第六章 Laplace方程

$Laplace$算子$\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$

基本解形式的导出

2023伯苓期末考过

由物理意义导出基本解只与r有关，所以写成E(r)

以$n=3$为例，在$\mathbb{R}^3$的静电场中，$Gauss$定律：在原点处放一个正电荷：$\Delta E(x) = \delta(x)$

其中$E(x)$只与x到原点的位置有关，即$E(x) = E(r), r = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$

解E(r)的ode方程，得到基本解主项

常规求导可得$\Delta E(r) = \frac{d^2E}{dr^2} + \frac{n-1}{r}\frac{dE}{dr}$，设$\frac{dE}{dr} = W(r)$，则为$\frac{dW}{dr} + \frac{n-1}{r}W(r) = \Delta E(r) = \delta(r)$

两端同乘$r^{n-1} \in C^\infty(\mathbb{R})$，则$\frac{dW}{dr}r^{n-1} + W(r)(n-1)r^{n-2} = \delta(r) \Rightarrow \frac{d}{dr}[r^{n-1}W(r)] = \delta(r)$

则当$r>0$时，$\frac{d}{dr}[r^{n-1}W(r)] = 0 \Rightarrow r^{n-1}W(r) = c$，即$\frac{dE}{dr} = W(r) = cr^{1-n}, n \geqslant 2$

即$E(r) = \begin{cases} c_n r^{2-n}, & n \geqslant 3 \\ c_2 \ln \frac{1}{r}, & n = 2 \end{cases}$，接下来求$c_n, n \geqslant 2$

注：(1)此时$r = |x| = |x - 0|$，因此时在原点处放置正电荷，若在x_0处放置单位正电荷$\Rightarrow r = |x - x_0|$

(2)$Laplace$方程的基本解常用$\Gamma(x$【动点】$, x_0$【定点】$) = \Gamma(|x - x_0|)$或$\Gamma(x, y)$表示

位势积分

推导位势积分公式：利用第二类Green公式，取v=1/r，2015考过

第二类Green公式

设$\Omega \subset \mathbb{R}^3$是一有界区域，且$\partial\Omega$光滑，设$u$和$v$都在$C^2(\Omega) \cap C^1(\overline{\Omega})$中，则

$$\iiint_\Omega (u\Delta v - v\Delta u)\,dx = \iint_{\partial\Omega} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] dS$$，n是$\partial\Omega$的单位外法向量

step 1: 取$v = \frac{1}{r}$，在Ω_ε上运用$Green$公式

令$v = \frac{1}{r}$，代入$Green$公式，设$B_\varepsilon(Q)$是以Q为圆心，ε为半径小球，$\Omega_\varepsilon = \Omega \backslash B_\varepsilon$，$P$是动点，$|PQ| = r$

则$v = \frac{1}{r} \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$，在$\Omega_\varepsilon$上使用$Green$公式，$u$仍为一般的$C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$

则$\iiint_{\Omega_\varepsilon} \left[u\Delta\left(\frac{1}{r}\right) - \frac{1}{r}\Delta u\right] dx = \iint_{\partial\Omega_\varepsilon} \left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right] dS$

step 2: 把这个空心小球的两部分拆开：$\iiint_{\Omega_\varepsilon} \left[u\Delta\left(\frac{1}{r}\right) - \frac{1}{r}\Delta u\right] dx = \iint_{\partial\Omega} \left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right] dS - \iint_{\partial B_\varepsilon(Q)} \left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial r}\right] dS \quad (*)$

其中因为$\frac{1}{r}$是基本解的主项，即$\Delta c_3\left(\frac{1}{r}\right) = \delta$，则在$\Omega_\varepsilon$中有$\Delta\left(\frac{1}{r}\right) = 0 \Rightarrow \iiint_{\Omega_\varepsilon} u\Delta\left(\frac{1}{r}\right) dx_p = 0$，则等式$(*)$左端$= -\iiint_{\Omega_\varepsilon} \frac{1}{r}\Delta u dx$

