MTS 361 –Metric Spaces

Course Outline

Metric Spaces: Definition and examples

Open sets, closed sets, Cantor set. Neighbourhoods, Limit points, Closure, Dense subset

Interior, exterior, frontier,

Convergence in Metric spaces

Continuity, compactness and connectedness.

Heine-Borel theorem, Bolzano–Wierstrass theorem.

1 Metric Spaces: Definition and Examples

Recall that the real number system $\mathbb{R}$ is richly endowed both with algebraic structure and topological structure. We summirize the algebraic properties by saying that $\mathbb{R}$ together with the binary operations of addition and multiplication is a field. However, the algebra in $\mathbb{R}$ is entwined with an order relation which leads to the notion of distance between any two numbers, and it is this notion of distance that generates a topological structure in $\mathbb{R}$. In this course we shall develop some of these topological properties in a more general setting. In many areas of mathematics, the concept of *neighbouring* or *near* is important because it allows us the possibility of considering limit processes and continuity. The most natural and concrete way of introducing this concept is by defining a distance function or metric in a set. Thus, the metric should be defined so that it behaves in a manner consistent with our idea of how distance should behave; in particular, it should agree with the important properties of the distance function defined on the real line $\mathbb{R}$.

Analysis is primarily concerned with limit processes and continuity. These concepts are given by the following definitions.

Definition 1 *The sequence* $\{x_n\}_{n=1}^{\infty}$ *of real numbers is said to be convergent if there exists a real number* x *(called the limit of the sequence) such that, given* $\epsilon > 0$, *a positive integer* n_0 *can be found with the property that*

$$n \geq n_0 \Longrightarrow |x_n - x| < \epsilon.$$

That is, $x_n \to x$ *as* $n \to \infty$ *or* $\lim_{n\to\infty} x_n = x$.

Definition 2 *A real-valued function* f *defined on a non-empty subset* X *of the real line is said to be continuous at* x_0 *in* X *if for each* $\epsilon > 0$ *there exist a* $\delta > 0$ *such that*

$$x \in X \text{ and } |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon,$$

and f *is said to be continuous if it is continuous at each point of* X.

The above definitions are dependent for their meaning in the concept of the absolute value of the difference between two real numbers. We observe that the definitions also hold for complex sequences and complex-valued functions with complex domain. It is therefore necessary to define some notion of distance which will be applicable to the elements of arbitrary sets.

Thus, a metric space is an initial point to start. It is a non-empty set equipped with a concept of distance which is suitable for the treatment of convergent sequences in the set and continuous functions defined on the set. The basic facts about metric spaces and the motivation they provide for development of topological spaces will be well examined. First, we make the following definition:

Definition 3 *A metric space is a nonempty set X together with a real-valued function $d : X \times X \to \mathbb{R}$ which satsifies the following four conditions:*

For all $x, y, z \in X$

M1 $d(x, y) \geq 0$ *(positive property)*

M2 $d(x, y) = 0$ *if and only if* $x = y$

M3 $d(x, y) = d(y, x)$ *(symmetric property)*

M4 $d(x, y) \leq d(x, z) + d(z, y)$, *(triangle inequality).*

The function d is called a metric on X, and $d(x, y)$ is to be thought of as the distance from x to y. A nonempty set X equipped with a metric d is denoted by (X, d) and is called *a metric space.* Usually, we simply say that X is a metric space, if we need to specify the metric, we say that (X, d) is a metric space.

Remark 4 *(i) Different metrics could be defined on the same set giving rise to different metric spaces.*

(ii) Note that if distances are generally greater going via an additional point, then they are greater going via any number of additional points $z_1, z_2, ..., z_n$: *for, by repeated use of* $M4$.

$$\begin{aligned}
d(x,y) &\leq d(x,z_1) + d(z_1,y) \\
&\leq d(x,z_1) + d(z_1,z_2) + d(z_2,y) \\
&\leq d(x,z_1) + d(z_1,z_2) + d(z_2,z_3) + d(z_3,y) \\
&\leq ... \\
&\leq d(x,z_1) + d(z_1,z_2) + d(z_2,z_3) + ... + d(z_n,y).
\end{aligned}$$

Theorem 5 *(Inverse Triangle Inequality). For all elements* x, y, z *in a metric space* (X, d), *we have*

$$|d(x,y) - d(x,z)| \leq d(y.z).$$

Proof. Since the absolute value $|d(x,y) - d(x,z)|$ is the largest of the two numbers $d(x,y) - d(x,z)$ and $d(x,z) - d(x,y)$, it suffices to show that they are both less or equal to $d(y,z)$. By the triangle inequality

$$d(x,y) \leq d(x,z) + d(z,y)$$

and hence

$$d(x,y) - d(x,z) \leq d(z,y) = d(y,z).$$

To get the other inequality, we use the triangle inequality again,

$$d(x,z) \leq d(x,y) + d(y,z)$$

and hence

$$d(x,z) - d(x,y) \leq d(y,z).$$

■

1.1 Examples of Metric Spaces

Given below are examples of metric spaces. The verifications that the functions defined are metric are, in some cases, left as exercises, but in some cases, generous hints are also usually given.

Example 6 *For a non-empty set X the discrete (or trivial) metric d is defined on $X \times X$ by*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

This is the roughest of metrics; given any $x \in X$, it is simply a measure of coincidence with x. This example shows that every non-empty set can be provided with a metric. The verification that d is a metric on X is trivial and so left as an exercise .

Example 7 *Let $X = \Re$ - the set of real numbers. Consider the properties of absolute value of $x \in X$. That is,*

(i) $|x| \geq 0$,

(ii) $|x| = 0 \iff x = 0$;

(ii) $|-x| = |x|$;

(iii) $|x + y| \leq |x| + |y|$.

Now define a metric $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$d(x, y) = |x - y| \quad \text{for all} \quad x, y \in \mathbb{R}.$$

Then d is a metric $\mathbb{R}$ often called the usual metric on $\mathbb{R}$ or the Euclidean metric on $\mathbb{R}$. Thus, the real line $\mathbb{R}$ as a metric space is always understood to have this as its metric. The fact that this is a metric space follows directly from the properties of absolute values above.

To see this, we have

Example 8 *<u>For M1</u> We have by (i) that for any $x, y \in \mathbb{R}$,*

$$d(x, y) = |x - y| \geq 0$$

thus verifying M1

For M2. *We have by (ii) that for any* $x, y \in \mathbb{R}$,

$$d(x,y) = |x-y| = 0 \iff x-y=0 \iff x=y.$$

For M2 We have by (iii),

$$d(x,y) = |x-y| = |-(y-x)| = |y-x| = d(y,x).$$

For M4, we have by (iv),

$$\begin{aligned} d(x,y) &= |x-y| \\ &= |(x-z)+(z-y)| \\ &\leq |x-z|+|z-y| = d(x,z)+d(z,y). \end{aligned}$$

Example 9 *Let $X = \mathbb{C}-$ the set of complex numbers. Note that $z \in \mathbb{C}$ is of the form $z = a + ib$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ by*

$$d(z, w) = |z - w|\,,\ \ z, w \in \mathbb{C}.$$

Then $(\mathbb{C}, d)$ is a metric space.

Most of the metric spaces we will consider are also linear spaces. In particular, most of these cases, the metric is generated by a simpler function called a norm which assigns a length to each vector in the linear space.

Definition 10 *Given a liner space X over $\mathbb{R}$ (or $\mathbb{C}$), a norm $\|.\|$ for X is a function on X which assigns to each element a real number, (or formally, $\|.\| : X \to \mathbb{R}$), satisfying the following properties:*

(i) $\|x\| \geq 0$,

(ii) $\|x\| = 0 \iff x = 0$;

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$ for any scalar α

and for all $x, y \in X$;

(iv) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality)

A linear space X with a norm $\|.\|$ is denoted by $(X, \|.\|)$ is called a normed linear space. Observe that different norms can be defined on the same linear space giving rise to different normed linear spaces.

