VECTOR AND TENSOR ANALYSIS (MTS315)

Course outline:

Vector algebra: vector, dot and cross products

Equation of curves and surfaces.

Differentiation of vectors and applications:

Gradient, divergence and curl.

Vector Integration: Lines, surface and volume integrals.

Divergence Theorem, Green's and Stoke's theorems

Tensor products of vector spaces: Tensor algebra, Symmetry, Cartesian tensors.

Recall:

1. $\bar{a} = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 K$

Constitute the real vector space R^3 with addition defined by

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

and Scalar multiplication defined by

$$\lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$$

λ being a scalar (a real number). For instance, the resultant of forces $\bar{a}$ and $\bar{b}$ is the sum $\bar{a} + \bar{b}$ of the two forces.

2. The dot product of two vectors is defined by $\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$ where θ is the angle between $\bar{a}$ and $\bar{b}$. This gives for ~~one~~ the norm or length $|\bar{a}|$ of $\bar{a}$, the formula

$$|\bar{a}| = \sqrt{\bar{a}\,\bar{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

as well a formula for θ. If $\bar{a}\cdot\bar{b} = 0$, $\bar{a}$ and $\bar{b}$ are orthogonal. The dot product is suggested by the work, $W = \bar{p}\cdot\bar{d}$ done by a force $\bar{p}$ in a displacement $\bar{d}$.

(3) The vector product or Cross product $\bar{v} = \bar{a}\times\bar{b}$ is a vector of length

$$|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$$

and perpendicular to both $\bar{a}$ and $\bar{b}$ such that $\bar{a}, \bar{b}, \bar{v}$ form a right-handed triple. In terms of components with respect to right-handed coordinates.

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})$$

DIFFERENTIATION OF VECTORS

1. GRADIENT OF A SCALAR FIELD

The gradient grad f of a given scalar function $f(x, y, z)$ is the vector function defined by

$$\text{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$$

we assume f is differentiable.

The differential Operator ∇ (del or nabla) is defined as

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} K \qquad 1.1$$

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K \qquad 1.2$$

~~Directional~~

2. DIRECTIONAL DERIVATIVES

The rate of change of f at any point P in any fixed direction given by a vector $\bar{b}$ is defined by

$$D_b f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s} \qquad (1.3)$$

where s = distance between P and Q.

where Q is a variable point on the ray C given by

$$\bar{r}(s) = x(s)i + y(s)j + z(s) = \bar{P}_0 + s\bar{b} \quad (1.4)$$

$$(s \geq 0, |\bar{b}| = 1)$$

$\bar{P}_0$ is the position vector of P.

Equation (1.3) shows that $D_b f = \frac{df}{ds}$ is the derivative of the function $f(x(s), y(s), z(s))$ with respect to the length s of C. Hence, assuming that f has continuous partial derivatives and applying the chain rule, we obtain,

$$D_b f = \frac{df}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' \quad (1.5)$$

$$= \frac{df}{ds} = \bar{b} \cdot \text{grad} f \quad (|b| = 1) \quad (1.6)$$

where primes denote derivatives with respect to s (which are taking at $s = 0$)

$$\bar{r}' = x'i + y'j + z'k.$$

Note: If the direction is given by a vector $\bar{a}$ of any length ($\neq 0$), then

$$D_b f = \frac{df}{ds} = \frac{1}{|\bar{a}|}\bar{a} \cdot \text{grad} f$$

EX

Find the directional derivative of $f(x,y,z) = 2x^2+3y^2+z^2$ at the point $P:(2,1,3)$ in the direction of the vector $\bar{a} = i - 2k$.

Solution:

$$\text{grad} f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left(2x^2+3y^2+z^2\right)$$

$$= 4x i + 3y j + 2z k$$

at $P:(2,1,3)$

$$\text{grad} f = 8i + 6j + 6k$$

The directional derivative is given by

$$D_{\bar{a}} f = \frac{df}{ds} = \frac{1}{|\bar{a}|}\bar{a}\cdot \text{grad} f$$

$$= \frac{1}{\sqrt{5}}(i-2k)\cdot(8i+6j+6k)$$

$$= \frac{8-12}{\sqrt{5}} = \frac{-4}{\sqrt{5}} \quad //$$

THEOREM 1: Let $f(P) = f(x,y,z)$ be a scalar function having continuous first partial derivatives. Then grad f exist and its length and direction are independent of the particular choice of Cartesian coordinates in the space. If at a point P the gradient of f is not the zero vector, it has the direction of ~~maximum~~ maximum increase of f at P.

Proof:

$$\text{from } D_{\bar{b}} f = \frac{df}{ds} = \bar{b} \cdot \text{grad} f$$

$$\Rightarrow D_{\bar{b}} f = |\bar{b}| \, |\text{grad} f| \cos\theta \qquad *1$$

where θ is the angle between $\bar{b}$ and grad f. Since, f is a scalar function, its value at point P depends on P but not on the particular choice of coordinates. ~~And~~ This also holds for the length of arc length s on ray C and therefore holds for $D_{\bar{b}} f$.

*1 shows that $D_{\bar{b}} f$ is maximum when

$\cos\theta = 1$, $\theta = 0$ and then $D_{\bar{b}} f = |\text{grad} f|$.
It follows that the length and direction of grad f are independent of the coordinates.

Since $\theta = 0$ iff $\bar{b}$ has the direction of grad f, ~~Hence, the gradient of~~ f the latter ~~has~~ is the ~~the~~ direction of maximum increase of f at P, provided grad $f \neq 0$ at P.

~~GRADIEEN~~

GRADIENT AS SURFACE NORMAL VECTOR.

Another basic use of the gradient results in connection with surface S in space given by

$$f(x, y, z) = c = \text{Const.} \qquad *_2$$

A curve C in space can be given by

$$\bar{r}(t) = x(t)i + y(t)j + z(t)k$$

If we want C to lie on S, its components must satisfy ($*_2$); thus

$$f(x(t), y(t), z(t)) = c \qquad *3$$

A tangent vector of C is

$$\bar{r}'(t) = x'(t)i + y'(t)j + z'(t)K$$

If C lies on S, this vector tangent to S.

At a fixed point P on S, these tangent vectors of all curve on S through P will generally form a plane, called the tangent plane of S at P. Its normal (the straight line through P and perpendicular to the tangent plane) is called the surface normal of S at P. A vector parallel to it is called a surface normal vector of S at P. ~~Now~~,

Differentiating $*_3$, we obtain

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = (\text{grad} f)\cdot \bar{r}' = 0$$

This means orthogonality of grad f and all the vector $\bar{r}'$ in the tangent plane.

<u>Example</u>: Find a unit normal vector $\bar{a}$ of the cone of revolution $z^2 = 4(x^2+y^2)$ at the point P: (1, 0, 2).

Solution:

The cone is the level surface $f=0$ of
$f(x,y,z) = 4x^2 + 4y^2 - z^2$

Thus, $\operatorname{grad} f = 8xi + 8yj - 2zK$

at P, $\operatorname{grad} f = 8i - 4K$

$\therefore$ The unit vector of the cone at P is

$$\bar{n} = \frac{1}{|\operatorname{grad} f|} \operatorname{grad} f = \frac{2}{\sqrt{5}} i - \frac{1}{\sqrt{5}} K$$

and the other one is $-\bar{n}$.

NOTE:
For a vector function $\bar{V}(P)$, if
$\bar{V}(P) = \operatorname{grad} f(P)$, the function $f(P)$ is called a potential function or a potential of $\bar{V}(P)$.
Such a $\bar{V}(P)$ and corresponding vector field are conservative.

Example:

Let a particle A of mass M be fixed at a point P_0 and let a particle B of mass m be free to take up various position P in space. Then A attracts B. According to the law of ~~gravit~~ Newton's law of gravitation the corresponding gravitational force $\bar{P}$ is directed from P to P_0 and its magnitude is proportional to $\frac{1}{r^2}$, where r is the distance between P and P_0.

*5

$$|\bar{p}| = \frac{C}{r^2}, \qquad \cancel{C = GMm} \quad C = GM_n.$$

Where G is the gravitational constant.
Hence, $\bar{P}$ defines a vector field in space.

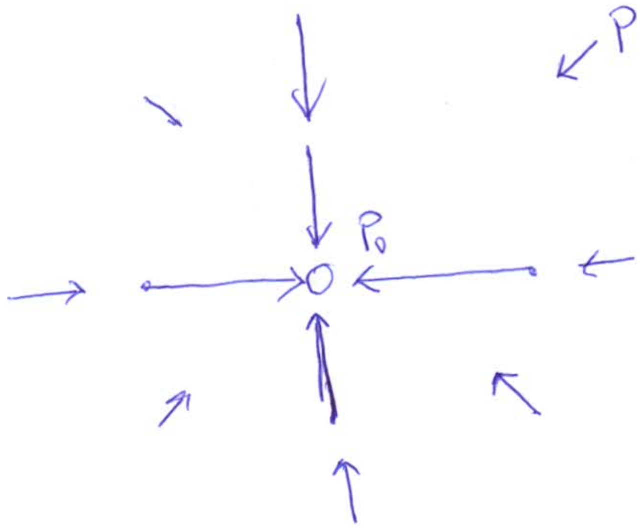


Fig1: Gravitational field.

If we introduce Cartesian coordinates such that P_0 has the coordinates x_0, y_0, z_0 and P has the coordinates x, y, z, then by the Pythagorean theorem

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \qquad (\geq 0)$$

Assuming that $r > 0$ and introducing the vector

$$\bar{r} = [x - x_0, y - y_0, z - z_0] = (x - x_0)i + (y - y_0)j + (z - z_0)k,$$

we have $|\bar{r}| = r$, and $\left(\frac{-1}{r}\right)\bar{r}$ is a unit vector in the direction of $\bar{P}$; the minus sign indicates that P is directed from P to P_0 (Fig. 1). From this and (*5) we obtain

$$\bar{P} = |\bar{P}|\left(-\frac{1}{r}\bar{r}\right) = -\frac{c}{r^3}\bar{r} = -c\left(\frac{x - x_0}{r^3}\right)i - c\left(\frac{y - y_0}{r^3}\right)j - c\left(\frac{z - z_0}{r^3}\right)k \qquad *6$$

This vector function describes the gravitational force acting on B.

Hence, by Newton's law of gravitation, the force of attraction between two particles is

$$\bar{P} = -\frac{c}{r^3}\bar{r} = -c\left(\frac{x - x_0}{r^3}i + \frac{y - y_0}{r^3}j + \frac{z - z_0}{r^3}k\right) \qquad *7$$

Here r is the distance between the two particles at $P_0 : (x_0, y_0, z_0)$ and $P : (x, y, z)$; thus

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

The key observation now is

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-2(x - x_0)}{2\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right]^{3/2}} = -\frac{(x - x_0)}{r^3} \qquad *8a$$

and similarly

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{(y-y_0)}{r^3} \quad , \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{(z-z_0)}{r^3} \qquad *8b$$

From this we see that $\bar{p}$ is the gradient of the scalar function $\quad f(x,y,z) = \frac{c}{r} \quad (r>0)$.

Thus, f is a potential of that gravitational field (and further potentials are $f+k$ with constant k).

Furthermore, we show next that f satisfies the most important partial differential equation of physics, the so called **Laplace's equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \qquad *9$$

Indeed, to see this, all we have to do is differentiate (*8),

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x-x_0)^2}{r^5}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(y-y_0)^2}{r^5}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(z-z_0)^2}{r^5},$$

and then add these three expressions. Their common denominator is r^5. Hence the three terms $-\frac{1}{r^3}$ contribute $-3r^2$ to the numerator, and the three other terms give $3(x-x_0)^2 + 3(y-y_0)^2 + 3(z-z_0)^2 = 3r^2$, so that the numerator is 0 and we obtain (*9), as claimed.

In (*9), the expression on the left is called the **Laplacian** of f and is denoted by $\nabla^2 f$ or Δf. The differential operator

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad *10$$

(read "nabla squared" or "delta") is called the **Laplace operator**. Using this operator, we may write (*9) in the form

$$\nabla^2 f = 0 \qquad *11$$

It can be shown that the field of force produced by any distribution of masses is given by a vector function that is the gradient of a scalar function f, and f satisfies (*9) in any region of space that is free of matter.

Divergence of a Vector Field

Let $\bar{v}(x,y,z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of $\bar{v}$. Then the function

$$\operatorname{div} \bar{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad \cdots \quad *12$$

is called the **divergence** of $\bar{v}$ or the divergence of the vector field defined by $\bar{v}$. Another common notation for the divergence of $\bar{v}$ is $\nabla \cdot \bar{v}$,

$$\operatorname{div} \bar{v} = \nabla \cdot \bar{v} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (v_1 i + v_2 j + v_3 k)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

with the understanding that the "product" $\left(\frac{\partial}{\partial x}\right) v_1$ is the dot product means the partial derivative $\frac{\partial v_1}{\partial x}$, etc. This is a common notation, but nothing more. Note that $\nabla \cdot \bar{v}$ means the scalar $\operatorname{div} \bar{v}$, whereas ∇f means the vector $\operatorname{grad} f$.

For example, if

$\bar{v} = 4xz\, i + 2xy\, j - 3yz^2 k$, then $\operatorname{div} \bar{v} = 4z + 2x - 6yz$.

Theorem (Invariance of the divergence)

The values of $\operatorname{div} \bar{v}$ depend only on the points in space (and, of course, on $\bar{v}$) but not on the particular choice of the coordinates in (*12), so that with respect to other Cartesian coordinates x^*, y^*, z^* and corresponding components v_1^*, v_2^*, v_3^* of $\bar{v}$

the function div $\bar{v}$ is given by

$$\operatorname{div} \bar{v} = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*} \quad \cdots \quad *13$$

Note: If $f(x,y,z)$ is a twice differentiable scalar function, then

$$\operatorname{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

and by (*12)

$$\operatorname{div}(\operatorname{grad} f) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right)$$

$$\Rightarrow \quad \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Since the term on the R.H.S is the **Laplacian** of f, we therefore have that :

$$\operatorname{div}(\operatorname{grad} f) = \nabla^2 f \quad \cdots \quad *14$$

Example 1 (Gravitational force)
The gravitational force $\bar{p}$ is the gradient of the scalar function $f(x,y,z) = \frac{c}{r}$, which satisfies Laplace's equation $\nabla^2 f = 0$. According to (*14), this means that

$$\operatorname{div} \bar{p} = 0 \quad (r > 0), \text{ since } \bar{p} = \operatorname{grad} f$$

The physical significance of the divergence of a vector field is shown in the following example from hydrodynamics

Example 2 (Motion of a compressible fluid. Physical meaning of divergence) : We consider the motion of a fluid in a region R having no **sources** or **sinks** in R (no points of inflow or outflow)

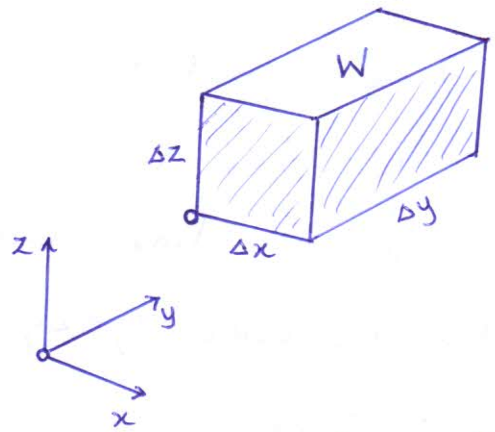


Fig. 2. Physical interpretation of the divergence

Consider the flow through a small rectangular box W of dimensions $\Delta x, \Delta y, \Delta z$ with edges parallel to the coordinate axes (as in Fig 2 above).

W has the volume $\Delta V = \Delta x \, \Delta y \, \Delta z$. Let $\bar{v} = [v_1, v_2, v_3]$

$\bar{v} = v_1 i + v_2 j + v_3 k$ be the velocity vector of the motion, We set $\bar{u} = \rho \bar{v} = [u_1, u_2, u_3] = u_1 i + u_2 j + u_3 k$

Assume that $\bar{u}$ and $\bar{v}$ are continuously differentiable vector functions of x, y, z, and t (that is, they have first partial derivatives, which are continuous). Let us calculate the change in the mass included in W by considering the flux across the boundary, that is, the total loss of mass leaving W per unit time.