下考虑$\iint_{\partial B_\varepsilon(Q)} \left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial r}\right] dS$：$r = \varepsilon$，则为$-\frac{1}{\varepsilon^2}\iint_{\partial B_\varepsilon(Q)} u dS - \frac{1}{\varepsilon}\iint_{\partial B_\varepsilon(Q)} \frac{\partial u}{\partial r} dS$

step 3: 由【积分中值定理】得，$\iint_{\partial B_\varepsilon(Q)} u dS = u(Q^*)4\pi\varepsilon^2$，$\iint_{\partial B_\varepsilon(Q)} \frac{\partial u}{\partial r} dS = \frac{\partial u}{\partial r}(\tilde{Q})4\pi\varepsilon^2$，其中$Q^*, \tilde{Q} \in \partial B_\varepsilon(Q)$

则$\lim_{\varepsilon\to 0} \iint_{\partial B_\varepsilon(Q)} \left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial r}\right] dS = -4\pi u(Q)$，此时原方程化为$-\iiint_\Omega \frac{1}{r}\Delta u dx = \iint_{\partial\Omega} \left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\right] dS + 4\pi u(Q)$

整理得到位势积分公式：$u(Q) = \frac{1}{4\pi}\left[\iint_{\partial\Omega} \frac{1}{r}\frac{\partial u}{\partial n} dS - \iint_{\partial\Omega} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS - \iiint_\Omega \frac{1}{r}\Delta u dx\right]$

$n > 3$时的位势积分公式：$u(Q) = \frac{1}{(n-2)|S^{n-1}|}\left[\int_{\partial\Omega} r^{2-n}\frac{\partial u}{\partial n} dS_P - \int_{\partial\Omega} u\frac{\partial}{\partial n}(r^{2-n}) dS_P - \int_\Omega r^{2-n}\Delta u dx_P\right]$

$\forall\varphi \in C_0^\infty(\Omega)$，$r = |P - Q|$，$\varphi(Q) = \langle\delta(Q-P), \varphi(P)\rangle_P = \left\langle\Delta\left(c_3\left(\frac{1}{r}\right)\right), \varphi(P)\right\rangle_P = c_3\left\langle\frac{1}{r}, \Delta\varphi(P)\right\rangle_P = c_3\iiint_\Omega \frac{1}{r}\Delta\varphi(P) dx_P$

又由位势积分公式，以及$\varphi \in C_0^\infty(\Omega) \Rightarrow \varphi|_{\partial\Omega} \equiv 0$，得$\varphi(Q) = -\frac{1}{4\pi}\iiint_\Omega \frac{1}{r}\Delta\varphi dx = c_3\iiint_\Omega \frac{1}{r}\Delta\varphi(P) dx_P \Rightarrow c_3 = -\frac{1}{4\pi}$

一般情况下：$c_n = \frac{-1}{(n-2)|S^{n-1}|}$，位势积分公式也可写成：$u(Q) = \int_{\partial\Omega} (-\Gamma(P,Q))\frac{\partial u}{\partial n} dS_P + \int_{\partial\Omega} u\frac{\partial}{\partial n}(\Gamma(P,Q)) dS_P + \int_\Omega \Gamma(P,Q)\Delta u dx_P$

平均值公式

球面平均值

设u是调和函数，$\Omega = B_R(Q)$，则有$u(Q) = \frac{1}{4\pi R^2}\iint_{S(Q,R)} u(P) dS_P$

证明：由位势积分$u(Q) = \frac{1}{4\pi}\iint_{\partial B} \frac{\partial u}{\partial n} dS + \frac{1}{4\pi R^2}\iint_{\partial B} u dS$

$claim$：设$\Omega \subset \mathbb{R}^3$是连通开集，$\partial\Omega$光滑，$\Delta u = 0$在$\Omega$上有$\iint_{\partial B} \frac{\partial u}{\partial n} dS = 0$

法一：利用$Green$第二公式，令$v = 1$即得

法二：利用分部积分，$0 = \int_\Omega \Delta u dx = \int_\Omega \sum_{j=1}^n \partial_{x_j}^2 u dx = \sum_{j=1}^n \int_\Omega \partial_{x_j}^2 u dx = \sum_{j=1}^n \left(\int_{\partial\Omega} \partial_{x_j} u \, \vec{n}_j dS - \int_\Omega (\partial_{x_j} 1)(\partial_{x_j} u) dx\right) = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS$

