f(x, y, z)$ has continuous second order partial derivatives then $\text{Curl}(\nabla f) = 0$.

(ii) If $\vec{F}$ is a conservative vector field then $\text{Curl } \vec{F} = 0$.

(iii) If $\vec{F}$ is defined on all of $\mathbb{R}^3$ whose component have continuous first order partial derivative and $\text{Curl } \vec{F} = 0$, then $\vec{F}$ is a conservative vector field.

<u>Example 2.1.2</u> Determine if $\vec{F} = x^2y\,i + xyz\,j - x^2y^2\,k$ is a conservative vector field.

<u>Solution</u>

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -x^2y^2 \end{vmatrix}$$

$$= -2x^2y\,i + yz\,k - (-2xy^2\,j) - xy\,i - x^2\,k$$

$$= -(2x^2y + xy)\,i + 2xy^2\,j + (yz - x^2)\,k$$

$$\neq 0$$

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then curl $\vec{F} \neq 0$ and so this vector field is not conservative

2.1.3 Given the vector field $\vec{F} = Pi + Qj + Rk$ the divergence is defined to be

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Alternatively $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$.

Example 2.1.4 Compute $\operatorname{div} \vec{F}$ for $\vec{F} = x^2 y i + xyz j - x^2 y^2 k$

Solution

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-x^2 y^2)$$

$$= 2xy + xz.$$

Note that $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$

Assignment

Verify that $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$ for the vector field $\vec{F} = yz^2 i + xy j + yz k$

In fluid Mechanics, if we take $\vec{F}$ as the velocity field of a flowing fluid, then $\operatorname{div} \vec{F}$ represents the net rate of the change of the mass of the fluid flowing to diverge from a point. If $\operatorname{div} \vec{F} = 0$, the $\vec{F}$ is called incompressible.

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3.0 Surface Integral

3.1.0 Parametric surfaces.

In this section, we consider how to parameterize a surface using the idea of parameterized curve.

Let $t \in [a, b]$, then

$$\vec{r}(t) = x(t)i + y(t)j + z(t)k$$

this results to a position vector for the points on the curve.

Taking the points (u, v) out of some two-dimensional space D, then

$$\vec{r}(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$$

this results to a position vector for the points on the surface S.

where $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$

Example 3.1.1 Determine the surface given by the parametric representation

solution Let $x = u$, $y = u\cos v$ $z = u\sin v$

$$y^2 + z^2 = u^2\cos^2 v + u^2\sin^2 v = u^2(\cos^2 v + \sin^2 v)$$

$$= u^2 = x^2.$$

To eliminate the parameters and given the equation in x, y, z, then

$$x^2 = y^2 + z^2.$$

this is for Quadric Surfaces

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Example 3.1.2 Give parametric representation for each of the following surfaces.

(i) the elliptic paraboloid $x = 5y^2 + 2z^2 - 10$

(ii) ~~the elliptic paraboloid~~

the sphere $x^2 + y^2 + z^2 = 30$.

(iii) the cylinder $y^2 + z^2 = 25$

Solution (i) the elliptic paraboloid $x = 5y^2 + 2z^2 - 10$

Let $x = f(y, z)$, then $x = 5y^2 + 2z^2 - 10$

where $y = y$, $z = z$

the parametric representation is also

$$\vec{r}(y, z) = (5y^2 + 2z^2 - 10)\,i + y\,j + z\,k.$$

(ii) the sphere $x^2 + y^2 + z^2 = 30$.

In spherical coordinates, we have equation of a sphere of radius a given as

$$\rho = a.$$

and so the equation of this sphere is

$$\rho = \sqrt{30}$$

By converting cartesian coordinates into spherical coordinates, we have

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

so that

$$\vec{r}(\theta, \varphi) = \sqrt{30} \sin\varphi \cos\theta\, i + \sqrt{30} \sin\varphi \sin\theta\, j + \sqrt{30} \cos\varphi\, k.$$

where $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$.

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(ii) the cylinder $y^2 + z^2 = 25$.

In a cylindrical coordinates the equation of a cylinder of radius a is given by

$$r = a.$$

So that $r = 5$.

By converting the cartesian coordinates to cylindrical coordinates, the

$$x = x, \quad y = r\sin\theta \quad z = r\cos\theta.$$

then $\vec{r}(x, \theta) = x i + 5\sin\theta + 5\cos\theta k$

where $0 \leq \theta \leq 2\pi$

3.1.3 Equation of tangent on a plane.

Let $\vec{r}(u, v) = x(u, v) i + y(u, v) j + z(u, v) k$

we define

$$\vec{r}_u(u, v) = \frac{\partial x}{\partial u}(u, v) i + \frac{\partial y}{\partial u}(u, v) j + \frac{\partial z}{\partial u}(u, v) k$$

$$\vec{r}_v(u, v) = \frac{\partial x}{\partial v}(u, v) i + \frac{\partial y}{\partial v}(u, v) j + \frac{\partial z}{\partial v}(u, v) k.$$

Provided $\vec{r}_u \times \vec{r}_v \neq 0$ which shows that $\vec{r}_u \times \vec{r}_v$ is orthogonal to the surface S.

Example 3.1.4 find the equation of the tangent plane of the surface given by

$$r(u, v) = u i + 2v^2 j + (u^2 + v) k.$$

at the points $(2, 2, 3)$

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Solution →

~~Let~~ $\vec{r}_u(u,v) = i + 2uk$ $\quad \vec{r}_v(u,v) = 4vj + k$.

So that

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{vmatrix} = -8uvi - j + 4vk$$

Given the point $(2, 2, 3)$, then

$u = 2 \Rightarrow u = 2$

$2 = 2v^2 \Rightarrow v = \pm 1$

$3 = u^2 + v$

from $3 = u^2 + v \Rightarrow 3 = 2^2 + v \Rightarrow 3 = 4 + v \Rightarrow v = -1$

So $u = 2$, $v = -1$. ~~At~~ this point, the normal vector is $\vec{n} = 16i - j - 4k$.

from $-8uvi - j + 4vk$

$\vec{n} = -8(2)(-1)i - j + 4(-1)k$

The tangent plane is

$16(x-2) - (y-2) - 4(z-3) = 0$

$16x - y - 4z = 18$.

3.1.4 Surface Area of the parametric Surface

Let $\vec{r}(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$ be a parametric surface. Provided S is traced out exactly once as (u,v) ranges over the points in D, the surface area of S is given by

$$A = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA.$$

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Finally, we need to determine $\vec{r}_\theta \times \vec{r}_\varphi$.

$$\vec{r}_\theta(\theta, \varphi) = -4\sin\varphi\sin\theta\, i + 4\sin\varphi\cos\theta\, j.$$

$$\vec{r}_\varphi = (\theta, \varphi) = 4\cos\varphi\cos\theta\, i + 4\cos\varphi\sin\theta\, j - 4\sin\varphi\, k.$$

~~by cross product~~

by cross product.

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} i & j & k \\ -4\sin\varphi\sin\theta & 4\sin\varphi\cos\theta & 0 \\ 4\cos\varphi\cos\theta & 4\cos\varphi\sin\theta & -4\sin\varphi \end{vmatrix}$$

$$= -16\sin^2\varphi\cos\theta\, i - 16\sin\varphi\cos\varphi\sin^2\theta\, k - 16\sin^2\varphi\sin\theta\, j - 16\sin\varphi\cos\varphi\cos^2\theta\, k$$

$$= -16\sin^2\varphi\cos\theta\, i - 16\sin^2\varphi\sin\theta\, j - 16\sin\varphi\cos\varphi(\sin^2\theta + \cos^2\theta)\, k$$

$$= -16\sin^2\varphi\cos\theta\, i - 16\sin^2\varphi\sin\theta\, j - 16\sin\varphi\cos\varphi\, k$$

Taking the magnitude.

$$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \sqrt{256\sin^4\varphi\cos^2\theta - 256\sin^4\varphi\sin^2\theta + 256\sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{256\sin^4\varphi(\cos^2\theta + \sin^2\theta) + 256\sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{256\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)}$$

$$= 16\sqrt{\sin^2\varphi} = 16|\sin\varphi| = 16\sin\varphi.$$

Finally surface area $\Rightarrow A = \iint_D 16\sin\varphi\, dA$

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$$\int_0^{2\pi}\int_0^{\pi/3} 16\sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} -16\cos\varphi \Big|_0^{\pi/3} d\theta = \int_0^{2\pi} 8 \, d\theta = \underline{\underline{16\pi}}$$

4.0 Surface Integrals.

In this section, we integrate functions over surface S, in three-dimensional space.

~~Evaluating~~ the surface integral depends on how ~~the~~ surface is given.

Given the surface S by $z = g(x,y)$. Then

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dA$$

Alternatively, evaluating surface integral for surfaces given by parameterization.

$$\vec{r}(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k$$

In this case, the surface integral is

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \left\| \vec{r}_u \times \vec{r}_v \right| dA.$$

where D is the rang of the parameters that face out the surface S.

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Note that, we can parameterize a surface given by $z = g(x,y)$ as

$$\vec{r}(x,y) = xi + yj + g(x,y)k$$ and this can be defined as

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

<u>Example 4.1.1</u> Evaluate $\iint_S 6xy\, dS$ where S is the portion of the plane $x+y+z=1$ that lies in front of the yz-plane.

<u>Solution</u> Since S lies in front of the yz-plane then equation of the surface is of the form $x = g(y,z)$ and often as $x = 1-y-z$.

the ranges for y and z are

$$0 \le y \le 1, \quad 0 \le z \le 1-y.$$

therefore

$$\iint_S f(x,y,z)\, dS = \iint_D f(g(y,z), y, z)\sqrt{1+\left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}\, dA.$$

$$\iint_S 6xy\, dS = \iint_D 6(1-y-z)y\sqrt{1+(-1)^2+(-1)^2}\, dA$$

$$= \sqrt{3}\iint_D 6(y-y^2-zy)\, dA.$$

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$$= 6\sqrt{3}\int_0^1\int_0^{1-y} y - y^2 - zy \, dz\, dy$$

$$= 6\sqrt{3}\int_0^1 \left[yz - zy^2 - \frac{1}{2}z^2 y\right]_0^{1-y} dy$$

$$= 6\sqrt{3}\int_0^1 \frac{1}{2}y - y^2 + \frac{1}{2}y^3 \, dy$$

$$= 6\sqrt{3}\left(\frac{1}{4}y^2 - \frac{1}{3}y^3 + \frac{1}{8}y^4\right)\Big|_0^1 = \frac{\sqrt{3}}{4}$$

<u>Example 4.1.2</u> Evaluate $\iint z\,dS$ where S is the upper half of sphere of radius 2.

<u>Solution</u> By parameterization of sphere; we have

$$\vec{r}(\theta, \varphi) = 2\sin\varphi\cos\theta\, i + 2\sin\varphi\sin\theta\, j + 2\cos\varphi\, k$$

with $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi/2$.

$$\vec{r}_\theta(\theta, \varphi) = -2\sin\varphi\sin\theta\, i + 2\sin\varphi\cos\theta\, j$$

$$\vec{r}_\varphi(\theta, \varphi) = 2\cos\varphi\cos\theta\, j + 2\cos\varphi\sin\theta\, j - 2\sin\varphi\, k$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} i & j & k \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \end{vmatrix}$$

$$= -4\sin^2\varphi\cos\theta\, i - 4\sin^2\varphi\sin\theta\, j - 4\sin\varphi\cos\varphi\, k$$

$$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi}$$

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$$= \sqrt{16\sin^4\varphi(\cos^2\theta + \sin^2\theta) + 16\sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{16\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)}$$

$$= 4\sqrt{\sin^2\varphi} = 4|\sin\theta| = 4\sin\theta.$$

therefore $\iint_S z\,dS = \iint_D 2\cos\varphi\,(4\sin\varphi)\,dA$

$$= \int_0^{2\pi}\int_0^{\pi/2} 4\sin(2\varphi)\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\left(-2\cos(2\varphi)\right)\Big|_0^{\pi/2} d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

Assignment

Evaluate $\iint_S y\,dS$ where S is the portion of the cylinder $x^2+y^2=3$ that lies between $z=0$, $z=6$.

4.2.0 Surface Integral of Vector fields

Definition 4.2.1 A surface S is closed if it is the boundary of some solid region E. e.g. sphere.

Definition 4.2.2 A closed surface has a positive orientation if we choose the set of unit normal vectors that point outward from the region E. while the negative orientation will be the set of unit normal vectors that point towards the region E.

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Given $\vec{r}(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$

the unit normal vector is given as

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}.$$

Given a vector field $\vec{F}$ with unit normal vector, then the surface integral of $\vec{F}$ over the surface S is given by

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S F \cdot \vec{n}\, ds$$

where the right hand integral is a standard surface integral. this is sometimes called the flux of $\vec{F}$ across S.