Consider the flow through the left face of W, whose area is $\Delta x \, \Delta z$. The components v_1 and v_3 of $\bar{v}$ are parallel to that face and contribute nothing to this flow. Hence the mass fluid entering through that face during a short time interval Δt is given approximately by

$$(\rho v_2)_y \,\Delta x \,\Delta z \,\Delta t = (u_2)_y \,\Delta x \,\Delta z \,\Delta t,$$

where the subscript y indicates that this expression refers to the left face. The mass of fluid leaving the box W through the opposite face during the same time interval is approximately $(u_2)_{y+\Delta y} \Delta x \Delta z \Delta t$, where the subscript $y+\Delta y$ indicates that this expression refers to the right face.

The difference $\Delta u_2 \Delta x \Delta z \Delta t = \dfrac{\Delta u_2}{\Delta y} \Delta V \Delta t$ $\quad [\Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y]$

is the approximate loss of mass. Two similar expressions are obtained by considering the other two pairs of parallel faces of W. If we add these three expressions, we find that the total loss of mass in W during the time interval Δt is approximately

$$\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)\Delta V \,\Delta t$$

where $\quad \Delta u_1 = (u_1)_{x+\Delta x} - (u_1)_x \quad$ and

$$\Delta u_3 = (u_3)_{z+\Delta z} - (u_3)_z$$

This loss of mass in W is caused by the time rate of change of the density and is thus equal to $-\dfrac{\partial \rho}{\partial t} \Delta V \Delta t$

If we equate both expressions, divide the resulting equation by $\Delta V \Delta t$, and let $\Delta x, \Delta y, \Delta z$ and Δt approach zero, then we obtain $\quad \operatorname{div} \bar{u} = \operatorname{div}(\rho \bar{v}) = -\dfrac{\partial \rho}{\partial t} \quad$ or

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \bar{v}) = 0 \quad \ldots \quad *15$$

This important relation is called the condition for the conservation of mass or the **continuity equation** of a compressible fluid flow.

If the flow is steady, that is, independent of time, then $\frac{\partial \rho}{\partial t} = 0$ and the continuity equation is

$$\operatorname{div}(\rho \bar{v}) = 0 \quad \ldots \quad *16$$

If the density ρ is constant, so that the fluid is incompressible, then equation (*16) becomes

$$\operatorname{div} \bar{v} = 0 \quad \ldots \quad *17$$

Equation (*17) is known as the **condition of incompressibility**. It expresses the fact that the balance of outflow and inflow for a given volume element is zero at any time.

Clearly, the assumption that the flow has no sources or sinks in R is essential to our argument

~~the ended end~~

From this discussion you should conclude and remember that, roughly speaking, the **divergence** measures outflow minus inflow.

Curl of a Vector Field

Let x, y, z be right-handed Cartesian coordinates, and let $\bar{v}(x,y,z) = v_1 i + v_2 j + v_3 k$ be a differentiable vector function. Then the function

$$\operatorname{curl} \bar{v} = \nabla \times \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \ldots \quad *18$$

$$\Rightarrow \nabla \times \bar{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) k$$

is called the **curl** of the vector function $\bar{v}$ or the curl of the vector field defined by $\bar{v}$

Note: ① For a left-handed Cartesian coordinate system, the determinant in (*18) has a minus sign in front.

② Instead of $\operatorname{curl} \bar{v}$ the notation

$$\operatorname{rot} \bar{v}$$

(suggested by "rotation") is also used.

Example 1 (Curl of a vector function)

Let $\bar{v} = 4yz\, i + 3zx\, j - 2z\, k$, then by (*18)

$$\operatorname{curl} \bar{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4yz & 3zx & -2z \end{vmatrix}$$

$$= \left(\frac{\partial(-2z)}{\partial y} - \frac{\partial(3zx)}{\partial z}\right) i + \left(\frac{\partial(4yz)}{\partial z} - \frac{\partial(-2z)}{\partial x}\right) j + \left(\frac{\partial(3zx)}{\partial x} - \frac{\partial(4yz)}{\partial y}\right) k$$

$$= (0 - 3x)i + (4y - 0)j + (3z - 4z)k$$

$$\text{Curl } \bar{v} = -3xi + 4yj - zk$$

The curl plays an important role in many applications. A typical basic example is illustrated below

Example 2 (Rotation of a rigid body. Relation to the curl)

A rotation of a rigid body B about a fixed axis in space can be described by a vector $\bar{w}$ of magnitude ω in the direction of the axis of rotation, where $\omega (>0)$ is the angular speed of the rotation, and $\bar{w}$ is directed so that the rotation appears clockwise if we look in the direction of $\bar{w}$.

The velocity field of the rotation can be expressed in the form
$$\bar{v} = \bar{w} \times \bar{r}$$
where $\bar{r}$ is the position vector of a moving point with respect to a Cartesian coordinate system having the origin on the axis of rotation.

Let us choose right-handed Cartesian coordinates such that
$$\bar{w} = \omega k;$$
that is, the axis of rotation is the z-axis. Then
$$\bar{v} = \bar{w} \times \bar{r} = -\omega y i + \omega x j \quad , \text{where } \bar{r} = xi + yj + zk$$

and, therefore

$$\text{curl } \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial (wx)}{\partial z}\right) i + \left(\frac{\partial(-wy)}{\partial z} - \frac{\partial(0)}{\partial x}\right) j + \left(\frac{\partial(wx)}{\partial x} - \frac{\partial(-wy)}{\partial y}\right) k$$

$$= 0 + 0 + (\omega - (-\omega))k$$

$$\text{Curl } \bar{v} = 2\omega k$$

Hence, in the case of a rotation of a rigid body, the curl of the velocity field has the direction of the axis of rotation, and its magnitude equals twice the angular speed ω of the rotation.

Note: ① For any twice continuously differentiable scalar function f,

$$\text{Curl}(\text{grad } f) = 0 \quad \ldots \quad *19$$

Hence, if a vector function is the gradient of a scalar function, its curl is the zero vector. Since the curl characterizes the rotation in a field, we also say more briefly that gradient fields describing a motion are **irrotational**. (If such a field occurs in some other connection, not as a velocity field, it is usually called **conservative**)

② Other than equation (*19), another key formula for any twice continuously differentiable scalar function is

$$\text{div}(\text{curl } \bar{v}) = 0 \quad \ldots \quad *20$$

It is plausible because of the interpretation of the curl as a rotation and the divergence as a flux.

A proof of (*20) follows readily from the definitions of curl and div; the six terms cancel in pairs. That is

$$LHS \equiv \operatorname{div}(\operatorname{curl} \bar{v}) = \operatorname{div}\left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k\right]$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$$= \left(\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_3}{\partial y \partial x}\right) + \left(-\frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_2}{\partial z \partial x}\right) + \left(\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_1}{\partial z \partial y}\right)$$

$$= 0 + 0 + 0$$

$$= 0 \equiv R.H.S$$

$$\therefore \operatorname{div}(\operatorname{curl} \bar{v}) = \underline{\underline{0}}$$

VECTOR INTEGRAL CALCULUS

Line Integrals

Line integral is the natural generalization of a definite integral

$$\int_a^b f(x)\,dx \qquad \ldots \quad (**1)$$

In equation (**1) we integrate the integrand $f(x)$ from $x=a$ along the x-axis to $x=b$. In a line integral we shall integrate a given function, called the **integrand**, along a curve C in space (or in the plane). Hence curve integral would be a better term, but line integral is standard.

We represent the curve C by a parametric representation

$$\bar{r}(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k \quad (a \le t \le b) \quad \ldots (**2)$$

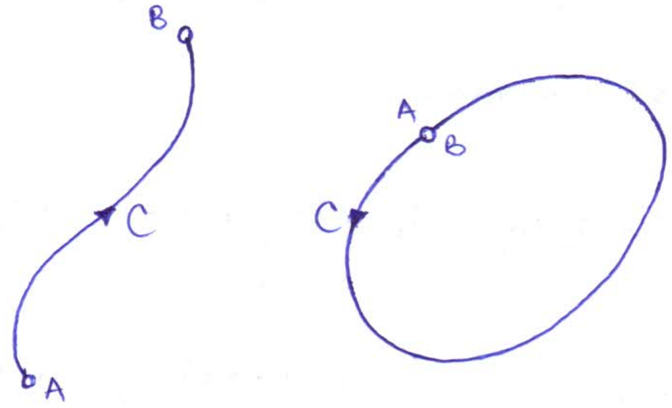


Fig. 1 : Oriented curve

C is called the **path of integration**, $A: \bar{r}(a)$ its initial point, and $B: \bar{r}(b)$ its terminal point. C is now oriented. The direction from A to B, in which t increases, is called the **positive direction** on C. The direction can be indicated by an arrow. When the points A and B coincide, then C is called a **closed path**.

C is called a **smooth curve** if C has a unique tangent at each of its points whose direction varies continuously as we move along C.

Technically: C has a representation (**2) such that $\bar{r}(t)$ is differentiable and the derivative $\bar{r}'(t) = \frac{d\bar{r}}{dt}$ is continuous and different from the zero vector at every point of C.

Definition and Evaluation of Line Integrals

A line integral of a vector function $F(\bar{r})$ over a curve C is defined by

$$\int_C F(\bar{r}) \cdot d\bar{r} = \int_a^b F(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\, dt \qquad \ldots \quad (**3)$$

In terms of components, with $d\bar{r} = [dx, dy, dz]$ and $' = \frac{d}{dt}$, eqtn (**3) becomes

$$\int_C F(\bar{r}) \cdot d\bar{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt \qquad \ldots \quad (**4)$$

If the path of integration C in (**3) is a closed curve, then instead of $\int_C$ we also write $\oint_C$

Note: (1) The integrand is a scalar, not a vector, because we take the dot product. Indeed, $F \cdot \frac{d\bar{r}}{dt}$ with $t = s$ (the arc length of C) is the tangential component of F.

(2) Line integrals such as (**3) arise naturally in mechanics, where they give the work done by a force F

in a displacement along C. We may thus call the line integral (**3) the **work integral**

③ We see that the integral in (**3) on the right is a definite integral of a function of t taken over the interval $a \le t \le b$ on the t-axis in the positive direction (the direction of increasing t). This definite integral exists for continuous F and piecewise smooth C, because this makes $F \cdot \bar{r}'$ piecewise continuous.

Other Forms of Line Integrals

The line integrals

$$\int_C F_1\, dx, \quad \int_C F_2\, dy, \quad \int_C F_3\, dz \qquad \ldots \quad (**5)$$

are special cases of (**3) when $\bar{F} = F_1 i$ or $F_2 j$ or $F_3 k$, respectively. Another form is

$$\int_C f(\bar{r})\, dt = \int_a^b f(\bar{r}(t))\, dt \qquad \ldots \quad (**6)$$

with C as in (**2). But this definition can also be regarded as a special case of (**3), with $F = F_1 i$ and $F_1 = f/\frac{dx}{dt}$, so that $f = F_1 x'$, as in (**4).

Example 1: Find the value of the line integral when $F(\bar{r}) = [-y, -xy] = -yi - xyj$ and C is the circular arc in fig 2 below from A to B.

④

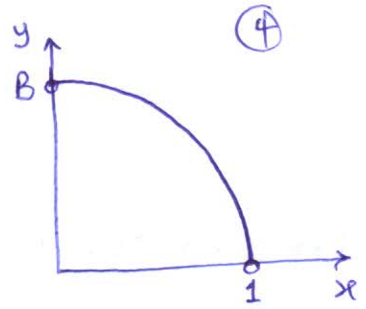


Fig. 2

Solution

We may represent C by $\bar{r}(t) = \cos t\, i + \sin t\, j \qquad (0 \le t \le \pi/2)$

thus $x(t) = \cos t$, $y(t) = \sin t$, so that

$$F(\bar{r}(t)) = -y(t)i - x(t)y(t)j = -\sin t\, i - \cos t \sin t\, j$$

By differentiation

$$r'(t) = -\sin t\, i + \cos t\, j \quad , \text{ so that}$$

using $\displaystyle\int_C F(\bar{r}) \cdot d\bar{r} = \int_0^{\pi/2} F(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\, dt$

$$= \int_0^{\pi/2} (-\sin t\, i - \cos t \sin t\, j) \cdot (-\sin t\, i + \cos t\, j)\, dt$$

$$= \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\, dt$$

$$= \int_0^{\pi/2} \sin^2 t\, dt - \int_0^{\pi/2} \cos^2 t \sin t\, dt$$

$$= \frac{\pi}{4} - \frac{1}{3} \simeq \underline{\underline{0.4521}}$$

where $\displaystyle\int_0^{\pi/2} \sin^2 t\, dt = \frac{1}{2}t - \frac{1}{4}\sin 2t \Big]_0^{\pi/2} = \frac{\pi}{4}$

$\displaystyle\int_0^{\pi/2} \cos^2 t \sin t\, dt$, let $u = \cos t \Rightarrow du = -\sin t\, dt \Rightarrow dt = \frac{du}{-\sin t}$

$$\Rightarrow \int_0^{\pi/2} u^2 \cancel{\sin t} \cdot \frac{du}{-\cancel{\sin t}} = -\int_0^{\pi/2} u^2\, du = \frac{u^3}{3}\Big]_0^{\pi/2} = \frac{(\cos t)^3}{3}\Big]_0^{\pi/2} = \frac{1}{3}$$

Example 2: (Line integral in space)

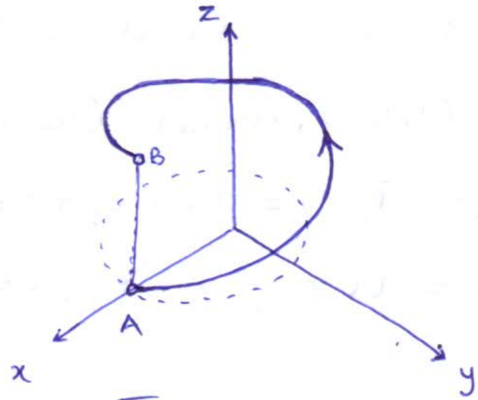


Fig. 3

Find the value of the line integral when $F(\bar{r}) = zi + xj + yk$ and C is the helix in fig 3 above

$$\bar{r}(t) = \cos t\, i + \sin t\, j + 3t\, k \qquad (0 \le t \le 2\pi)$$

Solution

We have $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 3t$. Thus

$$F(r(t)) \cdot r'(t) = (3t\, i + \cos t\, j + \sin t\, k) \cdot (-\sin t\, i + \cos t\, j + 3k)$$

$$= 3t(-\sin t) + \cos t(\cos t) + \sin t\,(3)$$

$$= 3t(-\sin t) + \cos^2 t + 3\sin t$$

Hence Using $\int_C F(\bar{r}) \cdot d\bar{r} = \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t)\,dt$

$$= \left[3t\cos t - 3\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t - 3\cos t \right]_0^{2\pi}$$

$$= 6\pi + \pi - 3 + 3 = 7\pi \simeq \underline{\underline{21.99}}$$

Example 3: (Dependence of a line integral on path (Same endpoints))

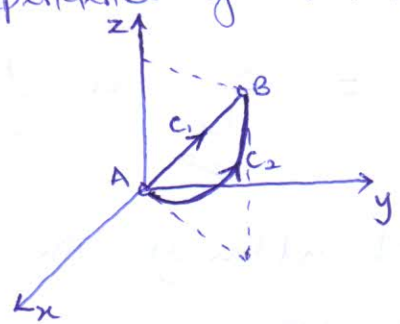


Fig 4

Evaluate the line integral with $F(\bar{r}) = 5zi + xyj + x^2zk$ along two different paths with the same initial point $A: (0,0,0)$ and the same terminal point $B: (1,1,1)$, namely (as in Fig 4)

a) C_1 : the straight-line segment $\bar{r}_1(t) = ti + tj + tk$, $0 \le t \le 1$, and

b) C_2 : the parabolic arc $\bar{r}_2(t) = ti + tj + t^2k$, $0 \le t \le 1$.