球体平均值

设u是调和函数，$\Omega = B_R(Q)$，则有$u(Q) = \frac{1}{|B_R|}\iiint_{B_R(Q)} u(P) dx_P$

证明：只需按照球体平均值公式一层一层积分

$\frac{4}{3}\pi R^3 \cdot u(Q) = \int_0^R 4\pi r^2 dr \cdot u(Q) = \int_0^R 4\pi r^2 \cdot u(Q) dr = \int_0^R \iint_{\partial B_r} u(P) dS_P dr = \iiint_{B_R(Q)} u(P) dx_P$

证明极值原理
只用到了平均值公式

极值原理

2015考过

设$\Omega \subset \mathbb{R}^3$是一连通开集，$u$在$\Omega$中调和，若$u$不恒等于常数，则$u$必不能在$\Omega$内达到上、下确界

推论：u在Ω中调和，则最大值、最小值一定在边界上取到

双层反证，假设在内部能取到最大值M，圈个小圈用平均值公式，证明小圈上任一点的取值也一定恒为M
反证方法当然是圈个小圈原积分区域分成两部分放缩，比较直观，可以画图辅助理解

证明：反证：假设$u \neq const$，且在$Q \in \Omega$达到其上确界M；由平均值公式$M = u(Q) = \frac{1}{|B_\delta(Q)|}\int_{B_\delta(Q)} u(P) dS_P$，

$claim$：u在$B_\delta(Q) \subseteq \Omega$上恒等于$M$，若不然，假设$P_0 \in B_\delta(Q), u(P_0) = M_1 < M$，由连续性可知$\exists$邻域$V_{P_0} \subset B_\delta(Q), s.t. u|_{V_{P_0}} < M$

$M = \frac{1}{|B_\delta(Q)|}\left[\int_{V_{P_0}} u(P) dx_P + \int_{B_\delta(Q)\backslash V_{P_0}} u(P) dx_P\right] < \frac{1}{|B_\delta(Q)|}[M|V_{P_0}| + M(|B_\delta| - |V_{P_0}|)] < M$，矛盾！

考虑到δ可以变动，即u在以Q为球心的某个球体上$\equiv M$

任取一点$Q' \in \Omega$并以一条路径联结Q与Q'。于是可以此路径含于B内之一点为心作另一球体B_1，同理在B_1中$u \equiv M$。

仿此进行即可证明$u(Q') = M$，从而在Ω中$u \equiv M$而与假设矛盾。

主要来看$Dirichlet$问题：$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = f \end{cases}$

利用极值原理，证明$Laplace$方程的唯一性和稳定性

唯一性：设u_1, u_2均满足$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = f \end{cases}$，即$\begin{cases} \Delta u_1 = 0 \\ u_1|_{\partial\Omega} = f \end{cases}$且$\begin{cases} \Delta u_2 = 0 \\ u_2|_{\partial\Omega} = f \end{cases}$，令$u = u_1 - u_2$，则$\begin{cases} \Delta u = 0, x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$

而调和函数的最大/小值一定在边界取到，u在边界上的最大、最小值都是$0 \Rightarrow u_1 = u_2$

稳定性：方程的解连续依赖于边界条件

由极值原理$|u_1 - u_2| \leqslant \max_{\partial\Omega} |u_1 - u_2| = \max_{\partial\Omega} |f_1 - f_2|$

简单区域中的Dirichlet问题

2015伯苓考过

$Neumann$问题：$\begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = f \text{，其中} n \text{为外法向量} \end{cases} (P_1)$

设u为(P_1)的解，则$u + C$也为(P_1)的解，要证：所有解都能表示成$u + C$的形式

设v是P_1的任一解，则：$\begin{cases} \Delta(u - v) = 0 \\ \left.\frac{\partial(u-v)}{\partial n}\right|_{\partial\Omega} = 0 \text{，其中} n \text{为外法向量} \end{cases}$

将$u - v$和$u - v$代入$Green$第一公式：$\iint_{\partial\Omega} (u-v)\frac{\partial(u-v)}{\partial n} dS = \iiint_\Omega (u-v)\Delta(u-v) dx + \iiint_\Omega \sum_{j=1}^3 \left[\frac{\partial(u-v)}{\partial x_i}\right]^2 dx_i$