Remark 11 *Given a normed linear space* $(X, \|.\|)$, *it is clear that the function* $d : X \times X \to \mathbb{R}$ *defined by*

$$d(x, y) = \|x - y\|$$

is a metric for X, *and we call this the metric generated by the* $\|.\|$. *So, every normed linear space is a metric space under the metric generated by its norm.*

Example 12 $(\mathbb{R}, |.|)$ *and* $(\mathbb{C}, |.|)$. *The set of real numbers* $\mathbb{R}$ *(the set of complex numbers* $\mathbb{C}$*) is a normed linear space with norm given by the modulus, that is,*

$$\|x\| = |x| .$$

We call this the usual norm for $\mathbb{R}$ *(or* $\mathbb{C}$*) and it generates the usual metric*

$$d(x, y) = |x - y| .$$

These are the spaces we are familiar with in real and complex analysis respectively.

Example 13 *Let $X = \mathbb{R} \times \mathbb{R}$ and let d be the usual distance function defined in the plane*

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

for any two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R} \times \mathbb{R}$. Then $(\mathbb{R} \times \mathbb{R}, d)$ is a metric space. We refer to d as theusua metric on $\mathbb{R} \times \mathbb{R}$. We can define another function d' on $\mathbb{R} \times \mathbb{R}$ by

$$d'(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

for any two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R} \times \mathbb{R}$. Then $(\mathbb{R} \times \mathbb{R}, d')$ is easily seen to be a metric space.

Since d and d' are different functions, we get two distinct metric spaces $(\mathbb{R} \times \mathbb{R}, d)$ and $(\mathbb{R} \times \mathbb{R}, d')$. We shall show later in the nex section, that topologically these spaces are really the same but they are different metric spaces because of d and d'. we also note that these metrics satisfy the inequalities

$$d(x,y) \leq d'(x,y) \leq \sqrt{2}d(x,y)$$

for all points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R} \times \mathbb{R}$.

Example 14 *Let $X = \mathbb{R}^n$ for any natural number n and define $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by*

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\mathbf{x} = (x_1 ...., x_n)$ and $\mathbf{y} = (y_1 ...., y_n)$ are elements in $\mathbb{R}^n$. Then ($\mathbb{R}^n, d$) is called the euclidean $n-$space, and d is referred to as the euclidean metric on $\mathbb{R}^n$. Properties $M1 - M3$ are obvious and property $M4$, the triangle inequality, follows from the Cauchy-Schwartz inequality.

Example 15 *The set of all $n-$tuples of real numbers $\mathbb{R}$ with metric ρ given by*

$$\rho(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + ... + |x_n - y_n| .$$

Then $(\mathbb{R}, \rho)$ is a metric space.

Example 16 *If (X, d) and (Y, ρ) are metric spaces, then thier Cartesian product $X \times Y = \{(x, y) : \ x \in X, \ y \in Y\}$ whose metric is given by*

$$\tau\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \left[d\left(x_1, x_2\right)^2 + \rho\left(y_1, y_2\right)^2\right]^{\frac{1}{2}} .$$

Then $(X \times Y, \tau)$ is a metric space.

Example 17 *Let* $B[0,1] = \{f : [0,1] \to\to \mathbb{R} \,|\, f$ *is bounded*$\}$. *Note that* $f : [0,1] \to \mathbb{R}$ *is bounded if* $\exists k > 0$ *such that* $|f(x)| \leq k\ \forall x \in [0,1]$. *Now define* $f \pm g$ *by*

$$(f \pm g)(x) = f(x) \pm g(x) \text{ whenever } f, g \in B[0,1] \text{ and } x \in [0,1].$$

Also let

$$\|f\| = \int_0^1 |f(x)|\, dx$$

be the norm of f, *then* $\|.\|$ *induces a metric* d *on* $B[0,1]$ *where*

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Then $(B[0,1], d)$ *is a metric space.*

Example 18 *Let $C[0,1]$ be the set of all bounded continuous real functions in $[0,1]$. Define*

$$\|f\| = \textsf{sup}\left\{|f(x)| : x \in [0,1]\ \right\},$$

where $f \in C[0,1]$. Then the $\|.\|$induces a metric d on $C[0,1]$ defined by

$$d(f,g) = \textsf{sup}\left\{|f(x) - g(x)| : f, g \in C[0,1] \text{ and } x \in [0,1]\ \right\}.$$

Then $(C[0,1], d)$ is a metric space.

The verification that the above are metric spaces are easy and left as exercises to the readers.

Remark 19 *If we relax the condition that $d(x,y) = 0$ if and only if $x = y$ with the condition $d(x,y) = 0$ for some $x \neq y$, we call d a pseudometric.*

Thus the L^p norms are pseudometrics on the spaces of measurable functions whose p^{th} power are integrable.

Exercises

1. Given a metric space (X, d), prove that for all $x, y, z \in X$, $|d(x, z) - d(y, z)| \leq d(x, y)$.

2. Given a non-empty set X and a function $d : X \times X \to \mathbb{R}$ with properties

   (i) $d(x, y) = 0 \nRightarrow x = y$,

   (ii) $d(x, y) \leq d(x, z) + d(y, z)$.

Prove that d is a metric for X.

3. Use the fact that $d(x,x) \le d(x,y) + d(y,x)$ for any x, y in a metric space (X,d) to deduce that $d(x,y) \ge 0$.

4. Let $X = \mathbb{R}^2$ and for $\mathbf{x}, \mathbf{y} \in X$ define $d(\mathbf{x}, \mathbf{y})$ by

$$d(\mathbf{x},\mathbf{y}) = d((x_1,x_2),(y_1,y_2)) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2 \\ |x_1| + |x_2 - y_2| + |y_1| & \text{if } x_2 \neq y_2. \end{cases}$$

Then, prove that (X,d) is a metric space.

5. Prove that $d(x,y) \le d'(x,y) \le \sqrt{2}d(x,y)$ for all points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R} \times \mathbb{R}$, where d is the usual metric and d' is the rectangular metric.

6. Verify that each of the following functions is a metric on $\mathbb{R}^n$:

i $d'(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ for all points $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$. d' is called the rectangular metric on $\mathbb{R}^n$.

ii $d^*(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ for all points $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$.

7. Prove that the following inequalities hold for all points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

$$d^*(x, y) \leq d'(x, y) \leq \sqrt{n} d^*(x, y)$$

for all points $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n \times \mathbb{R}^n$, where d^* is the metric in $6(b)$ above and d is the euclidean metric on $\mathbb{R}^n$.

8. Suppose (X, d_1) and (X, d_2) are metric spaces and k is a positive real numbers. Which of the followings are metric spaces?

i (X, d_1^2)

ii (X, kd_1)

iii $(X, d_1 + d_2)$

iv $(X, d_1 d_2)$

v $(X, \max(d_1, d_2))$

vi $(X, \min(d_1, d_2))$

9. Let X be a nonempty set and suppose that $d : X \times X \longrightarrow \mathbb{R}$ satisfies

$$d(x, y) = 0 \text{ if and only if } x = y$$

and

$$d(x, y) \leq d(x, z) + d(y, z) \quad \text{for all } x, y, z \in X.$$

Prove that (X, d) is a metric space.

10. Let (X, d) be a metric space and suppose $\rho : X \times X \longrightarrow \mathbb{R}$ is defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

for all points $x, y \in X$. Prove that (X, ρ) is a metric space, that it is bounded, and that $\rho(x, y) \leq d(x, y)$ for all $x, y \in X$.

2 Open sets, Closed sets and Neighbourhoods

2.1 Open Sets

Definition 20 *Let* (X, d) *be a metric space and let* x_0 *be a point of* X *and* $r > 0$ *is a real number. We define the open sphere (ball)* $S_r(x_0)$ *with centre* x_0 *and radius* r *as the subset of* X *given by*

$$S_r(x_0) = \{x : d(x, x_0) < r\}.$$

Remark 21 *(i)* $S_r(x_0) \neq \emptyset$ *since* $x_0 \in S_r(x_0)$.

(ii) For the metric space (X, d) where

$$d(x, y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{if } x \neq y, \end{cases}$$

we have $S_1(x_0) = \{x_0\}$.

(iii) For $S_r(x_0) \subset \mathbb{R}$, $S_r(x_0) = (x_0 - r, x_0 + r)$ which is a bounded open interval in $\mathbb{R}$. Hence, any bounded open interval on the real line is an open sphere.