Solution

a) By substituting $\bar{r}_1$ into F we obtain

$$F(\bar{r}_1(t)) = 5ti + t^2j + t^3k \quad , \quad \bar{r}_1' = i + j + k$$

. Hence the integral over C_1 is

$$\int_{C_1} F(\bar{r})\cdot d\bar{r} = \int_0^1 F(\bar{r}_1(t))\cdot \bar{r}_1'(t)\, dt$$

$$= \int_0^1 (5ti + t^2j + t^3k)\cdot(i+j+k)\,dt$$

$$= \int_0^1 (5t + t^2 + t^3)\,dt = \left[\frac{5t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4}\right]_0^1$$

$$= \frac{5}{2} + \frac{1}{3} + \frac{1}{4} = \frac{37}{12}$$ //

b) Similarly, by substituting $\bar{r}_2$ into F and calculating $\bar{r}_2'$, we have

$$F(\bar{r}_2(t)) = 5t^2i + t^2j + t^4k \quad , \quad \bar{r}_2' = i + j + 2tk$$

$$\Rightarrow \int_{C_2} F(\bar{r})\cdot d\bar{r} = \int_0^1 F(\bar{r}_2(t))\cdot \bar{r}_2'(t)\,dt = \int_0^1 (5t^2i + t^2j + t^4k)\cdot(i + j + 2tk)$$

$$= \int_0^1 (5t^2 + t^2 + 2t^5)\,dt = \left[\frac{5t^3}{3} + \frac{t^3}{3} + \frac{2t^6}{6}\right]_0^1$$

$$= \frac{5}{3} + \frac{1}{3} + \frac{2}{6} = \frac{28}{12} \text{ or } \frac{7}{3}$$

The two results are different, although the endpoints are the same. This shows that the value of a line integral (**3) will in general depend not only on F and on the endpoints A, B of the path but also on the path along which we integrate from A to B.

Motivation of the Line Integral (**3): Work Done by a Force

The work W done a constant force F in the displacement along a straight segment $\bar{d}$ is $W = F \cdot \bar{d}$. This suggests that we define the work W done by a variable force F in the displacement along a curve $C: \bar{r}(t)$ as the limit of sums of works done in displacements along small chords of C. We shall now show that this definition amounts to defining W by the line integral (**3).

For this we choose points $t_0 (=a) < t_1 < \dots < t_n (=b)$. Then the work ΔW_m done by $F(\bar{r}(t_m))$ in the straight displacement from $\bar{r}(t_m)$ to $\bar{r}(t_{m+1})$ is

$$\Delta W_m = F(\bar{r}(t_m)) \cdot [\bar{r}(t_{m+1}) - \bar{r}(t_m)] \approx F(\bar{r}(t_m)) \cdot \bar{r}'(t_m)\Delta t_m ,$$

$(\Delta t_m = t_{m+1} - t_m)$.

The sum of these n works is $W_n = \Delta W_0 + \dots + \Delta W_{n-1}$. If we choose points and consider W_n for every n arbitrarily but so that the greatest Δt_m approaches zero as $n \to \infty$, then the limit of W_n as $n \to \infty$ is the line integral (**3). This integral exists because of our general assumption that F is continuous and C is piecewise smooth; this makes $\bar{r}'(t)$

continuous, except at finitely many points where C may have corners or cusps.

Example 4 (Work done equals the gain in kinetic energy)

Let F be a force, so that (**3) is work. Let t be time, so that $\frac{d\bar{r}}{dt} = \bar{v}$, velocity.

Eqtn (**3) is then written as

$$W = \int_C F \cdot d\bar{r} = \int_a^b F(\bar{r}(t)) \cdot \bar{v}(t)\, dt \quad \ldots \text{(i)}$$

By Newton's second law (force = mass × acceleration)

$$F = m\bar{r}''(t) = m\bar{v}'(t),$$

where m is the mass of the body displaced. Substituting into (i) gives

$$W = \int_a^b m\bar{v}' \cdot \bar{v}\, dt = \int_a^b m\left(\frac{\bar{v}\cdot\bar{v}}{2}\right)' dt = \frac{m}{2}|\bar{v}|^2 \Big|_{t=a}^{t=b}$$

On the right, $\frac{m}{2}|\bar{v}|^2$ is the kinetic energy. Hence the work done equals the gain in kinetic energy.

General Properties of the Line Integral (**3)

(1) $\int_C kF \cdot d\bar{r} = k\int_C F \cdot d\bar{r}$ (k constant) ... (**7)

(2) $\int_C (F + G) \cdot d\bar{r} = \int_C F \cdot d\bar{r} + \int_C G \cdot d\bar{r}$ ... (**8)

(3) $\int_C F \cdot dr = \int_{C_1} F \cdot d\bar{r} + \int_{C_2} F \cdot d\bar{r}$ ... (**9)

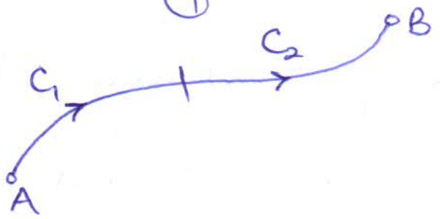


Fig 5

In eqtn (**9), the path C is subdivided into two arcs C_1 and C_2 that have the same orientation as C (fig 5). In (**8) the orientation of C is the same in all three integrals. If the sense of integration along C is reversed, the value of the integral is multiplied by -1 (i.e, $\int_B^A = -\int_A^B$)

THEOREM 1 (Direction-preserving transformations of parameter)
Any representations of C that gives the same positive direction on C also yield the same value of the line integral (**3)

Proof : We represent C in (**3) using another parameter t^* given by a function $t = \phi(t^*)$ that has a positive derivative and is such that $a^* \le t^* \le b^*$ corresponds to $a \le t \le b$. Then, writing $\bar{r}(\phi(t^*)) = \bar{r}^*(t^*)$ and using the chain rule, we have $dt^* = \left(\frac{dt^*}{dt}\right)dt$ and thus

$$\int_C F(r^*)\cdot d\bar{r}^* = \int_{a^*}^{b^*} \left[F(r^*(t^*)) \cdot \frac{d\bar{r}^*}{dt^*}\right] dt^*$$

$$= \int_{a^*}^{b^*} F(r(\phi(t^*))) \cdot \frac{d\bar{r}}{dt} \frac{dt}{dt^*}\, dt^*$$

$$= \int_a^b F(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\, dt = \int_C F(\bar{r})\cdot d\bar{r} .$$

Example 5: With regards to example (3a) above, if $t = 2t^* + 1$, evaluate the line integral. $\int_{C_1} F(\bar{r}) \cdot d\bar{r} = \int_0^1 (5t + t^2 + t^3)\,dt$

Solution

Given $t = 2t^* + 1 \Rightarrow t^* = \frac{t-1}{2}$

when $t = 0$, $t^* = \frac{0-1}{2} = -\frac{1}{2}$

$t = 1$, $t^* = \frac{1-1}{2} = 0$, Also,

$dt = 2dt^*$

$$\therefore \int_0^1 (5t + t^2 + t^3)\,dt = \int_{-\frac{1}{2}}^0 \left(5(2t^*+1) + (2t^*+1)^2 + (2t^*+1)^3\right) 2dt^*$$

$$= 2\int_{-\frac{1}{2}}^0 (8t^{*3} + 16t^{*2} + 20t^* + 7)\,dt$$

$$= \left[2\left(\frac{8t^{*4}}{4} + \frac{16t^{*3}}{3} + \frac{20t^{*2}}{2} + 7t^*\right)\right]_{-\frac{1}{2}}^0$$

$$= -2\left[\frac{1}{8} - \frac{2}{3} + \frac{5}{2} - \frac{7}{2}\right]$$

$= \frac{37}{12}$ This is the same result gotten in example (3a)

Line Integrals Independent of Path

$$\int_C F(\bar{r}) \cdot d\bar{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz) \quad \cdots \quad (**10)$$

The line integral (**10) is defined to be independent of path in a domain D in space if for every pair of endpoints A, B in D the integral (**10) has the same value for all paths in D that begin at A and end at B.

THEOREM 2: (Independence of path)

A line integral (**10) with continuous F_1, F_2, F_3 in a domain D in space is independent of path in D if and only if $F = [F_1, F_2, F_3]$ is the gradient of some function f in D,

$$F = \text{grad } f \quad , \quad \cdots \quad (a)$$

in components

$$F_1 = \frac{\partial f}{\partial x} \quad , \quad F_2 = \frac{\partial f}{\partial y} \quad , \quad F_3 = \frac{\partial f}{\partial z} \quad \cdots \quad (b)$$

<u>Proof</u>

Let (a) hold for some function f in D. Let C be any path in D from any point A to any point B, given by

$$\bar{r}(t) = x(t)i + y(t)j + z(t)k \quad , \quad a \le t \le b$$

Then from eqtn (b), the chain rule

$$\int_A^B (F_1\,dx + F_2\,dy + F_3\,dz) = \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt$$

$$= \int_a^b \frac{df}{dt}\,dt = f[x(t), y(t), z(t)]\Big|_{t=a}^{t=b}$$

$$= f(B) - f(A).$$

This shows that the value of the integral is simply the difference of the values of f at the two endpoints of C and is, therefore, independent of the path C.

Note: $$\int_A^B (F_1\,dx + F_2\,dy + F_3\,dz) = f(B) - f(A), \quad (F = \text{grad } f) \cdots (**11)$$

Eqtn (**11) is the analog of the usual formula for definite integrals in calculus. $\int_a^b g(x)\,dx = G(x)\Big|_a^b = G(b) - G(a)$, $[G'(x) = g(x)]$

formula (**11) should be applied whenever a line integral is independent of path.

Example 6: (Independence of path)

Show that the integral $\int_C F \cdot dr = \int_C (2x\,dx + 2y\,dy + 4z\,dz)$ is independent of path in any domain in space and find its value if C has the initial point $A: (0,0,0)$ and terminal point $B: (2,2,2)$

Solution

By inspection we find that

$$F = [2x, 2y, 4z] = 2xi + 2yj + 4zk = \text{grad } f, \text{ where } f = x^2 + y^2 + 2z^2.$$

Theorem 2 now implies independence of path. To find the value of the integral, we can choose the convenient straight path

$C: \bar{r}(t) = t(i+j+k) \quad , \quad 0 \le t \le 2$

we get $\bar{r}'(t) = i+j+k$; thus $F \cdot \bar{r}' = 2t + 2t + 4t = 8t$

$$\int_C (2x\,dx + 2y\,dy + 4z\,dz) = \int_0^2 F \cdot \bar{r}'\,dt = \int_0^2 8t\,dt = \underline{\underline{16}}$$

Example 7 : (Independence of path. Determination of a potential)

Evaluate the integral $I = \int_C (3x^2\,dx + 2yz\,dy + y^2\,dz)$ from $A: (0,1,2)$ to $B: (1,-1,7)$ by showing that F has a potential and applying (** 11).

<u>Solution</u>

If F has a potential f, we should have

$f_x = F_1 = 3x^2 \quad , \quad f_y = F_2 = 2yz \quad , \quad f_z = F_3 = y^2$

We show that we can satisfy these conditions. By integration and differentiation,

$f = \frac{3x^3}{3} = x^3 + g(y,z) \quad , \quad f_y = g_y = 2yz \quad , \quad g = \int g_y = y^2 z + h(z),$

$f_z = y^2 + h' = y^2, \quad h' = 0 \quad , \quad h = 0$, say.

This gives $f(x,y,z) = x^3 + y^2 z$ and by (** 11)

$$I = f(1,-1,7) - f(0,1,2)$$
$$= \left[(1)^3 + (-1)^2(7)\right] - \left[(0)^3 + (1)^2(2)\right]$$
$$= 1 + 7 - (0+2) = \underline{\underline{6}}$$

Integration Around Closed Curves and Independence of Path

A single closed curve is made up of two paths with common endpoints.

THEOREM 3 (Independence of path)

The integral (**10) is independent of path in a domain D if and only if its value around every closed path in D is zero.

Proof

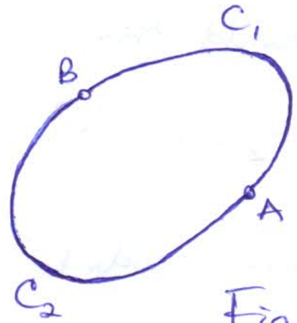


Fig 5. Proof of Theorem 3

If we have independence of path, integration from A to B along C_1 and along C_2 in Fig. 5 gives the same value. Now C_1 and C_2 together make up a closed curve C, and if we integrate from A along C_1 to B as before, but then in the opposite sense along C_2 back to A (so that this integral is multiplied by -1), the sum of the two integrals is zero, but this is the integral around the closed curve C.

Conversely, assume that the integral around any closed path C in D is zero. Given any points A and B and any two curves C_1 and C_2 from A to B in D, we see that C_1 with the orientation reversed and C_2 together form a closed path C. By assumption, the integral over C is zero. Hence the integrals over C_1 and C_2, both taken from A to B, must be equal. This proves the theorem.

Exactness and Independence of Path

Theorem 2 relates path independence of the line integral (**10) to the gradient and Theorem 3 to integration around closed curves. A third idea and theorem (Theorem 4, below) relate path independence to the exactness of the differential form

$$F_1\,dx + F_2\,dy + F_3\,dz \qquad \ldots \qquad (**12)$$

under the integral sign in eqn (**10). This form (**12) is called **exact** in a domain D in space if it is the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

of a differentiable function $f(x,y,z)$ everywhere in D, that is, if we have $F_1\,dx + F_2\,dy + F_3\,dz = df$.

Comparing these two formulas, we see that the form (**12) is exact if and only if there is a differentiable function $f(x,y,z)$ in D such that everywhere in D,

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z} \qquad \ldots \quad (i)$$

In vectorial form these three equations (i) can be written

$$F = \operatorname{grad} f \qquad \ldots \qquad (**13)$$

Hence, by theorem 2, the integral (**10) is independent of path in D if and only if the differential form (**12) has continuous components F_1, F_2, F_3 and is exact in D.

A domain D is called **simply connected** if every closed curve in D can be continuously shrunk to any point in D without leaving D.

For example, the interior of a sphere or a cube, the interior of a sphere with finitely many points removed, and the domain between two concentric spheres are simply connected, while the interior of a torus (a doughnut) and the interior of a cube with one space diagonal removed are not simply connected.

THEOREM 4 (Criterion for exactness and independence of path)

Let F_1, F_2, F_3 in the line integral (**10),

$$\int_C F(\vec{r})\cdot d\vec{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

be continous and have continuous first partial derivatives in a domain D in space. Then:

ⓐ If (**10) is independent of path in D - and thus the differential form (**12) under the integral sign is exact - then in D,

$$\operatorname{Curl} F = 0 \quad ; \quad \cdots \quad (**\,14)$$

in components $\dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z}, \quad \dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}, \quad \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y} \quad \cdots \quad (**\,15)$

ⓑ If (**15) holds in D and D is simply connected, then (**10) is independent of path in D.

Proof

(a) If (** 10) is independent of path in D, then $F = \text{grad} f$ by (** 11) and $\text{curl } F = \text{curl}(\text{grad} f) = 0$, so that (** 14) holds.

(b) We claim that the converse is also true, that is, if (** 14) holds, then (** 10) is independent of path in D, provided D is simply connected.

The proof needs Stokes's theorem and can now be given as follows. Let C be any simple closed path in D. Since D is simply connected, we can find a surface S in D bounded by C. Stokes's theorem is applicable and gives

$$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \oint_C F \cdot \bar{r}'\,ds = \iint_S (\text{curl} F) \cdot \bar{n}\,dA$$

for proper direction on C and normal vector $\bar{n}$ on S. Now, regardless of the choice of C, the integral over S is zero because, by (** 14), its integrand is identically zero. From this and theorem 3 it follows that the integral (** 10) is independent of path in D. This completes the proof.

Note: For a line integral in the plane

$$\int_C F(\bar{r}) \cdot d\bar{r} = \int_C (F_1\,dx + F_2\,dy),$$

Curl F has just one component and (** 15) reduces to the single relation

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \quad \ldots \quad (** 16$$

Example 8 (Exactness and independence of path. Determination of a potential)

Using (** 15), show that the differential form under the integral sign of

$$I = \int_C \left[2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz\right]$$ is

exact, so that we have independence of path in any domain, and find the value of I from A: (0, 0, 1) to B: (1, π/4, 2).