$\Rightarrow \iiint_\Omega \sum_{j=1}^3 \left[\frac{\partial(u-v)}{\partial x_i}\right]^2 dx_i = 0$，即$\frac{\partial(u-v)}{\partial x_i} = 0$，$i = 1,2,3, \Rightarrow u - v = c$

例：设$\Omega = B_R(Q), u \in C^2(\Omega) \cap C(\overline{\Omega})$，

证明：(1)$u(Q) = \frac{1}{4\pi R^2}\iint_{\partial B_R} u dS_P + \frac{1}{4\pi}\iiint_{B_R} \left(\frac{1}{R} - \frac{1}{r}\right)\Delta u dx$

(2)若$\Delta u \geqslant 0$，则$u(Q) \leqslant \frac{1}{4\pi R^2}\iint_{\partial B_R} u dS_P$

(1)【证明方法与平均值公式类似：位势积分$+ Green$公式】

由于$\Omega = B_R(Q)$，则$u(Q) = \frac{1}{4\pi}\left[\frac{1}{R}\iint_{\partial B_R(Q)} \frac{\partial u}{\partial n} dS + \frac{1}{R^2}\iint_{\partial B_R(Q)} u dS - \iiint_{B_R(Q)} \frac{1}{r}\Delta u dx\right]$

要证$u(Q) = \frac{1}{4\pi}\left[\frac{1}{R^2}\iint_{\partial B_R} u dS_P + \iiint_{B_R} \left(\frac{1}{R} - \frac{1}{r}\right)\Delta u dx\right]$

即证$\frac{1}{R}\iint_{\partial B_R(Q)} \frac{\partial u}{\partial n} dS = \iiint_{B_R(Q)} \frac{1}{R}\Delta u dx$，$Green$第二公式代入$v = 1$即得$\iint_{\partial B_R(Q)} \frac{\partial u}{\partial n} dS = \iiint_{B_R(Q)} \Delta u dx$

(2)由(1)中所证，$\iint_{\partial B_R(Q)} \frac{\partial u}{\partial n} dS = \iiint_{B_R(Q)} \Delta u dx \geqslant 0$且$\frac{1}{R} - \frac{1}{r} \leqslant 0$

即得$u(Q) \leqslant \frac{1}{4\pi R^2}\iint_{\partial B_R} u dS_P$(代入(1)所证等式)

$Green$函数

和证明平均值公式一样，Green公式+位势积分表示u(Q)，就可以消掉一项

对于$Dirichlet$问题：$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$，考虑另外一个$Dirichlet$问题：$\begin{cases} \Delta v = 0 \\ v|_{\partial\Omega} = \frac{1}{r(P,Q)} \end{cases}$

将Q视为动点，P视为定点，则由$Green$公式有

$$0 = \frac{1}{4\pi}\iiint_\Omega [u\Delta v - v\Delta u] dx = \frac{1}{4\pi}\iint_{\partial\Omega} \left[g\frac{\partial v}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n}\right] dS \quad (1)$$

$$\text{位势积分：} u(Q) = \frac{1}{4\pi}\left[\iint_{\partial\Omega} \frac{1}{r}\frac{\partial u}{\partial n} dS - \iint_{\partial\Omega} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS - \iiint_\Omega \frac{1}{r}\Delta u dx\right]$$

$$= \frac{1}{4\pi}\left[\iint_{\partial\Omega} \frac{1}{r}\frac{\partial u}{\partial n} dS - \iint_{\partial\Omega} g\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS\right] \quad (2)$$

$(1) + (2)$得$u(Q) = \frac{1}{4\pi}\iint_{\partial\Omega} g(P)\boxed{\frac{\partial}{\partial n}\left(v - \frac{1}{r}\right)} dS = \iint_{\partial\Omega} g(P)\frac{\partial}{\partial n}G(P,Q) dS_P$

其中$G(P,Q) = -\frac{1}{4\pi}\left(\frac{1}{r(P,Q)} - v(P,Q)\right)$，称为$Green$函数，是有一个奇点的调和函数($P = Q$处有奇点)