Let $F = Pi + Qj + Rk$, then

$$\iint_S F \cdot ds = \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_D (Pi + Qj + Rk) \cdot \frac{-g_x i - g_y j + k}{\sqrt{(g_x)^2 + (g_y)^2 + 1}} \sqrt{(g_x)^2 + (g_y)^2 + 1}\, dA$$

$$= \iint_D (Pi + Qj + Rk) \cdot (-g_x i - g_y j + k)\, dA$$

$$= \iint_D -Pg_x - Qg_y + R\, dA.$$

Note that $\sqrt{(g_x)^2 + (g_y)^2 + 1} = \|\nabla f\|$

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So $\iint_S F \cdot dS = \iint_S F \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \left(\frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \right) \|\vec{r}_u \times \vec{r}_v\| \, dA$

$$= \iint F \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

Example No 2.3 Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $F = yj - zk$ and S is the surface given by the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$ at $y = 1$. Assume that S has positive orientation.

<u>Solution</u> Since S has a positive orientation then the unit normal vectors is needed to point outwards from the region enclosed by S.

Starting with paraboloid S_2. the surface is of the form $y = g(x, y)$. So

$$f(x, y, z) = y - g(x, z) = y - x^2 - z^2$$

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then $\nabla f = \langle f_x \; f_y \; f_y \rangle$.

$$= \langle -2x, -1, -2z \rangle$$

$$\vec{n} = \frac{-\nabla f}{\|-\nabla f\|} = \frac{\langle 2x, -1, 2z \rangle}{\|\nabla f\|}$$

therefore

$$\iint_{S_1} \vec{F} \cdot d\vec{s} = \iint_D (y\,j - z\,k) \cdot \left(\frac{\langle 2x, -1, 2z \rangle}{\|\nabla f\|} \right) \|\nabla f\| \, dA$$

$$= \iint_D -y - 2z^2 \, dA$$

$$= \iint_D -(x^2 - z^2) - 2z \, dA$$

$$= -\iint_D x^2 + 3z^2 \, dA$$

Let $x = r\cos\theta$, $z = r\sin\theta$

$0 \le \theta \le 2\pi$, $0 \le r \le 1$.

$$\iint_{S_1} F \cdot d\vec{s} = -\iint_D x^2 + 3z^2 \, dA$$

$$= -\int_0^{2\pi} \int_0^1 (r^2\cos^2\theta + 3r^2\sin^2\theta) \, r \, dr \, d\theta$$

$$= -\int_0^{2\pi} \int_0^1 (\cos^2\theta + 3\sin^2\theta) \, r^3 \, dr \, d\theta$$

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$$= -\int_0^{2\pi} \left[\frac{1}{2}\left(1+\cos(2\theta)\right) + \frac{3}{2}\left(1-\cos 2\theta\right) \right] \left(\frac{1}{4} r^4\right)\Big|_0^1 d\theta$$

$$= -\frac{1}{8}\int_0^{2\pi} 4 - 2\cos 2\theta \, d\theta = -\frac{1}{8}\left(4\theta - \sin 2\theta\right)\Big|_0^{2\pi}$$

$$= -\pi$$

$S_2 \rightarrow$ Disk.

Since the disk is really just the portion of the plane $y=1$ i.e in front of the disk radius 1 in the xz-plane. then

$$y = 1 = g(x,z).$$

$$\sqrt{(g_x)^2 + 1 + (g_y)^2}$$

$$\iint_{S_2} \vec{F} \cdot d\vec{s} = \iint_{S_2} y \, ds = \iint_D 1\sqrt{0+1+0} \, dA = \iint_D dA$$

Hence $\iint_{S_2} F \cdot ds = \pi$ Since $\iint_D dA$ is the double Integral that will give the area of the region D.

$$\oiint_S \vec{F} \cdot d\vec{s} = \iint_{S_1} F \cdot ds + \iint_{S_2} F \cdot ds = -\pi + \pi = 0$$

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4.3.0 Stoke's Theorem

Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also Let $\vec{F}$ be a vector field then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot d\vec{s}$$

Example 4.3.1 Use Stoke's Theorem to evaluate $\iint_S \text{curl} F\cdot ds$ where $\vec{F} = z^2 i - 3xy j + x^3y^3 k$.

and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S is oriented upwards.

Solution

Since $z = 1$, then $1 = 5 - x^2 - y^2$

$\Rightarrow x^2 + y^2 = 4$. at $z = 1$.

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The boundary curve will be the circle of radius 2 in the plane at $z=1$. By parameterization

$$\vec{r}(t) = 2\cos t\, i + 2\sin t\, j + k \qquad 0 \le t \le 2\pi$$

$$\iint_S \text{curl}\, \vec{F} \cdot dS = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(r(t)) \cdot \vec{r}'(t)\, dt$$

$$\vec{F}_r(\vec{r}(t)) = (1)^2 i - 3(2\cos t)(2\sin t) j + (2\cos t)^3 (2\sin t)^3 k$$

$$= i - 12\cos t \sin t\, j + 64\cos^3 t \sin^3 t\, k$$

$$\vec{r}'(t) = -2\sin t\, i + 2\cos t\, j$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = -2\sin t - 24\sin t \cos^2 t$$

~~So that~~

$$\iint_S \text{curl}\, F \cdot dS = \int_0^{2\pi} -2\sin t - 24\sin t \cos^2 t\, dt$$

$$= 2\cos t + 8\cos^3 t \Big|_0^{2\pi}$$

$$= 0$$

4.4.0 Divergence Theorem

Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\vec{F}$ be a vector field whose component have continuous first order partial derivatives. then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\, \vec{F}\, dV$$

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$$\operatorname{div} \vec{F} = y - y + 1 = 1$$

with $\operatorname{div} \vec{F} = \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z}$

The integral is

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint \operatorname{div} \vec{F} \, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 4r - 3r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{3}{4} r^4 \right) \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{5}{4} d\theta = \frac{5}{2}\pi .$$

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5.0 LAPLACE & FOURIER TRANSFORM.

<u>Definition 5.1.1.</u> For a complex-valued function x defined for $t > 0$, the laplace transform of $x(t)$ is defined by

$$X(s) = \int_0^\infty x(t) e^{-st} \, dt$$

for all $s \in \mathbb{R}$. Alternatively, we may use the ~~expressi~~ expression

$$\mathcal{L}\{x(t)\} = X(s)$$

to denote the laplace transform

<u>Example 5.1.2</u> Let $x(t) = e^{-at}$ for $a \in \mathbb{R}$. then

$$X(s) = \int_0^\infty e^{-(s+a)} \, dt = \frac{1}{s+a}, \quad s > a.$$

<u>Example 5.1.3</u> Let $x(t) = \sin wt$, for $w \in \mathbb{R}$, then

$$X(s) = \int_0^\infty \sin wt \, e^{-st} \, dt$$

Integrating by part twice, we obtain

$$\int_0^\infty \sin w(t) e^{-st} \, dt = \frac{w}{s^2} - \frac{w^2}{s^2} \int_0^\infty \sin(wt) e^{-st} \, dt$$

Also

$$\frac{s^2 + w^2}{s^2} \int_0^\infty \sin wt \, e^{-st} = \frac{w}{s^2} \Rightarrow \int_0^\infty \sin wt \, e^{-st} \, dt = \frac{w}{s^2 + w^2}$$

Thus,

$$\mathcal{L}\{\sin(wt)\} = \frac{w}{s^2 + w^2}$$

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5.1.4 Properties of the Laplace Transform

(1) Linearity of the Laplace Transform

$$\mathcal{L}\{c\,x(t)\} = c\,\mathcal{L}\{x(t)\} = c\,X(s) \quad c \in \mathbb{R}$$

$$\mathcal{L}\{A\,x(t) + B\,y(t)\} = A\,X(s) + B\,Y(s) \quad A, B \in \mathbb{R}$$

5.1.5 Derivative of the Laplace Transform

The first derivative of Laplace transform is given by

$$\mathcal{L}\{x'(t)\} = s\,X(s) - x(0)$$

Proof: From the definition

$$\mathcal{L}\{x'(t)\} = \int_0^\infty e^{-st}\,x'(t)\,dt$$

by Integration by part

$$\Rightarrow \int_0^\infty e^{-st}\,x'(t)\,dt = x(t)\,e^{-st}\Big|_0^\infty + s\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= -x(0) + s\int_0^\infty e^{-st}\,dt.$$

Hence

$$\mathcal{L}\{x'(t)\} = -x(0) + s\,X(s)$$
$$= s\,X(s) - x(0)$$

The nth Derivative of the Laplace transform is given by

$$f^n(t) = s^n\mathcal{L}\{x(t)\} - s^{n-1}x(0) - \cdots - x^{n-1}(0)$$

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by. linearity property, we have

$$\mathcal{L}\{\cos(\omega t)\} = \mathcal{L}\left\{\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}\right]$$

by shift property, we have

$$\frac{1}{2}\left[\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{(s-i\omega)(s+i\omega)}\right] = \frac{1}{2}\left[\frac{2s}{(s^2+\omega^2)}\right] = \frac{s}{s^2+\omega^2}$$

5.1.10 <u>Inverse Laplace Transform</u>

If for a given function $X(s)$, we can find a function $x(t)$ such that $\mathcal{L}\{x(t)\} = X(s)$, then the inverse Laplace transform is denoted

$$\mathcal{L}^{-1}\{X(s)\} = x(t)$$

Also, by linearity property, we have

$$\mathcal{L}^{-1}\{c\,X(s)\} = c\,x(t), \quad c \in \mathbb{R}$$

$$\mathcal{L}^{-1}\{A\,X(s) + B\,Y(s)\} = A\,x(t) + B\,y(t)$$

$A, B \in \mathbb{R}$.

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5.1.6 **First shift theorem.**

The theorem of the Laplace transform states that

$$\mathcal{L}\{x(t)e^{at}\} = X(s-a)$$

Proof By definition

$$\mathcal{L}\{x(t)e^{at}\} = \int_0^\infty x(t)e^{-(s-a)t}\,dt$$
$$= X(s-a)$$

5.1.7 **Scaling property**

The scaling property of the Laplace transform states

$$\mathcal{L}\{x(at)\} = \frac{1}{a} X\left(\frac{s}{a}\right).$$

proof By definition

$$\mathcal{L}\{x(at)\} = \int_0^\infty e^{-st} x(at)\,dt = \frac{1}{a}\int_0^\infty e^{-\left(\frac{x}{a}\right)y} x(y)\,dy$$
$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$

5.1.8 **Laplace Transform of t^n**

The laplace transform of t^n, can be defined to be

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Example 5.1.9 Evaluate the laplace transform of $x(t) = \cos(wt)$ for $w \in \mathbb{R}$.

Solution Let $\cos(wt) = \dfrac{e^{iwt} + e^{-iwt}}{2}$.

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5.1.11 Convolution Theorem.

The Convolution, in the interval $[0, \infty)$ is defined as

$$(x * y)(t) = \int_0^t x(\tau)\, y(t-\tau)\, d\tau$$

Theorem 5.1.12 The Laplace transform of two functions under Convolution is

$$\mathcal{L}\{(x * y)(t)\} = \mathcal{L}\{x(t)\} \cdot \mathcal{L}\{y(t)\}$$
$$= X(s)\, Y(s)$$

proof from the definition of the Laplace transform we have

$$\mathcal{L}\left[\int_0^t x(\tau)\, y(t-\tau)\, d\tau\right] = \int_0^\infty \left(\int_0^t x(\tau)\, y(t-\tau)\, d\tau\right) e^{-st}\, dt.$$

$$\int_0^\infty \left[\int_0^t x(\tau)\, y(t-\tau)\, d\tau\right] e^{-st}\, dt = \int_0^\infty \left[\int_0^\infty u(t-\tau)\, x(\tau)\, y(t-\tau)\, d\tau\right] e^{-st}\, dt.$$

by introducing unit step function $u(t-\tau)$

$$\int_0^\infty \left[\int_0^\infty u(t-\tau)\, x(\tau)\, y(t-\tau)\, d\tau\right] e^{-st}\, dt = \int_0^\infty x(\tau) \left[\int_0^\infty u(t-\tau)\, y(t-\tau)\, e^{-st}\, dt\right] d\tau$$

$$= \int_0^\infty u(t-\tau)\, y(t-\tau)\, e^{-st}\, dt = e^{-s\tau}\, Y(s)$$

therefore

$$\int_0^\infty x(\tau) \left[\int_0^\infty u(t-\tau)\, y(t-\tau)\, e^{-st}\, dt\right] d\tau =$$

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$$= \int_0^\infty x(\tau) e^{-s\tau} Y(s)\, d\tau = Y(s) \int_0^\infty e^{-s\tau} x(\tau)\, d\tau = X(s) Y(s)$$

<u>Example 5.1.13</u> Solve $f(t) + \int_0^t (t-u) f(u)\, du = \sin 2t$.