Example 22 *Let $X = \mathbb{R}$ with the usual metric d defined by*

$$d(x, y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{if } x \neq y, \end{cases}$$

for arbitrary points $x, y \in \mathbb{R}$. Describe the open spheres (balls)

$$(i)\ \ S_{\frac{1}{2}}(1) \quad (ii)\ \ S_1(1) \quad (iii)\ \ S_2(1) \quad (iv)\ \ S_{\frac{1}{2}}(2)$$

$$(v)\ \ S_1(5) \quad (vi)\ \ S_{\frac{3}{2}}(4).$$

Solution 23 (i) *By definition*

$$S_{\frac{1}{2}}(1) = \left\{ y \in \mathbb{R} : d(y, 1) < \frac{1}{2} \right\}.$$

We know fron the definition of d that for all $x, y \in \mathbb{R}$, $d(x, y)$ has two values namely 0 or 1 Thus, $d(y, 1) < 1$ implies that it cannot be equal to 1. Hence,

$d(y, 1)$ *must be equal to* 0*. Therefore, it follows that*

$$S_{\frac{1}{2}}(1) = \{y \in \mathbb{R} : d(y, 1) = 0\}.$$

Since d *is a metric, it thus follows that* $d(x, y) = 0 \iff x = y$*. In particular,* $d(y, 1) = 0$ *implies that* $y = 1$*. Hence,*

$$S_{\frac{1}{2}}(1) = \{y \in \mathbb{R} : d(y, 1) = 0\} = \{y \in \mathbb{R} : y =\} = \{1\}.$$

Hence, the open ball $S_{\frac{1}{2}}(1) = \{1\}$ *is a singleton set* $\{1\}$.

(ii)

$$S_1(1) = \{y \in \mathbb{R} : d(y, 1) < 1\} = \{y \in \mathbb{R} : d(y, 1) = 0\}$$

Again, since d is a metric, we have that for each pair $x.y \in \mathbb{R}$, $d(x,y) = 0 \iff x = y$. In particular, $d(y,1) = 0 \implies y = 1$. Hence

$$S_1(1) = \{y \in \mathbb{R} : d(y,1) = 0\} = \{y \in \mathbb{R} : y = 1\} = \{1\}.$$

(vi)

$$S_{\frac{3}{2}}(4) = \left\{y \in \mathbb{R} : d(y,4) < \frac{3}{2}\right\}.$$

But $d(x,y) = 0$ or 1 for all $x, y \in \mathbb{R}$. In either case, $d(x,y)$ is always less than $\frac{3}{2}$ for all $x, y \in \mathbb{R}$. Hence,

$$S_{\frac{3}{2}}(4) = \left\{y \in \mathbb{R} : d(y,4) < \frac{3}{2}\right\} = \{y \in \mathbb{R}\} = \mathbb{R}.$$

$(iii) - (v)$ *are left as exercises.*

Definition 24 *A subset G of a metric space (X, d) is open if given any point $x \in G, \exists\ r > 0$ such that $S_r(x) \subset G$.*

Remark 25 *On $\mathbb{R} = (-\infty, \infty)$:*

(i) $\{x\}$ *is not open since for* $r > 0, S_r(x) \not\subseteq \{x_0\}$

(ii) $[0, 1]$ *is not open since* $0 \in [0, 1]$ *and* $r > 0$, $S_r(0) \not\subseteq [0, 1]$. *Also for* $r > 0, S_r(1) \not\subseteq [0, 1]$.

(iii) $(0, 1)$ *is open.*

(iv) any open interval in $\mathbb{R}$, bounded or not is an open set in $\mathbb{R}$. Furthermore, the open intervals are the only intervals of $\mathbb{R}$ which are open sets.

Theorem 26 *In any metric space* (X, d), $\emptyset$ *and* X *are open.*

Proof. *Since* $\emptyset$ *has no point, each point in* $\emptyset$ *is the centre of an open sphere which is contained in* $\emptyset$. *This implies that* $\emptyset$ *is open. Clearly* X *is open since every open sphere centred at each points of* X *is contained in* X. ■

Remark 27 *A set is open or not only with respect to a specific metric space containing it and not on its own. For example, with respect to the metric space* $\mathbb{R}$, $[0, 1]$ *is not open. However, with respect to the metric space* $([0, 1], d)$ *it is open.*

Theorem 28 *In any metric space* (X, d)*, each open sphere is an open set.*

Proof. Let $S_r(x_0)$ be an open sphere in X, and let $x \in S_r(x_0)$. Then we shall produce an open sphere centred at x and contained in $S_r(x_0)$. Since $d(x, x_0) < r$, set $r_1 = r - d(x, x_0) > 0$. Then, we shall show that $S_{r_1}(x) \subseteq S_r(x_0)$. Now, let $y \in S_{r_1}(x)$, then $d(y, x) < r_{1.}$ Then by triangle inequality we obtain that

$$\begin{aligned} d(y, x_0) &\leq d(y, x) + d(x, x_0) \\ &< r_1 + d(x, x_0) \\ &= [r - d(x, x_0)] + d(x, x_0) \\ &= r. \end{aligned}$$

This implies that $y \in S_r(x_0)$. We have shown that

$S_{r_1}(x) \subset S_r(x_0)$. This implies that $S_r(x_0)$ is open.

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Theorem 29 *Let (X, d) be a metric space. A subset G of X is open iff G is a union of open spheres.*

Proof. First, suppose G is open, we shall show that G is a union of open spheres. If $G = \emptyset$, then G is a union of empty class of open spheres and so the conclusion is trivial. Now, suppose $G \neq \emptyset$, then each of its points is the centre of an open sphere which is contained in G. Hence, G is the union of all the open sphere contained in it.

Assume that G is the union of a class $\mathcal{S}$ of open spheres. We shall show that G is open. If $\mathcal{S} = \emptyset$, then G is also empty, and by Theorem 26 , G is open. Suppose $\mathcal{S} \neq \emptyset$, then it follows that $G \neq \emptyset$. Let $x \in G$, since G is the union of open spheres in $\mathcal{S}$, x belongs to an open sphere $S_r(x_0)$ in $\mathcal{S}$. By Theoren 28, x is the centre of an open sphere $S_{r_1}(x) \sqsubseteq S_r(x_0)$. Since $S_r(x_0) \sqsubseteq G$, $S_{r_1}(x) \sqsubseteq G$ and we have an open sphere centred on x and contained in G. G is therefore open. $\blacksquare$

Theorem 30 *Let X be a metric space. Then*

(i) any union of open sets is open;

(ii) any finite intersection of open sets is open.

Proof. (i) Suppose $\{G_i\}$ is an arbitrary class of open sets. We shall show that $G = \cup_i G_i$ is open. If $\{G_i\} = \emptyset$, then $G = \cup_i G_i = \emptyset$ and so G is open. Next, suppose $\{G_i\}$ is not empty. Then G_i being open is the union of open sphere. Hence G is the union of all such open spheres and so G is open.

(ii) Let $\{G_i\}$ be a finite class of open sets. we must show that $G = \cap_i G_i$ is open. Suppose $\{G_i\} = \emptyset$. Then $G = \cap_i G_i^c = \cup_i G_i^c = X$. Hence G is open for X is open. Now assume that $\{G_i\}$ is not empty. Let $\{G_i\} =$

$\{G_1, G_2, ..., G_n\}$,where $n \in \mathbb{N}$. If $G = \cap_{i=1}^{n} G_i = \emptyset$, then G is open. Suppose $G \neq \emptyset$, then $\exists x \in G$ and so $x \in G_i \ \forall i = 1, 2, ..., n$.

Since each G_i is open, $\exists r_i$, $i = 1, ..., n$ such that $S_{r_i}(x) \subseteq G_i$, where $r_i > 0$. Let $r = \min\{r_1, ..., r_n\}$. Then $S_r(x) \subseteq S_{r_i}(x) \subseteq G_i$, $i = 1, ..., n$. This implies that $S_r(x) \subseteq \cap_{i=1}^{n} G_i = G$. Hence G is open and the proof is complete.

■

Remark 31 *In any metric space, the class of open sets is closed under the formation of arbitrary unions and finite intersections However, if we take arbitrary intersection of open sets, the set obtained is not necessarily open. To see this, consider the class* $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right) : n = 1, 2, ...\right\}$ *of open intervals in* $\mathbb{R}$. *Observe that* $\cap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, *is closed since a singleton set is closed.*

2.2 Closed Sets

Definition 32 *Let (X, d) be a metric space and let $A \subseteq X$. A point $x \in X$ is called a limit point of A if each open sphere centred at x contains at least one point of A different from x.*

Example 33 *Let $X = \mathbb{R}$.*

(1) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$. Then, the point $0 \in \mathbb{R}$ is a limit point of A.

