25.01.21 MTS 341 (2020/2021) Mathematical Method II (03) 08033598608

Course Contents: Sturm-Liouville problem; orthogonal polynomials and functions

Fourier Series and integrals, Fourier transforms

Partial differential equations; 1st and 2nd order equations : general and particular solutions, linear equations with constant coefficients

- classification of 2nd order linear equations
- Solution of heat, wave and Laplace equations by method of separable variables
- Eigen functions expansions and FT.

Okunwowoja 09070878247

Ajot — 08060064656

PRQT: MTS 232 (Ord. diff. equations), MTS 241 (Mathematical Method 1)

11—1; Monday
COLPHEC 3

§1. Orthogonal Polynomials & functions

- Inner product | inner product space
- orthogonal set / orthonormal set
- orthogonal projection / decomposition
- orthogonal functions / Polynomials
- Least square approximation by polynomials.

1.1 Inner product and inner product space

Definition 1.1.1

The inner product $\langle \cdot , \cdot \rangle$ on a vector space X is a mapping of $X \times X$ into the scalar field K ($= \mathbb{R}$ or $\mathbb{C}$) satisfying the following.
For all vectors x, y and z and scalars α we have:

1. $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4. $\langle x, x \rangle \geq 0$ and equal if and only if $x = \theta$.

The inner product on X defines a norm on X given by $\|x\| = \langle x, x \rangle$ and a distance function, or metric, on X given by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}$$

When an inner product space is complete, the space is called a Hilbert space and is denoted by H.

1.2 Orthogonal and orthonormal sets

Definition 1.2.1

Two elements x and y in an inner product space X are said to be orthogonal if $\langle x, y \rangle = 0$. A set of vectors is called an orthonormal set if these vectors are pairwise orthogonal and of norm 1.

$$\bar{u}_i \cdot \bar{u}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Example 1.2.2. Euclidean space $\mathbb{R}^n$.

Given vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ an inner product is defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{j=1}^{n} x_j y_j$$

This makes $\mathbb{R}^n$ into a Hilbert space

Example 1.2.3.

Show that the following set is an orthogonal set:

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$$

We check by direct calculations:

$$\vec{u}_1 \cdot \vec{u}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} = 0.$$

Example 1.2.4. Show that $\vec{u}_1 = \frac{1}{\sqrt{11}}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{u}_3 = \frac{1}{\sqrt{66}}\begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$ form an orthonormal set.

Solution. Direct computation show

$\|\vec{u}_1\| = \sqrt{(\frac{3}{\sqrt{11}})^2 + (\frac{1}{\sqrt{11}})^2 + (\frac{1}{\sqrt{11}})}$

$$\vec{u}_1 \cdot \vec{u}_1 = \vec{u}_2 \cdot \vec{u}_2 = \vec{u}_3 \cdot \vec{u}_3 = 1$$

$$\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_2 \cdot \vec{u}_3 = \vec{u}_1 \cdot \vec{u}_3 = 0$$

Theorem 1.2.5. Any orthogonal set is linearly independent

Proof. Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal set, and suppose

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p = \vec{0} \qquad (*)$$

Multiplying (using inner product) both sides of the equation by $\vec{u}_i$, we obtain

$$\vec{u}_i \cdot (c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_i\vec{u}_i + \cdots + c_p\vec{u}_p) = 0$$

$$c_i \vec{u}_i \cdot \vec{u}_i = 0$$

Since $\vec{u}_i \neq \vec{0}$, it follows $c_i = 0$. Therefore, the only solution for $(*)$ is the trivial one. This is the definition of linear independence. ∎

Definition 1.2.6: A basis of a subspace is said to be an orthogonal basis if it is an orthogonal set.

Theorem 1.2.7.