<u>Solution</u> Taking the Laplace transform of both sides we have

$$\mathcal{L}\{f(t)\} + \mathcal{L}\{t * f(t)\} = \mathcal{L}\{\sin 2(t)\}$$

we have

$$\mathcal{L}\{f(t)\} + \frac{1}{s^2} \cdot \mathcal{L}\{f(t)\} = \frac{2}{s^2+4}$$

Solving for $\mathcal{L}\{f(t)\}$, gives

$$\mathcal{L}\{f(t)\}\left[1 + \frac{1}{s^2}\right] = \frac{2}{s^2+4}$$

$$\mathcal{L}\{f(t)\}\left[\frac{s^2+1}{s^2}\right] = \frac{2}{s^2+4}$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^2+4} \times \frac{s^2}{s^2+1} = \frac{2s^2}{(s^2+4)(s^2+1)}$$

Taking the Inverse Laplace transform

$$f(t) = \frac{4}{3} \sin(2t) - \frac{2}{3} \sin(t).$$

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5.2.0 FOURIER TRANSFORM

Definition 5.2.1 Let $f : \mathbb{R} \to \mathbb{R}$. The Fourier transform of $f \in L^1(\mathbb{R})$ denoted by $\mathcal{F}[f](x)$ is given by the integral

$$\mathcal{F}[f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-ixt)\, dt$$

for $x \in \mathbb{R}$ for which the integral exists.

Definition 5.2.2: Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that $\int_{-\infty}^{\infty} [f]\, dt$ converges and in any finite interval f, f' are piecewise continuous with atmost finitely many maxima/minima/discontinuous. Let $F = \mathcal{F}[f]$. Then if f is continuous at $x \in \mathbb{R}$, we have

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) \exp(itx)\, dx$$

Theorem 5.2.3 Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous, f', g' piecewise continuous. If

$$\mathcal{F}[f](x) = \mathcal{F}[g](x) \quad \forall x$$

then $f(t) = g(t), \quad \forall t$.

proof From the inversion, we have

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}[t](x) \exp(itx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}[g](x) \exp(itx)\, dx = g(t).$$

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Example 5.2.4 Compute the fourier transform of

$$f(x) = \begin{cases} e^{-at} & \text{for } t \geq 0 \\ e^{at} & \text{for } t < 0 \end{cases}$$

Solution By Definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ixt}\, dt$.

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(a-ix)t}\, dt + \int_{0}^{\infty} e^{-(a+ix)t}\, dt \right],$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(a-ix)t}}{a-ix} \right]_{-\infty}^{0} + \left[\frac{e^{-(a+ix)t}}{-(a+ix)} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{1}{a-ix} - 0 \right) + \left(0 - \frac{1}{-(a+ix)} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a-ix} + \frac{1}{a+ix} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{a+ix+(a-ix)}{(a-ix)(a+ix)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2a}{(a-ix)(a+ix)} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{2a}{a^2+x^2} \right] = f(x)$$

Example 5.2.5 Let $a > 0$, compute the fourier transform of $f(x) = e^{-at^2}$.

Solution By defn $F[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ixt}\, dt$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-at^2} \cdot e^{ixt}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(ixt + at^2)}\, dt$$

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Lecture 3

Changing variables & Jacobians

3.1 Introduction

The slope of the $f = f(x, y)$ surface depends on the direction in which one moves from the point (x, y), and in Lecture 1 we learnt that the partial derivative must be defined in terms of the change along a particular direction or axis.

For a function $f(x, y)$ we decided that the obvious directions to choose are along the x and y axes, keeping y and x fixed, respectively.

The question now is are these *really* the obvious directions?

For example, consider the function

$$f(x, y) = e^{-(x^2+y^2)} \cos(4(x^2 + y^2))$$

shown in Figure 3.1. Does it really make sense to impose a square "x-constant, y-constant" cake-rack mesh onto this function? Probably not! It would be in more sympathy with the function to use a mesh with radial symmetry as in Figure 3.2.

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Figure 3.1: The function $\exp -(x^2 + y^2) \cos(4(x^2 + y^2))$.

Figure 3.2: A radial "dartboard" mesh reflects the symmetry rather better than the x, y constant cake rack.

To achieve this we must make an appropriate **transformation** to a **new set of variables**. This raises the questions of

- How to choose the new variables and thus describe the transformation.
- How to describe the function in the new variables.
- How to find the partial derivatives with respect to these new variables.

3.1.1 The general problem

The more general problem is illustrated in Figure 3.3. Subfigure (a) shows a surface $f(x, y)$ with lines of constant x and y underneath. The partial derivatives wrt x and y are found by slicing or moving along these directions.

Figure 3.3(b) shows the exactly the same surface with lines of constant u and v underneath. The partial derivatives wrt u and v are found by slicing along these new directions.

Figure 3.3: (a) Surface $f(x, y)$ above a set of x-constant and y-constant lines. (b) Surface $F(u, v)$ above a set of u-constant and v-constant curves but plotted in the x, y frame. The surface has exactly the same shape.

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3.1.2 Choosing the new variables

There is no routine method for doing this, but usually the problem symmetry drops very large hints as to the best transformation.

Later we will consider some of the most standard transformations, viz between Cartesian and

- Plane Polar Coordinates (2D): for radial symmetry
- Spherical Polar Coordinates (3D): for spherical symmetry, and
- Cylindrical Polars Coordinates (3D): for cylindrical symmetry.

We also consider what would make a bad transformation.

3.2 Rewriting the function and finding the derivatives

First though we should consider the general problems of rewriting the function and finding the partial derivatives with respect to arbitrary new variables (u, v).

There are two main cases to consider:

1. The straightforward case. **OLD variables in terms of NEW**.

$$x = x(u, v) \quad \text{and} \quad y = y(u, v) \quad . \tag{3.1}$$

2. The more complicated case. **NEW variables in terms of OLD**.

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \quad . \tag{3.2}$$

3.2.1 CASES 1A and 1B: OLD variables in terms of NEW

We are transforming from (x, y) to (u, v) coordinates, and the transformation is given as "old in terms of new" variables. That is:

$$x = x(u, v) \quad \text{and} \quad y = y(u, v) \quad . \tag{3.3}$$

Case 1A:
If we know what the function $f(x, y)$ is explicitly, we can replace x and y in the function to get the new function as

$$f = f(x(u, v), y(u, v)) = F(u, v) \quad . \tag{3.4}$$

Then one can work out the partial derivatives with respect to u and v directly from the new function $F(u, v)$.

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♣ Example of Case 1A.

Let $f(x, y) = x/y$, and let the transformation be $x = u$ and $y = u/v$.

$$\Rightarrow f = F(u, v) = \frac{u}{(u/v)} = v \quad \text{and} \quad \Rightarrow \frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial v} = 1 \ . \tag{3.5}$$

Case 1B: In this case we don't have an explicit form for $f(x, y)$, and, therefore, cannot find an explicit form for function $F(u, v)$.

But we can still write expressions for the partial derivatives using the Chain Rule for partials from Lecture 2:

Case 1B: Use the Chain Rule for partials to obtain the derivatives

$$\left(\frac{\partial f}{\partial u}\right) = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) \tag{3.6}$$

$$\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right) \tag{3.7}$$

♣ Example of Case 1B.

We are told $f = f(x, y)$, and $x = u$ and $y = u/v$.

Given this transformation, the best we can do is to write for **any** function $f(x, y)$

$$\left(\frac{\partial f}{\partial u}\right) = \left(\frac{\partial f}{\partial x}\right)(1) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{1}{v}\right) = \left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\frac{1}{v} \tag{3.8}$$

$$\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)(0) + \left(\frac{\partial f}{\partial y}\right)\left(-\frac{u}{v^2}\right) = -\left(\frac{\partial f}{\partial y}\right)\frac{u}{v^2} \ . \tag{3.9}$$

To check this result, let's use (as in the example of Case 1A) $f(x, y) = x/y$. Then

$$\left(\frac{\partial f}{\partial u}\right) = \frac{1}{y}(1) + \left(-\frac{x}{y^2}\right)\left(\frac{1}{v}\right) = \left(\frac{v}{u}\right) - \left(\frac{u}{v}\right)\left(\frac{v^2}{u^2}\right) = \left(\frac{v}{u}\right) - \left(\frac{v}{u}\right) = 0 \tag{3.10}$$

$$\left(\frac{\partial f}{\partial v}\right) = \frac{1}{y}(0) + \left(-\frac{x}{y^2}\right)\left(-\frac{u}{v^2}\right) = \frac{u^2 v}{u^2 v} = 1 \tag{3.11}$$

This is exactly as found earlier at Equation 3.5.

3.2.2 A source of confusion introduced, and cleared up

Before continuing, it is worth checking that you are happy that the functions $f()$ and $F()$ have different names. They work differently on their parameters, so should be named differently.

COLLINS

But notice that the chain rule for partials seems to deliver an expression for $\partial f/\partial u$ and $\partial f/\partial v$.

But you might complain that f is a function of (x, y) NOT (u, v), and so we should have written $\partial F/\partial u$ and $\partial F/\partial v$. You are correct. So how are we getting away with it?

We are using f as a "go-between" value $f = f(x, y)$ and $f = F(u, v)$. This is correct because at corresponding values of (x, y) and (u, v) the functions do return exactly the same value. (Look back at Figure 3.3.)

Some books introduce another symbol (z, say) as the "go-between". They write $z = f(x, y)$ and $z = F(u, v)$ and then write

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} , \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} , \tag{3.12}$$

and so on — but this seems over-fussy.

3.2.3 CASES 2A and 2B: NEW variables in terms of OLD

We still wish to transform the function $f = f(x, y)$ to (u, v) coordinates. But now the transformation is given as "new in terms of old" variables, that is:

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \ . \tag{3.13}$$

One's first thought is that this should be the better way of defining a transformation, but it turns out to be really inconvenient!

New-ito-Old prevents us from finding either $F(u, v)$ or using the Chain Rule for Partials. We cannot immediately work out $\partial x/\partial u$, etc.

We proceed in one of two ways.

Case 2A: Try to invert the transformation.
Use $u = u(x, y)$ and $v = v(x, y)$ as simultaneous equations from which to find $x = x(u, v)$ and $y = y(u, v)$.
If you can invert, the problem becomes a Case 1 problem.

♣ Example of Case 2A

Suppose you are told that

$$u = x \quad \text{and} \quad v = x/y \ . \tag{3.14}$$

These are new in terms of old. But it is obvious that

$$x = u, \quad \text{and} \quad \Rightarrow v = u/y, \quad \Rightarrow y = u/v \ . \tag{3.15}$$

We now have Old-in-terms-of-New, and can move to Case 1 above.

COLLINS

Case 2B: When the transformation is not invertible Write down the Chain Rule for Partials the "wrong" way round! That is, we want $\partial f/\partial u$, so start with $\partial f/\partial x$.

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \tag{3.16}$$

$$\frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial y} \tag{3.17}$$

Treat these as simultaneous equations in the unknown $\partial f/\partial u$ and $\partial f/\partial v$ and re-arrange. Ie:

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial v}{\partial x}\right) / \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.18}$$

$$\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial u}{\partial y}\right) / \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.19}$$

So we can recover the partial derivatives with respect to the new variables without an explicit expression for the function in the new variables.

3.2.4 SUMMARY

To summarize:

1: If you have the transformation as **OLD variables in terms of NEW** either

- **1A:** find the function explicity and find the partials from it directly, or
- **1B:** find just the partials using the Chain-rule-for-partials directly.

2: If you have the transformation as **NEW variables in terms of OLD** either

- **2A:** invert the given transformation to get OLD variables in terms of NEW, and then go to CASE 1, or
- **2B:** use the Chain-rule-for-partials for the OLD variables to give simultaneous equations for the partial derivatives with respect to to the NEW variables.

COLLINS

♣ Example #1

Q: The elevation of a hill is described by

$$z = z(x, y) = \exp\left\{-(x^2 + y^2)\right\} \ . \tag{3.20}$$

By deriving the function in plane polars, find $\partial z/\partial r$ and $\partial z/\partial \phi$ and comment on their values. Determine the value of r where the hill is steepest.

A: Plane polars are (r, ϕ) where $x = r\cos\phi$, $y = r\sin\phi$.

This is old in terms of new, and we know the function, ⇒CASE 1A.

The hill is therefore

$$z = \exp\left\{-r^2\right\} \ . \tag{3.21}$$

So

$$\frac{\partial z}{\partial r} = -2r\exp\left\{-r^2\right\} \quad \text{and} \quad \frac{\partial z}{\partial \phi} = 0 \ . \tag{3.22}$$

The change of height is all in the radial direction. $\partial z/\partial \phi = 0$ means that if you move at constant r (that is, "round" the hill) you will not change height at all.