Solution

We have $F_1 = 2xyz^2$, $F_2 = x^2z^2 + z\cos yz$

$F_3 = 2x^2yz + y\cos yz$

$(F_3)_y = 2x^2z + \cos yz - yz\sin yz = (F_2)_z$

$(F_1)_z = 4xyz = (F_3)_x$

$(F_2)_x = 2xz^2 = (F_1)_y$

To find f, we integrate F_2 (which is "long," so that we save work) and then differentiate to compare with F_1 and F_3

$$f = \int F_2\,dy = \int (x^2z^2 + z\cos yz)\,dy = x^2z^2y + \sin yz + g(x,z)$$

$$f_x = 2xz^2y + g_x = F_1 = 2xyz^2, \quad g_x = 0, \quad g = h(z)$$

$$f_z = 2x^2zy + y\cos yz + h' = F_3 = 2x^2zy + y\cos yz, \quad h' = 0$$

So that, taking $h = 0$, we have

$$f(x,y,z) = x^2yz^2 + \sin yz.$$

From this and (**11) we get

$$I = f(B) - f(A) = f(1, \pi/4, 2) - f(0,0,1)$$

$$= \left[(1)(\pi/4)(2)^2 + \sin \tfrac{\pi}{4}\cdot 2\right] - \left[(0)^2(0)(1)^2 + \sin(0)(1)\right]$$

$$= \pi + \sin \tfrac{1}{2}\pi - 0$$

$$= \underline{\underline{\pi + 1}}$$

Example 9 (On the assumption of simple connectedness in Theorem 4)

Let $F_1 = -\dfrac{y}{x^2+y^2}$, $F_2 = \dfrac{x}{x^2+y^2}$, $F_3 = 0$.

Differentiation shows that (**15) is satisfied in any domain of the xy-plane not containing the origin, for example, in the domain $D: \frac{1}{2} < \sqrt{x^2+y^2} < \frac{3}{2}$ shown in Fig 6 below

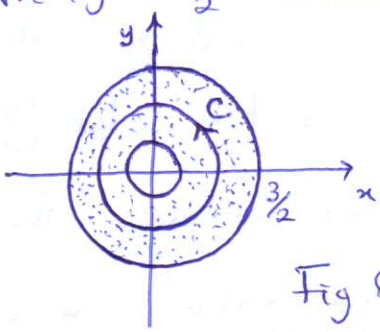


Fig 6

Indeed, F_1 and F_2 do not depend on z, and $F_3 = 0$, so that the first two relations of (**15) are trivially true, and the third is verified by differentiation:

$$\frac{\partial F_2}{\partial x} = \frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial F_1}{\partial y} = \frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

Clearly, D in Fig 6 is not simply connected. If the integral

$$I = \int_C (F_1\,dx + F_2\,dy) = \int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

were independent of path in D, then $I = 0$ on any closed curve in D, for example, on the circle $x^2+y^2=1$.

But setting $x = r\cos\theta$, $y = r\sin\theta$, and noting that the circle is represented by $r=1$, we have

$$x = \cos\theta, \quad dx = -\sin\theta\,d\theta, \quad y = \sin\theta, \quad dy = \cos\theta\,d\theta,$$

so that $-y\,dx + x\,dy = \sin^2\theta\,d\theta + \cos^2\theta\,d\theta = (\sin^2\theta + \cos^2\theta)\,d\theta = d\theta$

and counterclockwise integration gives

$$I = \int_0^{2\pi} \frac{d\theta}{1} = \underline{\underline{2\pi}}$$

Since D is not simply connected, we cannot apply Theorem 4 and conclude that I is independent of path in D.

Although $F = \operatorname{grad} f$, where $f = \arctan\left(\frac{y}{x}\right)$ (verify!), we cannot apply Theorem 2 either because the polar angle $\theta = \arctan\left(\frac{y}{x}\right)$ is not single-valued, as it is required for a function in calculus.

DOUBLE INTEGRAL

The double integral of $f(x,y)$ over the region R is denoted by

$$\iint_R f(x,y)\,dx\,dy \quad \text{or} \quad \iint_R f(x,y)\,dA \quad \cdots \quad (1^*)$$

Properties of Double Integral

For any functions f and g of (x,y), defined and continuous in a region R, we have

1. $$\iint_R k f(x,y)\,dx\,dy = k\iint_R f(x,y)\,dx\,dy$$

2. $$\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$$

3. $$\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy \quad \text{, where } R = R_1 + R_2$$

4. Mean value theorem for double integrals: there exists at least one point (x_0, y_0) in R such that we have

$$\iint_R f(x,y)\,dx\,dy = f(x_0,y_0)A,$$

where A is the area of R.

Evaluation of Double Integrals

Double integrals over a region R may be evaluated by two successive integrations as follows. Suppose that R can be described by inequalities of the form

$a \le x \le b \quad , \; g(x) \le y \le h(x)$ as in the figure below

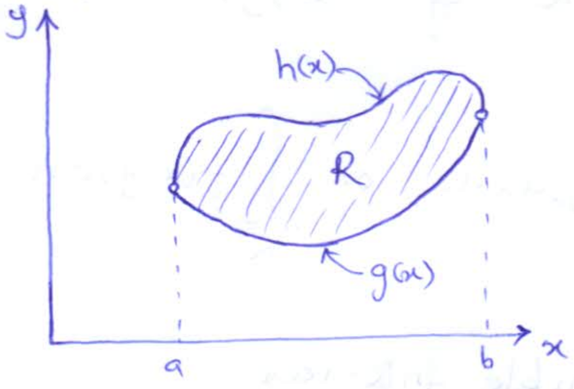


Fig 1

So that $y = g(x)$ and $y = h(x)$ represent the boundary of R.

Then $$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx \qquad \cdots \quad (2^*)$$

We first integrate the inner integral

$$\int_{g(x)}^{h(x)} f(x,y)\,dy$$

In this integration we keep x fixed, that is, we regard x as a constant. The result of this integration will be a function of x, say, $F(x)$. Integrating $F(x)$ over x from a to b, we then obtain the value of the double integral in (2^*).

Similarly, if R can be described by inequalities of the form $c \le y \le d \quad , \; p(y) \le x \le q(y)$ as in the figure below

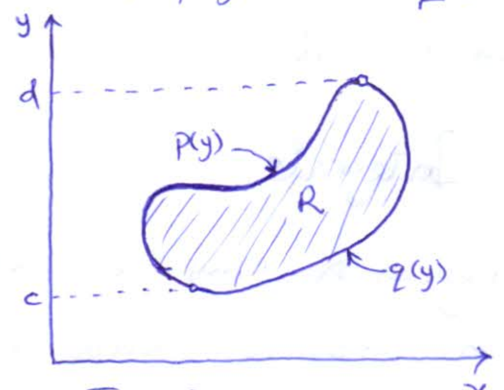


Fig. 2

we obtain

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx \right] dy \qquad \cdots (3^*)$$

we now integrate first over x (treating y as a constant) and then the resulting function of y from c to d.

Applications of Double Integrals

1. The area A of a region R in the xy-plane is given by

$$A = \iint_R dx\,dy$$

2. The volume V beneath the surface $z = f(x,y)$ (>0) and above a region R in the xy-plane is

$$V = \iint_R f(x,y)\,dx\,dy$$

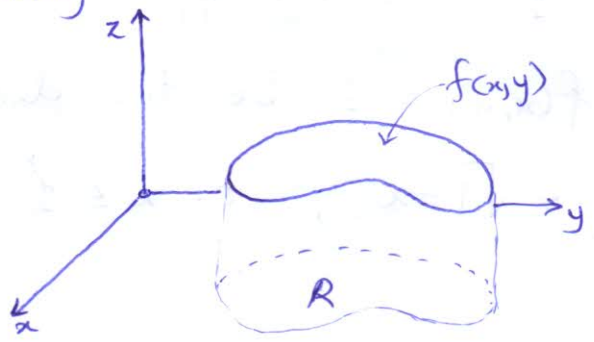


Fig. 3

3. Let $f(x,y)$ be the density (= mass per unit area) of a distribution of mass in the xy-plane. Then the total mass M in R is $\quad M = \iint_R f(x,y)\,dx\,dy$;

4. The center of gravity of the mass in R has the coordinates $\bar{x}$, $\bar{y}$, where

$$\bar{x} = \frac{1}{M}\iint_R x f(x,y)\,dx\,dy \quad \text{and} \quad \bar{y} = \frac{1}{M}\iint_R y f(x,y)\,dx\,dy$$

the moments of inertia I_x and I_y of the mass in R about the x- and y-axes, respectively, are

$$I_x = \iint_R y^2 f(x,y)\,dx\,dy \quad , \quad I_y = \iint_R x^2 f(x,y)\,dx\,dy \ ;$$

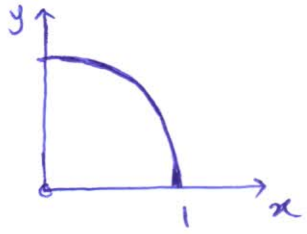
and the polar moment of inertia I_o about the origin of the mass in R is

$$I_o = I_x + I_y = \iint_R (x^2+y^2) f(x,y)\,dx\,dy .$$

Example 1 : Center of gravity. Moments of inertia

Let $f(x,y) = 1$ be the density of mass in the region

$R: \ 0 \le y \le \sqrt{1-x^2} \ , \quad 0 \le x \le 1$



Find the center of gravity and the moments of inertia, I_x, I_y and I_o

Solution

The total mass in R is obtained as the double integral

$$M = \iint_R 1\,dx\,dy = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} dy\right] dx = \int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2\theta\,d\theta = \pi/4$$

$(x = \sin\theta)$, which is the area of R.

The coordinates of the center of gravity are

$$\bar{x} = \frac{1}{\pi/4}\iint_R x \cdot 1\, dx\,dy = \frac{4}{\pi}\int_0^1\left[\int_0^{\sqrt{1-x^2}} x\,dy\right]dx = \frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx$$

$$= -\frac{4}{\pi}\int_0^1 z^2\,dz = \frac{4}{3\pi}$$

$(\sqrt{1-x^2} = z)$, and $\bar{y} = \bar{x}$, for reasons of symmetry. Furthermore,

$$I_x = \iint_R y^2\,dx\,dy = \int_0^1\left[\int_0^{\sqrt{1-x^2}} y^2\,dy\right]dx = \frac{1}{3}\int_0^1 \left(\sqrt{1-x^2}\right)^3 dx$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{\pi}{16},$$

$$I_y = \frac{\pi}{16}$$

$$I_0 = I_x + I_y = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8} \simeq \underline{0.3927}$$

Change of Variables in Double Integrals. Jacobian

Practical problems often require a change of the variables of integration in double integrals.

From the definite integral the formula for the change from x to u is

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(x(u))\,\frac{dx}{du}\,du$$

where we assume that $x = x(u)$ is continuous and has a continuous derivative in some interval $\alpha \le u \le \beta$ such that $x(\alpha) = a$, $x(\beta) = b$ [or $x(\alpha) = b$, $x(\beta) = a$] and $x(u)$ varies between a and b when u varies between α and β.

The formula for a change of variables in double integrals from x, y to u, v is

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \quad \cdots \quad (4^*)$$

that is, the integrand is expressed in terms of u and v, and $dx\,dy$ is replaced by $du\,dv$ times the absolute value of the **Jacobian**

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \cdots \quad (5^*)$$

Here we assume that the functions $x = x(u,v)$, $y = y(u,v)$ affecting the change are continuous and have continuous partial derivatives in some region R^* in the uv-plane such that for every (u,v) in R^* the corresponding point (x,y) lies in R and vice versa.

Also, polar coordinates r and θ, can be introduced by setting $x = r\cos\theta$, $y = r\sin\theta$. Then

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

and

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \quad \cdots \quad (6^*)$$

where R^* is the region in the $r\theta$-plane corresponding to R in the xy-plane.

Example 2 : Double integral in polar coordinates

Using eqtn (6^*), obtain I_x in example 1 above

7

$$I_x = \iint_R y^2\,dx\,dy \quad , \text{using } x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\Rightarrow y^2 = r^2\sin^2\theta \quad , \quad dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$

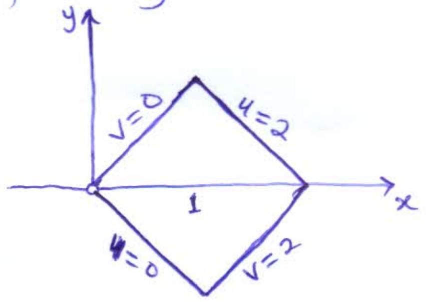
$$\Rightarrow I_x = \int_0^{\pi/2}\int_0^1 r^2\sin^2\theta\; r\,dr\,d\theta = \int_0^{\pi/2}\sin^2\theta\,d\theta \cdot \int_0^1 r^3\,dr$$

$$= \frac{\pi}{4}\cdot\frac{1}{4} = \frac{\pi}{16}$$

Example 3 Change of variables in double integral

Evaluate the following double integral over the square R in the figure below

$$\iint_R (x^2+y^2)\,dx\,dy$$



Solution

The shape of R suggests the transformation $x+y=u$, $x-y=v$.

then $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$, the Jacobian is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

R corresponds to the square $0 \le u \le 2$, $0 \le v \le 2$, and, therefore

$$\iint_R (x^2+y^2)\,dx\,dy = \int_0^2\int_0^2 \frac{1}{2}(u^2+v^2)\,\frac{1}{2}\,du\,dv$$

$$= \frac{1}{4}\int_0^2\left[\int_0^2 (u^2+v^2)\,du\right]dv = \frac{1}{4}\int_0^2\left[\frac{u^3}{3}\Big|_0^2 + uv^2\Big|_0^2\right]dv$$

$$= \frac{1}{4}\int_0^2 \left(\frac{8}{3} + 2v^2\right) dv = \frac{1}{4}\left[\frac{8v}{3}\Big|_0^2 + \frac{2v^3}{3}\Big|_0^2\right]$$

$$= \frac{1}{4}\left[\frac{8(2)}{3} + \frac{2(8)}{3}\right] = \frac{1}{\cancel{4}}\left[\cancel{4}\left(\frac{8}{3}\right)\right]$$

$$= \frac{8}{3}$$

GREEN'S THEOREM IN THE PLANE

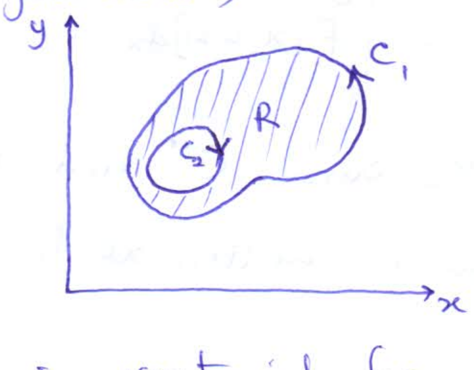
The Green's theorem helps to transform a double integral over a plane region into a line integral over the boundary of the region and conversely.

THEOREM 1 : Green's theorem in the plane
(Transformation between double integrals and line integrals)

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy) \quad \ldots \quad (7^*)$$

here we integrate along the entire boundary C of R such that R is on the left as we advance in the direction of integration (as in figure below)



Equation (7^*) in vectorial form is written as

$$\iint_R (\text{curl}\, F) \cdot \bar{k}\, dx\,dy = \oint_C F \cdot d\bar{r} \quad \ldots \quad (8^*)$$

$$(F = F_1 i + F_2 j)$$

Proof

We first prove Green's theorem for a special region R that can be represented in both the forms

$$a \le x \le b \ , \ u(x) \le y \le v(x) \quad \text{and} \quad c \le y \le d \ , \ p(y) \le x \le q(y)$$

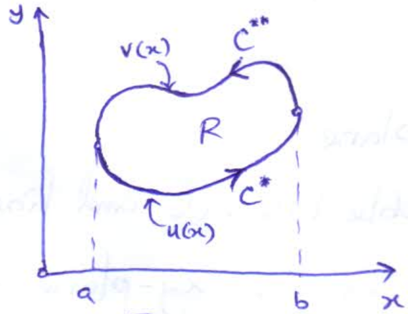


Fig (a)

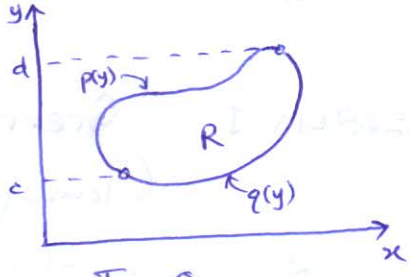


Fig (b)

Using (2^*) we obtain ~~the~~ for the second term on the left side of (7^*) (without the minus sign)

$$\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y}\,dy \right] dx \quad \dots \quad (i)$$

$$\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y}\,dy = F_1(x,y)\Big|_{y=u(x)}^{y=v(x)} = F_1[x, v(x)] - F_1[x, u(x)]$$

inserting this into (i) we find

$$\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = \int_a^b F_1[x, v(x)]\,dx - \int_a^b F_1[x, u(x)]\,dx$$

$$= -\int_b^a F_1[x, v(x)]\,dx - \int_a^b F_1[x, u(x)]\,dx$$

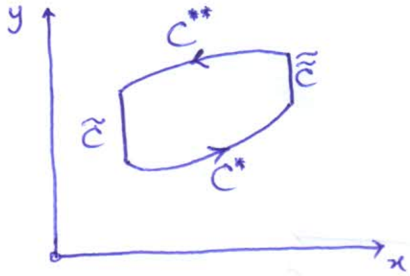
Since $y = v(x)$ represents the curve C^{**} and $y = u(x)$ represents C^*, the last two integrals may be written as line integrals over C^{**} and C^*; therefore

$$\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = -\int_{C^{**}} F_1(x,y)\,dx - \int_{C^*} F_1(x,y)\,dx \quad \dots \quad (ii)$$

$$= -\oint_C F_1(x,y)\,dx$$

This proves (7^*) in Green's theorem if $F_2 = 0$.