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace W. Then, for each $\vec{w} \in W$, its coordinate $[\vec{w}]_{\mathcal{B}}$ relative to this orthogonal basis can be expressed as

$$[\vec{w}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}, \quad c_i = \frac{\vec{w}\cdot\vec{u}_i}{\|\vec{u}_i\|^2}, \quad i = 1, 2, \ldots, p. \qquad (*_2)$$

In other words,

$$\begin{aligned}\vec{w} &= c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p \\ &= \frac{\vec{w}\cdot\vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 + \frac{\vec{w}\cdot\vec{u}_2}{\|\vec{u}_2\|^2}\vec{u}_2 + \cdots + \frac{\vec{w}\cdot\vec{u}_p}{\|\vec{u}_p\|^2}\vec{u}_p. \qquad (*_3)\end{aligned}$$

Proof.

Example 1.2.8.

Consider $\vec{u}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{u}_3 = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$ form an orthogonal basis for $\mathbb{R}^3$, find the coordinate of $\vec{w} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$ relative to this basis.

Solution: ~~Recall that if the basis were~~ Since the basis is indeed orthogonal, we use formula $(*_2)$:

$$c_1 = \frac{\vec{w}\cdot\vec{u}_1}{\|\vec{u}_1\|^2} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} \Big/ \left\|\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\right\|^2 = 1$$

$$c_2 = \frac{\vec{w}\cdot\vec{u}_2}{\|\vec{u}_2\|^2} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} \Big/ \left\|\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\|^2 = -2$$

$$c_3 = \frac{\vec{w}\cdot\vec{u}_3}{\|\vec{u}_3\|^2} = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}\cdot\begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} \Big/ \left\|\begin{matrix} -1 \\ -4 \\ 7 \end{matrix}\right\|^2 = -1$$

$$\therefore [\vec{w}]_{\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}} = \begin{pmatrix} 1 \\ -2 \\ \ldots \end{pmatrix}$$

Scanned with CamScanner

§1.3 Orthogonal Projections

Definition 1.3.1. Given a vector $\vec{u}$, the orthogonal projection of $\vec{y}$ onto $\vec{u}$, denoted by

$$\hat{y} = \mathrm{Proj}_{\vec{u}}(\vec{y}),$$

is defined as the vector $\hat{y}$ parallel to $\vec{u}$ such that

$$\vec{y} = \hat{y} + \vec{z}, \quad \vec{z} \perp \vec{u}$$

Since $\hat{y}$ is parallel to $\vec{u}$, we have

$$\hat{y} = \alpha \vec{u}$$

Hence we may write

$$\vec{y} = \alpha\vec{u} + \vec{z}, \quad \vec{z} \perp \bar{u}.$$

This is also called an orthogonal decomposition. Multiplying the above expression by $\vec{u}$, we find

$$\vec{y}\cdot\vec{u} = (\alpha\vec{u} + \vec{z})\cdot\vec{u} = \alpha\vec{u}\cdot\vec{u} \Rightarrow \alpha = \frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}}$$

Consequently, we find the explicit formula for the orthogonal projection:

$$\mathrm{Proj}_{\bar{u}}(\bar{y}) = \hat{y} = \alpha\bar{u} = \left(\frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}}\right)\bar{u} \quad \cdots (*_4)$$

In general, for any subspace W, the orthogonal projection of $\bar{y}$ onto W, denoted by

$$\hat{y} = \mathrm{Proj}_W(\vec{y})$$

is defined as the vector in W such that

$$(\bar{y} - \hat{y}) \perp W.$$

In other words,

$$\vec{y} = \hat{y} + z, \quad \hat{y} \in W$$

This means that any vector $\bar{y}$ can be decomposed into two components: one is the projection $\hat{y}$ onto W (which is in W) and other component is perpendicular to W.

To derive an explicit expression, we suppose that W has an orthogonal basis $\mathcal{B} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_p\}$. Then we may write, since $\hat{y} \in W$,

$$\hat{y} = \text{Proj}_W(\vec{y}) = c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_p\bar{u}_p$$

and

$$\bar{y} = \hat{y} + \vec{z} = (c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_p\bar{u}_p) + \vec{z}, \quad \vec{z} \perp W.$$

By multiplying by $\vec{u}_i$, we find

$$\bar{y}\cdot\bar{u}_i = (c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_p\bar{u}_p + \bar{z})\cdot\bar{u}_i = c_i\bar{u}_i\cdot\bar{u}_i \Longrightarrow c_i = \frac{\bar{y}\cdot\bar{u}_i}{\bar{u}_i\cdot\bar{u}_i}$$