The gradient is a function of r alone, so we can find the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}r}\text{Slope} = \frac{\mathrm{d}}{\mathrm{d}r}(-2r\exp\left\{-r^2\right\}) = (-2 + 4r^2)\exp\left\{-r^2\right\} \tag{3.23}$$

This is zero when $r = 1/\sqrt{2}$ and Slope $= -\sqrt{2}\exp\{-1/2\}$.

♣ Example #2.

Here is one that can be solved using all approaches.

Q: Find the partial derivatives $\partial f/\partial u$ and $\partial f/\partial v$ when

$$f = f(x, y) = x^2 - y^2 \quad \text{and} \quad u = (x + y) \quad v = (x - y) \ . \tag{3.24}$$

A: Transformation is new in terms of old. ⇒CASE 2.

Try CASE 2A: Can we invert the transformation? Yes! Adding then substracting we find

$$x = \frac{1}{2}(u + v) \quad y = \frac{1}{2}(u - v) \ . \tag{3.25}$$

Now goto CASE 1. We know the function, ⇒**CASE 1A.**

$$f = F(u, v) = x^2 - y^2 = \frac{1}{4}\left((u + v)^2 - (u - v)^2\right) = uv \tag{3.26}$$

COLLINS

So the derivatives are

$$\frac{\partial f}{\partial u} = v \quad \text{and} \quad \frac{\partial f}{\partial v} = u \tag{3.27}$$

That's the problem solved. But now notice that if **CASE *A** works, we can always find **just the derivatives** using **CASE *B**.

For example, having inverted the transformation to get

$$x = \frac{1}{2}(u+v) \quad y = \frac{1}{2}(u-v) \quad .$$

we could use CASE 1B to find just the derivatives:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = (2x).\frac{1}{2} + (-2y).\frac{1}{2} = x - y = v \tag{3.28}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = (2x).\frac{1}{2} + (-2y).\frac{-1}{2} = x + y = u \tag{3.29}$$

Now pretend that we did not (or could not) invert the transformation. We could still find the derivative using Case 2B.

Write the chain rule for partials the other way around:

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \tag{3.30}$$

$$\frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial y} \tag{3.31}$$

and for simplicity fill in the values you know

$$2x = \left(\frac{\partial f}{\partial u}\right)(1) + \left(\frac{\partial f}{\partial v}\right)(1) \tag{3.32}$$

$$-2y = \left(\frac{\partial f}{\partial u}\right)(1) + \left(\frac{\partial f}{\partial v}\right)(-1) \tag{3.33}$$

Now use as simultaneous equations for $\partial f/\partial u$ and $\partial f/\partial v$:

$$2x - 2y = 2\left(\frac{\partial f}{\partial u}\right) \quad \Rightarrow \quad \left(\frac{\partial f}{\partial u}\right) = (x-y) = v \quad , \tag{3.34}$$

and subtracting gives, once again, the same results:

$$2x + 2y = 2\left(\frac{\partial f}{\partial v}\right) \quad \Rightarrow \quad \left(\frac{\partial f}{\partial v}\right) = (x+y) = u \quad . \tag{3.35}$$

COLLINS

♣ **Example #3.** Harder!
Q: A quantity $V = V(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \ . \tag{3.36}$$

Express this equation in plane polar coordinates (r, ϕ) where $x = r\cos\phi$, $y = r\sin\phi$.

A: We want first to write down expressions for $\partial V/\partial x$ and $\partial V/\partial y$ which will involve $\partial r/\partial x$, $\partial r/\partial y$, $\partial\phi/\partial x$, and $\partial\phi/\partial y$. This is the wrong way round for the transformation. So

- Either invert the transformation (Case 2A), or
- Write the CRfP the "other way round" (Case 2B) and use sim eqs.

Both will work, but let's use 2B ...

$$\frac{\partial V}{\partial r} = \left(\frac{\partial V}{\partial x}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial V}{\partial y}\right)\frac{\partial y}{\partial r} = \left(\frac{\partial V}{\partial x}\right)\cos\phi \qquad + \left(\frac{\partial V}{\partial y}\right)\sin\phi \tag{3.37}$$

$$\frac{\partial V}{\partial \phi} = \left(\frac{\partial V}{\partial x}\right)\frac{\partial x}{\partial \phi} + \left(\frac{\partial V}{\partial y}\right)\frac{\partial y}{\partial \phi} = \left(\frac{\partial V}{\partial x}\right)(-r\sin\phi) + \left(\frac{\partial V}{\partial y}\right)(r\cos\phi) \tag{3.38}$$

Sorting out the simultaneous equations (do check!) gives

$$\frac{\partial V}{\partial x} = \cos\phi\frac{\partial V}{\partial r} - \frac{\sin\phi}{r}\frac{\partial V}{\partial \phi} \tag{3.39}$$

$$\frac{\partial V}{\partial y} = \sin\phi\frac{\partial V}{\partial r} + \frac{\cos\phi}{r}\frac{\partial V}{\partial \phi} \tag{3.40}$$

We can drop the V and consider $\frac{\partial}{\partial x}$ as an *operator*

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \tag{3.41}$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \tag{3.42}$$

COLLINS

So the operator

$$\frac{\partial^2}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) \tag{3.43}$$

$$= \cos^2\phi\frac{\partial^2}{\partial r^2} - \cos\phi\sin\phi\left[-\frac{1}{r^2}\frac{\partial}{\partial\phi} + \frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right] \tag{3.44}$$

$$- \frac{1}{r}\sin\phi\left[-\sin\phi\frac{\partial}{\partial r} + \cos\phi\frac{\partial^2}{\partial\phi\partial r}\right] + \frac{-\sin\phi}{r}\left[-\frac{\cos\phi}{r}\frac{\partial}{\partial\phi} - \frac{\sin\phi}{r}\frac{\partial^2}{\partial\phi^2}\right]$$

which can be tidied up a little. Note how the operators move through to the right, operating as they go.

Let's follow through a couple of terms. The following is straightforward:

$$\left(\cos\phi\frac{\partial}{\partial r}\right)\left(\cos\phi\frac{\partial}{\partial r}\right) = \cos^2\phi\frac{\partial^2}{\partial r^2} \tag{3.45}$$

but this requires the product rule

$$\left(\cos\phi\frac{\partial}{\partial r}\right)\left(-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) = -\cos\phi\sin\phi\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial\phi} - \cos\phi\sin\phi\left(\frac{1}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \tag{3.46}$$

The tidied expression for y is (you should check!)

$$\frac{\partial^2}{\partial y^2} = \sin\phi^2\frac{\partial^2}{\partial r^2} - \cos\phi\sin\phi\left[\frac{2}{r^2}\frac{\partial}{\partial\phi} - \frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right] \tag{3.47}$$

$$+\frac{1}{r}\cos\phi\left[\cos\phi\frac{\partial}{\partial r} + \sin\phi\frac{\partial^2}{\partial\phi\partial r}\right] + \frac{1}{r^2}\cos\phi^2\frac{\partial^2}{\partial\phi^2} \tag{3.48}$$

Adding them up:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]. \tag{3.49}$$

So Laplace's equation in plane polar coordinates is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} = 0. \tag{3.50}$$

COLLINS

3.3 Jacobians (Jacobian determinants)

Recall the equations that appeared from solving the simultaneous equations in case 2B:

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial v}{\partial x}\right) \Big/ \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.51}$$

$$\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial u}{\partial y}\right) \Big/ \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.52}$$

Notice that the denominator is the same in both expressions. It has special significance, and is called the Jacobian.

It is common to write the Jacobian as a determinant, but there is also another useful notation.

$$J = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} \tag{3.53}$$

As we shall see later, the Jacobian is especially useful when changing variables from (x, y) to (u, v) in multiple integrals.

3.3.1 Multiple definitions of the Jacobian determinant?

In some books one reads that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} \tag{3.54}$$

while in others

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad . \tag{3.55}$$

This is puzzling, until you realize that they are **identical**! However, perhaps the first way is to be preferred for reasons we see now.

3.3.2 Jacobians and their inverses

In examples given later you will see that

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)} \tag{3.56}$$

We shall gain an intuitive understanding of why this must be so when we come to use Jacobians in multiple integration. However it is quite easy to prove using the **Jacobian matrix**. This is not on the syllabus.

COLLINS

The Chain rule for partials says

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \tag{3.57}$$

or, switching the order of the products,

$$\frac{\partial f}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial f}{\partial y} \tag{3.58}$$

We can treat these as operator equations: in steps ...

$$\frac{\partial}{\partial u} f = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} f \qquad \frac{\partial}{\partial v} f = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} f \tag{3.59}$$

$$\Rightarrow \frac{\partial}{\partial u} f \equiv \frac{\partial x}{\partial u}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} f \qquad \frac{\partial}{\partial v} f \equiv \frac{\partial x}{\partial v}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} f \tag{3.60}$$

In matrix notation, this operator is

$$\begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \begin{bmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{bmatrix} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \tag{3.61}$$

The matrix $\boldsymbol{J}_{\frac{x,y}{u,v}}$ is a **Jacobian matrix**.

Now repeat with the Chain Rule around the other way. You will arrive at the following

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \begin{bmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{bmatrix} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \boldsymbol{J}_{\frac{u,v}{x,y}} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} \tag{3.62}$$

But it follows that

$$\begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \boldsymbol{J}_{\frac{u,v}{x,y}} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} \tag{3.63}$$

so that

$$\boldsymbol{J}_{\frac{x,y}{u,v}} \boldsymbol{J}_{\frac{u,v}{x,y}} = \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{J}_{\frac{x,y}{u,v}} = \left[\boldsymbol{J}_{\frac{u,v}{x,y}}\right]^{-1} . \tag{3.64}$$

But a result from linear algebra tells us that, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square, $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$, and hence $|\boldsymbol{A}||\boldsymbol{A}^{-1}| = 1$. The result for Jacobian determinants follows immediately.

3.4 Three standard transformations

There are certain transformations which occur very frequently, viz:
(1) Cartesian to plane polar coordinates;
(2) Cartesian to spherical polar coordinates; and
(3) Cartesian to cylindrical polar coordinates.

COLLINS

3.4.1 Cartesian to plane polars

This is useful for problems with radial symmetry in the plane.

Figure 3.4: Cartesian to plane polars. Many texts use θ rather than ϕ.

The transformation is

$$x = r\cos\phi,\ \ y = r\sin\phi \tag{3.65}$$

$$\Rightarrow \frac{\partial x}{\partial r} = \cos\phi;\ \ \frac{\partial x}{\partial \phi} = -r\sin\phi;\ \ \frac{\partial y}{\partial r} = \sin\phi;\ \ \frac{\partial y}{\partial \phi} = r\cos\phi \tag{3.66}$$

Writing the CRfP in operator form gives

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \tag{3.67}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y}\ . \tag{3.68}$$

Using these are simutaneous equations gives

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\ . \tag{3.69}$$

(You could also find these by inverting the transformation.)

The Jacobian

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = r\cos^2\phi + r\sin^2\phi = r\ . \tag{3.70}$$

and

$$\frac{\partial(r,\phi)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix} = (\cos\phi)(\frac{\cos\phi}{r}) - (\sin\phi)(-\frac{\sin\phi}{r}) = \frac{1}{r}\ . \tag{3.71}$$

COLLINS

3.4.2 3D Cartesian to cylindrical polars

Cylindrical polars add a z-axis, perpendicular to the plane, which is identical to the Cartesian z-axis.

Figure 3.5: Cartesian x, y, z to cylindrical polars r, ϕ, z

The transformation is

$$x = r\cos\phi; \quad y = r\sin\phi; \quad z = z. \tag{3.72}$$

So

$$\frac{\partial}{\partial r} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \tag{3.73}$$

$$\frac{\partial}{\partial \phi} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y} \qquad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{3.74}$$

and using these as simultaneous equations (or inverting the transformation):

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \tag{3.75}$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \qquad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{3.76}$$

The Jacobian is now a 3×3 determinant

$$\frac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -r\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r, \tag{3.77}$$

and (do check!)

$$\frac{\partial(r,\phi,z)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \phi}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \phi}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \frac{1}{r}\ . \tag{3.78}$$

Most entries can be copied over from the plane polar case.

COLLINS

3.4.3 Cartesian to spherical polars

Figure 3.6: Cartesian to spherical polars

The transformation involves the radius r of a sphere $r^2 = x^2 + y^2 + z^2$, the polar angle $0 \le \theta \le \pi$ and the azimuthal angle $0 \le \phi \le 2\pi$

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi; \quad z = r\cos\theta. \tag{3.79}$$

Hence

$$\frac{\partial}{\partial r} = \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z} \tag{3.80}$$

$$\frac{\partial}{\partial \theta} = r\cos\theta\cos\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z} \tag{3.81}$$

$$\frac{\partial}{\partial \phi} = -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} \tag{3.82}$$

and

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.83}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.84}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{3.85}$$

The Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

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(3.86)

and as an exercise you should show that

$$\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \frac{1}{r^2 \sin\theta} \ . \tag{3.87}$$

3.5 A "good" transformation

Cylindrical and spherical polars are particularly useful when solving problems involving cylindrical and spherical regions. But what in general makes a "good" transformation.