(2) If $A = [0, 1]$. Then 0 and 1 are limit points of A. Indeed, every element x, such that $0 \leq x \leq 1$ are limit points.

(3) If $A = (0, 1)$. Then 0 and 1 are limit points.

(4) If $A = \{..., -2, -1, 0, 1, 2, 3, ...\}$ .Then A has no limit point. To see this, take $x = \frac{1}{4} \in \mathbb{R}$, then $S_{\frac{1}{16}}(\frac{1}{4})$ does not contain any element of A.

Definition 34 *A set F in a metric space (X, d) is said to be closed if F contains each of its limit points.*

Example 35 *In the example above, (2) is closed while (1) and (3) are not closed.*

Theorem 36 *Let (X, d) be a metric space, then $\emptyset$ and X are closed sets.*

Proof. The empty set $\emptyset$ has no limit points, so it contains them all and is therefore closed.

Since X contains all points, it automatically contains its own limit points and thus closed. ■

Theorem 37 *Let (X, d) be a metric space. A subset F of X is closed if and only if its complement $X \sim F$ is open.*

Proof. Suppose $X \sim F$ is open. We shall show that F is closed. We know that F fails to be closed if it does not contain all its limit points. That is, if F has a limit point in $X \sim F$. Since $X \sim F$ is open, each of its points is the centre of an open sphere which is completely contained in $X \sim F$. Hence, no point of $X \sim F$ is a limit point of F. This implies that F contains all its limit points and so F is closed.

Conversely, suppose F is closed. We shall show that $X \sim F$ is open. If $X \sim F$ $= \emptyset$, then we know that it is open since $\emptyset$ is open. So, assume $X \sim F \neq \emptyset$.

Let $x \in X \sim F$. Since F is closed, $x \notin F$ and so x is not a limit point of F. Hence, $\exists$ an open sphere $S_r(x)$ such that $S_r(x) \cap F = \emptyset$. This says that $S_r(x) \subseteq X \sim F$. This implies that $X \sim F$ is open. ■

Definition 38 *Let x_0 be a point in a metric space (X, d) and r is a nonnegative real number, the closed sphere $S_r[x_0]$ with centre x_0 and radius r is the subset of X defined by*

$$S_r[x_0] = \{x : d(x, x_0) \leq r\}.$$

Remark 39 *(i) $S_r[x_0]$ is not empty because it contains only its centre when* $r = 0$.

(ii) Closed spheres on $\mathbb{R}$ are precisely the closed intervals.

(iii) Observed that though open spheres on $\mathbb{R}$ are open intervals, there are open intervals which are not open spheres, e.g., $(-\infty, +\infty)$.

Theorem 40 *In any metric space* (X, d), *each closed sphere is a closed set.*

Proof. Let $S_r[x_0]$ be a closed sphere in X. By Theorem 37, it suffices to show that its complement $X \sim S_r[x_0]$ is open. If $X \sim S_r[x_0] = \emptyset$, it is clearly open since $\emptyset$ is open. Next, assume $X \sim S_r[x_0] \neq \emptyset$. Let $x \in X \sim S_r[x_0]$. Since $d(x, x_0) > r$, set $r_1 = d(x, x_0) - r > 0$. Take r_1 as the radius of an open sphere $S_{r_1}[x]$ centred on x, and we show that $X \sim S_r[x_0]$ is open by showing that $S_{r_1}[x] \subseteq X \sim S_r[x_0]$. Let $y \in S_{r_1}[x]$ so that $d(y,x) < r_1$. Because of this and the fact that $d(x_0,x) \leq d(x_0,y) + d(y,x)$, we see that

$$\begin{aligned} d(y, x_{0)} &\geq d(x, x_0) - d(y, x) \\ &> d(x, x_0) - r_1 \\ &= d(, x_0) - [d(x, x_0) - r] \\ &= r, \end{aligned}$$

so that $y \in X \sim S_r[x_0]$. ■

Theorem 41 *Let (X, d) be a metric space. Then*

(i) any intersection of closed sets in X is closed; and

(ii) any finite union of closed sets in X is closed.

Proof. (i) Let $\{F_i\}_{i=1}^{\infty}$ be a collection of closed sets in X. Consider the $\cap_{i=1}^{\infty} F_i$. It will be closed if and only if $X \sim \cap_{i=1}^{\infty} F_i$ is open. But

$$X \sim \cap_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} (X \sim F_i) \quad \text{(by De Morgans law)}.$$

Hence $X \sim \cap_{i=1}^{\infty} F_i$ is the union of a collection of open sets since each $X \sim F_i$, $i = 1, 2, ...$ is open. By Theorem 30 (i) $\cup_{i=1}^{\infty} (X \sim F_i)$ is open and so $\cap_{i=1}^{\infty} F_i$ is closed.

(ii) Let $\{F_i\}_{i=1}^{k}$ be any finite member of closed sets in X. Consider the $\cup_{i=1}^{k} F_i$. It will be closed if and only if $X \sim \cup_{i=1}^{k} F_i$ is open. But

$$X \sim \cup_{i=1}^{k} F_i = \cap_{i=1}^{k} (X \sim F_i) \quad \text{(by De Morgans law)}.$$

Hence $X \sim \cup_{i=1}^{k} F_i$ is the intersection of a finite member of open sets, since each $X \sim F_i$, $i = 1, 2, ..., k$ is open. Then, by Theorem 30 (ii) $\cap_{i=1}^{k} (X \sim F_i)$ is open and so $\cup_{i=1}^{k} F_i$ is closed. ■

Remark 42 *Observe that in Theorem 41 (ii) the finiteness condition is important because the union of an infinite family of closed sets need not necessarily be closed:*

In $\mathbb{R}$ *with the usual metric, consider the family of closed intervals*

$$\left\{\left[\frac{1-k}{k}, \frac{k-1}{k}\right] : k \in \mathbb{N}\right\}. \quad \text{Now } \cup_{k=1}^{\infty} \left\{\left[\frac{1-k}{k}, \frac{k-1}{k}\right]\right\} = (-1, 1)$$

an open interval which is not a closed set.

Example 43 *Closed intervals in* $\mathbb{R}$ *are closed sets and closed spheres (balls) in* $\mathbb{R}^n$ *are closed sets in* $\mathbb{R}^n$*. To prove the latter assertion, consider the closed sphere* $S_r[x_0]$*. Let* x *be an arbitrary point in* $\mathbb{R}^n \sim S_r[x_0]$*. We shall construct an open sphere with* x *as centre which is entirely contained in* $\mathbb{R}^n \sim S_r[x_0]$*.*

Now $d(x_0, x) > r$ and hence $r' = d(x, x_0) - r$ is a positive real number. We construct the open sphere with x as the centre and r' as radius. We shall show that $S_{r'}(x)$ is contained entirely in $\mathbb{R}^n \sim S_r[x_0]$. In fact, let $y \in S_{r'}(x)$. Then we have

$$d(x, y) < d(x_0, x) - r$$

and hence

$$\begin{aligned} d(x_0, y) &= d(x_0, y) + d(y, x) - d(x, y) \\ &> d(x_0, x) - d(x, y) \\ &> r. \end{aligned}$$

Thus $y \in \mathbb{R}^n \sim S_r[x_0]$ and hence $S_r[x_0]$ is closed.