Therefore,

$$\hat{y} = \text{Proj}_W(\bar{y}) = c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_p\bar{u}_p$$

$$= \frac{\vec{y}\cdot\vec{u}_1}{\|\bar{u}_1\|^2}\bar{u}_1 + \frac{\bar{y}\cdot\bar{u}_2}{\|\bar{u}_2\|^2}\bar{u}_2 + \cdots + \frac{\bar{y}\cdot\bar{u}_p}{\|\bar{u}_p\|^2}\bar{u}_p \quad (*_5)$$

Example 1.3.2. Let

$$\vec{u}_1 = \begin{pmatrix}3\\1\\1\end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix}-1\\2\\1\end{pmatrix}, \quad \vec{y} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$$

Find (a) $\text{Proj}_{\vec{u}_1}(\vec{y})$ (b) $\text{Proj}_{\vec{u}_2}(\vec{y})$, (c) $\text{Proj}_W(\bar{y})$, where $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$

Solution. We see that

$$\vec{u}_1\cdot\vec{u}_2 = \begin{pmatrix}3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix} = 0$$

Thus $\vec{u}_1$ and $\vec{u}_2$ form an orthogonal basis for $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$

(a) By $(*_4)$,

$$\text{Proj}_{\vec{u}_1}(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 = \frac{6+3-1}{11}\begin{pmatrix}3\\1\\1\end{pmatrix} = \frac{8}{11}\begin{pmatrix}3\\1\\1\end{pmatrix}$$

(b) Analogously, $\text{Proj}_{\vec{u}_2}(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2 = \frac{-2+6-1}{6}\begin{pmatrix}-1\\2\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\2\\1\end{pmatrix}$

(c) Using $(*_5)$ and aswer from part (a) & (b) $\text{Proj}_W(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2$

$$= \text{Proj}_{\vec{u}_1}(\vec{y}) + \text{Proj}_{\vec{u}_2}(\vec{y}) = \frac{8}{11}\begin{pmatrix}3\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

In general, suppose that W has an orthogonal basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$. Then

$\text{Proj}_W(\vec{y}) = \text{Proj}_{\vec{u}_1}(\vec{y}) + \text{Proj}_{\vec{u}_2}(\bar{y}) + \cdots + \text{Proj}_{\vec{u}_p}(\vec{y})$

1.4. Orthogonal functions and Polynomials

Let V be the 'vector space' of continuous integrable functions defined on some interval $[a,b]$. Many inner product can be defined on V. Here we define the inner product of two functions g and h in one of two ways:

$$\langle g,h\rangle = \int_a^b g(x)\, h(x)\, w(x)\, dx \quad \text{or} \qquad (1)$$

$$\langle g,h\rangle = \sum_{i=1}^{n} g(x_i)\, h(x_i)\, w(x_i), \qquad (2)$$

where $w(x)$ is a positive function, called a <u>weighting function</u>. With the inner product defined, we say that the two functions $g(x)$ and $h(x)$ are <u>orthogonal</u> if

$$\langle g,h\rangle = 0$$

<u>Example 1.4.1</u>

It is easy to verify, for example, that the function $g(x)=1$, $h(x)=x$ are orthogonal if the inner product is

$$\langle g,h\rangle = \int_{-1}^{1} g(x)\, h(x)\, dx$$

The functions $g(x) = \sin nx$, $h(x) = \sin mx$, integers n and m, are orthogonal if

$$\langle g,h\rangle = \int_0^{2\pi} g(x)\, h(x)\, dx, \quad \text{and} \quad n \neq m.$$

<u>Orthogonal Polynomials</u>: Let $\mathcal{P}$ be the vector space of all polynomials with real coefficients: $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, Basis for $\mathcal{P}$ are $1, x, x^2, \dots, x^n, \dots$

Suppose that $\mathcal{P}$ is endowed with an inner product.