If you were asked what is the nicest region shape to fit into Cartesian coordinates Oxy you would answer a rectangle.

Similarly, if you drew your Ouv coordinate system at right angles and asked the same question, you would again answer a square.

So given some arbitrary region shape in the xy plane, the best sort of transformation is one which maps it onto a rectangle in the uv plane, as illustrated in Figure 3.7(a). A specific example is given in Figure 3.7(b) for plane polars. The quadrant's extent is awkard to describe in Cartesians, but easy in describe in plane polars.

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Figure 3.7: Useful transformations tend to take an awkward region shape in Oxy into a rectangle in Ouv.

3.5.1 Will any transformation work? When is one bad?

Another question is "will any transformation work?" The answer is no. If we start with two (or n) independent variables (x, y), we must ensure that the new two (or n) are also independent.

The test for independence is rather simple.

If $u(x, y)$ and $v(x, y)$ are functionally INDEPENDENT then

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0. \tag{3.88}$$

Recall that the Jacobian is the denominator in a relationship relating partial derivatives with respect to different sets of variables. Trouble is expected if this is zero.

Proof: For reading only

The proof is straightforward. If $u(x, y)$ and $v(x, y)$ are functionally dependent, then $u = u(v)$, which in turn means that a function $z = z(u, v) = 0$ exists. Differentiating z with respect to x and y gives:

$$\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{3.89}$$

$$\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{3.90}$$

For consistency between these two equations we must have $u_x = \beta u_y$ and $v_x = \beta v_y$, where β is some constant. Thus

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = (u_x v_y - v_x u_y) = (\beta u_y v_y - \beta v_y u_y) = 0 \quad . \tag{3.91}$$

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♣ Example

Q: Test whether there is a functional dependence between the variables (u, v) if

$$u = x^2 + y + 1 \quad \text{and} \quad v = x^4 + 2x^2 y + y^2 + x^2 - y. \tag{3.92}$$

A:

$$\begin{aligned} \frac{\partial(u, v)}{\partial(x, y)} &= (u_x v_y - v_x u_y) \\ &= (2x)(2x^2 + 2y - 1) - (4x^3 + 4xy + 2x)(1) \\ &= (4x^3 + 4xy - 2x) - (4x^3 + 4xy + 2x) \\ &= 0 \end{aligned} \tag{3.93}$$

So, yes the variables are not independent. With a little work you'll spot that the dependence is $v = u^2 - 3u + 2$.

FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA

COLLEGE PHYSICAL SCIENCES

DEPARTMENT OF MATHEMATICS

MTS 241 (MATHEMATICAL METHOD I)

BY

NKWUDA, FRANCIS MONDAY

COLLINS

MODULE ONE

Real Valued Function of Real Variable

Recall that a function is a rule that establishes the relationship between two or more variables.

If X and Y are an arbitrary sets. For every $x \in X$ there is assigned a unique element $f(x) = y \in Y$, then f is called **a function**. This is written as $f : X \to Y$, where the set X is the **domain of the function** and Y is the **codomain of the function**. The value of f at x is written as $f(x)$ and is called image or range of the function. When the set X is actually a subset of $\mathbb{R}^n$, then we say that $f : \mathbb{R}^n \to Y$ is a function of several variables (even if the dimension n is huge). If the set Y is $\mathbb{R}$, the set of real numbers, then we say that $f : X \to \mathbb{R}$ is a real-valued function. The special cases of real-valued function of a single variable is the primary topic of discussion in calculus I and graphical methods do enhance intuition on the attitude of such functions.

Definition: A function whose range is a set of real numbers is called a real valued function. In other words, it is a function that assigns a real number to each member of its domain.

Pictorial examples of real valued function:

Figure 1: Example of Real valued function.

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In the above example, every element in the set D is related to exactly one element in the set $Y \subset \mathbb{R}$

The range $= \{5, 6, 7, 9\}$

The following are said to be real valued function since their range is the set of real numbers or some subset of the real numbers.

(i) $f(x)2$, the range is $\{2\} \subset \mathbb{R}$.

(ii) $f(x) = 11x$, the range is $\mathbb{R}$.

(iii) $f(x) = e^x$, the range is $(0, +\infty) \subset \mathbb{R}$

(iv) $f(x) = \frac{1}{x^2}$, the range is $(0, +\infty) \subset \mathbb{R}$

Now, given the domain of a function, we find the range of the function $g(t) = 6t^2 + 5$ at

(i) $t = 0$, (ii) $t = 2$ and (iii) $t = 3$

Solution

The given function $g(t) = 6t^2 + 5$

At $t = 0, g(0) = 6(0)^2 + 5 = 5$

At $t = 1, g(1) = 6(1)^2 + 5 = 29$

At $t = 3, g(3) = 6(3)^2 + 5 = 59$

Therefore, the domain$= \{0, 2, 3\}$ and the range $= \{5, 29, 59\}$

Types of Functions:

(i) One -to - one (Injective) function: is the type of function that each element in the domain has a distinct image in the co-domain. In another way, a function $f : X \to Y$ is said to be one-to- one or injective if $f(x_1) = f(x_2)$ in Y implies that $x_1 = x_2$ in X or equivalently, $x_1 = x_2$ in X implies that $f(x_1) = f(x_2)$

Figure 2: Example of one –to -one function.

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For instance, Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is a set of real numbers) be a function defined by $f(x) = 3x + 1$, is f one -to- one function?

Solution.

$f(x) = f(y)$ implies that $3x + 1 = 3y + 1 = 3x = 3y$ or $x = y$

Hence, f is injective.

(ii) Onto (Surjective)function: is a function that every element in the codomain has at least one pre-image in the domain.

Example : Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is a set of real numbers) be a function defined by $f(x) = 3x + 1$, is f surjective?

Solution.

Let $y \in \mathbb{R}$ be arbitrary. Is there $x \in \mathbb{R}$ such that $3x + 1 = y$

Yes, set $x = \frac{y-1}{3}$, then $3x+1 = \frac{3(y-1)}{3} + 1 = y-1+1 = y$. Hence f is surjective.

(iii) Bijective function: a function $f : X \to Y$ is said to bijective if it is both injective and onto function.

Example: a function defined by $f(x) = 3x+1$ is a bijective function since it is both one to one and onto.

(iv) Many -to- one function: A function is said to be many to one function if there are at least two elements in the domain whose images are the same.

Example : a function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2+1$

Solution.

$f(1) = (1)^2 + 1 = 2$

$f(-1)^2 + 1 = 2$

$f(2)^2 + 1 = 5$

$f(-2)^2 + 1 = 5$ and so on

(v) Constant function: $f : \mathbb{R} \to \mathbb{R}$ defined by $b = f(x) = D.a \in \mathbb{R}$.

Domain is $f = \mathbb{R}$

Range $f = \{D\}$

(vi) Composite function: Let $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ defined by

$f(x) = x^2+1, g(x) = 2x+5$.

Find (i) $(gof)(x)$ (ii) $(gof)(0)$ (iii) $(gof)(2)$ (iv) $(fog)(x)$ (v) $(fog)(3)$

Solution.

(i) $(gof)(x) = g[f(x)] = g[x^2+1] = 2(x^2+1)+5$(by the definition of g)

$\Rightarrow 2x^2+2+5 = 2x^2+7$

(ii) $(gof)(0) = 2 \times 0^2 + 7 = 0+7 = 7$

(iii) $(gof)(2) = 2 \times 2^2 + 7 = +7 = 15$

(iv) $(fog)(x) = f[g[x]] = f[2x+5] = (2x+5)^2 + 1$(by definition of f)

$\Rightarrow 4x^2+20x+25+1 = 4x^2+20x+26$

$(fog)(3) = 4 \times 3^2 + 20 \times 3 + 26 = 36+60+26 = 122$

Exercises

(1). Let X, Y be any two sets, which of the following relations $f : X \to Y$ is a function? When f is a function, determine whether it is injective, surjective or bijective.

(i) $X = \{2,6,8,12\}, Y = \{a,b,c\}.f = \{(2,a),(6,b),(12,c)\}$

(ii) $X = \{a,b,c,d\}, Y = \{5,12,7,10\}.f = \{(a,7),(b,12),(c,10),(d,5)$

(iii) $X = \{6,4,7,11\}, Y = \{a,b,c\}.f = \{(6,a),(4,c),(7,b),(11,c)$

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(iv) $X = Y = \mathbb{R}$ (the set of real numbers)$f = \{x, 3x+5 : x \in X$

(v) $X =$ the set of integers, $Y =$the set of positive integers. $f = \{n, n^2 : n \in X\}$

(2). Let $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = x^2,\ if\ x \geq 0\ or\ \frac{1}{2},\ if\ x < 0$

Find (i) $(gof)(0)$ (ii) $(gof)(-1)$, (iii) $(fog)(x)(x > 0)$, (iv)$(gof)(x)(x, 0)$

(3). Let $X = [-1, 1]$ and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$. $h : X \to \mathbb{R}$ be defined by $f(x) = sinx, g(x) = sin(\pi x), h(x) = logx$

Examine whether

(i) f is injective, surjective

(ii) g is injective, surjective

(iii) h is injective, surjective.

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MODULE ONE

Review of Differentiation

The Gradient Function

Consider the curve whose equation is given by $y = f(x)$

Let m be the gradient of the secant line AB, then

$m = \frac{increase\ in\ y}{increase\ in\ x}$

From the idea of gradient of a line, we have that

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Suppose we put $x_2 - x_1 = \triangle x$ ($\triangle\ x \neq 0,$ Since $x_2 \neq x_1$) we have $x_2 = x_1 + \triangle x$

$y_1 = f(x_1),\ y_2 = f(x_2) = f(x_1 + \triangle x)$

Therefore $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$

Let the point A remain fixed, and let the point B move closer to A along the curve. As B moves closer to A along the curve, $\triangle x$ gets closer to zero. $\triangle x$ is called the increment of x, and is read 'δx'. As B approaches A along the curve, the slope of the secant approaches the slope of the tangent at A.

If $\triangle x$ is made to approach zero, the function

$m = \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$ is called the gradient function of $y = f(x)$

The Derivative of a function

Recall that the gradient of the curve $y = f(x)$ is given by

$m_1 = \frac{f(x + \triangle x) - f(x)}{\triangle x}$

The expression m_1 is the gradient of the secant AB. As B moves closer to A, $\triangle x$ becomes smaller and it tends to zero. Let m_2 be the gradient of the tangent at a point A. As B approaches A along the curve, m_1 tends to m_2.

In other words, the gradient of the secant AB tends to the gradient of the tangent to the curve at A. We say that m_2 is the limiting value of m_1 as $\triangle x$ tends to zero. This limiting value is written as

$\lim\limits_{\triangle x \to 0} \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$

and it is denoted simply as $f'(x)$. Thus

$f'(x) = \lim\limits_{\triangle x \to 0} \frac{f(x_1 + \triangle x) - f(x)}{\triangle x}$

Consider again the function $y = f(x)$. A small increment in x will cause a small increment in y. If we represent the small increment in x by $\triangle x$, then we can also represent a small increment in y by $\triangle y$. Thus if $y = f(x)$, then $y + \triangle y = f(x + \triangle x)$

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As B moves closer to $A, \triangle x \to 0$.

If m_1 is the gradient of the secant AB and m_2 is the gradient of the tangent at A. m_2 is the limiting value of m_1 as $\triangle x \to 0$. $m_2 = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$

The expression $m_2 = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$ is denoted by $\frac{dy}{dx}$, which is read $'dee\ y\ dee\ x'$. $\frac{dy}{dx}$ is the limiting value of the slope of the secant AB as B gets closer to A.

The process of finding the derivative of a function is called **differentiation**. The notations which are commonly used for the derivative of a function are:

$f'(x)$ read $f-$prime of x;

$\frac{d}{dx}f$ read $dee-deex$ of f;

$\frac{df}{dx}$ read $dee-f-by-deex$;

If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$. The notations $\frac{dy}{dx}$ and $f'(x)$ will often be used interchangeably. Sometimes, $\frac{dy}{dx}$ is called the **differential coefficient of** y **with respect to (wrt)** x.

As we proceed, different meanings associated with $\frac{dy}{dx}$ will be unfolded, but for the time being $\frac{dy}{dx}$ is either the gradient function of the curve $y = f(x)$ or the rate of change of y with respect to x.

Differentiation by First Principle

The technique adopted in the last section in finding the derivative of a function from the consideration of the limiting value is called **differentiation from first principle**.

Examples

(i)Find the derivative of $f(x) = x^2$ from the first principle.