2.3 The Cantor Set

One important example of a closed set on $\mathbb{R}$ is the famous set first studied by George Cantor (1845-1918)

Example 44 *We denote by F_0 the closed interval $[0,1]$. From F_0, we remove the open middle interval $(\frac{1}{3},\frac{2}{3})$, that is the open middle thierd of F_0. Let the remaining closed set be denoted by F_1; thus $F_1=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. From each of the two disjoint closed sets $[0,\frac{1}{3}],[\frac{2}{3},1]$, we next remove the open middle third i.e. $(\frac{1}{9},\frac{2}{9})$ and $(\frac{7}{9},\frac{8}{9})$ and denote the remaining closed set by F_2; i.e.*

$$F_2=[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1].$$

We continue in this way infinitum, at each stage, removing the open middle third of each of the closed intervals remainig from the previous stage. In this *way we obtain an infinite sequence of closed sets* $\{F_n\}$ *such that* $F_n \supset F_{n+1}$ *for all* $n = 0, 1.2, \ldots.$ *Write* $F = \bigcap_{n=1}^{\infty} F_n$. *Then* F *is known as Cantor's set, Since each* F_n *is closed, it follows from Theorem 41 that* F *is closed.*

Note that F *is the set that remains when we have removed the open intervals*

$$(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9}), \ldots.$$

To see this, denote by M_n *the union of the open middle thirds removed at the* n^{th} *stage; thus*

$$M_1 = (\frac{1}{3}, \frac{2}{3}), \quad M_2 = (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9}), \quad \textit{etc.}$$

and

$$F_1 = F_0 \sim M_1, F_2 = F_0 \sim M_2, \ldots, F_n = F_0 \sim M_n, \ldots.$$

Then

$$F = \bigcap_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} (F_0 \sim M_n) = F_0 \sim \bigcup_{n=1}^{\infty} M_n.$$

2.4 Neighbourhoods

Definition 45 *Let X be a metric space with metric d and let $p \in X$. A subset N of X is called a neighbourhood of p if N contains an open set G which contains p. That is, $\exists$ an open set G such that $p \in G \subset N$.*

Example 46 *Let (X, d) be any metric space and let $p \in X$. Then X is a neighbourhood of p.*

Example 47 *Let* $X = \mathbb{R}$ *and for* $x, y \in X$, *define* $d(x, y) = |x - y|$. *Write* $I = [-1, 1]$ *and let* 0 $\in \mathbb{R}$; *then* I *is a neighbourhood of* 0.

Theorem 48 *A subset* A *of a metric space* (X, d) *is open* $\iff$ *A contains a neighbourhood of each of its points.*

Proof. First, let us assume that A contains a neighbourhood of each of its points. For each $p \in A, \exists\ N_p$ such that $p \in N_p \subset A$. Hence $\exists$ open set G_p such that

$$p \in G_p \subset N_p \subset A.$$

Write

$$G = \bigcup \{G_p : p \in A\}.$$

We shall show that $G = A$.

To see this, first let $x \in G$; then $x \in G_p$ for some p. This implies that $x \in G_p \subset N_p \subset A$. Hence, $x \in A$ and so $G \subset A$. i.e. $\exists$ open set $G_x \ni x \in G_x \subset N_x \subset A$.

Therefore, $x \in G$ and so $G = A$.

Conversely, let A be open. We shall show that A contains a neighbourhood of each of its points. But since A is open and contains itself, we take A as a neighbourhood of its points. Hence, A contains a neighbourhood of each of its points. $\blacksquare$

2.5 Accumulation (or Limit Points)

Definition 49 *Let (X, d) be a metric space and let $A \subseteq X$. A point $p \in X$ is called a point of accumulation (or limit point) for A if every neighbourhood of p intersects A nontrivially. In other words, p is a point of accumulation for A if every neighbourhood of p contains at least one point of A distinct from p.*

Example 50 *(1) Let $X = \mathbb{R}$ and let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$. Then 0 is a limit point for A.*

(2) Let $A = [0, 1)$. Then 0 and 1 are limit points of A. Indeed, every element $0 \leq x \leq 1$ is a limit point.

(3) Let $A = (0, 1)$. Then 0 and 1 are limit points.

(4) Let $A = \{..., -2, -1, 0, 1, 2, ...\}$ *. Then* A *has no limit point. To see this, let* $x = \frac{1}{4} \in \mathbb{R}$*. Then* $S_{\frac{1}{2}}(\frac{1}{16})$ *does not contain an element of* A*. Also, if* $x \in A$*, say* $x = -1$*, then* $S_{-1}(\frac{1}{2})$ *does not contain an element of* A*.*

Let us denote by A' the set of all points of accumulation for A. Then we have the following results:

Theorem 51 *Let* A *be a subset of a metric space* (X, d)*, then the set* A *is closed* $\iff$ $A' \subset A$*. That is,* A *is closed* $\iff$ A *contains all its points of accumulation.*

Proof. First, let A be closed. We must show that A contains all its points of accumulation. That is,, we must show that $A' \subset A$. Let p$\in$ A', and suppose

$p \in A$, then $p \in X \sim A$. But $X \sim A$ is open since A is closed. Since $X \sim A$ is open, it contains a neighbourhood of each of its points, in particular, it contains a neighbourhood of p. This neighbourhood is lying entirely in $X \sim A$ which is impossible since p is a point of accumulation for A. Hence, $p \in A$.

Conversely, let $A' \subset A$. We must show that A is closed. That is, $X \sim A$ is open. To see, we shall show that $X \sim A$ contains a neighbourhood of each of its points. Let $p \in X \sim A$, then $p \notin A$ and therefore $p \notin A'$. That is, p is not a point of accumulation for A. Hence, there exists at least one neighbourhood of p which does not intersect A. Thus $p \in N \in X \sim A$. Hence, we have shown that $X \sim A$ contains a neighbourhood of each of its points and so $X \sim A$ is open. i.e. A is closed. ■

Theorem 52 *Let A be any subset of a metric space (X, d) and denote by A' the collection of all accumulation points of A. Then $A \cup A'$ is closed.*

Proof. *We shall show that* $X \sim (A \cup A')$ *is open. To do this, we shall show that* $X \sim (A \cup A')$ *contains a neighbourhood of each of its points. Let* p *be any point in* $X \sim (A \cup A')$, *we shall show that* $X \sim (A \cup A')$ *contains a neighbourhood of* p. *Now,*

$$X \sim (A \cup A') = (X \sim A) \cap (X \sim A') \qquad \text{by De Morgans.}$$

So $p \in X \sim A$ *and* $p \in X \sim A'$. *Hence* $p \notin A$ *and* $p \notin X \sim A'$. *i.e.* p *is not a point of accumulation for* A. *This implies that* $\exists$ *at least one neighbourhood* $\mathcal{U}$ *of* p *disjoint from* A. *i.e.* $\mathcal{U} \cap \mathcal{A} = \emptyset$. *Also* $\mathcal{U} \cap \mathcal{A}' = \emptyset$. *To see this, let* $x \in \mathcal{U} \cap \mathcal{A}'$, *then* x *will be a point of accumulation for* A. *Every neighbourhood of* x *will intersects* A. *In particular,* $\mathcal{U}$ *intersects* A, *a contradiction, hence* $\mathcal{U} \cap \mathcal{A}' = \emptyset$. *Thus* $\mathcal{U} \cap \mathcal{A} = \emptyset$ *and* $\mathcal{U} \cap \mathcal{A}' = \emptyset$. *Hence* $\mathcal{U}$ *does not intersect* $A \cup A'$. *Thus* $\mathcal{U}$ *is a neighbourhood of* p *and is contained entirely in* $X \sim (A \cup A')$. ■

Definition 53 *Let* (X, d) *be a metric space and* A *a subset of* X*. By closure of* A *we mean the smallest closed set containing* A*.*

Denote the closure of A by $\overline{A}$.

Theorem 54 *Let* A *be a subset of a metric space* (X, d)*. Then,* $\overline{A} = A \cup A'$*.*

Proof. Let $K = \{\cap E_\alpha : E_\alpha$ is a closed set and that $E_\alpha \supset A\}$. Then K is closed since it is the intersection of a collection of closed sets. Also $K \supset A$ since $A \subset E_\alpha$ for all $\alpha \in \Omega$. Let $x \in A$, this implies that $x \in E_\alpha$ for all $\alpha \in \Omega$. Since $A \subset \alpha \in \Omega$ for all $\alpha \implies x \in \cap E_\alpha = K$. Now if F is a closed set such that $F \supset A$, then we shall show that $F \supset K$. Clearly, F is one of the $E_{\alpha' s}, \alpha \in \Omega$. Since F is bigger than their intersection, then $F \supset K$. ∎

Definition 55 *Let* (X, d) *be a metric space. Then* A *is said to be everywhere dense in* X *if* $\overline{A} = X$*. More generally, a set* A *is dense in a set* B *where*A, B *are subsets of* X *if* $\overline{A} = B$*.*

Example 56 *(1)* $(0, 1)$ *is dense in* $[0, 1]$*.*

(2) The set $\mathbb{Q}$ *of rational numbers is everywhere dense in* $\mathbb{R}$*.*

2.6 Interior of a set

Definition 57 *Let* (X, d) *be a metric space and let* $A \subseteq X$*. A point* $x \in A$ *is called an interior point of* A *if* x *is contained in an open sphere (ball) contained in* A*.*

Thus x is an interior point of A if A is a neighbourhood of x. We denote the set of all interior point of A by $Int(A)$ Thus $Int(A) = \{x \in A : S_r(x) \subseteq A, \text{ for some } r >$

Example 58 *(1) If $A = (0, 1) \subset \mathbb{R}$, then every point of A is an interior point of A but of course if $A = [0, 1]$, then either 0 nor 1 is an interior point;*

(2) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$. Then A has no interior point

(3) If A is any open set in (X, d), then every point of A is an interior point.