<u>Definition 1.4.2</u>. ~ (relative to the inner product) are polynomials $p_0, p_1, p_2, \dots$ such that degree $p_n = n$ (p_0 is a nonzero constant) and $\langle p_n, p_m\rangle = 0$ for $n \neq m$.

Example 1.4.3. The polynomials $p_0(x)=1$, $p_1(x)=x$, and $p_2(x)=3x^2-1$ constitute a sequence of orthogonal polynomials under the inner product

$$\langle g,h\rangle = \int_{-1}^{1} g(x)h(x)\,dx.$$

We know from example 1.4.1 that $\langle p_0,p_1\rangle = 0$. Also we obtain that

$$\langle p_0,p_2\rangle = \int_{-1}^{1} 1\cdot(3x^2-1)\,dx = x^3-x\Big|_{-1}^{1} = 0, \quad \langle p_1,p_2\rangle = \int_{-1}^{1} x\cdot(3x^2-1)\,dx =$$

$$\frac{3}{4}x^4-\frac{1}{2}x^2\Big|_{-1}^{1} = 0.$$

The following facts can be proved about a finite sequence of orthogonal polynomials $p_0(x), p_1(x), \ldots, p_k(x)$:

(i) If $p(x)$ is any polynomial of degree at most k, then one can write

$$p(x) = d_0p_0(x) + d_1p_1(x) + \cdots + d_kp_k(x)$$

with the coefficients $d_0, \ldots, d_k$ uniquely determined when $\langle p_i,p_i\rangle \neq 0$ for all i. Let us take the inner product of $p_i(x)$ with both sides of the above equation:

$$\langle p,p_i\rangle = d_0\langle p_0,p_i\rangle + \cdots + d_k\langle p_k,p_i\rangle$$
$$= 0+\cdots+0+d_i\langle p_i,p_i\rangle+0+\cdots+0, \quad \langle p_j,p_i\rangle = 0 \text{ for all } j\neq i.$$

Hence we have

$$d_i = \frac{\langle p,p_i\rangle}{\langle p_i,p_i\rangle}.$$

(ii) If $p(x)$ is any polynomial of degree less than k, then

$$\langle p,p_k\rangle = 0$$

By Property i), $p(x) = d_0p_0(x)+d_1p_1(x)+\cdots+d_\ell p_\ell(x)$, where $\ell < k$ is the degree of p. Taking the inner product of p_k with both sides of this equation, we obtain that $\langle p,p_k\rangle = 0$.

(iii) If the inner product is given by (1), then $p_k(x)$ has k simple real zeros in the interval (a,b).

(iv) The orthogonal polynomials satisfy a three-term recurrence relation:

$$p_{i+1}(x) = A_i(x)(x-B_i)p_i(x) - C_ip_{i-1}(x), \quad i=0,1,\ldots,k-1$$

where $p_{-1}(x)=0$ and

$$A_i = \frac{\text{Leading coefficient of } p_{i+1}}{\text{leading coefficient of } p_i}$$

$$B_i = \frac{\langle xp_i,p_i\rangle}{\langle p_i,p_i\rangle}, \qquad C_i = \begin{cases} \text{arbitrary} & \text{if } i=0 \\ \dfrac{A_i\cdot\langle p_i,p_i\rangle}{A_{i-1}\cdot\langle p_{i-1},p_{i-1}\rangle} & \text{if } i>0 \end{cases}$$

In the case where the polynomials are monic (with leading coefficient 1), the following recurrent holds:

$$P_{-1}(x) = 0$$

$$P_0(x) = 1$$

$$P_1(x) = \left(x - \frac{\langle xP_0, P_0\rangle}{\langle P_0, P_0\rangle}\right) P_0(x)$$

$$P_{i+1}(x) = \left(x - \frac{\langle xP_i, P_i\rangle}{\langle P_i, P_i\rangle}\right) P_i(x) - \frac{\langle P_i, P_i\rangle}{\langle P_{i-1}, P_{i-1}\rangle} P_{i-1}(x), \quad i = 1, 2, \ldots$$

This property allows us to generate an orthogonal polynomial sequence provided $\langle P_i, P_i\rangle \neq 0$ for all i.