Solution

$f(x) = x^2$

$f(x+\triangle x) = (x+\triangle x)^2 = x^2 + 2x\triangle x + (\triangle x)^2$

$f(x+\triangle x) - f(x) = x^2 + 2x\triangle x + (\triangle x)^2 - x^2 = 2x\triangle x + (\triangle x)^2$

Thus

$\frac{f(x+\triangle x)-f(x)}{\triangle x} = 2x + \triangle x$

$\lim\limits_{\triangle x \to 0} \frac{f(x+\triangle x)-f(x)}{\triangle x} = 2x$

Therefore $f'(x) = 2x$.

(ii) Find the derivative of $y = x^3$ from the first principle

Solution.

$y = x^3$

$y + \triangle y = (x+\triangle x)^3 = x^3 + 3x^2\triangle x + 3x(\triangle x)^2 + (\triangle x)^3$

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$\triangle y = (x+\triangle x)^3 - y = (x+\triangle x)^3 - x^3$

$= 3x^2 \triangle x + 3x(\triangle x)^2 + (\triangle x)^3$

$\frac{\triangle y}{\triangle x} = \frac{3x^2\triangle x + 3x(\triangle x)^2 + (\triangle x)^3}{\triangle x} = 3x^2 + 3x\triangle x + (\triangle x)^2$

Therefore, $\frac{dy}{dx} = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = 3x^2$

Hence, $\frac{dy}{dx} = 3x^2$.

(iii) Find from the first principle, the derivative of $y = ax + b$, a and b are constants.

Solution.

Given $y = ax + b$

$y + \triangle y = a(x+\triangle x) + b = ax + a\triangle x + b$

$\triangle y = ax + a\triangle x + b - (ax+b)$

$= ax + a\triangle x + b - ax - b = a\triangle x$

Therefore $\frac{\triangle y}{\triangle x} = a$

So that $\frac{dy}{dx} = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = a \Rightarrow \frac{dy}{dx} = a$

(iv) Find the derivative of $y = \frac{1}{x}$

Solution.

$y = \frac{1}{x}$

$y + \triangle y = \frac{1}{x+\triangle x}$

$\triangle y = \frac{1}{x+\triangle x} - y = \frac{1}{x+\triangle x} - \frac{1}{x}$

$= \frac{x-(x+\triangle x)}{(x+\triangle x)x} = \frac{-\triangle x}{x(x+\triangle x)}$

$\lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = -\frac{1}{x^2}$

Hence, $\frac{dy}{dx} = \frac{1}{x^2}$

In what follows, we are going to prove some theorems that gives basic formulas for differentiation.

Theorem 1: If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

Proof. Let $y = x^n$

$y + \triangle y = (x+\triangle x)^n$

$= x^n + nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^n$

$\triangle y = x^n + nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^n - x^n$

$= nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^n$

$\frac{\triangle y}{\triangle x} = nx^{n-1}\triangle x + \frac{n(n-1)}{1.2}x^{n-2}(\triangle x)^2 + \ldots + (\triangle x)^{n-1}$

Therefore, $\frac{dy}{dx} = \lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = nx^{n-1}$

Hence if, $y = x_n$, then $\frac{dy}{dx} = nx^{n-1}$

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Examples

(i) $y = ax^n$:

$\frac{dy}{dx} = nax^{n-1}$

(ii) $y = -2x^3$:

$\frac{dy}{dx} = -2.3x^{3-1} = -6x^2$

(iii) $\frac{1}{x^3} = x^{-3}$: $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$

Theorem 2: (The Derivative of a sum) Let f, U and V be differential functions of x such that $f(x) = U(x) + V(x)$ then

$f'(x) = u'(x) + v'(x)$

In other words, If $y = u + v$, where u and v are differential functions of x

then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Hence, the derivative of sum is the sum of the derivatives.

Proof. Let $y = u(x) + v(x)$, where u and v are differentiable functions of x. Then applying first principle rule gives the desired result.

Examples. Find the derivative of each of the following:

(i) $2x^3 - 5x^2 + 2$ (ii) $3x^2 + \frac{1}{x}$ (iii) $\frac{x^3+2x^2+1}{x}$

Solutions.

(i)Let $y = 2x^3 - 5x^2 + 2$

Therefore, $\frac{dy}{dx} = 6x^2 - 10x$

(ii) Let $y = 3x^2 + \frac{1}{x} = 3x^2 + x^{-1}$

$\frac{dy}{dx} = 6x - \frac{1}{x^2}$

(iii) Let $y = \frac{1}{x^3} = x^{-3}$: $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4} = x^2 + 2x + x^{-1}$

$\frac{dy}{dx} = 2x + 2 - \frac{1}{x^2}$

Derivative of function of a function (Composite function).

Theorem 3: Suppose $y = f(u)$ and $u = g(x)$, then

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

The above formula is called **Chain Rule** and the prove is left as Exercise for the reader.

Examples. Find the derivative of each of the following:

(i) $y = (3x^2 - 2)^3$ (ii) $y = \sqrt{(1 - 2x^3)}$ (iii) $y = \frac{5}{(6-x^2)^3}$.

Solutions.

(i) Given $y = (3x^2 - 2)^3$, Let $u = 3x^2 - 2$, then $y = u^3$

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$\frac{dy}{du} = 3u^2$, $\frac{du}{dx} = 6x$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 3u^2.6x = 18xu^2$

Substituting for u, we have

$\frac{dy}{dx} = 18x(3x^2-2)^2$

(ii) Given $y = \sqrt{(1-2x^3)}$. Let $u = 1-2x^3$. Then $y = u\frac{1}{2}$

$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$, $\frac{du}{dx} = -6x^2$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2\sqrt{u}}, -6x^2 = \frac{3x^2}{\sqrt{u}}$

Hence, substituting for u, gives

$\frac{dy}{dx} = -\frac{3x^2}{\sqrt{1-2x^3}}$

(iii) $y = \frac{5}{(6-x^2)^3}$.

Let $u = 6-x^2$, Then $y = \frac{5}{u^3} = 5u^{-3}$.

$\frac{dy}{dx} = -15u^{-4}$, $\frac{du}{dx} = -2x$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -15u^{-4}.-2x = 30xu^{-4} = \frac{30x}{u^4}$

Hence, substituting for u, you have

$\frac{dy}{dx} = \frac{30x}{(6-x^2)^4}$

The Derivative of a Product

We shall now consider the derivative of $y = uv$ where u and v are functions of x.

Theorem 4: Suppose $y = uv$ where u and v are functions of x, then

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

The above formula is called **Product Rule**. **Examples:** Find the derivative of the following;

(i) $y = (3+2x)(1-x)$ (ii) $y = (1-2x+3x^2)(4-5x^2)$ (iii) $y = \sqrt{x}(1+2x)^2$

Solutions.

(i) $y = (3+2x)(1-x)$. Let $u = 3+2x,\ v = 1-x$

$\dfrac{du}{dx} = 2,\ \frac{dv}{dx} = -1$

Therefore, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (3+2x)(-1) + (1-x)(2)$

$= -(3+2x) + 2(1-x)$

$= -3-2x+2-2x$

$\frac{dy}{dx} = -1-4x$

(ii) $y = (1-2x+3x^2)(4-5x^2)$. Let $u = 1-2x+3x^2,\ v = 4-5x^2$

$\frac{du}{dx} = -2+6x,\ \frac{dv}{dx} = -10x$

COLLINS

Therefore

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (1-2x+3x^2)(-10x) + (4-5x^2)(-2+6x)$

$= -10x(1-2x+3x^2) + (6x-2)(4-x^2)$

(iii)$y = \sqrt{x}(1+2x)^2$. Let $u = \sqrt{x},\ v = (1+2x)^2$.

$\frac{du}{dx} = \frac{1}{2\sqrt{x}},\ \frac{dv}{dx} = 4(1+2x)$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= \sqrt{x}.\ 4(1+2x) + (1+2x)^2.\frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = 4\sqrt{x}(1+2x) + \frac{1}{2\sqrt{x}}(1+2x)^2$

The Derivative of a Quotient

Theorem 6: Let $y = \frac{u}{v}$ where u and v are functions of x and $v(x) \neq 0$ for all x, then

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}$

Let $y = \frac{u}{v}$ where u and v are functions of x and $v(x) \neq 0$ for all x

The above formula is called **Quotient Rule**.

Examples. Find the derivative of each of the following;

(i) $\frac{1+x^2}{1-x^2}$ (ii) $\frac{3+2x-x^2}{\sqrt{1+x}}$ (iii) $\frac{2+x}{x^2+2x+7}$

Solutions.

(i) $y = \frac{1+x^2}{1-x^2}$. Let $u = 1+x^2,\ v = 1-x^2$

$\frac{du}{dx} = 2x,\ \frac{dv}{dx} = -2x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}$

$= \frac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2}$

$= \frac{2x-2x^3+2x+2x^3}{(1-x^2)^2}$

$= \frac{4x}{(1-x^2)^2}$

(ii) $y = \frac{3+2x-x^2}{\sqrt{1+x}}$. Let $u = 3+2x-x^2,\ v = (1+x)^{\frac{1}{2}}$.

$\frac{du}{dx} = 2-2x = 2(1-x),\ \frac{dv}{dx} = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2(1+x)^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{(\sqrt{1+x}).2(1-x)-(3+2x-x^2).\frac{1}{2(1+x)^{\frac{1}{2}}}}{1+x}$

$= \frac{4(1+x)(1-x)-(3+2x-x^2)}{2(1+x)(1+x)^{\frac{1}{2}}} = \frac{4-4x^2-3-2x+x^2}{2(1+x)^{\frac{3}{2}}} = \frac{1-2x-3x^2}{2(1+x)^{\frac{3}{2}}}$

(iii) $y = \frac{2+x}{x^2+2x+7}$, Put $u = 2+x,\ v = x^2+2x+7$

$\frac{du}{dx} = 1,\ \frac{dv}{dx} = 2x+2$

$\frac{dy}{dx} = \frac{(x^2+2x+7).1-(2+x)(2x+2)}{(x^2+2x+7)^2}$

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$= \frac{x^2+2x+7-2(x+2)(x+1)}{(x^2+2x+7)^2} = \frac{x^2+2x+7-2x^2-6x-4}{(x^2+2x+7)^2} = \frac{-x^2-4x+3}{(x^2+2x+7)^2}$

$\frac{dy}{dx} = \frac{3-4x-x^2}{(x^2+2x+7)^2}$

Derivative of Trigonometric, logarithmic, Exponential and Hyperbolic Functions are summarized in the table below;

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S/N	y	$\frac{dy}{dx}$
1	$cosx$	$-sinx$
2	$sinx$	$cosx$
3	$tanx$	sec^2x
4	$secx$	$secx\ tanx$
5	$cosecx$	$-cosecx\ cotx$
6	$cotx$	$-cosec^2x$
7	log_ax	$\frac{1}{x}log_ae$
8	$ln\ x,$	$\frac{1}{x}$
9	a^x	a^xlna
10	e^x	e^x
11	e^{ax}	ae^{ax}
12	$cosh\ x$	$sinh\ x$
13	$sinhx$	$coshx$
14	$tanhx$	$sech^2x$
15	$tan^{-1}x$	$\frac{1}{1+x^2}$
16	$cot^{-1}x$	$\frac{-1}{1+x^2}$
17	$cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$
18	$sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
19	$sec^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$
20	$tanh^{-1}x$	$\frac{1}{1-x^2}$
21	$cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}$
22	$sinh^{-1}x$	$\frac{1}{\sqrt{x^2+1}}$

Table 1.0

COLLINS

Examples: Find the derivative of each of the following;

(a) $cos\ 2x$ (b) $sec\ 6x$ (c) $cosec\ \frac{3}{4}\ x^2$

Solutions.

(a) Given that $y = cos\ 2x$, let $u = 2x, \Rightarrow y = cos\ u$.

Therefore by chain rule,

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -sin\ u.2 = -2sin\ 2x$

(b) Let $y = sin\frac{1}{3}x$

$\frac{dy}{dx} = cos\frac{1}{3}x.\frac{d}{dx}(\frac{1}{3}x) = \frac{1}{3}cos\frac{1}{3}x$

(c) Let $y = cosec\ \frac{3}{4}x^2$. Put $u = \frac{3}{4}$, So $y = cosec\ u$

$\frac{dy}{du} = -cosec\ u.cot\ u$ and $\frac{du}{dx} = \frac{3}{2}x$

$\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = -\frac{3}{2}x\ cot\ \frac{3}{4}x^2.cosec\ \frac{3}{4}x^2.$

(d) $log_a(5x+1)$

Solution

Given that $y = log_a(5x+1)$, let $u = 5x+1$, then $y = log_a u$

$\frac{dy}{du} = \frac{1}{u}log_a e,\ \frac{du}{dx} = 5$

Thus $\frac{dy}{dx} = \frac{du}{dx}.\frac{dy}{du} = 5.\frac{1}{u}log_a e = \frac{5}{5x+1}log_a e$

(e) $y = sinh\ 3x$

$\frac{dy}{dx} = cosh\ 3x\frac{d}{dx}(3x) = 3cosh\ 3x$

Application of Differentiation

The differential Calculus is a powerful tool for solving problems or determining the maximum or minimum values of a function. The business man wants to maximize profit and minimize cost, the government determines the tax rate that optimize tax revenue. Hence, the nature determines many maximum or minimum problems.