The result remains valid if C has portions parallel to the y-axis (such as $\tilde{C}$ and $\tilde{\tilde{C}}$ in figure below).



Indeed, the integrals over these portions are zero because in (ii) on the right we integrate with respect to x. Hence we may add these integrals to the integrals over C^* and C^{**} to obtain the integral over the whole boundary C in (ii).

Similarly, using (3^*), we obtain for the first term in (7^*) on the left by means of the second representation of the special region (Fig 6).

$$\iint_R \frac{\partial F_2}{\partial x}\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} \frac{\partial F_2}{\partial x}\,dx\right] dy$$

$$= \int_c^d F_2(q(y), y)\,dy + \int_d^c F_2(p(y), y)\,dy$$

$$= \oint_C F_2(x,y)\,dy. \qquad \ldots \text{(iii)}$$

Together with (ii) this gives (7^*), that is,

(ii) $\Rightarrow$
$$-\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = \oint_C F_1(x,y)\,dx \qquad \ldots \text{(iv)}$$

(iv) + (iii)
$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy)$$

this proves Green's theorem for special regions.

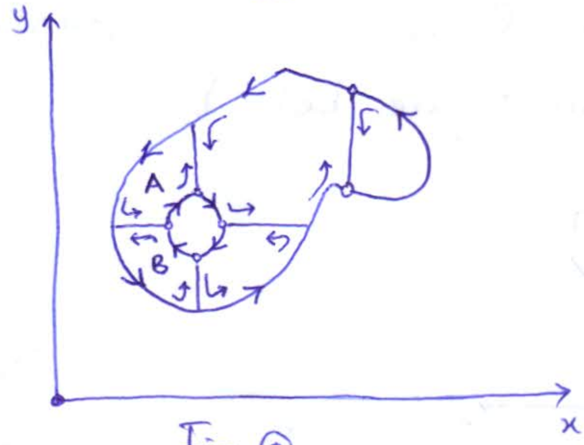


Fig. ©

We now prove the theorem for a region R that itself is not a special region but can be subdivided into finitely many special regions (fig ©). In this case we apply the theorem to each subregion and then add the results; the left-hand members add up to the integral over R while the right-hand members add up to the line integral over C plus integrals over the curves introduced for subdividing R. Each of the latter integrals occurs twice, taken once in each direction. Hence these two integrals cancel each other (i.e, integral over A and B in fig © above), and we are left with the line integral over C.

The proof thus far covers all regions that are of interest in practical problems. To prove the theorem for the most general region R satisfying the conditions in the theorem, we must approximate R by a region of the type just considered and then use a limiting process.

Example 4 : Verification of Green's theorem in the plane

Given $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2+y^2=1$. Show that the Green's theorem holds.

Solution

$$\frac{\partial F_1}{\partial y} = 2y - 7, \quad \frac{\partial F_2}{\partial x} = 2y + 2$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R \left[(2y+2) - (2y-7)\right] dx\,dy = \iint_R (2y+2-2y+7)\,dx\,dy$$

$$= 9\iint_R dx\,dy = 9\pi$$

Since the circular disk R has area π $(\pi r^2 = \pi\cdot(1)^2 = \pi)$.

On the right in (7^*) we represent C (oriented counterclockwise!) by $\bar{r}(t) = [\cos t, \sin t]$ Then $\bar{r}'(t) = [-\sin t, \cos t]$

On C we thus obtain

$$F_1 = \sin^2 t - 7\sin t, \quad F_2 = 2\cos t\sin t + 2\cos t$$

Hence the integral in (7^*) on the right becomes

$$\oint_C (F_1 x' + F_2 y')\,dt = \int_0^{2\pi} \left[(\sin^2 t - 7\sin t)(-\sin t) + 2(\cos t\sin t + \cos t)(\cos t)\right] dt$$

$$= 0 + 7\pi + 0 + 2\pi = 9\pi$$

$$\therefore \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = 9\pi = \oint_C (F_1\,dx + F_2\,dy)$$ which verifies the Green's theorem in the plane.

Example 5 : Area of a plane region as a line integral over the boundary

In (7^*) we first choose $F_1 = 0$, $F_2 = x$ and then $F_1 = -y$, $F_2 = 0$.

This gives $\iint_R dx\,dy = \oint_C x\,dy$ and $\iint_R dx\,dy = -\oint_C y\,dx$, respectively.

The double integral is the area A of R. By addition we have

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx) \quad \ldots \quad (9^*)$$

where we integrate as indicated in Green's theorem. This interesting formula expresses the area of R in terms of a line integral over the boundary. It has various applications; for instance, the theory of certain **planimeters** (instruments for measuring area) is based on it.

For an **ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $x = a\cos t$, $y = b\sin t$ we get $x' = -a\sin t$, $y' = b\cos t$; thus from (9^*) we obtain the familiar result

$$A = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\,dt = \frac{1}{2}\int_0^{2\pi}\left[ab\cos^2 t - (-ab\sin^2 t)\right]dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,(\cos^2 t + \sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}\cdot abt\Big|_0^{2\pi}$$

$$= \frac{1}{\cancel{2}}\cdot ab\cdot \cancel{2}\pi = \pi ab.$$

Example 6: Area of a plane region in polar coordinates

Let r and θ be polar coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$. Then

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta$$
$$dy = \sin\theta\,dr + r\cos\theta\,d\theta$$

and (9^*) becomes a formula that is well known from calculus, namely,

$$A = \frac{1}{2}\oint_C r^2\,d\theta \quad \ldots \quad (10^*)$$

As an application of (10^*), we consider the **cardioid** $r = a(1-\cos\theta)$, where $0 \le \theta \le 2\pi$. We find

$$A = \frac{a^2}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}a^2$$

Surfaces for Surface Integrals

Representations of Surfaces

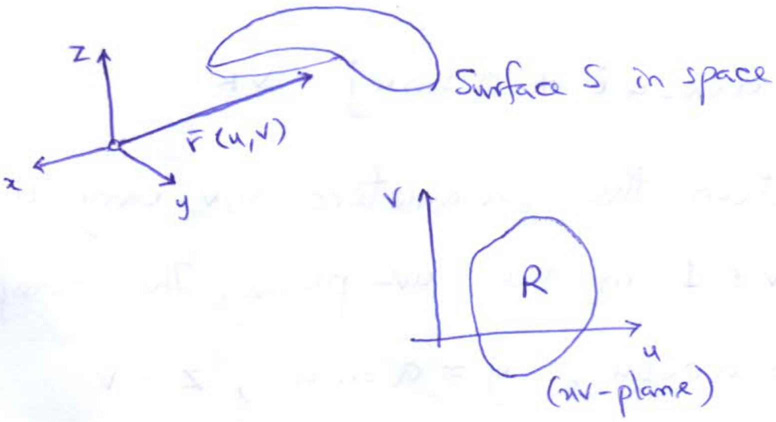


Fig 4. Parametric representations of a surface

For surfaces S in surface integrals, it is often more practical to use a parametric representation. Surfaces are two-dimensional. Hence we need two parameters, which we call u and v. Thus a parametric representation of a surface S in space is of the form

$$\bar{r}(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k \quad , (u,v) \text{ in } R \quad \cdots (11^*)$$

where R is some region in the uv-plane. This mapping (11^*) maps every point (u,v) in R onto the point of S with position vector $\bar{r}(u,v)$

Example 7 : Parametric representation of a cylinder

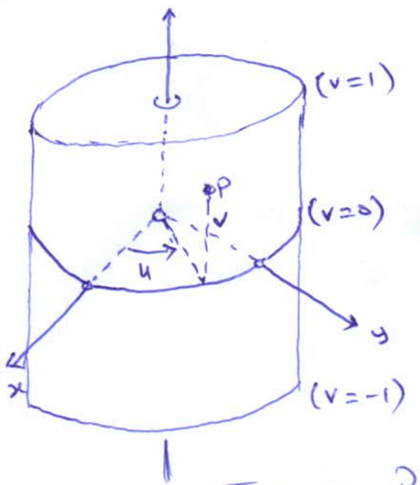


Fig 5. Parametric representation of a cylinder

The circular cylinder ~~$x^3+y^3=a^2$~~ $x^2+y^2=a^2$, $-1 \le z \le 1$, has radius a, height 2, and the z-axis as axis. A parametric representation is

$$\bar{r}(u,v) = a\cos u\, i + a\sin u\, j + vk$$

In this representation the parameters u,v vary in the rectangle. $R: 0 \le u \le 2\pi$, $-1 \le v \le 1$ in the uv-plane. The components of $\bar{r}(u,v)$ are $x = a\cos u$, $y = a\sin u$, $z = v$

The curves v = constant are parallel circles. The curves u = constant are vertical straight lines. The point P in fig 5 corresponds to $u = \frac{\pi}{3} = 60^\circ$, $v = 0.7$.

Note: ① For a sphere $x^2+y^2+z^2=a^2$, its parametric representation is of the form

$$\bar{r}(u,v) = a\cos v\cos u\, i + a\cos v\sin u\, j + a\sin v\, k \quad \ldots (i)$$

where the parameters u, v vary in the rectangle R in the uv-plane given by the inequalities $0 \le u \le 2\pi$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$. The curves u = constant and v = constant are the "meridians and parallels" on S. This representation is used in geography for measuring the latitude and longitude of points on the globe.

Another parametric representation of the sphere also used in mathematics is

$$\bar{r}(u,v) = a\cos u\sin v\, i + a\sin u\sin v\, j + a\cos v\, k \quad \ldots (ii)$$

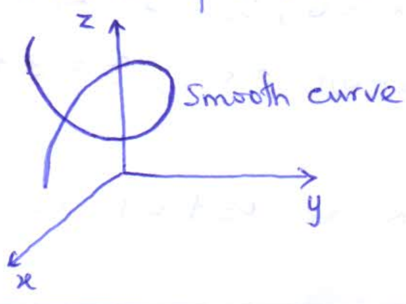
where $0 \le u \le 2\pi$, $0 \le v \le \pi$

② For a circular cone $z = \sqrt{x^2+y^2}$, $0 \le z \le H$, its parametric representation is of the form

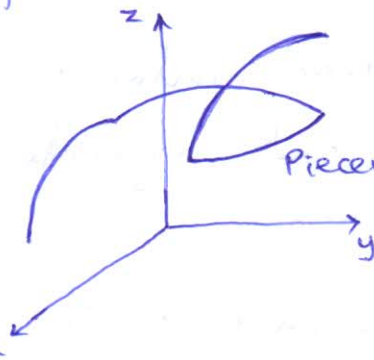
$$\bar{r}(u,v) = u\cos v\, i + u\sin v\, j + u k \quad \ldots (iii)$$

where u, v vary in the rectangle $R: 0 \le u \le H$, $0 \le v \le 2\pi$.

③ A vector-valued function $\bar{r}$ defined on an interval I is **smooth** if $\bar{r}$ has a continuous derivative on I and $\bar{r}'(t) \ne \bar{0}$ for each interior point t. A curve C is **smooth** if it has a smooth parametrization



④ A continuous vector-valued function $\bar{r}$ defined on an interval I is **piecewise smooth** if I is composed of a finite number of subintervals on each of which $\bar{r}$ is smooth and if $\bar{r}$ has one-sided derivatives at each interior point of I. A curve C is **piecewise smooth** if it has a piecewise smooth parametrization



Piecewise smooth curve

⑤ Suppose (x_0, y_0, z_0) and (x_1, y_1, z_1) are distinct points in space, and consider the parametric equations

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad z = z_0 + (z_1 - z_0)t \quad \cdots (iv)$$

Since $(x, y, z) = (x_0, y_0, z_0)$ at $t = 0$ and $(x, y, z) = (x_1, y_1, z_1)$ for $t = 1$, it follows that eqtn (iv) gives parametric equations for the line through (x_0, y_0, z_0) and (x_1, y_1, z_1). Moreover,

$$\bar{r}(t) = [x_0 + (x_1 - x_0)t]\mathbf{i} + [y_0 + (y_1 - y_0)t]\mathbf{j} + [z_0 + (z_1 - z_0)t]\mathbf{k} \quad \text{for } 0 \le t \le 1$$

is a smooth parametrization of the line segment from (x_0, y_0, z_0) to (x_1, y_1, z_1).

Example: Find a smooth parametrization of the line segment from $(0, 3, -2)$ to $(6, \frac{1}{2}, -2)$

Soln

$x_0 = 0, \; y_0 = 3, \; z_0 = -2, \; x_1 = 6, \; y_1 = \frac{1}{2}, \; z_1 = -2$

$x = x_0 + (x_1 - x_0)t = 0 + (6 - 0)t = 6t, \quad y = 3 + (\frac{1}{2} - 3)t = 3 - \frac{5}{2}t$

$z = -2 + (-2 - (-2))t = -2$

$\therefore \bar{r}(t) = 6t\,\mathbf{i} + (3 - \frac{5}{2}t)\mathbf{j} - 2\mathbf{k}$ for $0 \le t \le 1$ is a smooth parametrization.

⑥ Tangents and Normals to curves

Let C be a smooth curve and $\bar{r}$ a (smooth) parametrization of C defined on an interval I. Then for any interior point t of I, the **tangent vector** $T(t)$ at the point $\bar{r}(t)$ is defined by

$$\bar{T}(t) = \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|} = \frac{\frac{d\bar{r}}{dt}}{\left\|\frac{d\bar{r}}{dt}\right\|} \quad \ldots \text{(v)}$$

Let C be a smooth curve, and let $\bar{r}$ be a (smooth) parametrization of C defined on an interval I such that $\bar{r}'$ is smooth. Then for any interior point t of I for which $\bar{T}'(t) \neq \bar{0}$, the **normal vector** $\bar{N}(t)$ at the point $\bar{r}(t)$ is defined by

$$\bar{N}(t) = \frac{\bar{T}'(t)}{\|\bar{T}'(t)\|} = \frac{\frac{d\bar{T}}{dt}}{\left\|\frac{d\bar{T}}{dt}\right\|} \quad \ldots \text{(vi)}$$

⑦ Let $\bar{F}$ be a continuous vector field defined on a smooth oriented curve C. Then the **line integral** of $\bar{F}$ over C, denoted $\int_C \bar{F}\cdot d\bar{r}$, is defined by

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \bar{F}(x,y,z)\cdot\bar{T}(x,y,z)\,ds \quad \ldots \text{(vii)}$$

where $\bar{T}(x,y,z)$ is the tangent vector at (x,y,z) for the given orientation of C.

8 Tangent Plane and Surface Normal

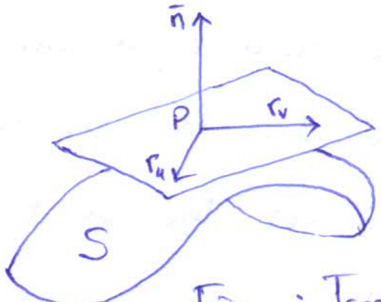


Fig : Tangent plane and normal vector

If S is given by the parametrization

$\bar{r}(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$, the idea is that we get a curve C on S by taking a pair of continuous functions (not both constant) $u = u(t)$, $v = v(t)$,

So that C has the position vector $\tilde{r}(t) = \bar{r}(u(t), v(t))$.