Example 1.4.4. Legendre polynomials. The inner product is given by

$$\langle g, h\rangle = \int_{-1}^{1} g(x)h(x)\,dx$$

starting with $P_0(x) = 1$, we get

$$\langle P_0, P_0\rangle = \int_{-1}^{1} 1\,dx = 2$$

$$\langle xP_0, P_0\rangle = \int_{-1}^{1} x\,dx = 0$$

Hence

$$P_1(x) = (x - 0)\,P_0(x) = x$$

and

$$\langle P_1, P_1\rangle = \int_{-1}^{1} x^2\,dx = 2/3$$

$$\langle xP_1, P_1\rangle = \int_{-1}^{1} x^3\,dx = 0$$

So now we have

$$P_2(x) = (x - 0)P_1(x) - \frac{2/3}{2} P_0(x) = x^2 - \frac{1}{3}$$

Continuing this process we would get

$$P_0(x) = 1,$$

$$P_1(x) = x,$$

$$P_2(x) = x^2 - 1/3$$

$$P_3(x) = x^3 - \frac{3}{5}x$$

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35},$$

Scanned with CamScanner

1.4.5 Applications: Least squares approximation by polynomials

Given a function $f(x)$ defined on some interval (a,b), we want to approximate it by a polynomial of degree at most k. Here we measure the difference between $f(x)$ and a polynomial $p(x)$ by

$$\langle f(x)-p(x), f(x)-p(x)\rangle,$$

where the inner product is defined by either (1) or (2). And we would like to seek a polynomial of degree at most k to minimize the above inner product. Such a polynomial is a least squares approximation to $f(x)$ by polynomials of degree not exceeding k.

The least-squares approximation of a function f by polynomials in subspace of polynomial functions of degree no more than k is its orthogonal projection onto the subspace. The coordinates of this projection along the axes $p_0, \ldots, p_k$ are thus $\frac{\langle f, p_0\rangle}{\langle p_0, p_0\rangle}, \ldots, \frac{\langle f, p_k\rangle}{\langle p_k, p_k\rangle}$.

Below is the illustration on the use of orthogonal polynomials for obtaining least-squares approximations w.r.t. continuous version of inner products

Example 1.4.6 Calculate the polynomial at degree at most 3 that best approximate e^x over the interval $[-1,1]$ in the least-squares sense.

Solution. ~~Here~~ We proceed by finding an orthogonal sequence of polynomials $p_0(x), \ldots, p_k(x)$ for the chosen inner product such that $\langle p_i, p_j\rangle = 0$ whenever, $i \neq j$. Then every polynomial of degree at most k can be written uniquely as $p(x) = d_0 p_0(x), \ldots, d_k p_k(x)$ where $d_i = \frac{\langle p, p_i\rangle}{\langle p_i, p_i\rangle}$

Here we obtain a best approximation by orthogonally projecting e^x onto the subspace of functions spanned by Legendre polynomials $p_0, \ldots, p_3$. In other words,

$$p(x) = \sum_{i=0}^{3} d_i p_i(x)$$

where $d_i = \frac{\langle e^x, p_i\rangle}{\langle p_i, p_i\rangle}$

Compute the following inner products:

$$\langle p_0, p_0\rangle = \int_{-1}^{1} 1\, dx = 2,$$

$$\langle p_1, p_1\rangle = \int_{-1}^{1} x^2 dx = 2/3,$$

$$\langle p_2, p_2\rangle = \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx = \frac{8}{45},$$

$$\langle p_3, p_3\rangle = \int_{-1}^{1} \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right) dx = \frac{8}{175},$$

$$\langle e^x, p_0\rangle = \int_{-1}^{1} e^x dx = e - \frac{1}{e}, \quad \langle e^x, p_1\rangle = \int_{-1}^{1} e^x . x\, dx = 2/e$$

$$\langle e^x, p_2\rangle = \int_{-1}^{1} e^x\left(x^2 - \frac{1}{3}\right) dx = \frac{2}{3}e - \frac{14}{3e}, \quad \langle e^x, p_3\rangle = \int_{-1}^{1} e^x\left(x^3 - \frac{3}{5}x\right) dx = -2e + \frac{74}{5e}$$