Definition: Turning points are those points along the graph of the function which have a zero gradient or the points with horizontal tangents to the graph of the functions.

It is where the gradient of the curve vanishes. i.e, where $\frac{dy}{dx} = 0$

Maximum and Minimum Values

Definition: A function f is said to have a local (or relative) maximum at a $x = x_0$, if $f(x0) \geq f(x_0 + h)$ for all positive and negative values of h sufficiently near zero, and it is said to have a local (or relative)minimum at a point $x = x_0$, if $f(x_0) \leq f(x_0 + h)$ for all values of h near zero.

The conditions for the function to attain their minimum or maximum values are as follows

(1) For minimum, $\frac{d^2y}{dx^2} > 0$

COLLINS

(2) For maximum, $\frac{d^2y}{dx^2} < 0$

Example: A monopolist's revenue function is $R = 240x + 57x^2 - x^3$

Find the output level that gives maximum revenue. What is the maximum revenue?

Solution.

$R = 240x + 57x^2 - x^3$

$\frac{dR}{dx} = 240 + 114x - 3x^2$

At turning point $\frac{dR}{dx} = 0$ i.e,$x^2 - 38x - 80 = 0 \Rightarrow (x-40)(x+2) = 0 \Rightarrow x = 40$ or $x = -2$

The output level that gives maximum revenue is 40. Thus

Maximum Revenue $R = 240x + 57x^2 - x^3 = 240(40) + 57(40) - 40^3 = 36,800$

Therefore, the maximum revenue is $36,800$.

Integration

Integration is the opposite of differentiation.

Recall that during differentiation, we multiplied the power by the function and subtract 1 from the power. But in integration, we add 1 to the power and divide the function by the addition of power and 1. That is, in carrying out integration, the first step is to determine whether the function to be integrated fits one of the basic integration formula. If it doesn't, the next step is to try convert the function into a form to which the basic formula may be applied.

In determining which rule is applicable, try the power rule first. This rule states that :

$\int x^n = \frac{x^{n+1}}{n+1}$.

Examples : Evaluate $\int 3x^5, \Rightarrow 3.\frac{x^{5+1}}{5+1} = 3.\frac{x^6}{6} = \frac{x^6}{2} = \frac{1}{2}x^6$

Indefinite Integral: We know that after the integration of an indefinite integral, there is need to add a constant of integration as follows;

$\int x^n = \frac{x^n+1}{n+1} + C$ where C is a constant

Definition: A function F is called the integral of a function f on an interval if $F'(x) = f(x)$ for every value of $x \in I$.

If F is defined by $F(x) = 4x^2 + x^2 + 5$, then $F'(x) = 12x^2 + 2x$. Thus, if f is the function defined by $f(x) = 12x^2 + 2x$, we say that f is the derivative of F and that F is an integral of f.

In general, if a function F is an integral of a function f on an intervals I and G defined by

$G(x) = F(x) + C$

COLLINS

where C is an arbitrary constant, then

$G'(x) = F'(x) = f(x)$

and G is also an integral of f on the interval I. In fact, we shall show below that if F is any particular integral of f on an interval I, then all possible integral of f on I are defined by $F(x)+C$, where is an arbitrary constant. To do this we need these two preliminary theorems.

Theorem 1: If f is a function such that $f'(x) = 0$ for all values of x in the interval, then f is constant on I.

Proof. Assume that f is not constant on the interval I. Then, there exists two distinct numbers, x_1, x_2 on I where $x_1 < x_2$ such that $f(x_1) \neq f(x_2)$. Because, by hypothesis, $f'(x) = 0$ for all $x \in I$, then $f'(x) = 0$ for all $x \in [x_1, x_2]$. Hence, f is differentiable at all $x \in [x_1, x_2]$ and f is continuous on $[x_1, x_2]$. Therefore, the hypothesis of the mean theorem is satisfied, and we conclude that there exists a number $c \in (x_1, x_2)$, such that

$f'(c) = \frac{f(x_1)-f(x_2)}{x_1-x_2}$ (1)

But $f'(x) = 0$ for all $x \in [x_1, x_2]$, then $f'(c) = 0$, and from (1) it follows that $f(x_1) = f(x_2)$. Hence a contradiction, and so f is constant on I.

Theorem 2: If f and g are functions such that $f'(x) = g'(x)$ for all values of x in the interval I, then there is a constant k such that

$f(x) = g(x) + K$ for all $x \in I$

Basic Functions of Integral (Standard Integrals).

1. $\int dx = x + C$
2. $\int af(x)dx = a\int f(x)dx$, where a is a constant.
3. $\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$
4. $\int \sum_{i=1}^{n} c_i f_i(x)dx = \sum_{i=1}^{n} c_i \int f_i(x)dx$, where $n \in \mathbb{N}$, and $c_i, i = 1, \ldots, n$ are arbitrary constants.
5. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, if $n \neq -1$
6. $\int sinaxdx = -\frac{cosax}{a} + C$, $a \neq 0$
7. $\int cosaxdx = \frac{sinax}{a} + C, a \neq 0$
8. $\int tanaxdx = \frac{ln|secax|}{a} + C, a \neq 0$
9. $\int \frac{1}{x}dx = ln|x| + C$
10. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C, a \neq 0$

The above formulas can be seen as Integration of basic functions. Some complicated functions could be done knowing fully well the Integrals of these basic functions.

Example. Evaluate $\int (3x+5)dx$

COLLINS

Solution.

$\int(3x+5)dx = 3\int xdx + 5\int dx$ by formula (4)

$= 3(\frac{x^2}{2}+C_1)+5(x+C_2)$, by formula (5) and (1).

$= \frac{3}{2}x^2+5x+(3C_1+5C_2)$

$= \frac{3}{2}x^2+5x+C$

where $C = 3C_1+5C_2$. C is also a constant, since $3C_1$ and $5C_2$ are also arbitrary constants.

Integration by Partial Fractions

Our Concern here will be on rational functions which could be resolved by partial fractions into sum of integrals of the forms we have seen so far.

Expressions of the form $\int \frac{7x+8}{2x^2+11x+5}dx$ do not appear in our list of standard integral, but do occur in many mathematical applications.

So, the above equation can be expressed in partial fractions which are simpler to solve

$\frac{7x+8}{2x^2+11x+5}dx = \frac{7x+8}{(x+5)(2x+3)}dx = \frac{3}{x+5}+\frac{1}{2x+3}$

Example. Evaluate $\int \frac{3x^2+18x+3}{3x^2+5x-2}dx$

Solution. Since the integral's denominator is not greater than that of the numerator, we divide out so that

$\frac{3x^2+18x+3}{3x^2+5x-2}dx = 1+\frac{13x+5}{3x^2+5x-2}$

$\frac{13x+5}{(3x-1)(x+2)} = \frac{4}{3x-1}+\frac{3}{x+2}$

Therefore $\frac{13x+5}{3x^2+5x-2} = 1+\frac{4}{3x-1}+\frac{3}{x+2}$

Hence,$\int \frac{13x+5}{3x^2+5x-2}dx = \int dx + \frac{4}{3}\frac{4}{3x-1}dx + 3\frac{1}{x+2}dx$

$= x+\frac{4}{3}ln|3x-1|+3ln|x+2|+C, x \neq \frac{1}{3}, -2$

Integration by Parts

This is applied to integrate a product where either function is not derivative of the other.

For instance $\int x^2 ln|x|dx.ln|x|$ is not the derivative of x^2 neither is x^2 the derivative of $ln|x|$.

Therefore, it is necessary that we find another methods of obtaining such integral.

If u and v are functions of x, then we know that

$\frac{d}{dx}(uv) = u\frac{dv}{dx}+v\frac{du}{dx}$.

If we integral both sides with respect to x, we have

$uv = \int u\frac{dv}{dx}+\int v\frac{du}{dx}$. To make the memorization of formula easier. We have $\int udv = uv - \int vdu$ (2)

The equation (2) above is called Integration by parts formula.

Example. Evaluate $\int x^2 ln|x|dx$

Solution.

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From $\int x^2 ln|x|dx$, we set $u = ln|x|$ and $dv = x^2dx$, and obtain $du = \frac{1}{x}dx$ and $v = \int x^2dx = \frac{x^3}{3}$

Using the integration by parts formula

$\int udv = uv + \int vdu$

$\int x^2 ln|x|dx = \int udv$

$= ln|x|(\frac{x^3}{3}) - \frac{1}{3}\int x^3 \frac{1}{x}dx$

$= \frac{x^3}{3}ln|x| - \frac{1}{3}(\frac{x^3}{3}) + C$

$= \frac{x^3}{3}[ln|x| - \frac{1}{3}] + C, x \neq 0.$

Method of Substitution

Suppose we are interested in finding a function of the form

$\int 2x\sqrt{1+x^2}dx$ (3)

If we make the substitution $u = 1 + x^2$, then it follows that

$du = 2xdx$ and (3) becomes $\int u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} + C$

Now we replace u by $(1 + x^2)$, and obtain finally that

$\int 2x\sqrt{1+x^2}dx = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$

The justification of this procedure used in the illustration above is given by the theorem that follows. The theorem is analogous to the chain rule for differentiation, and hence may be called the chain rule for integration.

Theorem. Let g be a differentiable function of x, and let the range of g be an interval I. Suppose that f is a function defined on I and F is an anti derivative of f on I. Then if $u = g(x)$,

$\int f(g(x)g'(x))dx = \int f(u)du = F(u) + C = F(g(x)) + C$

The proof of the above theorem is left as Exercise to the reader.

Definite Integral

Before we develop this section we give the following important theorems.

Theorem. (The Fundamental theorem of Calculus) Let $f : [a,b] \to \mathbb{R}$ be continuous, then the function defined on $[a,b]$ into $\mathbb{R}$ by

$g(x) = \int_a^x f(t)dt$, $a \leq x \leq b$ is continuous and differentiable in (a,b) and $g'(x) = f(x)$.

Theorem. Let $f : [a,b] \to \mathbb{R}$ be continuous then

$\int_a^b f(x)dx = F(b) - F(a)$ where F is any integral of f i.e., a function such that $F' = f$.

Definition. Let $f : [a,b] \to \mathbb{R}$, the definite integral of f from a to b is denoted by $\int_a^b f(x)dx$. and is given by

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$\int_a^b f(x)dx = \lim\limits_{\|\triangle\|\to 0} \sum_{i=1}^{n} f(\xi_i)\,\triangle_i x$
if this limit exists.

Example. Evaluate $\int_0^4 \sqrt{2x+1}dx$.

Solution. We know that $\int \sqrt{2x+1}dx = \frac{1}{3}(2x+1)^{\frac{3}{2}}$. Hence

$\int_0^4 \sqrt{2x+1}dx = \frac{1}{3}(2x+1)^{\frac{3}{2}}|_0^4$

$= \frac{1}{3}(9)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}}$

$= \frac{1}{3}(27-1)$

$= \frac{26}{3}$

Mean Value Theorem

In this section, we discuss some of the most useful theorem in Differential Calculus known as the mean-value theorems given again by the two famous French mathematicians Cauchy and Lagrange. Lagrange proved a result only by using the first two conditions of Rolle's theorem.

Hence, it is called Lagrange's Mean-Value theorem. Cauchy gave another mean-value theorem in which he used two functions instead of one function as in the case of Rolle's theorem and Lagrange's Mean-Value theorem. You will see later that Lagrange's theorem is a particular of Cauchy's mean value theorem. Finally, we discuss the generalised form of these two theorems. We begin with Mean-Value theorem given by J. L. Lagrange.

Lagrange's Mean-Value Theorem

If a function $f : [a,b] \to \mathbb{R}$ is

(i) continuous on $[a,b]$ and

(ii) derivable on$]a,b[$, then there exists at least a point $c \in]a,b[$ such that

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Now the function ϕ, being the sum of two continuous and derivable functions is itself

(i) continuous on $[a,b]$

(ii) derivable on $]a,b[$, and

(iii) $\phi(a) = \phi(b)$.

Therefore, by Rolle's theorem a real number $c \in]a,b[$ Such that, $\phi.(c) = 0$.

But $\phi.(x) = f'(x) + A$

So $\phi = \phi'(c) = f'(c) + A$

which means that $f'(c) = -A = \frac{f(b)-f(a)}{b-a}$

In the statement of the above theorem, sometimes b is replaced by $a+h$, so that the number c between a

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and b can be taken as $a+\theta h$ where $0<\theta<1$. According them, the theorem can be restated as follows: Let f be defined and continuous on [a, a + h] and derivable on $]a,a+h[$, then there exists θ,$0<\theta<1$ such that

$f(a+h)=f(a)+hf'(a+\theta h)$.