The tangent vector of C is given by

$$\tilde{r}'(t) = \frac{d\tilde{r}}{dt}$$

$$= \frac{\partial \bar{r}}{\partial u} u' + \frac{\partial \bar{r}}{\partial v} v' \quad \ldots \text{(viii)}$$

Hence the partial derivatives r_u and r_v at P are tangential to S at P, and we assume that they are linearly independent, so that they span the tangent plane of S at P. Then there vector product gives a normal vector $\bar{N}$ of S at P,

$$\bar{N} = r_u \times r_v \neq 0 \quad \ldots \text{(ix)}$$

The corresponding unit normal vector $\bar{n}$ of S at P is given as

$$\bar{n} = \frac{1}{|\bar{N}|}\bar{N} = \frac{1}{|r_u \times r_v|} r_u \times r_v \quad \ldots \text{(x)}$$

Also, if S is represented by $g(x,y,z) = 0$, then

$$\bar{n} = \frac{1}{|\text{grad } g|} \text{grad } g \quad \ldots \text{(xi)}$$

Surface Integrals

For a given vector function F we define the surface integral over a surface S as

$$\iint_S F \cdot \bar{n}\, dA = \iint_R F\,[\bar{r}(u,v)] \cdot \bar{N}(u,v)\, du\, dv \quad \ldots (1')$$

Note the integrand is a scalar due to the dot products. Indeed, $F \cdot \bar{n}$ is the normal component of F. This integral arises naturally in flow problems, where it gives the flux across S when $F = \rho \bar{v}$. We may thus call the surface integral (1′) the flux integral.

In terms of components we write

$$F = F_1 i + F_2 j + F_3 k$$

$$\bar{n} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$$

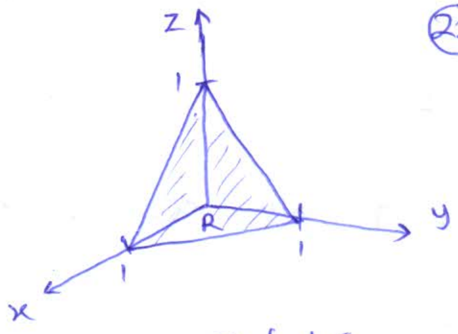
$$\bar{N} = N_1 i + N_2 j + N_3 k$$

Equation (1′) then becomes

$$\iint_S F \cdot \bar{n}\, dA = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$

$$= \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3)\, du\, dv \quad \ldots (2')$$

Example 1 Evaluate the surface integral when $F = [x^2, 0, 3y^2]$ and S is the portion of the plane $x + y + z = 1$ in the first octant as in the figure below



Solution

Writing $x = u$ and $y = v$, we have $z = 1 - x - y = 1 - u - v$

Hence we can represent the plane $x + y + z = 1$ in the form $\bar{r}(u,v) = ui + vj + (1-u-v)k$. We obtain the first-octant portion S of this plane by restricting $x = u$ and $y = v$ to the projection R of S in the xy-plane. R is the triangle bounded by the two coordinate axes and the straight line $x + y = 1$:

thus $0 \le x \le 1-y$, $0 \le y \le 1$. Using

$$\bar{N} = r_u \times r_v = [1, 0, -1] \times [0, 1, -1] = [1, 1, 1]$$

Hence $F(s) \cdot N = [u^2, 0, 3v^2] \cdot [1, 1, 1] = u^2 + 3v^2$

$$\Rightarrow \iint_S F.n\,dA = \iint_R (u^2 + 3v^2)\,du\,dv$$

$$= \int_0^1 \int_0^{1-v} (u^2 + 3v^2)\,du\,dv = \int_0^1 \left[\frac{1}{3}(1-v)^3 + 3v^2(1-v)\right] dv$$

$$= \frac{1}{6}$$

Other forms of Surface Integrals

Let Σ be the graph of a function having continuous partial derivatives and defined on a region R in the xy plane that is composed of a finite number of vertically or horizontally simple regions. Let g be continuous on Σ. Then surface integral of g over Σ, denoted $\iint_{\Sigma} g(x,y,z)\,ds$, is defined by

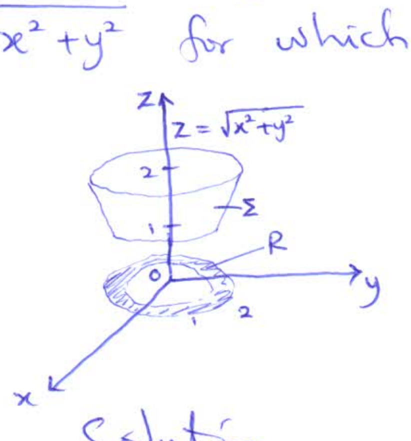
$$\iint_{\Sigma} g(x,y,z)\,ds = \lim_{\|\wp\| \to 0} \sum_{k=1}^{n} g(x_k, y_k, z_k)\,\Delta S_k \quad \cdots (3')$$

If Σ is the graph of f on R, then

$$\iint_{\Sigma} g(x,y,z)\,dS = \iint_{R} g(x,y,f(x,y))\sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\;dA \quad \cdots (4')$$

i.e $z = f(x,y)$

Example 2 Evaluate $\iint_{\Sigma} z^2\,ds$, where Σ is the portion of the cone $z = \sqrt{x^2+y^2}$ for which $1 \le x^2+y^2 \le 4$



<u>Solution</u>

If R is the ring $1 \le x^2+y^2 \le 4$ and if

$f(x,y) = \sqrt{x^2+y^2}$ for (x,y) in R, $g(x,y,f(x,y)) = z^2$

then Σ is the graph of f on R. Because

24

$$\sqrt{f_x^2(x,y) + f_y^2(x,y) + 1} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}$$

$$= \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} = \sqrt{1+1} = \sqrt{2}$$

Using

$$\iint_{\Sigma} z^2 \, dS = \iint_{R} g(x,y,f(x,y)) \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1} \, dA$$

$$= \iint_{R} (x^2+y^2)\sqrt{2} \, dA = \sqrt{2}\int_0^{2\pi}\int_1^2 r^2 r \, dr \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^2 r^3 \, dr \, d\theta = \sqrt{2}\int_0^{2\pi} \frac{r^4}{4}\Big|_1^2 \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{2^4}{4} - \frac{1^4}{4}\right) d\theta = \sqrt{2}\int_0^{2\pi} \frac{15}{4} \, d\theta$$

$$= \frac{15\sqrt{2}}{4}\int_0^{2\pi} 1 \, d\theta = \frac{15\sqrt{2}}{4} \, \theta \Big|_0^{2\pi}$$

$$= \frac{15\sqrt{2}}{\cancel{4}\,2} \cdot \cancel{2}\pi$$

$$\iint_{\Sigma} z^2 \, dS = \frac{15\pi\sqrt{2}}{2}$$

Triple Integrals

Defn: Let D be the solid region between the graphs of two continuous functions F_1 and F_2 on a vertically or horizontally simple region R in the xy plane. If f is continuous on D, we write

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D f(x,y,z)\,dV \quad \dots \quad (5')$$

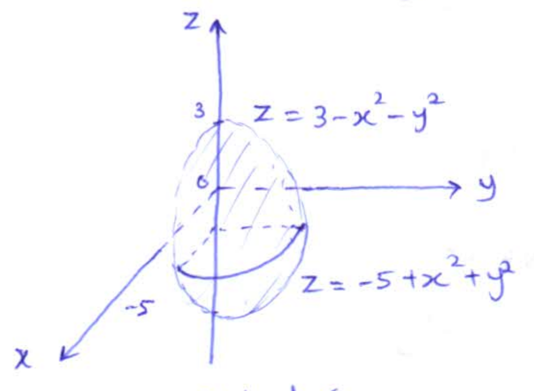
for the unique number that lies between $L_f(P)$ and $U_f(P)$ for every partition P of any parallelepiped containing D. The number $\iiint_D f(x,y,z)\,dV$ is called the **triple integral** of f on D.

$$\iiint_D f(x,y,z)\,dV = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \left(\int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\,dz \right) dx \right] dy \quad \dots \quad (6')$$

Example 1: Let D be the solid region bounded by the portions of the two circular paraboloids

$$z = 3 - x^2 - y^2 \quad \text{and} \quad z = -5 + x^2 + y^2$$

for which $x \geq 0$ and $y \geq 0$ (figure below). Evaluate $\iiint_D y\,dV$.



Solution

To be able to use (6'), we must determine the region R in the xy plane such that D is the solid region between the two paraboloids on R. For this purpose we first determine where the two paraboloids intersect. At any point (x, y, z) of intersection, x and y must satisfy

$$3 - x^2 - y^2 = -5 + x^2 + y^2$$

which is equivalent to $x^2 + y^2 = 4$. But if $x^2 + y^2 = 4$, then

$$z = 3 - x^2 - y^2 = 3 - 4 = -1$$

So the intersection lies in the plane $z = -1$. We find that the corresponding region R in the xy plane is the horizontal simple region in the first quadrant that lies inside the circle $x^2 + y^2 = 4$, and hence between the graphs of $x = 0$ and $x = \sqrt{4 - y^2}$ for $0 \le y \le 2$. Since

$$3 - x^2 - y^2 \geq -5 + x^2 + y^2 \quad \text{for } (x, y) \text{ in } R$$

we have

$$\iiint_D y \, dv = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} y \, dz \, dx \, dy$$

$$= \int_0^2 \int_0^{\sqrt{4-y^2}} yz \Big|_{-5+x^2+y^2}^{3-x^2-y^2} dx \, dy$$

$$= \int_0^2 \int_0^{\sqrt{4-y^2}} y(8 - 2x^2 - 2y^2) \, dx \, dy$$

$$= \int_0^2 \left[y(8 - 2y^2)x - \frac{2}{3} x^3 y \right] \Big|_0^{\sqrt{4-y^2}} dy$$

$$= \int_0^2 \left[y(8 - 2y^2)\sqrt{4 - y^2} - \frac{2}{3} y (4 - y^2)^{3/2} \right] dy$$

$$= \frac{4}{3}\int_0^2 y\,(4-y^2)^{3/2}\,dy$$

$$= -\frac{4}{15}(4-y^2)^{5/2}\Big|_0^2$$

$$= -\frac{4}{15}(4-(2)^2)^{5/2} - \left(-\frac{4}{15}(4-(0)^2)^{5/2}\right)$$

$$= 0 + \frac{4}{15}(4)^{5/2}$$

$$= \frac{128}{15}$$

Divergence Theorem of Gauss

The transformation from triple integrals into surface integral over the boundary surface of a region in space is done by the divergence theorem, which involves the divergence of a vector function

$$F = F_1 i + F_2 j + F_3 k$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \ldots \quad (a)$$

THEOREM 1 (Divergence theorem of Gauss)

(Transformation between volume integrals and surface integrals)

Let D be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $F(x,y,z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing D. Then

$$\iiint_D \operatorname{div} F \, dV = \iint_S F \cdot \bar{n} \, dA \quad \ldots \quad (7')$$

where $\bar{n}$ is the outer unit normal vector of S

Formula (7') in components. Using (a) and $\bar{n} = [\cos\alpha, \cos\beta, \cos\gamma]$ we can write (7') as

$$\iiint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA \quad \cdots (8')$$

which can also be written as

$$\iiint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iint_S (F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy) \quad \cdots (9')$$

Example 1 Evaluate $I = \iint_S (x^3 dy\,dz + x^2 y\, dz\,dx + x^2 z\, dx\,dy)$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$

Solution

Comparing I with eqtn (9') we have $F_1 = x^3$, $F_2 = x^2 y$, $F_3 = x^2 z$,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

Introducing polar coordinates r, θ defined by $x = r\cos\theta$, $y = r\sin\theta$ (thus, cylindrical coordinates r, θ, z), we have $dx\,dy\,dz = r\,dr\,d\theta\,dz$

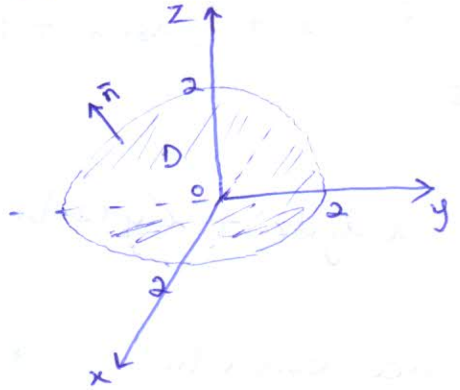
and we obtain

$$I = \iiint_D 5x^2\, dx\,dy\,dz = 5\int_{z=0}^{b} \int_{r=0}^{a} \int_{\theta=0}^{2\pi} x^2 r\, dr\, d\theta\, dz$$

$$= 5\int_{z=0}^{b} \int_{r=0}^{a} \int_{\theta=0}^{2\pi} r^2 \cos^2\theta\, r\, dr\, d\theta\, dz = 5b \int_0^a \int_0^{2\pi} r^3 \cos^2\theta\, dr\, d\theta$$

$$= 5b\, \frac{a^4}{4} \int_0^{2\pi} \cos^2\theta\, d\theta = \frac{5}{4} \pi b a^4$$

Example 2 Let D be the region bounded by the xy plane and the hemisphere shown in the figure below, and let $F(x,y,z) = x^3 i + y^3 j + z^3 k$. Evaluate $\iint_\Sigma F \cdot \bar{n}\, ds$ where Σ is the boundary of D.



Solution

Since a direct calculation of the surface integral would be complicated, we turn to the Divergence theorem. Now

$$\operatorname{div} F(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$$

$$\Rightarrow \iint_\Sigma F \cdot \bar{n}\, dS = \iiint_D \operatorname{div} F(x,y,z)\, dV = \iiint_D 3(x^2+y^2+z^2)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (3\rho^2)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^5}{5} \sin\phi \Big|_0^2 d\phi\, d\theta = \frac{96}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\, d\phi\, d\theta$$

$$= \frac{96}{5}\int_0^{2\pi} -\cos\phi \Big|_0^{\pi/2} d\theta = \frac{96}{5}\int_0^{2\pi} 1\, d\theta = \frac{192}{5}\pi$$

Stokes's Theorem

It is used to transform line integrals into surface integrals and conversely.

THEOREM 1 Stokes's theorem

(Transformation between surface integrals and line integrals)

Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. Let $F(x,y,z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

$$\iint_S (\text{curl } F)\cdot \bar{n}\, dA = \oint_C F\cdot \bar{r}'(s)\, ds \quad \ldots \quad (10')$$

where $\bar{n}$ is a unit normal vector of S and, depending on $\bar{n}$, the integration around C is taken in the sense shown in the figure below

Furthermore $\bar{r}' = \dfrac{d\bar{r}}{ds}$ is the unit tangent vector and s the arc length of C.

Formula (10') in components is given as

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{N}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\bar{N}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{N}_3\right] du\,dv$$
$$= \oint_{\bar{C}} (F_1\,dx + F_2\,dy + F_3\,dz) \quad \ldots \quad (11')$$

where R is the region with boundary curve $\bar{C}$ in the uv-plane corresponding to S represented by $\bar{r}(u,v)$, and $\bar{N} = [N_1, N_2, N_3] = r_u \times r_v$

Example 1 Verify the Stokes's theorem for $F = yi + zj + xk$ as S the paraboloid $z = f(x,y) = 1 - (x^2+y^2)$, $z \geq 0$.

<u>Solution</u>

The curve C is the circle $\bar{r}(s) = \cos s\, i + \sin s\, j$. It has the unit tangent vector $\bar{r}'(s) = -\sin s\, i + \cos s\, j$. Consequently, the line integral in eqtn (10') on the right is simply

$$\oint_C F \cdot d\bar{r} = \int_0^{2\pi} [(\sin s)(-\sin s) + 0 + 0]\, ds = -\pi .$$

On the other hand, in (10') on the left we need

$$\text{curl} F = [-1,-1,-1] \text{ and } \bar{N} = \text{grad}\,(z - f(x,y)) = [2x, 2y, 1]$$

So that $(\text{curl} F) \cdot \bar{N} = -2x - 2y - 1$.

$$\iint_S (\text{curl} F) \cdot \bar{n}\, dA = \iint_R (-2x - 2y - 1)\, dx\, dy$$

$$= \iint_{\tilde{R}} (-2r\cos\theta - 2r\sin\theta - 1)\, r\, dr\, d\theta$$

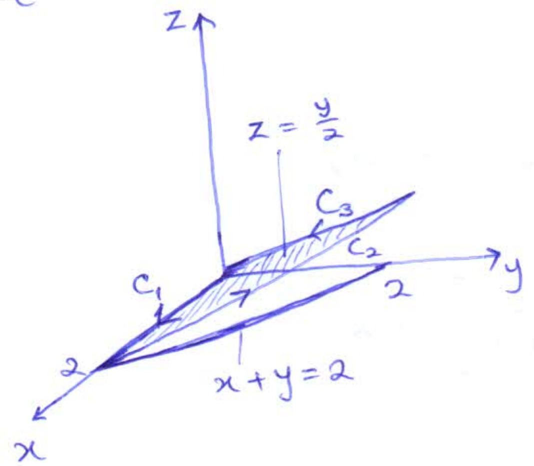
where $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$. Now the projection R of S in the xy-plane is given in polar coordinates by $\tilde{R}: r \leq 1$, $0 \leq \theta \leq 2\pi$. The integration of the cosine and sine terms over θ from 0 to 2π gives zero. The remaining term $-1 \times (r)$ has the integral $(-\frac{1}{2})2\pi = -\pi$, in agreement with the previous result.