Then

$$d_0 = \frac{1}{2}\left(e - \frac{1}{e}\right) \simeq = 1.1752012.$$

$$d_1 = \frac{3}{2}\cdot\frac{2}{e} \simeq 1.1036383; \quad d_2 = \frac{45}{8}\left(\frac{2}{3}e - \frac{14}{3e}\right) \simeq 0.53672153$$

$$d_3 = \frac{175}{8}\left(-2e + \frac{74}{5e}\right) \simeq 0.17613908.$$

So the least-squares approximation to e^x on $(-1, 1)$ is

$$p(x) = d_0 p_0(x) + d_1 p_1(x) + d_2 p_2(x) + d_3 p_3(x)$$

$$= 0.99629402 + 0.99795487x + 0.53672153x^2 + 0.17613908x^3.$$

Reading Assignments: Read extensively on orthonormal matrix

Fourier Series, Integrals and Transform

2.1. Periodic functions:

If the value of each ordinate $f(t)$ repeats at equal intervals in the abscissa, then $f(t)$ is said to be a ~

If $f(t) = f(t+T) = f(t+2T) = \dots$ then T is called the period of the function $f(t)$. For example:

$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \dots$ so $\sin x$ is a periodic function with the period 2π.

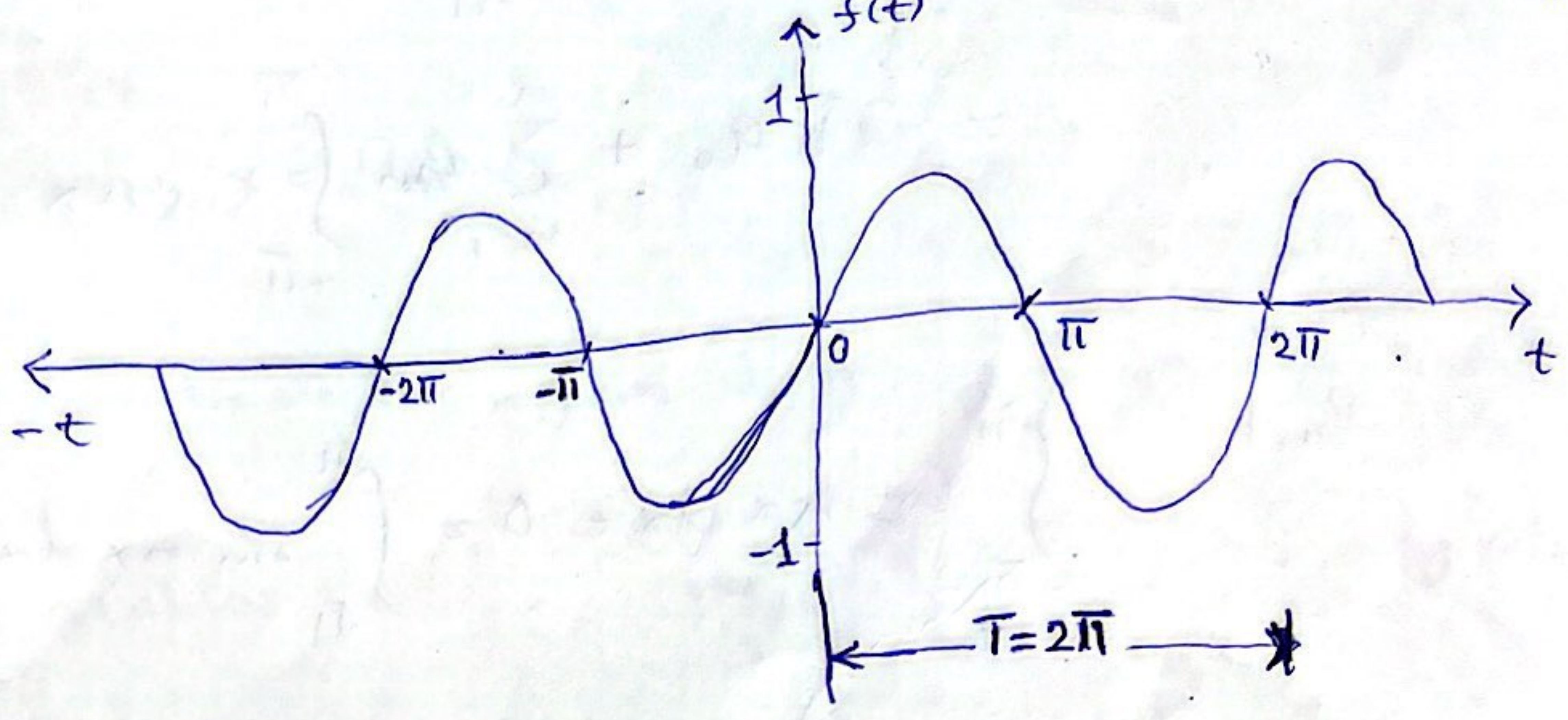

Fig 1: — Trigonometric function with period

2.2. Fourier Series

When the French mathematician Joseph Fourier (1768—1830) was trying to solve a problem in heat conduction, he needed to express a function as an infinite series of sine and cosine functions:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{——— (1)}$$

The series in Equation 1 is called a trigonometric series or Fourier series and it turns out that expressing a function as a Fourier series is sometimes more advantageous that expanding it as a power series.

We start by assuming that the Fourier series converges and has a continuous function $f(x)$ as its sum on the interval $[-\pi, \pi]$, that is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad -\pi \le x \le \pi \quad \text{—— (2)}$$

Our aim is to find formula for the coefficients a_n and b_n in terms of f. Recall that for a power series $f(x) = \sum c_n (x-a)^n$ we found a formula for the coefficients in terms of derivatives:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Here, we use integrals. If we integrate both sides of Equation (2) and assume that it's permissible to integrate the series term-by-term, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx)\,dx$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\,dx$$

But $\int_{-\pi}^{\pi} \cos nx\,dx = 0 = \int_{-\pi}^{\pi} \sin nx\,dx$

So,

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$$

and solving for a_0 gives

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad (3)$$

