Matched Subspace Detection Using Compressively Sampled Data

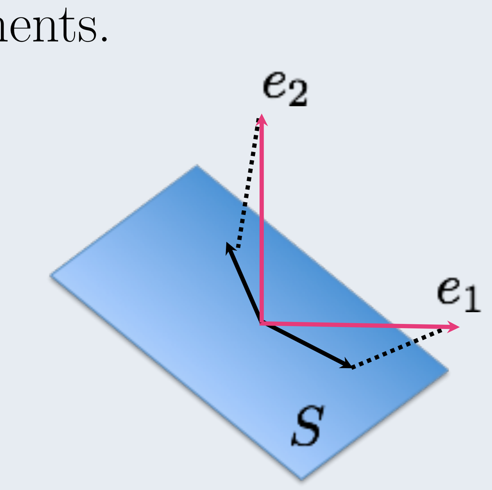
Dejiao Zhang & Laura Balzano

Department of Electrical Engineering and Computer Science, University of Michigan

Motivation and Objective

We model many signals in science and engineering using low-dimensional subspaces. We consider the problem of deciding whether a high-dimensional vector in $\mathbb{R}^n$ lies in a given $d-$dimensional ($d \ll n$) subspace $\mathcal{S}$ by using only a few compressive measurements.

By leveraging random matrix theory, we show that a reliable test statistic can be constructed even with a few compressive measurements.



Problem Formulation

We seek to detect whether the unknown vector $v \in \mathbb{R}^n$ lies in an d ($d \ll n$) dimensional subspace $\mathcal{S}$ given only a small number of compressive measures of the form $x = Av + \xi$, where $A \in \mathbb{R}^{m \times n} (m \leq n)$ is the given sampling matrix and $\xi \in \mathbb{R}^m$ is additive noise.

Given $x = Av + \xi$ with x, A and U known, where $U \in \mathbb{R}^{n \times d}$ whose orthonormal columns span $\mathcal{S}$

$\mathcal{H}_0 : v \in \mathcal{S}$ vs $\mathcal{H}_1 : v \notin \mathcal{S}$

Let $v = v_\perp + v_\parallel$ where $v_\parallel \in \mathcal{S}$ and $v_\perp \in \mathcal{S}^\perp$.

Full data with no noise: $\mathcal{H}_0 : \|v_\perp\|_2 = 0$ vs $\mathcal{H}_1 : \|v_\perp\|_2 > 0$

Undersampled data with noise ? $v_\perp$ **is not available.**

Can we obtain a reliable detector by looking at the projection residual in terms of undersampled data, *i.e.*,

$$T = \|(\mathcal{I} - \mathcal{P}_{AU})x\|_2^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta, \quad \eta \geq 0 \tag{1}$$

where $\mathcal{P}_{AU}$ denotes the projection operator onto the column space of AU.

Main Results

Assumptions

- The rows of sampling matrix A are independent sub-gaussian random vectors with mean zero, $\mathbb{E}\left[A_i A_i^T\right] = \frac{m}{n}\mathbb{I}_n$ and $\max_i \|A_i\|_{\Psi_2} \leq K$.
- The entries of noise vector are i.i.d sub-gaussian random variables with $\mathbb{E}[\xi_i] = 0$, $\mathrm{Cov}(\xi_i) = \sigma^2$ and $\|\xi_i\|_{\Psi_2} \leq K_1$.

Theorem 1 (Noiseless Data)

If $m > 2ed$, then with probability at least $1 - 3\exp\left[-\tau_1 m\right]$ we obtain

$$\frac{m}{2en}\|v_\perp\|^2 \leq \|(\mathcal{I} - \mathcal{P}_{AU})\, Av\|^2 \leq e\frac{m}{n}\|v_\perp\|^2$$

(The energy outside the subspace is preserved.)