The basic properties of interiors are the following:

(i) $Int(A)$ is an open subset of A which contains every open subset of A (this is often expressed by saying that the interior of A is the largest open subset of A);

(ii) A is open $\iff A = Int(A)$;

(iii) $Int(A)$ equals the union of all open subsets of A.

The proof of the above facts are easy and so left as exercise.

Exercise 59 *(1) Let A and B be subsets of a metric space (X, d). Show that*

(i) $Int(A) \cup Int(B) \subseteq Int(A \cup B)$;

(ii) $Int(A) \cap Int(B) = Int(A \cap B)$.

(2) Give an example of two sets A and B of the real line such that $Int(A) \cup Int(B) \neq Int(A \cup B)$.

2.7 Boundary point

Definition 60 *Let (X, d) be a metric space and $A \subset X$. We say that a point $x \in X$ is a frontier point (i.e. a boundary point) of A if every neighbourhood N_x of x satisfies $N_x \cap Int(A) = \emptyset$, $N_x \cap A = \emptyset$. That is, x is a frontier point if $x \subset \overline{A} \cap \overline{A^c}$. The set of all frontier points of $A \subset X$ is denoted by $Fr(A)$ or ∂A and is called the frontier or boundary of A.*

The following theoren is left as exercise

Theorem 61 *Let (X, d) be a metric space and let A, B, C be subsets of X, then*

1. $\partial A = \overline{A} - Int(A)$

2. B *is closed* $\iff$ $Fr(B) \subset B$

3 C *is both open and closed* $\iff$ $\partial C = \emptyset$

4 ∂A *is a closed set.*

3 Convergence in Metric Spaces

One of our aims in studying metric spaces is to study convergent sequences in a more general setting than that of classical analysis. We have the following definition:

Definition 62 *Let* (X, d) *be a metric space and let* $\{x_n\}$ *a sequence of points in* X. *The sequence* $\{x_n\}$ *is said to be convergent if* $\exists$ *a point* x *in* X *such that either*

(1) for each $\epsilon > 0, \exists$ *a positive integer* n_0 *such that* $n \geq n_0 \implies d(x_n, x) < \epsilon$; *or equivalently,*

(2) for each open sphere $S_\epsilon(x)$ *centred on* x, $\exists$ *a positive integer* n_0 *such that* x_n *is in* $S_\epsilon(x)$ *for all* $n \geq n_0$.

Remark 63 *(i) Observe that condition (1) is a direct generalization of convergence of sequences of real numbers while condition (2) is saying that each open sphere centred on* x *contains all points of the sequence from some place on.*

(ii) the statement that $\{x_n\}$ *is convergent can equivalently be expressed as follows:* $\exists$ *a point* x *in* X *such that* $d(x_n, x) \to 0$. *This is symbolize by writing* $x_n \to x$, *and we express it by saying that* x_n *approaches* x *or that* x_n *converges to* x.

(iii) the point x *is called the limit of the sequence* $\{x_n\}$ *and we sometimes write lim* $x_n = x$ *or* $\lim x_n = x$

Definition 64 *Let* $\{x_n\}$ *be a sequence in a metric space* (X, d). *Then* x_n *is called a Cauchy sequence if for any* $\epsilon > 0, \exists\, n_0 \in \mathbb{N}$ *such that* $d(x_m, x_n) < \epsilon$ *if* $m, n \geq n_0$.

Definition 65 *A metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *converges to a point in* X.

Example 66 *(1) The real line $\mathbb{R}$ is a complete metric space*

(2) The complex plane $\mathbb{C}$ is a complete metric space. To see this, let $\{z_n\}$ be a Cauchy sequence of complex numbers, where $z_n = a_n + ib_n$. Then $\{a_n\}$ and $\{b_n\}$ are themselves Cauchy sequences of real numbers, since

$$|a_m - a_n| \leq |z_m - z_n|$$

and

$$|b_m - b_n| \leq |z_m - z_n| \, .$$

Since $\mathbb{R}$ is complete, then $\exists$ real numbers a and b such that $a_n \to a$ and

$b_n \to b$. Now put $z = a + ib$, the we see that $z_n \to z$ by means of

$$\begin{aligned} |z_n - z| &= |(a_n + ib_n) - (a + ib)| \\ &= |(a_n - a) + i(b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &\to 0. \end{aligned}$$

Observe that the completeness of $\mathbb{C}$ depends directly on the completeness of $\mathbb{R}$.

The following result relates these concepts to one another.

Theorem 67 *A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) can converge to at most one point in X.*

Proof. Let $\{x_n\}$ converges to x and x', where x, $x' \in X$, the we shall show that $x = x'$. Given $\epsilon > 0, \exists$ integers N, N' such that

$n \geq N \implies d(x_n, x) < \frac{\epsilon}{2}$. Also $n \geq N' \implies d(x_n, x') < \frac{\epsilon}{2}$.Now, take $n \geq \max\{N, N'\}$, we obtain that

$$\begin{aligned} d(x, x') &\leq d(x_n, x) + d(x_n, x') \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we conclude that $d(x, x') = 0$. This implies that $x = x'$ and the proof is complete. ■

Theorem 68 *In a metric space (X, d), the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x if and only if every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x.*

Proof. Since $x_n \longrightarrow x$ as $n \longrightarrow \infty$ implies that $\exists N \in \mathbb{Z}^+$ such that $d(x_n, x) < \epsilon \; \forall n \geq N$, where $\epsilon > 0$ is pre-assigned. Also, since $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence, $\exists M \in \mathbb{Z}^+$ such that $k_n \geq N$ if $n \geq M$. Hence, $n \geq M \implies d(x_{n_k}, x) < \epsilon$. Thus, this implies that $x_{n_k} \to x$ as $k \to \infty$.

The converse is trivial since $\{x_n\}_{n=1}^{\infty}$ itself is a subsequence. ■

Theorem 69 *If is a convergent sequence $\{x_n\}$ in a metric space (X, d) has infinitely many distinct points, then its limit is a limit point of the set of points of the sequence.*

Proof. Trivial and therefore left as exercise. ■

Theorem 70 *Let (X, d) be a complete metric space, and Y be a subspace of X. Then Y is complete if and only if it is closed.*

Proof. First assume that Y is complete as a subspace of X, then we shall show that Y is closed. Let y be a limit point of Y. Then, for each positive integer n_1, $S_{\frac{i}{n}}(y)$ contains a point y_n in Y. Clearly $\{y_n\}$ converges to y in X and it is a Cauchy sequence in Y, and since Y is complete, $y \in Y$. Therefore, Y is closed.

Conversely, assume that Y is closed, then we shall show that it is complete. Let $\{y_n\}$ be a Cauchy sequence in Y. Clearly, it is also a Cauchy sequence in X, and since X is complete, $\{y_n\}$ converges to a point x in X. We shall show that x is in Y. If $\{y_n\}$ has only finitely many distinct points, then x is that point infinitely repeated and is thus in Y. Otherwise, if $\{y_n\}$ has infinitely many distinct points, then by Theorem 69, x is a limit point of the set of points of the sequence; it is therefore a limit point of Y, and since Y is closed, $x \in Y$.

■

4 Continuity in Metric Spaces

Definition 71 *Let* (X, d) *and* (Y, ρ) *be metric spaces, we say that a mapping* $f : X \to Y$ *is continuous at* $x_0 \in X$ *if either of the following equivalent conditions is satisfied:*

(i) for each $\epsilon > 0 \; \exists \; \delta > 0$ *such that* $d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \epsilon$;

(ii) for each open sphere $S_\epsilon(f(x_0))$ *centred on* $f(x_0) \; \exists$ *an open sphere* $S_\delta(x_0)$ *centred on* x_0 *such that* $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$.