Note well that $\bar{N}$ is an upper normal vector of S, and $\bar{r}(s)$ orients C counterclockwise, as required in Stokes's theorem.

Example 2 Let C be the oriented triangle described in the figure below, which lies in the plane $z = \frac{y}{2}$. If

$$F(x,y,z) = -3y^2 i + 4zj + 6xk$$

Calculate $\int_C F \cdot d\bar{r}$



Solution

Direct calculation of the line integral would require evaluation of three separate line integrals, one over each of the line segments C_1, C_2, and C_3 composing C. However, if we apply stoke's theorem, then we need only evaluate $\iint_\Sigma (\text{curl } F) \cdot \bar{n} \, ds$, where Σ is the triangular surface having boundary C and is oriented by the normal directed upward. First we find that

$$\text{curl } F(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y^2 & 4z & 6x \end{vmatrix} = -4i - 6j + 6yk$$

$$F_1 = -4, \; F_2 = -6, \; F_3 = 6y$$

If $f(x,y) = \frac{y}{2}$, then Σ is the graph of f on the triangular region R in the first quadrant bounded by the coordinate axes and the line $x + y = 2$.

Now using

$$\iint_{\Sigma} (\text{curl}\, F)\cdot \bar{n}\, dS = \iint_{R} \left[-F_1 f_x(x,y) - F_2 f_y(x,y) + F_3\right] dA$$

$$= \int_0^2 \int_0^{2-x} \left[-(-4)(0) - (-6)\tfrac{1}{2} + 6y\right] dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (3+6y)\, dy\, dx$$

$$= 3\int_0^2 (y+y^2)\Big|_0^{2-x} dx$$

$$= 3\int_0^2 (6-5x+x^2)\, dx$$

$$= 3\left(6x - \frac{5}{2}x^2 + \frac{x^3}{3}\right)\Big|_0^2$$

$$= 3\left(6(2) - \frac{5}{2}(2)^2 + \frac{(2)^3}{3}\right)$$

$$= 14$$

Therefore $\int_C F\cdot d\bar{r} = \underline{\underline{14}}$

Example 3. Evaluate $\int_C \mathbf{F}\cdot\mathbf{r}'\,ds$, where C is the circle $x^2+y^2=4$, $z=-3$, oriented counterclockwise as seen by a person standing at the origin, and, with respect to right-handed Cartesian coordinates,

$$F = yi + xz^3j - zy^3k$$

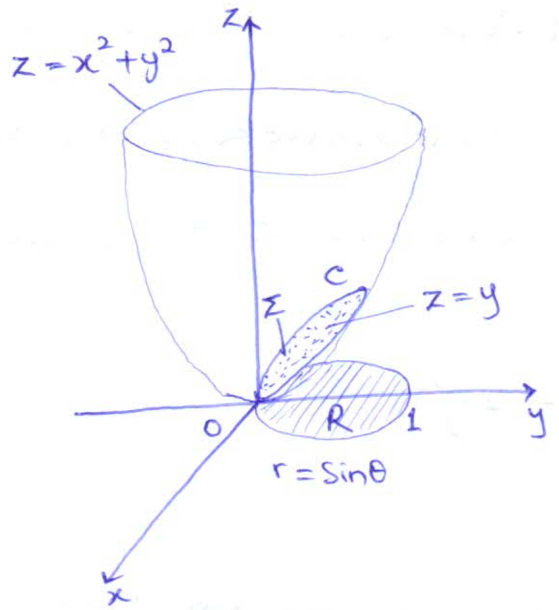
<u>Solution</u>

As a surface S bounded by C we can take the plane circular disk $x^2+y^2 \leq 4$ in the plane $z=-3$. Then $\bar{n}$ in Stokes's theorem points in the positive z-direction; thus $\bar{n}=\bar{k}$. Hence $(\operatorname{curl} F)\cdot\bar{n}$ is simply the component of $\operatorname{curl} F$ in the positive z-direction. Since F with $z=-3$ has the components $F_1=y$, $F_2 = xz^3 = x(-3)^3 = -27x$, $F_3 = 3y^3$, we thus obtain

$$(\operatorname{curl} F)\cdot\bar{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -27-1 = -28$$

Hence the integral over S in Stokes's theorem equals -28 times the area 4π of the disk S. This yields the answer $-28\cdot 4\pi = -112\pi \simeq -352$.

Example 4 Let C be the intersection of the paraboloid $z=x^2+y^2$ and the plane $z=y$, and give C its counterclockwise orientation as viewed from the positive z axis. Evaluate $\int_C (xy\,dx + x^2\,dy + z^2\,dz)$.



Solution

Let $F(x,y,z) = xy\,i + x^2 j + z^2 k$

and let Σ be the portion of the plane $z = y$ that lies inside the paraboloid.

Using $\displaystyle\iint_\Sigma (\text{curl}\, F)\cdot \vec{n}\, dS = \int_C F_1\, dx + F_2\, dy + F_3\, dz$

Notice that if (x,y,z) is on C, then $x^2+y^2 = y$, and this is an equation of the circular cylinder having equation $r = \sin\theta$ in cylindrical coordinates. Therefore if R is the region in the xy plane bounded by the circle $r = \sin\theta$, then Σ is the graph of $z = y$ on R. When we orient Σ by the normal directed upward, the induced orientation on C is counterclockwise, as prescribed. Since

$$\text{Curl}\, F(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & z^2 \end{vmatrix} = xk$$

$F_1 = 0\ ,\ F_2 = 0\ ,\ F_3 = x$

and $f(x,y) = y$

$$\Rightarrow \int_C (xy\,dx + x^2 dy + z^2 dz) = \iint_\Sigma (\text{curl } F)\cdot \bar{n}\, ds$$

$$= \iint_R [-F_1 f_x(x,y) - F_2 f_y(x,y) + F_3]$$

$$= \iint_R [-0(0) - 0(1) + x]\, dA$$

$$= \iint_R x\, dA = \int_0^{\pi} \int_0^{\sin\theta} (r\cos\theta) r\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{r^3}{3} \cos\theta \Big|_0^{\sin\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi} \sin^3\theta \cos\theta\, d\theta$$

$$= \frac{1}{12} \sin^4\theta \Big|_0^{\pi}$$

$$= 0$$

Tensor Analysis

Indicial Notation

In this section, we are going to examine and see that indicial notation simplifies the bookkeeping process of the many mathematical expressions needed in continuum mechanics by allowing us to express them in a compact form.

Summation (Dummy Indices)

Consider the sum

$$S = a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n \quad (1)$$

or

$$S = \sum_{i=1}^{n} a_i x_i \quad (2)$$

or

$$S = \sum_{j=1}^{n} a_j x_j = \sum_{k=1}^{n} a_k x_k \quad (3)$$

The indices i, j, k in Eqs. (1) and (3) are called dummy indices. Thus, if an index is repeated once within a simple term, it is a dummy index indicating a summation. By convention, each dummy index has values 1, 2, 3.

For example

$$S = \sum_{i=1}^{3} a_i x_i \quad (\text{Sum of 3 terms})$$

Also, $$S = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j \quad (\text{Sum of } 3^2 \text{ terms}).$$ (4)

In order for us to alleviate writing the summation symbol continuously, we will employ Einstein's summation convention. With this convention, it is understood that the summation occurs over repeated indices without having to write the summation symbol. Hence Eqn (4) become

$$S = a_i x_i$$

and

$$S = a_{ij} x_i x_j$$ (5)

According to this convention, expressions such as $S = a_i b_i c_i$ (where i is repeated more than two times) are meaningless.

Multiple Equations (Free Indices)

Let us Consider

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}$$ (6)

Using dummy indices, we can reduce Eqn (6) to

$$\left.\begin{aligned} y_1 &= a_{1n} x_n \\ y_2 &= a_{2n} x_n \\ y_3 &= a_{3n} x_n \end{aligned}\right\} \quad (7)$$

Notice Eqn (7) are of the same form, differing only by the index 1, 2, 3. Thus, by introducing the concept of a free index, Eqn (7) can be written as

$$y_i = a_{in} x_n \quad (8)$$

here i is the free index and n is a dummy index (as described in the previous section). Free indices appear only once in each term of an equation, indicating multiple equations. They typically take on the values 1, 2, 3

For example, $y_i = a_{im} x_m$ represents three equations. If there are two free indices in an equation, then the expression represents nine equations. For example

$$T_{ij} = A_{im} A_{jm} \quad (9)$$

represents nine equations, each with three terms on the right-hand side. Taking the free indices $i=1$, $j=2$

$$T_{12} = A_{11} A_{21} + A_{12} A_{22} + A_{13} A_{23} \quad (10)$$

Kronecker Delta

The Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

i.e., $\delta_{11} = \delta_{22} = \delta_{33} = 1$ and $\delta_{12} = \delta_{13} = \delta_{21} = \dots = 0.$

The matrix $\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

is the identity matrix.

Let us consider some examples:

1. $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

2. $\delta_{1m} a_m = \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 = a_1$

   $\delta_{2m} a_m = \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 = a_2$

   $\delta_{3m} a_m = \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 = a_3$

   $\Rightarrow \delta_{im} a_m = a_i$

3. In general $\delta_{im} T_{mj} = T_{ij}.$

... are components of a vector and T_{ij} are

4. The Kronecker delta δ_{ij}, can be used to contract indices as follows

$$\delta_{im}\delta_{mj} = \delta_{ij}$$

$$\delta_{im}\delta_{mj}\delta_{jn} = \delta_{in}$$

5. If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are unit vectors perpendicular to each other,

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

or $\vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{e}_3 \cdot \vec{e}_3 = 1$ and $\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{e}_3 = \vec{e}_2 \cdot \vec{e}_1 = \dots = 0$.

Tensors

A tensor is a linear transformation, denoted by $\underset{\sim}{T}$. It transforms any vector into another vector. A second-order tensor represents nine scalar quantities

$$\underset{\sim}{T}\vec{a} = \vec{b} \qquad (12)$$

$\underset{\sim}{T}$ has the following properties

$$\underset{\sim}{T}(\vec{a}+\vec{b}) = \underset{\sim}{T}\vec{a} + \underset{\sim}{T}\vec{b}$$
$$\underset{\sim}{T}(\alpha\vec{a}) = \alpha\underset{\sim}{T}\vec{a} \qquad (13)$$

where α is a scalar

Components of a Tensor

Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be unit vectors in a rectangular Cartesian coordinate system. The Cartesian components of a vector $\vec{a}$ are given by

$$a_1 = \vec{a}\cdot\vec{e}_1,\ a_2 = \vec{a}\cdot\vec{e}_2, \text{ and } \vec{a}_3 = \vec{a}\cdot\vec{e}_3$$

or

$$a_i = \vec{a}\cdot\vec{e}_i \qquad (14)$$

Equivalently,

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = a_i\vec{e}_i \qquad (15)$$

Now consider a tensor $\underset{\sim}{T}$. For any vector $\vec{a}$,
$\vec{b} = \underset{\sim}{T}\tilde{a}$ is a vector given by

$$\vec{b} = \underset{\sim}{T}\vec{a} = \underset{\sim}{T}(a_i \vec{e}_i) = a_i \underset{\sim}{T}\vec{e}_i \qquad (16)$$

i.e., $\vec{b} = a_1 \underset{\sim}{T}\vec{e}_1 + a_2 \underset{\sim}{T}\vec{e}_2 + a_3 \underset{\sim}{T}\vec{e}_3$.

Note that $\underset{\sim}{T}\vec{e}_i$ are not perpendicular unit vectors.

Using Eqns (14) and (16), the components of $\vec{b}$ are

$$b_1 = \vec{e}_1 \cdot \vec{b} = \vec{e}_1 \cdot (a_i \underset{\sim}{T}\vec{e}_i) = a_i \vec{e}_1 \underset{\sim}{T}\vec{e}_i$$

$$= a_1 \vec{e}_1 \cdot \underset{\sim}{T}\vec{e}_1 + a_2 \vec{e}_1 \cdot \underset{\sim}{T}\vec{e}_2 + a_3 \vec{e}_1 \cdot \underset{\sim}{T}\vec{e}_3$$

$$b_2 = a_i \vec{e}_2 \cdot \underset{\sim}{T}\vec{e}_i$$

$$b_3 = a_i \vec{e}_3 \cdot \underset{\sim}{T}\vec{e}_i$$

or

$$b_i = a_j \vec{e}_i \cdot \underset{\sim}{T}\vec{e}_j \qquad (16)$$

Now, $\vec{b} = \underset{\sim}{T}\vec{a}$ in matrix format is

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

from which we identify

$$b_i = T_{ij} a_j \qquad (17)$$

Comparing Eqns. (16) and (17), we find that the components of $\underset{\sim}{T}$ are

$$T_{ij} = \vec{e}_i \cdot \underset{\sim}{T}\vec{e}_j \qquad (18)$$

Example

What is the vector $\underset{\sim}{T}\vec{e}_j$

Solution

$$\underset{\sim}{T}\vec{e}_j = T_{kj}\vec{e}_k \tag{19}$$

To understand this, Let us consider $\underset{\sim}{T}\vec{e}_2$ as an example

$$\underset{\sim}{T}\vec{e}_2 = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{12} \\ T_{22} \\ T_{32} \end{bmatrix} \tag{20}$$

Thus, the j in $\underset{\sim}{T}\vec{e}_j$ determines the column, the same way that j denotes the column in the expression T_{ij}.

By definition,

$$\begin{bmatrix} T_{12} \\ T_{22} \\ T_{32} \end{bmatrix} = T_{12}\vec{e}_1 + T_{22}\vec{e}_2 + T_{32}\vec{e}_3 = T_{K2}\vec{e}_K \tag{21}$$

Comparing Eqs. 20 and 21, we see

$$\underset{\sim}{T}\vec{e} = T_{K2}\vec{e}_K$$

Now, going back to Eqn. 18 with $i=3$ and $j=2$.

From Eqn 20 and the definition of the dot product,

$$T_{32} = \vec{e}_3 \cdot \underset{\sim}{T}\, \vec{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} T_{12} & T_{22} & T_{32} \end{bmatrix}$$

$$= (0)(T_{12}) + (0)(T_{22}) + (1)(T_{32})$$

$$= T_{32} \qquad (22)$$

Thus, the i in $\vec{e}_i$ in Eqn (18) determines the row, the same way that i denotes the row in the expression T_{ij}.

Note: The expression $\underset{\sim}{T}\vec{e}_j = T_{kj}\vec{e}_k$ will become useful when trying to determine the components of a tensor when the resulting transformation of original unit vectors is known.

Sum and Product of Tensors

Let $\underset{\sim}{T}$ and $\underset{\sim}{S}$ be two tensors. Then,

$$(\underset{\sim}{T} + \underset{\sim}{S})\vec{a} = \underset{\sim}{T}\vec{a} + \underset{\sim}{S}\vec{a} \qquad (23)$$

Examining the components of the sum,

$$(\underset{\sim}{T} + \underset{\sim}{S})_{ij} = T_{ij} + S_{ij} \qquad (24)$$

or, in matrix notation,

$$[\underset{\sim}{T} + \underset{\sim}{S}] = [\underset{\sim}{T}] + [\underset{\sim}{S}] \qquad (25)$$

where $[\underset{\sim}{T}]$ is the matrix representation of the tensor $\underset{\sim}{T}$.

For the product of two tensors,

$$(\underset{\sim}{T}\,\underset{\sim}{S})\vec{a} = \underset{\sim}{T}(\underset{\sim}{S}\vec{a})$$

$$(\underset{\sim}{T}\,\underset{\sim}{S})_{ij} = T_{im} S_{mj}$$

$$[\underset{\sim}{T}\,\underset{\sim}{S}] = [\underset{\sim}{T}][\underset{\sim}{S}]$$

Note: In general $\underset{\sim}{T}\,\underset{\sim}{S} \neq \underset{\sim}{S}\,\underset{\sim}{T}$.