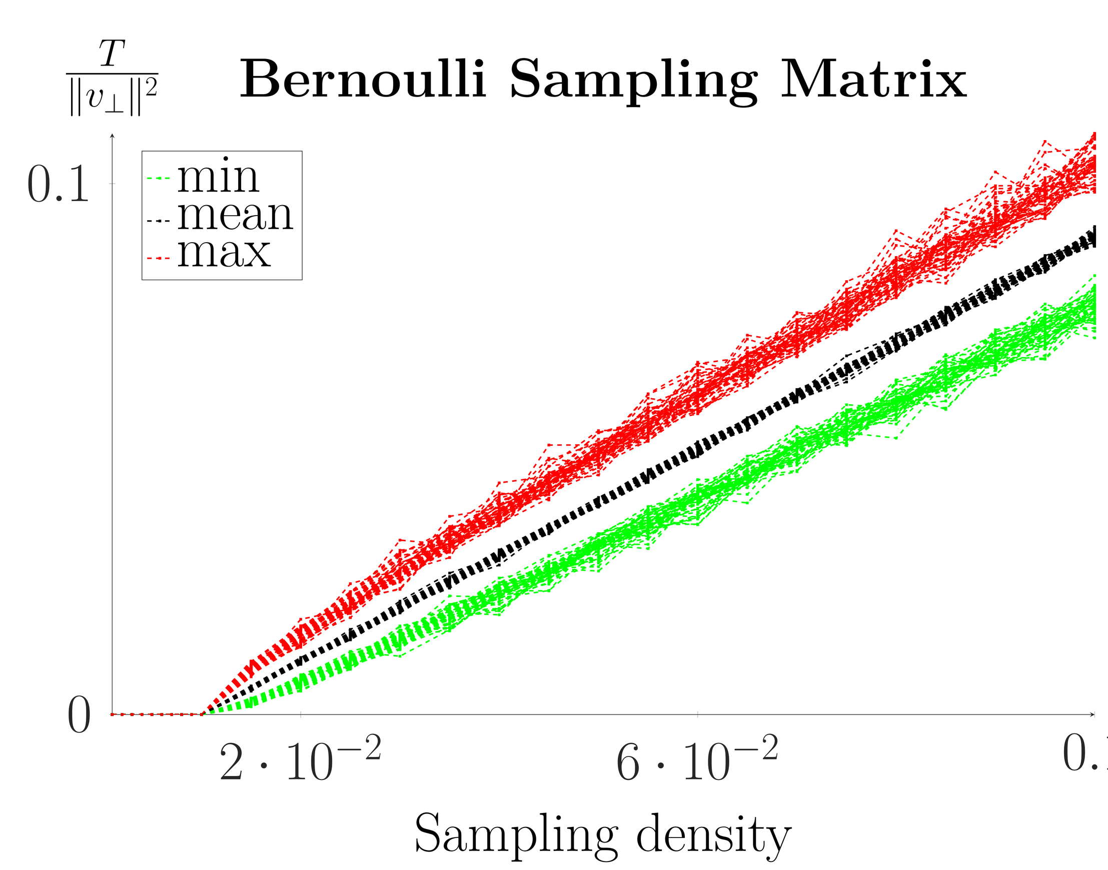


Figure 1: Illustration of Theorem over 100 simulations, each with a fixed subspace $\mathcal{S}$ and different sampling density m/n. The sampling matrices are generated as a sparse matrix such that $A_{ij} = \pm\sqrt{3/n}$, w.p. $1/6$ and $A_{ij} = 0$, w.p. $2/3$. For each sampling density, we sample 50 instances of $Av_\perp$, and then compute the mean, maximum and minimum of our defined test statistic.

Theorem 2 (Noisy Data)

Let $\eta = e(m - d)\sigma^2$, then the false positive rate is bounded:

$$\mathbb{P}\left(T \geq \eta | \mathcal{H}_0\right) \leq \exp\left[-\tau_2(m - d)\right].$$

Additionally, suppose $m > 2ed$ and

$$\|v_\perp\|^2 \geq 4e(e + 2)\left(1 - d/m\right) n\sigma^2$$

holds for any $v \notin \mathcal{S}$, then the false negative rate is bounded:

$$\mathbb{P}\left(T \leq \eta | \mathcal{H}_1\right) \leq \exp\left[-\tau_3 m\right] + \exp\left[-\tau_4(m - d)\right]$$

(A reliable detector can be obtained as long as $\|v_\perp\|_2^2$ scales with n.)