Remark 72 *Observe that the condition (i) is a generalization of the definition of convergence ealier given while condition (ii) translates condition (i) into the language of open spheres.*

Example 73 *Every mapping on a discrete metric space is continuous. To see this, consider a map f from a discrete metric space (X, d) into a metric space (Y, ρ). For $x_0 \in X$, $S_1(x_0) = \{x_0\}$ so, given $\epsilon > 0$ $\rho(f(x), f(x_0) < \epsilon$ when $d(x, x_0) < 1$.*

Theorem 74 *Let (X, d) and (Y, ρ) be metric spaces and f is a mapping of X into Y. Then f is continuous at x_0 if and only if $x_n \to x_0 \implies f(x_n) \to f(x_0)$.*

Proof. First, assume that f is continuous at x_0. We shall show that if $\{x_n\}$ is a sequence in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Let $S_\epsilon(f(x_0))$ be an open sphere centred at $f(x_0)$. By our assumption, $\exists$ an open sphere $S_\delta(x_0)$ centred at x_0 such that $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$.

Since $x_n \to x_0$, all $x_n's$ from some place on lie in $S_\delta(x_0)$. But since $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$, all $f(x_n)'s$ from some place on lie in $S_\epsilon(f(x_0))$. From this it follows that $f(x_n) \to f(x_0)$.

Conversely, assume that f is not continuous at x_0. We shall show that $x_n \to x_0 \not\Rightarrow f(x_n) \to f(x_0)$. By assumption, $\exists$ an open sphere $S_\epsilon(f(x_0))$ with the property that the image under f of each open sphere centred on x_0 is not contained in it. Now consider the sequence of open spheres $S_1(x_0), S_{\frac{1}{2}}(x_0), ..., S_{\frac{1}{n}}(x_0), ...$ Form a sequence $\{x_n\}$ such that $x_n \in S_{\frac{1}{n}}(x_0)$ and $f(x_n) \notin S_\epsilon(f(x_0))$. Clearly $x_n \to x_0$ and that $f(x_n) \nrightarrow f(x_0)$. ■

Example 75 *(i) Consider the real function on* $(\mathbb{R}^2, \|\|_2)$ *defined by*

$$f(\lambda,\mu)=\begin{cases}\frac{\lambda\mu}{\lambda^2+\mu^2}, & (\lambda,\mu)\neq(0,0)\\ 0 & (\lambda,\mu)=(0,0).\end{cases}$$

Now the sequence $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}$ *converges to* $(0,0)$ *but* $f\left(\frac{1}{n},\frac{1}{n}\right)=\frac{1}{2}\ \forall n\in\mathbb{N}$. *So* f *is not continuous at* $(0,0)$.

(ii) Consider the scalar function p_0 *on* $(C[0,1],\|\|_1)$ *defined by* $p_0(f)=f(0)$. *Now the sequence* $\{f_n\}$ *where*

$$f_n(t)=\begin{cases}1-nt & 0\leq t\leq\frac{1}{n}\\ 0 & \frac{1}{n}<t\leq 1.\end{cases}$$

satisfies $\|f_n\|_1=\frac{1}{2n}\to 0$ *as* $n\to\infty$, *so* $\{f_n\}$ *converges to the zero function, but* $p_0(f_n)=1\ \forall n\in\mathbb{N}$. *So* p_0 *is not continuous at the zero function.*

Definition 76 *A mapping of one metric space into another is said to be continuous if it is continuous at each point in the domain.*

Theorem 77 *Let (X, d) and (Y, ρ) be metric spaces and f is a mapping of X into Y. Then f is continuous if and only if $x_n \to x \implies f(x_n) \to f(x)$.*

Proof. This follows from the proof of Theorem 74. ■

Remark 78 *Observe that the above Theorem shows that convergence is preserved under continuous mapping.*

Our next result characterizes continuous mapping in terms of open spheres.

Theorem 79 *Let (X, d) and (Y, ρ) be metric spaces and f is a mapping of X into Y. Then f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in$Y$.*

Proof. First assume that f is continuous. If G is open in Y, we shall show that $f^{-1}(G)$ is open in X. If $f^{-1}(G) = \emptyset$ it is clearly open, so assume $f^{-1}(G) \neq \emptyset$. Let $x \in f^{-1}(G)$, then $f(x) \in G$.

Since G is open, $\exists$ an open sphere $S_\epsilon(f(x))$ centred at $f(x)$ and contained in G. By definition of continuity, $\exists$ an open sphere $S_\delta(f(x))$ such that $f(S_\delta(x)) \subseteq S_\epsilon(f(x))$. Since $S_\epsilon(f(x)) \subseteq G$, we also have $f(S_\delta(x)) \subseteq G$ and from this we easily see that $S_\delta(x) \subseteq f^{-1}(G)$. $S_\delta(x)$ is therefore an open sphere centred on x and contained in $f^{-1}(G)$, so $f^{-1}(G)$ is open.

Conversely, assume that $f^{-1}(G)$ is open whenever G is open. We shall show that f is continuous. We do this by showing that f is continuous at an arbitrary point $x \in X$. Let $S_\epsilon(f(x))$ be an open sphere centred on $f(x)$. This open sphere is an open set, so its inverse image is an open set which contains x. By this, $\exists$ an open sphere $S_\delta(x)$ which is contained in the inverse image. It is clear from this that $S_\delta(x) \subseteq S_\epsilon(f(x))$, so f is continuous. Since x was chosen to be arbitrary point in X, f is continuous. $\blacksquare$

Remark 80 *Observe that the above result established that continuous mappings pull open sets back to open sts.*

5 Compactness for Metric spaces

We recall the classical Bolzano-Weierstrass theorem: If X is a closed and bounded subset of the real line, then every infinite subset of X has a limit point in X.

In this section, we shall look for an equvalent formulation of the above result for metric spaces. First, we give the following definitions

Definition 81 *A metric space is said to have the Bolzano-Weierstrass property if every infinite subset has a limit point.*

Definition 82 *A metric space is said to be sequentially compact if every sequence in it has a convergent subsequence.*

Remark 83 *We shall show that each of Definitions 81 and 82 is equivalent to compactness for metric spaces.*

Theorem 84 *A metric space is sequentially compact if and only if it has the Bolzano-Weierstrass property.*

Proof. Let X be a metric space and assume that X is sequentially compact. We shall show that an infinite subset A of X has a limit point. Now, since A is infinite, a sequence $\{x_n\}$ of distinct points can be extracted from A. By the sequential compactness of X, this sequence has a subsequence which converges to a point x. By Theorem 69, x is a limit point of the set of points of the subsequence, and since this set is a subset of A, x is also a limit point of A.

Next, assume that every infinite subset of X has a limit point, and we shall show from this that X is sequentially compact. Let $\{x_n\}$ be an arbitrary sequence in X. If $\{x_n\}$ has a point which is infinitely repeated, then it has a constant subsequence, and this subsequence is clearly convergent. If no point of $\{x_n\}$ is infinitely repeated, then the st of points of this sequence is infinite. By assumption, the set A has a limit point x, and we can extract from $\{x_n\}$ a subsequence which converges to x. ■

Theorem 85 *Every compact metric space has the Bolzano-Weierstrass property.*

Proof. Let X be a compact metric space and A an infinite subset of X. Assume that A has no limit point, and from this we shall arrive at a contradiction. By assumption, each point of X is not a limit point of A, and so each point of

X is the centre of and open sphere which contains no point of A different from its centre. The class of all these spheres is an open cover, and by compactness there exists a finite subcover. Since A is contained in the set of all centres of spheres in this subcover, A is clearly finite. This contradicts the fact that A is infinite, and the proof is complete. ■

Theorem 86 *(Lebesgue's Covering Lemma) In a sequentially compact metric space, every open cover has a Lebesgue number.*

Proof. Omitted. ■

Theorem 87 *Every sequentially compact metric space is totally bounded.*

Proof. Let X. be a sequentially compact metric space, and let $\epsilon > 0$ be given. Choose a point $a_1 \in X$ and form the open sphere $S_\epsilon(a_1)$. If $S_\epsilon(a_1)$ contains every point of X, then the single- element $\{a_1\}$ is an $\epsilon-$net. If there are points outside of $S_\epsilon(a_1)$, let a_2 be such a point and form the two-element set $S_\epsilon(a_1) \cup S_\epsilon(a_2)$. If this union contains every point of X, then the two=element set $\{a_1, a_2\}$ is an $\epsilon-$net. If we continuue in this way, some union of the form

$$S_\epsilon(a_1) \cup S_\epsilon(a_2) \cup ... \cup S_\epsilon(a_n)$$

will necessarily contain every point of X; for if this process could be continued indefinitely, then the sequence $\{a_1, a_2, ..., a_n, ...\}$ would be a sequence with no convergent subsequence, contrary to the assumed sequential compactness of X. We see by this that some finite set of the form $\{a_1, a_2, ..., a_n\}$ is an $\epsilon-$net, so X is totally bounded. ■

Finally, we shall now show that compactness is implied by sequential compactness.