Identity Tensor

The identity tensor, $\underset{\sim}{I}$, is the linear transformation that transforms any vector into itself. Thus

$$\underset{\sim}{I}\vec{a} = \vec{a} \qquad (26)$$

from which it is evident that $\underset{\sim}{I}$ can be represented as

$$\underset{\sim}{I} = \delta_{ij} \qquad (27)$$

Transpose of a Tensor

The transpose of a tensor $\underset{\sim}{T}$ is defined as the tensor $\underset{\sim}{T}^T$ that satisfies

$$\vec{a} \cdot (\underset{\sim}{T} \vec{b}) = \vec{b} \cdot (\underset{\sim}{T}^T \vec{a}) \qquad (28)$$

Considering the unit vectors, for which Eqn (28) must also hold, $\vec{e}_i \cdot (\underset{\sim}{T} \vec{e}_j) = \vec{e}_j \cdot (\underset{\sim}{T}^T \vec{e}_i)$

which shows that

$$T_{ij} = (T^T)_{ji} \qquad (29)$$

Note: The transpose of the product of two tensors is the product of the two tensors transposed in reverse order.

$$(\underset{\sim}{T}\,\underset{\sim}{S})^T = \underset{\sim}{S}^T \underset{\sim}{T}^T \qquad (30)$$

Orthogonal Tensor

An orthogonal tensor, $\underset{\sim}{Q}$, is a linear transformation for which transformed vectors preserve their lengths and angles. Thus, given an orthogonal tensor $\underset{\sim}{Q}$,

$$|\underset{\sim}{Q}\vec{a}| = |\vec{a}| \qquad (31)$$

Note: In general, $\underset{\sim}{Q}\vec{a} \neq \vec{a}$. Only their lengths are equal. Furthermore,

$$\cos(\vec{a},\vec{b}) = \cos(\underset{\sim}{Q}\vec{a}, \underset{\sim}{Q}\vec{b}) \qquad (32)$$

and, so it follows

$$(\underset{\sim}{Q}\vec{a})\cdot(\underset{\sim}{Q}\vec{b}) = \vec{a}\cdot\vec{b} \qquad (33)$$

Note: In general, $\underset{\sim}{T}\,\underset{\sim}{T}^T \neq \underset{\sim}{T}^T\underset{\sim}{T} \neq \underset{\sim}{I}$,

but $\underset{\sim}{Q}\,\underset{\sim}{Q}^T = \underset{\sim}{Q}^T Q = \underset{\sim}{I}$.

From Eqn. (28)

$$(\underset{\sim}{Q}\vec{a})\cdot(\underset{\sim}{Q}\vec{b}) = \vec{b}\cdot(\underset{\sim}{Q}^T(\underset{\sim}{Q}\vec{a})) \qquad (34)$$

Substituting Eqn. (33) into Eqn. (34),

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot(\underset{\sim}{Q}^T(\underset{\sim}{Q}\vec{a}))$$

$$\vec{b}\cdot\vec{a} - \vec{b}\cdot\underset{\sim}{Q}^T(\underset{\sim}{Q}\vec{a}) = 0$$

$$\vec{b}\cdot(\underset{\sim}{I} - \underset{\sim}{Q}^T\underset{\sim}{Q})\vec{a} = 0$$

Therefore, for arbitrary (nonzero) vectors $\vec{a}$ and $\vec{b}$,

$$\underset{\sim}{I} = \underset{\sim}{Q}^T\underset{\sim}{Q} = \underset{\sim}{Q}\,\underset{\sim}{Q}^T \qquad (35)$$

Hence, for an orthogonal tensor $\underset{\sim}{Q}$, it's also true that

$$\underset{\sim}{Q} = \underset{\sim}{Q}^{-1} \qquad (36)$$

In indicial notation

$$Q_{im} Q_{jm} = Q_{mi} Q_{mj} = \delta_{ij} \qquad (37)$$

Note: The determinant of an orthogonal tensor is ± 1, where

$$\det(\underset{\sim}{Q}) = \begin{cases} +1 & \text{indicates a rotation} \\ -1 & \text{indicates a reflection} \end{cases}$$

Example

A rigid body is rotated $90°$ by the right-hand rule about the $\vec{e}_3$ axis (See the Fig. below).

1. Find the matrix representation of the tensor $\underset{\sim}{R}$ describing this rotation.
2. Examine whether $\underset{\sim}{R}$ is orthogonal
3. Find the determinant of $\underset{\sim}{R}$.
4. Suppose this rigid body experiences a $90°$ right-hand rotation about the original $\vec{e}_1$ axis by the right-

tensor $\underset{\sim}{S}$ describing this rotation.

(5) Find the final position of a point, P, originally at (1, 1, 0) after these two rotations.

Solution

(1.) $\underset{\sim}{R}$ is the transformation

$$\underset{\sim}{R}\,\vec{e}_1 = \vec{e}_2$$
$$\underset{\sim}{R}\,\vec{e}_2 = -\vec{e}_1 \qquad (38)$$
$$\underset{\sim}{R}\,\vec{e}_3 = \vec{e}_3$$

Recall Eqn. (19).

$$\underset{\sim}{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

with $\underset{\sim}{R}\,\vec{e}_1$, $\underset{\sim}{R}\,\vec{e}_2$, and $\underset{\sim}{R}\,\vec{e}_3$ as columns.

From Eqns (19) and (38)

$$\underset{\sim}{R}\,\vec{e}_1 = R_{11}\vec{e}_1 + R_{21}\vec{e}_2 + R_{31}\vec{e}_3 = \vec{e}_2$$
$$\underset{\sim}{R}\,\vec{e}_2 = R_{12}\vec{e}_1 + R_{22}\vec{e}_2 + R_{32}\vec{e}_3 = -\vec{e}_1 \qquad (39)$$
$$\underset{\sim}{R}\,\vec{e}_3 = R_{13}\vec{e}_1 + R_{23}\vec{e}_2 + R_{33}\vec{e}_3 = \vec{e}_3$$

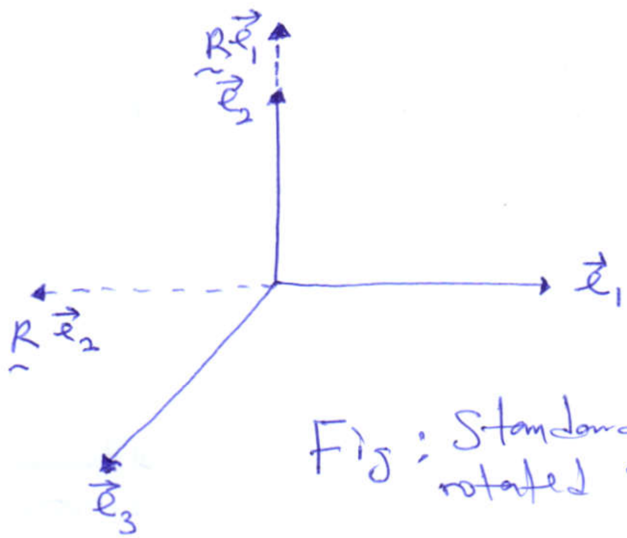


Fig: Standard Coordinate system rotated 90° about the $\vec{e}_3$ axis.

2. Recall Eqn. 35, $\underset{\sim}{R}$ is orthogonal iff ("if and only if")

$$\underset{\sim}{R}\, R^T = \underset{\sim}{I}.$$

$$[\underset{\sim}{R}^T] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So,

$$\underset{\sim}{R}\,\underset{\sim}{R}^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{\sim}{I}.$$

Therefore, $\underset{\sim}{R}$ is orthogonal.

3. $$\det(\underset{\sim}{R}) = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -(-1)(1) = 1.$$

Therefore, $\underset{\sim}{R}$ is a rotation.

4. $\underset{\sim}{S}\vec{e}_1 = \vec{e}_1$

$\underset{\sim}{S}\vec{e}_2 = \vec{e}_3$

$\underset{\sim}{S}\vec{e}_3 = -\vec{e}_2$

$\underset{\sim}{S}$ describes only the second rotation, not the combined total transformation. Following equation 39

$$[\underset{\sim}{S}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

5. The vector describing the location of point P is $\vec{r} = (1, 1, 0)^T$. Hence, the new location, p, of point P after both rotations is

$$\vec{r}\,' = \underset{\sim}{S}\,\underset{\sim}{R}\,r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, $p = (-1, 0, 1)$.

Note: The order of rotations matters, e.g., $\underset{\sim}{R}\,\underset{\sim}{S}\,\vec{r}$ yields $p = (-1, 1, 0)$. This exemplifies the fact that given two tensors $\underset{\sim}{T}$ and $\underset{\sim}{S}$, $\underset{\sim}{T}\,\underset{\sim}{S} \neq \underset{\sim}{S}\,\underset{\sim}{T}$.

Tensor Symmetry, Principal values, and Principal Directions

Symmetric Vs Antisymmetric Tensor

A tensor, $\underset{\sim}{T}^{symm}$, is defined as symmetric iff $\underset{\sim}{T}^{symm} = (\underset{\sim}{T}^{symm})^T$. This is different from orthogonal tensors. A symmetric tensor does not necessarily preserve lengths and angles. The components of a symmetric tensor satisfy

$$T_{ij} = T_{ji} \qquad (40)$$

This means that $T_{12} = T_{21}$, $T_{31} = T_{13}$, and $T_{23} = T_{32}$. Also, the diagonal elements of $\underset{\sim}{T}^{symm}$ and $(\underset{\sim}{T}^{symm})^T$ must be equal.

A tensor, $\underset{\sim}{T}^{asymm}$, is antisymmetric iff

$$\underset{\sim}{T}^{asymm} = -(\underset{\sim}{T}^{asymm})^T. \qquad (41)$$

The components of antisymmetric tensors satisfy

$$T_{ij} = -T_{ji} \qquad (42)$$

This means that $T_{12} = -T_{21}$, $T_{31} = -T_{13}$ and $T_{23} = -T_{32}$. Also, ~~the~~ diagonal elements of $\underset{\sim}{T}^{asymm}$ must be zero. Antisymmetric tensors are also known as asymmetric or skew tensors.

Any tensor $\underset{\sim}{T}$ can always be decomposed into the sum of a symmetric and antisymmetric tensor.

$$\underset{\sim}{T} = \underset{\sim}{T}^{symm} + \underset{\sim}{T}^{asymm} \qquad (43)$$

where

$$\underset{\sim}{T}^{symm} = \frac{\underset{\sim}{T} + \underset{\sim}{T}^{T}}{2}$$

$$\underset{\sim}{T}^{asymm} = \frac{\underset{\sim}{T} - \underset{\sim}{T}^{T}}{2} \qquad (44)$$

Eigenvalues and Eigenvectors

Consider a tensor $\underset{\sim}{T}$ and a vector $\vec{a}$. $\vec{a}$ is defined as an eigenvector of $\underset{\sim}{T}$ if it transforms under $\underset{\sim}{T}$ into a vector parallel to itself. This means

$$\underset{\sim}{T}\,\vec{a} = \lambda \vec{a} \qquad (45)$$

where λ is called the eigenvalue.

For definiteness, all eigenvectors will be of unit length.

Note: Any vector is an eigenvector of $\underset{\sim}{I}$ as $\underset{\sim}{I}\vec{a} = \vec{a}$ and $\lambda = 1$.

Now let $\vec{n}$ be a unit eigenvector. From Eqns 45 and 26,

$$\underset{\sim}{T}\,\vec{n} = \lambda\vec{n} = \lambda\underset{\sim}{I}\,\vec{n} \qquad (46)$$

with $\vec{n}\cdot\vec{n} = 1$. This implies

$$\left(\underset{\sim}{T} - \lambda\underset{\sim}{I}\right)\vec{n} = 0 \qquad (47)$$

or, in component form (let $\vec{n} = a_i\vec{e}_i$),

$$\left(T_{ij} - \lambda\delta_{ij}\right)a_j = 0 \qquad (48)$$

with $a_j a_j = 1$. In long form

$$(T_{11}-\lambda)a_1 + T_{12}a_2 + T_{13}a_3 = 0$$
$$T_{21}a_1 + (T_{22}-\lambda)a_2 + T_{23}a_3 = 0 \qquad (49)$$
$$T_{31}a_1 + T_{32}a_2 + (T_{33}-\lambda)a_3 = 0$$

and

$$a_1^2 + a_2^2 + a_3^2 = 1 \qquad (50)$$

Equations 48, 49 and 50 are used to solve for eigenvectors of a tensor. Eqn 49 are a system of linear homogeneous equations in a_1, a_2 and a_3. Recall from linear algebra that the homogeneous equation $\underset{\sim}{A}\vec{x} = 0$ has only the trivial solution $\vec{x} = (0,0,0)$ unless $\underset{\sim}{A}$ is a singular matrix, i.e., $\underset{\sim}{A}^{-1}$ does not exist and $\det(\underset{\sim}{A}) = 0$. Eqns 48 or 49 is of the form $\underset{\sim}{A}\vec{x} = 0$, and, in order to find nonzero eigenvectors, we must have $\det(\underset{\sim}{A}) = 0$. Thus,

$$\left|\underset{\sim}{T} - \lambda \underset{\sim}{I}\right| = 0 \qquad (51)$$

or, in long form,

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0 \qquad (52)$$

Eqn 52 is a cubic equation in λ. It is known as the characteristic equation of $\underset{\sim}{T}$.

Given $\underset{\sim}{T}$ with $T_{21} = T_{31} = 0$, show that $\vec{e}_1$ is an eigenvector of $\underset{\sim}{T}$ with T_{11} as the corresponding eigenvalue.

Solution

Recall Eqn. 45 that says, $\underset{\sim}{T}\vec{a} = \lambda\vec{a}$, Also recall Eqn 19. Then

$$\underset{\sim}{T}\cdot\vec{e}_1 = T_{11}\vec{e}_1 + T_{21}\vec{e}_2 + T_{31}\vec{e}_3$$

$$\Rightarrow \underset{\sim}{T}\vec{e}_1 = T_{11}\vec{e}_1,$$ which satisfies Eqn. 45

Example

Given $\left[\underset{\sim}{T}\right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$, find the eigenvalues, λ's, and corresponding eigenvectors, $\vec{n}$'s.

Solution

To find the eigenvalues, we will make use of the characteristic equation, Eqn. 52. For the given $\underset{\sim}{T}$, Eqn. 52 is

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \end{vmatrix} = 0 \qquad (53)$$

which becomes

$$(2-\lambda)\left[(3-\lambda)(-3-\lambda)-16\right]=0$$

$$(2-\lambda)(\lambda^2-25)=0 \qquad (54)$$

and this gives $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = -5$. To find the eigenvectors, we need to solve Eqn (48) for each λ. Eqn (48) for the given $\underset{\sim}{I}$ is

$$\begin{aligned}(2-\lambda)a_1 + 0a_2 + 0a_3 &= 0\\ 0a_1 + (3-\lambda)a_2 + 4a_3 &= 0\\ 0a_1 + 4a_2 + (-3-\lambda)a_3 &= 0\end{aligned} \qquad (55)$$

So for $\lambda_1 = 2$, Eqn. (55) become

$$0a_1 + 0a_2 + 0a_3 = 0$$

$$0a_1 + 1a_2 + 4a_3 = 0$$

$$0a_1 + 4a_2 + (-5)a_3 = 0$$

These are two equations with two unknowns, which when solved give, $a_2 = a_3 = 0$ and a_1 unspecified. But, eigenvectors are of unit length. Using Eqn. (50), $a_1^2 + a_2^2 + a_3^2 = 1$, so $a_1 = \pm 1$. Thus, the eigenvector

For $\lambda_2 = 5$, Eqn (55) become

$$-3a_1 + 0a_2 + 0a_3 = 0$$
$$0a_1 + (-2)a_2 + 4a_3 = 0$$
$$0a_1 + 4a_2 + (-8)a_3 = 0$$

which when solved, give $a_1 = 0$ and $a_2 = 2a_3$. Using Eqn (56)

$$(2a_3)^2 + a_3^2 = 1 \implies a_3 = \pm \frac{1}{\sqrt{9}}$$
$$a_2 = \pm \frac{2}{\sqrt{5}}$$

Therefore, the eigenvector, $\vec{n}_2$, corresponding to $\lambda_2 = 5$ is $\vec{n}_2 = \pm \frac{1}{\sqrt{5}}(2\vec{e}_2 + \vec{e}_3)$.

Similarly, for

$$\lambda_3 = -5, \quad \vec{n}_3 = \pm \frac{1}{\sqrt{5}}(-\vec{e}_2 + 2\vec{e}_3).$$