Certain important and useful results can be deduced from Lagrange Mean-Value theorem.

We state and prove these results as follows:

You already know that derivative of a constant function is zero.

Conversely, if the derivative of a function is zero, then it is a constant function. This can be formalised in the following way:

I) If a function f is continuous on $[a,b]$, derivable on $]a,b[$ and$f'(x)=0$ for all $x\in]a,b[$, then $f(x)=k$ for all $x\in[a,b]$, where k is some fixed real number.

To prove it, let λ be any point of $[a,b]$. Then $[a,\lambda]\subset[a,b]$

Thus, f is

i) continuous on $[a,\lambda]$

ii) derivable on $]a,\lambda$

Therefore, by Lagrange's mean value theorem, $c\in]a,\lambda[$ such that

$f'(c)=\frac{(f(\lambda)-f(a))}{(\lambda-a)}$

Now $f'(x)=0\ \forall\ x\in]a,b[$

$f'(x)=0\forall\ x\in]a,\lambda$

$f'(c)=0$

$f(l)=f(a)\ \forall\ l\in[a,b]$

But l is any arbitrary point of $[a,b]$.

Therefore $f(x)=f(a)=k$ for all $x\in[a,b]$.

Example : Prove that for any quadratic function x^2+mx+n, the value of θ in Lagranges mean value theorem is always $\frac{1}{2}$, whatever $1,m,n$ and h may be.

Solution: Let $f(x)=x^2+mx+n;x[a,a+h]$.

f being a polynomial function is continuous in $[a,a+h]$ and derivable in $]a,a+h[$. Thus, f satisfies the conditions of Lagranges mean value theorem.

Therefore, there exists $\theta(0<\theta<1)$ such that

$f(a+h)=f(a)+hf(a+\theta h)$

Putting the values of f and f you will get

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$(a+h)^2+m(a+h)=n=a2+ma+n+h[21(a+\theta h)+m]$

i.e. $\Rightarrow h^2=2\theta h^2$

which gives $\theta=\frac{1}{2}$, whatever a,h,l,m,n may be.

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Taylor Series of Several Variables

Recall that the single variable of Taylor series is as shown below:

Let f be an infinitely differentiable function in some open interval around $x = a$.

$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$ (1.1)

Where the nth term of equation (1.1) is called the nth order of Taylor polynomial or Taylor polynomial of order n.

For linear approximation in one variable: Take the constant and linear terms from Taylor series expansion (1.1) above. In an open interval around $x = a$ gives $f(x) \approx f(a) + f'(a)(x-a)$

For quadratic approximation in one variable: Take the constant, linear and quadratic terms from Taylor series (1.1) above, in an open interval around $x = a$. We have that

$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

Example : Find the Taylor series for $cos(x) = \frac{d}{dx}sin(x)$, where $sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Solution

$cos(x) = \frac{d}{dx}sin(x) = \frac{d}{dx}[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

For Taylor Series of Multi (several) Variables:

Let f be an infinitely differentiable function in some open neighbourhood around $(x,y) = (a,b)$. Then,

$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2}{2!} + ..$ (1.2)

The above equation (1.2) is called Taylor series of several variables and can be written in more compact form as shown below:

Let $x = (x,y)$ and let $a = (a,b)$. With this new vector notation, the Taylor series can be written as

$f(x) = f(a) + [(x-a).\bigtriangledown f(a)] + [(x-a).(H(x).(x-a))] + \ldots$

Where H is the matrix of second derivatives, called Hessian matrix

$$H(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix}$$

For linear approximation in several variables: Take the constant and linear terms from the Taylor series (1.2) above, in a neighbourhood of $(x,y) = (a,b)$.

$f(x) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

For quadratic approximation in several variables: Take the constant, linear and quadratic terms from the Taylor series (1.2) above. In a neighbourhood of $(x,y) = (a,b)$. Then, we have

$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2}{2!}$

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$= f(a) + [(x-a).\bigtriangledown f(a)] + [(x-a).\, H(x).(x-a)]$

Definition: If f is of the class of C^n, then

$T_f^n(x,y;a,b) = \sum_{r=0}^{n} \frac{d_{(a,b)}^{(r)} f(x-a,y-b)}{r!} = f(a,b) + \frac{d_{(a,b)}^{(2)} f(x-a,y-b)}{2!} + \frac{d_{(a,b)}^{(3)} f(x-a,y-b)}{3!} + \ldots + \frac{d_{(a,b)}^{(n)} f(x-a,y-b)}{n!}$

Theorem : f is called Taylor Series if and if the limit of its remainder is equal to zero. i.e; $f = \lim_{n\to\infty} R_f^n(x,y;a,b) = 0$

The remainder $R_f^n(x,y;a,b) = f(x,y) - T_f^n(x,y;a,b)$

Example 1: Let $f = sin(x,y) + xe^{-y}$. $(a,b) = (0,0)$

Find $T_f^3(x,y;0,0) = T$

Solution

$T_f^3(x,y;0,0) = T_f^2(x,y;0,0) + \frac{1}{3!} d_{(0,0)}^3 f(x,y) = 2x + y - xy + \frac{1}{6}[-x^3 - 3x^2y - y^3]$.

Example 2: Determine the second degree Taylor polynomial linear approximations $L(x,y)$ and $Q(x,y)$, for the following functions of x and y near the given point.

a. $f(x,y) = sin2x + cosy$ for (x,y) near the point $(0,0)$

b. $f(x,y) = xe^y + 1$ for (x,y) near the point $(1,0)$

Solution

$f(x,y) = sin2x + cosy$

a. To determine the second-degree Taylor polynomial (quadratic) approximation, $Q(x,y)$, we need the second partials of the function f as:

$f_{xx}(x,y) = -4sin2x$

$f_{xy}(x,y) = 0$

$f_{yy}(x,y) = -cosy$

Evaluating these second partials at the point $(0,0)$:

$f_{xx}(0,0) = -4sin2(0) = 0$

$f_{xy}(0,0) = 0$

$f_{yy}(0,0) = -cos0 = -1$

Then,

$Q(x,y) = L(x,y) + \frac{f_{xx}(0,0)}{2}(x-0)^2 + f_{xy}(0,0)^2 + f_{xy}(0-0)(x-0)(y-0) + \frac{f_{yy}(0,0)}{2}(y-0)^2$

$= 1 + 2x + \frac{0}{2}x^2 + (0)xy + + \frac{-1}{2}y^2$

$= 1 + 2x - \frac{y^2}{2}$

See the plot of the function f along with its quadratic approximation (the 2nd degree Taylor polynomial).

Solution

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b. $f(x,y) = xe^y + 1$

To determine the second-degree Taylor polynomial (quadratic) approximation, $Q(x,y)$, we need the second partials of f.

$f_{xx}(x,y) = 0$

$f_{xy}(x,y) = e^y$

$f_{yy}(x,y) = xe^y$

Evaluating these second partials at the point $(1,0)$;

$f_{xx}(1,0) = 0$

$f_{xy}(1,0) = e^0$

$f_{yy}(1,0) = (1)e^0 = 1$

Then,

$Q(x,y) = L(x,y) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + f_{xy}(1,0)$

$= 1 + x + y + \frac{0}{2}(x-1)^2 + (1)(x-1)y$

$= 1 + x + y + xy + \frac{y^2}{2}$

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Jacobian Function: Dependence and Independence

The set of equations

$$u = u(x,y) \tag{1}$$

$$v = v(x,y) \tag{2}$$

defines, in general, a transformation or mapping, which establishes a correspondence between points in the uv and xy planes. The equations (1 and 2) are called transformation equations. If to each point of the uv plane there is corresponds one and only one point of the xy plane, and conversely, we speak of a one- to -one transformation or mapping. In such a case, a set of points in the xy plane (such as a curve or region) is mapped into a set of points in the uv plane (curve or region) and conversely. The corresponding sets of points in the two planes are often called images of each other.

Jacobian of a Transformation

Under the transformation (1 and 2), R of the xy plane is, in general, mapped into a region R' of the uv plane. Then, if $\triangle A_{xy}$ and $\triangle A_{uv}$ denote, respectively, the areas of these regions, we can show that if u and v are continuously differentiable,

$$\lim \frac{\triangle A_{uv}}{\triangle A_{xy}} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \tag{3}$$

where lim denotes the limit as $\triangle A_{xy}$ (or $\triangle A_{uv}$) approaches zero and where the determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{4}$$

is called the Jacobian of the transformation (1 and 2).

Functional Dependence

Let $u = u(x,y)$ and $v = v(x,y)$. If we can write $u = g(v)$ or $v = h(u)$, then u and v are said to be functional dependent. If functional dependence between u and v exists, then we can consider $f(u,v) = 0$. So,

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{5}$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{6}$$

In matrix form, this is

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$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

Since the right hand side is zero, and we desire a non-trivial solution, the determinant of the coefficient matrix must be zero for functional dependency. i.e;

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \tag{8}$$

. **Remark**, since $detJ = detJ^T$, then it is equivalent to

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = 0 \tag{9}$$

That is, the Jacobian determinant J must be zero for functional dependence.

Example 1. Determine if

$$u = y+z \tag{3}$$

$$v = x+2z^2 \tag{4}$$

$$w = x-4yz-2y^2 \tag{5}$$

are functionally dependent.

Solution.

The determinant of the resulting coefficient matrix, by extension to three functions of three variables, is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -4(y+z) \\ 1 & 4z & -4y \end{vmatrix}$$

$= (-1)(-4y-(-4)(y+z))+(1)(4z)$

$= 4y-4y-4z+4z = 0.$

So, u,v,w are functionally dependent. In fact $w = v-2u^2$.

Example 2. Let

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$$x+y+z = 0 \tag{6}$$

$$x^2+y^2+z^2+2xz = 1 \tag{7}$$

Can x and y be considered as functions of z?

Solution. If $x = x(z)$ and $y = y(z)$, then $\frac{dx}{dz}$ and $\frac{dy}{dz}$ must exist. If we take

$f(x,y,z) = x+y+z = 0,$

$g(x,y,z) = x^2+y^2+z^2+2xz-1 = 0$

$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$dg = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$

$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial x}\frac{dx}{dz} + \frac{\partial f}{\partial y}\frac{dy}{dz} = 0$

$\frac{\partial g}{\partial z} + \frac{\partial g}{\partial x}\frac{dx}{dz} + \frac{\partial g}{\partial y}\frac{dy}{dz} = 0$

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial z} \\ -\frac{\partial y}{\partial z} \end{pmatrix} \tag{8}$$

then the solution matrix $(\frac{dx}{dz}, \frac{dy}{dz})^T$ can be obtained by Crammer's rule:

$$\frac{dx}{dz} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial z} & \frac{\partial f}{\partial y} \\ -\frac{\partial g}{\partial z} & \frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}} = \frac{\begin{vmatrix} -1 & 1 \\ -(2z+2x) & 2y \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2x+2z & 2y \end{vmatrix}} = \frac{-2y+2z+2x}{2y-2x-2z} = -1,$$

$$\frac{dy}{dz} = \frac{\begin{vmatrix} \frac{\partial f}{\partial x} & -\frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & -\frac{\partial g}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & -1 \\ 2x+2z & -(2z+2x) \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2x+2z & 2y \end{vmatrix}} = \frac{0}{2y-2x-2z},$$

Remark. Here that in the expression for $\frac{dx}{dz}$ that the numerator and denominator cancel; there is no special condition defined by the Jacobian determinant of the denominator being zero. In the second, $\frac{dy}{dz} = 0$ if $y-x-z \neq 0$, in which case the formula cannot give us the derivative.

Now, in fact, it is easily shown by algebraic manipulations (which for more general functions are not possible).

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Independent Function

Suppose that $u \equiv u(x,y)$ and $v \equiv v(x,y)$ are two functions of two independent variables, x and y, then in general, it is not possible to express u solely in terms of v, nor v solely in terms of u.

However, on occasions, it may be possible, as the following illustration.

1. If $u \equiv \frac{x+y}{x}$ and $v \equiv \frac{x-y}{y}$

then

$u \equiv 1 + \frac{x}{y}$ and $v \equiv \frac{x}{y} - 1$

which gives

$(u-1)(v+1) \equiv \frac{x}{y}.\frac{y}{x} \equiv 1.$

Hence,

$u \equiv 1 + \frac{1}{v+1}$ and $v \equiv \frac{1}{u-1} - 1.$

If u and v are not connected by an identical relationship, they are said to be **Independent Functions**

Theorem : Two functions, $u(x,y)$ and $v(x,y)$, are independent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0$$

Observe that the independent of given functions, means that the determinant of the partial derivative of the functions is not equivalent to zero.

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