Green公式+位势积分表示u(Q)，和上面方法一样

(P_2) $\begin{cases} \Delta u = f, x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$，试用$Green$函数求解此问题，类似地，考虑另外一个$Dirichlet$问题：$\begin{cases} \Delta v = 0 \\ v|_{\partial\Omega} = -\Gamma(P,Q) \end{cases}$

$Green$公式有$\iiint_\Omega v\Delta u dx = \iint_{\partial\Omega} v\frac{\partial u}{\partial n} dS_P$

位势积分有$u(Q) = \int_{\partial\Omega} (-\Gamma(P,Q))\frac{\partial u}{\partial n} dS_P + \int_\Omega \Delta u\Gamma(P,Q) dx_P$

则有$u(Q) = \iiint_\Omega \Delta u\underbrace{\boxed{(v + \Gamma(P,Q))}}_{Green\text{函数}} dx = \iiint_\Omega \Delta u G(P,Q) dx$

【叠加原理】(P) $\begin{cases} \Delta u = f, x \in \Omega \\ u|_{\partial\Omega} = g \end{cases}$，

(P_1) $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$，$(P_2)$ $\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$

[若u_1, u_2分别为$(P_1)(P_2)$的解，则$u_1 + u_2$为(P)的解]

则(P)有形式解$u(Q) = \int_{\partial\Omega} g(P)\frac{\partial}{\partial n}G dS_P + \int_\Omega f(P)G dx$，

其中$G(P,Q) = \Gamma(P,Q) + v(P)$，$v(P)$为调和函数

$Green$函数的性质

(1)边界上是0：$G(P,Q)|_{\partial\Omega} = 0$

(2)Δ作用是δ：$\Delta G(P,Q) = \Delta(\Gamma(P,Q) + \frac{1}{4\pi}v(P,Q)) = \delta(P,Q)$

(3)$Green$函数可以表示$Dirichlet$问题的形式解：$\Rightarrow u(Q) = \iint_{\partial\Omega} g(P)\frac{\partial}{\partial n}G(P,Q) dS_P$

(4)交换性$G(Q_1, Q_2) = G(Q_2, Q_1)$

(5)$n = 3$时：① $0 < -G(P,Q) < \frac{1}{r(P,Q)}$

② $\iint_{\partial\Omega} \frac{\partial G(P,Q)}{\partial n} dS_P = 1$

交换性证明

和证明位势积分的思路一样，在挖去两个小球的空间上用Green公式，再拆开写

证明：设$\Omega_\varepsilon = \Omega - B_\varepsilon(Q_1) - B_\varepsilon(Q_2)$，$u_1 = G(P, Q_1)$，$u_2 = G(P, Q_2)$

在Ω_ε上对u_1, u_2使用$Green$公式：由于$\Delta u_1 = \Delta u_2 = 0$，且$u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0$

则有$0 = \iint_{\partial B_\varepsilon(Q_1)+\partial B_\varepsilon(Q_2)} u_1\frac{\partial u_2}{\partial n} - u_2\frac{\partial u_1}{\partial n} dS$，

即$\iint_{\partial B_\varepsilon(Q_1)+\partial B_\varepsilon(Q_2)} G(P,Q_1)\frac{\partial G(P,Q_2)}{\partial n} dS = \iint_{\partial B_\varepsilon(Q_1)+\partial B_\varepsilon(Q_2)} G(P,Q_2)\frac{\partial G(P,Q_1)}{\partial n} dS$

即$\iint_{\partial B_\varepsilon(Q_1)} G(P,Q_1)\frac{\partial G(P,Q_2)}{\partial n} - G(P,Q_2)\frac{\partial G(P,Q_1)}{\partial n} dS = \iint_{\partial B_\varepsilon(Q_2)} G(P,Q_2)\frac{\partial G(P,Q_1)}{\partial n} - G(P,Q_1)\frac{\partial G(P,Q_2)}{\partial n} dS$