Theorem 88 *Every sequentially compact metric space is compact*

Proof. .Let X be a sequentially compact metric space, and let $\{G_i\}$ be an open cover. By Theorem , this open cover has a Lebesgue number a. We put $\epsilon = \frac{a}{3}$, and use Theorem 87 to fin an $\epsilon-$net

$$A = \{a_1, a_2, ..., a_n\}.$$

For each $k = 1, 2, ..., n$, we have $d(S_\epsilon(a_k)) \leq 2\epsilon = \frac{2a}{3} < a$. By the definition of Lebesgue number, for each k we can find a G_{i_k} such that $S_\epsilon(a_k) \subseteq G_{i_k}$. Since every point of X belongs to one of the $S_\epsilon(a_k)'s$, the class $\left\{G_{i_1}, G_{i_2}, ..., G_{i_n}\right\}$ is a finite subcover of $\{G_i\}$. X is therefore compact. ■

We have so far shown that if X is a metric space, the the following three conditions are all equivalent to one another

1. X is compact;

2. X is sequentially compact;

3. X has the Bolzano-Weierstrass property.

Theorem 89 *Any continuous mapping of a compact metric space into a metric space is uniformly continuous.*

Proof. Let f : be a continuous mapping of a compact metric space (X, d) to a metric space (Y, ρ). Let $\epsilon > 0$ be given. For each point $x \in X$, consider the image $f(x)$ and the open sphere $S_{\frac{\epsilon}{2}}(f(x))$ centred on this image with radius $\frac{\epsilon}{2}$. Since f is continuous, the inverse images of these open spheres is an open subset of X, and the class of all such inverse images is an open cover of X. Since X is compact, Theorem 86 guarantees that this open cover has a Lebesgue number δ. If x and x' are any two points in X for which $d(x, x') < \delta$, then the set $\{x, x'\}$ is a set with diametre less than δ, both points belong to the inverse image of some of the above open spheres, both $f(x)$ and $f(x')$ belong to one of these open spheres, and therefore $\rho(f(x), f(x')) < \epsilon$, which shows that f is indeed uniformly continuous. ■

6 Connectedness

Definition 90 *A metric space (X, d) is said to be connected if it can not be represented as the union of two disjoint non-empty open sets. That is, there* *do not exist non=empty open sets A and B such that*

$$X = A \cup B \quad \textit{and} \quad A \cap B = \emptyset.$$

Also observe that there do not exist two disjoint non-empty closed sets whose union is X. If such A and B exist, we say that X is disconnected and that A and B are said to give a disconnection.

Theorem 91 *A metric space (X, d) is disconnected iff X has a non-empty proper subset that is both open and closed.*

Proof. If $A \subseteq X$ is both open and closed, then $X = A \cup (X \sim A)$ gives a disconnection. Conversely, if $X = A \cup B$, $A \cap B = \emptyset$ and A, B are open and non-empty, then $B = X \sim A$ and $A = X \sim B$. Thus $X = A \cup (X \sim A)$. This implies that A is both open and closed. ■

Definition 92 *A subset S of a metric space (X, d) is disconnected if there exists two non-empty open sets A and B of X such that $A \cap S$ and $B \cap S$ are disjoint non-empty open sets whose union is S. i.e.,*

$$A \cap S \neq \emptyset, \quad B \cap S \neq \emptyset$$

$$(A \cap S) \cap (B \cap S) = \emptyset$$

$$(A \cap S) \cup (B \cap S) = S.$$

If S is disconnected, then the open sets A and B with the above properties are said to form a disconnection of S.

Definition 93 *A subset S of a metric space is said to be connected if it is not disconnected.*

Example 94 *The set $\mathbb{N}$ of natural numbers is disconnected in $\mathbb{R}$. To see this, take*

$$A = \left\{x \in \mathbb{R} : x < \frac{13}{12}\right\}, ...B = \left\{x \in \mathbb{R} : x > \frac{13}{12}\right\}.$$

Example 95 *The set $\mathbb{Q}^+$ of all positive rational numbers is disconnected. To see this, take*

$$A = \{x \in \mathbb{R} : x < 2\}, ...B = \{x \in \mathbb{R} : x > 2\}.$$

We may ask, are there any connected subsets of $\mathbb{R}$ or $\mathbb{R}^n$?. The answer is given in the following result:

Theorem 96 *A subspace of the real line $\mathbb{R}$ is connected if and only if it is an interval. In particular, $\mathbb{R}$ is connected.*

Proof. Assume X is connected subspace of $\mathbb{R}$. We shall show that X is an interval. Suppose X is not an interval, then $\exists\ x, y, z$ such that $x < y < z$ with $x, z \in X$ but $y \notin X$. We shall show that this gives a disconnection of X. In fact, consider the sets

$$[X \cap (-\infty, y)] \cup [X \cap (y, +\infty)] .$$

Then we see that

$$X = [X \cap (-\infty, y)] \cup [X \cap (y, +\infty)] .$$

This gives a disconnection of X and so X is disconnected. This is a contradiction. Hence, X is an interval.

Conversely, assume that X is an interval. Then we shall show that X is connected. Suppose not, then $\exists$ a disconnection $X = A \cup B$. Since A and B have to be non-empty, we can choose $x \in A$, $z \in B$ and since $A \cap B = \emptyset$.

We cannot have $x = z$; hence we have $x \neq z$. There are two possibilities, either $x < z$ or $z < x$. Take $x < z$ and so $\exists\ y$ such that $x < y < z$.

$X = A \cup B \implies$ that $\exists\ x, z$ such that $x \in A$, $z \in B$. Since X is an interval, $[x, z] \subset X$. Now, define

$$y = \sup\{[x, z] \cap A\} \quad \text{for fixed } x, z.$$

Then y exist as a real number. Since A is closed, we must have $y \in A$ because y is a point of accumulation for A.

[Note: if K is a bounded infinite set of real numbers and $\alpha = \textbf{sup}\, K$, then α is a point of accumulation for K. Similarly, if $\beta = \textbf{inf}\, K$, then β is a point of accumulation for K].

Hence, $y < z$ (note that we cannot have equality because $z \in B$). Again, by the definition of y, we have the following:

$y + \epsilon \in B$ for every $\epsilon > 0$ such that $y + \epsilon \leq z$ (use definition of sup). Thus we must have $y \in B$. This is because if we choose a sequence $\{\epsilon_n\}$ such that $\epsilon_n > 0$ and $y + \epsilon_n \leq z$ and $\epsilon_n \rightarrow 0$, then $y + \epsilon_n \rightarrow y$ and since all such must be in B since B is closed.

But we have shown earlier that $y \in A$ which is disjoint from B. This is a contradictiong arising out of our assumption that X is disconnected. Therefore, X is connected. $\blacksquare$

Theorem 97 *The image of a connected metric space under a continuous maps is connected.*

Proof. Let (X, d) and (Y, ρ) be metric spaces where (X, d) is connected and let $f : X \to Y$. We shall show that $f(X)$ is a connected subspace (Y, ρ). Suppose not, then $\exists$ two non-empty disjoint open sets G, H in Y such that

$$f(X) \subseteq [f(X) \cap G] \cup [f(X) \cap H].$$

We choose G and H so that their union contains $f(X)$ and their intersection with $f(X)$ are disjoint and non-empty. This leads to a contradiction because

$$X = f^{-1}[G] \cup f^{-1}[H]$$

will give a disconnection of X. ■

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