To determine a_n for $n \geq 1$ we multiply both sides of Equation (2) by $\cos mx$ (where m is an integer and $m \geq 1$) and integrate term-by-term from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \cos mx\,dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right) \cos mx\,dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx$$

A simple calculation shows that

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad \text{for all } m, n$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

So we get,

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \pi$$

Solving for a_m, and then replacing m by n, we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, 3, \ldots \qquad \text{—— (4)}$$

Similarly, if we multiply both sides of Equation (2) by $\sin mx$ and integrate from $-\pi$ to π, we get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, 3, \ldots \qquad \text{—— (5)}$$

Remarks: (1) Two functions ϕ_m and ϕ_n of the same form are orthogonal if $\int \phi_m \phi_n \, dx = 0$ for all $m \neq n$ and $\int \phi_m \phi_n \, dx = \alpha$ for all $m = n$.

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \forall \, m \neq \pm n$$

$$\int_{-\pi}^{\pi} (\cos nx)^2 \, dx \neq 0 \quad \forall \, m = n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad \forall \, m \neq n$$

$$\int_{-\pi}^{\pi} (\sin nx)^2 \neq 0 \quad \forall \, m = n$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \quad \forall \, m = n$$

$\frac{1}{2\pi}\left(\frac{4\pi^2}{2}\right) = \pi$

(2) We have derived Formulas (3), (4), and (5) assuming f is a continuous function such that Equation (2) holds and for which the term-by-term integration is legitimate. But we can still consider the Fourier series of a wider class of functions: A piecewise continuous function on $[a,b]$ is continuous except perhaps for a finite number of removable or jump discontinuities.

Example 1: Find the Fourier series representing

$$f(x) = x, \quad 0 < x < 2\pi$$

Solution: Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ——— (*)

Hence $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left(x\,\frac{\sin nx}{n} - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right)_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right) = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left(\frac{-2\pi\cos 2n\pi}{n}\right) = -\frac{2}{n}$$

Substituting the values of a_0, a_n, b_n in (*), we get

$$x = \pi - 2\left(\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \dots\right).$$