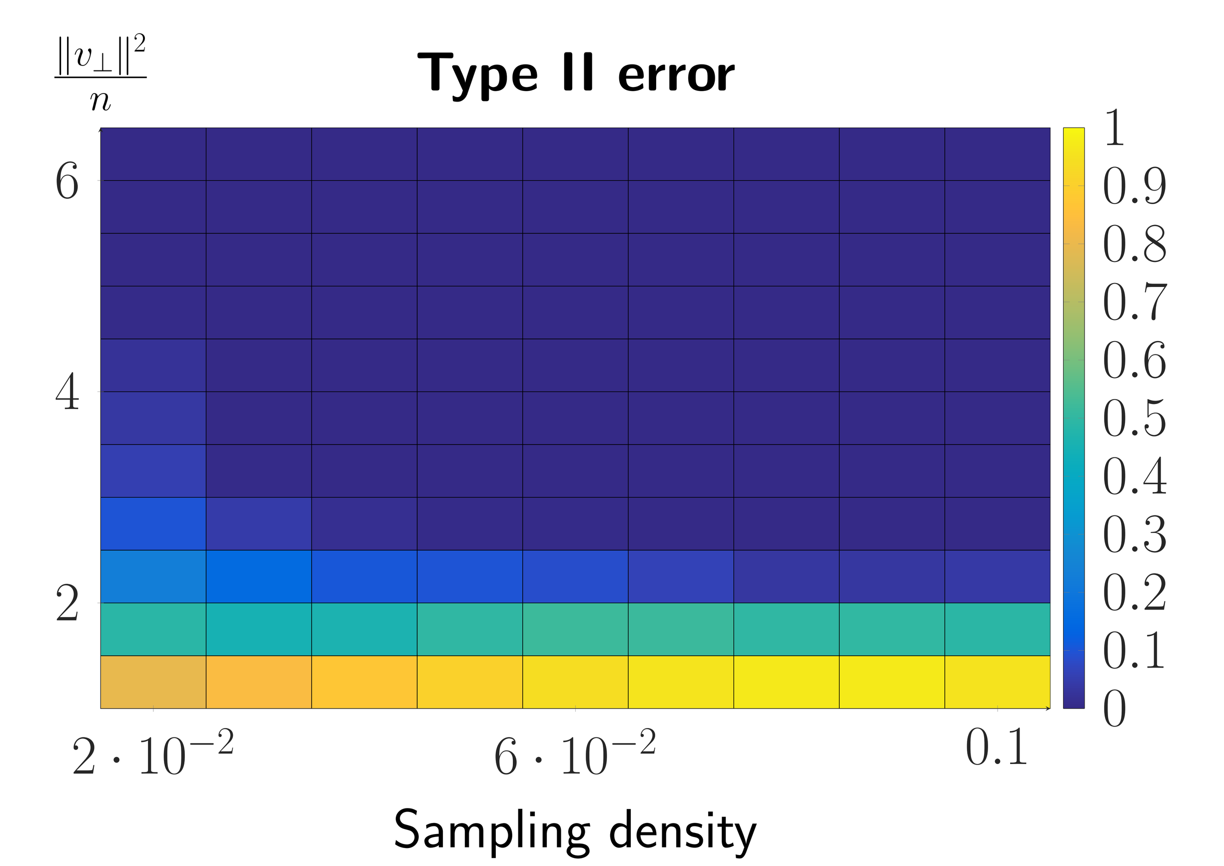


Figure 2: Illustration of Theorem 2 with $n = 5000$ and $d = 50$. The sampling matrices are generated in the same way of Fig 1. The entries of noise vectors are generated as i.i.d uniform random variables with mean zero and unit covariance. Type II error is averaged over 100 simulations with different sampling densities and different instances of $v_\perp$.

References

[1] Dejiao Zhang and Laura Balzano. Matched subspace detection using compressively sampled data. *ICASSP*, 2017.

[2] Martin Azizyan. and Aarti Singh. Subspace detection of high-dimensional vectors using compressive sampling. *IEEE Statistical Signal Processing Workshop*, 2012.