左边：由于$P \in \partial B_\varepsilon(Q_1)$，$\sup_{P\in\partial B_\varepsilon(Q_1)} |G(P,Q_1)| = \sup_{P\in\partial B_\varepsilon(Q_1)} |\Gamma(P,Q_1) + v(P)| = \sup_{P\in\partial B_\varepsilon(Q_1)} \left|v(P) - \frac{1}{4\pi\epsilon}\right| \leqslant \frac{1}{4\pi\epsilon} + c_1$

$\sup_{P\in\partial B_\varepsilon(Q_1)} \left|\frac{\partial G(P,Q_2)}{\partial n}\right| \leqslant c_2$，则$\left|\iint_{\partial B_\varepsilon(Q_1)} G(P,Q_1)\frac{\partial G(P,Q_2)}{\partial n} dS\right| \leqslant 4\pi\epsilon^2\left(\frac{1}{4\pi\epsilon} + c_1\right)c_2 \xrightarrow{\varepsilon\to 0} 0$

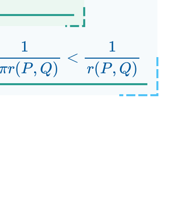
同样类似求位势积分的操作，在小球边界上化简形式，积分中值取极限

$$\iint_{\partial B_\varepsilon(Q_1)} -G(P,Q_2)\frac{\partial G(P,Q_1)}{\partial n} dS = \iint_{\partial B_\varepsilon(Q_1)} -G(P,Q_2)\left(-\frac{\partial}{\partial r}\right)\left[\frac{1}{4\pi r} + v(P)\right] dS$$

$$= \iint_{\partial B_\varepsilon(Q_1)} G(P,Q_2)\left[-\frac{1}{4\pi\varepsilon^2} + \frac{\partial}{\partial r}v(P)\right] dS$$

$$= G(P^*, Q_2) + 4\pi\varepsilon^2\left(G \cdot \frac{\partial v}{\partial r}\right)(\tilde{P})\text{，其中} P^*\tilde{P} \in \partial B_\varepsilon(Q_1)$$

$$\xrightarrow{\varepsilon\to 0} G(Q_1, Q_2)$$



同理，右式$\xrightarrow{\varepsilon\to 0} G(Q_2, Q_1)$，即得$G(Q_1, Q_2) = G(Q_2, Q_1)$

$0 < -G(P,Q) < \frac{1}{r(P,Q)}$证明

v在Ω上调和，其模长能被边界上的取值所限制住，两边放缩

证明：由$G(P,Q) = -\frac{1}{4\pi r} + v$，$v = \begin{cases} \Delta v = 0 \\ v|_{\partial\Omega} = \frac{1}{4\pi r} \end{cases}$，由极值原理，$\min_{\partial\Omega} \frac{1}{4\pi r} \leqslant v \leqslant \max_{\partial\Omega} \frac{1}{4\pi r}$

一方面，$-G(P,Q) = \frac{1}{4\pi r(P,Q)} - v(P) \geqslant \frac{1}{4\pi r(P,Q)} - \max_{\partial\Omega} \frac{1}{4\pi r(P,Q)} \to +\infty(P \to Q)$

则$\exists\delta > 0, s.t. -G(P,Q) > 0$，$\forall P \in B_\delta(Q)$

则$\forall P \in \Omega \backslash B_\delta$，有$\begin{cases} \Delta(-G(P,Q)) = 0 \\ -G(P,Q)|_{\partial\Omega} = 0 \\ G(P,Q)|_{\partial B_\delta(Q)} > 0 \end{cases}$，由极值原理得$G(P,Q) > 0$，$\forall P \in \Omega \backslash B_\delta$

另一方面$-G(P,Q) = \frac{1}{4\pi r(P,Q)} - v(P) \leqslant \frac{1}{4\pi r(P,Q)} - \min_{\partial\Omega} \frac{1}{4\pi r(P,Q)} \leqslant \frac{1}{4\pi r(P,Q)} < \frac{1}{r(P,Q)}$

$\int_{\partial\Omega} \frac{\partial G(P,Q)}{\partial n} dS_P = 1$证明

要证明=1的积分形式恰为边界取值恒为1的Dirichlet问题的形式解，而u恒为1满足方程

证明：对于(P_1) $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 1 \end{cases} \Rightarrow u(Q) = \int_{\partial\Omega} \frac{\partial G(P,Q)}{\partial n} dS_P$，而$u \equiv 1$一定是$(P_1)$的解

由极值原理可得(P_1)的唯一性，$\Rightarrow \iint_{\partial\Omega} \frac{\partial G(P,Q)}{\partial n} dS_P = 1$

半平面的$Green$函数

半平面上的问题$n = 2$时，静电源像法，求$Green$函数

形式解$u(Q) = \int_{\partial\Omega} \frac{\partial G(P,Q)}{\partial n} f(P) dS_P$

$\Gamma(P,Q)$：在Q点放置单位正电荷在P点产生的电势

$v(P)$：在Q点的对称点Q_1处放置负电荷产生的电位，$s.t.$在边界上的电势为0

就是针对这个问题把相应的G(P,Q)带进形式解u(Q)的表达式里去，求导，化简，2023考过：利用Green函数求解上半平面的Dirichlet问题

$\Rightarrow$在半平面问题中，在$Q(x, -y)$处放置单位负电荷

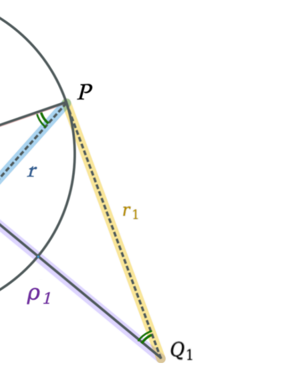
于是有$G(P,Q) = \Gamma(P,Q) + v(P) = -\frac{1}{2\pi}\ln\frac{1}{r(P,Q)} + \frac{1}{2\pi}\ln\frac{1}{r(P,Q_1)}$，想代入形式解中

计算$\frac{\partial G(P,Q)}{\partial n} = -\frac{1}{2\pi}\left[\frac{\partial}{\partial y_P}\ln r - \frac{\partial}{\partial y_P}\ln r_1\right] = -\frac{1}{2\pi}\left[\frac{1}{r}\frac{\partial r}{\partial y_P} - \frac{1}{r_1}\frac{\partial r_1}{\partial y_P}\right]$，其中$\frac{\partial r}{\partial y_P} = \frac{-(y - y_P)}{r}$，$\frac{\partial r_1}{\partial y_P} = \frac{y + y_P}{r_1}$

因为$r(P,Q) = [(x_P - x)^2 + (y_P - y)^2]^{\frac{1}{2}}$，$r(P,Q_1) = [(x_P - x)^2 + (y_P + y)^2]^{\frac{1}{2}}$，且$\frac{\partial}{\partial n} = -\frac{\partial}{\partial y_P}$

代入得$u(Q) = \int_{\partial\Omega} \frac{1}{2\pi}\left(\frac{y - y_P}{r^2} + \frac{y + y_P}{r_1^2}\right) f(x_P, y_P) dS_P = \int_{\partial\Omega} \frac{1}{\pi}\frac{y}{(x - x_P)^2 + y^2} f(x_P, 0) dS_P$【注意在边界上】

$\Rightarrow u(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x_P)\frac{y}{(x - x_P)^2 + y^2} dx_P$【形式解】



圆的$Green$函数

圆上的Laplace方程的Dirichlet问题

$\begin{cases} \Delta u = 0, x \in \Omega \\ u|_{\partial\Omega} = f \end{cases}$，利用静电源像法求$Green$函数$G(P,Q) = \Gamma(P,Q) + v(P)$

(1)Q点关于$\partial\Omega$的对称点是Q_1的位置

(2)Q_1处放置多少负电荷

如图所示，$r = |PQ|, r_1 = |PQ_1|, \rho = |OQ|, \rho_1 = |Q_1|$

取Q_1为Q的反演点，即$\triangle OPQ \sim \triangle OPQ_1$

即$\frac{r}{R} = \frac{r_1}{\rho_1}, \frac{R}{\rho_1} = \frac{\rho}{R} \Rightarrow \rho_1\rho = R^2 \Rightarrow$确定$Q_1$的位置

$\frac{r}{r_1} = \frac{R}{\rho_1} = \frac{\rho}{R} = \frac{1}{R/\rho} \Rightarrow$在$Q_1$点放置$\frac{R}{\rho}$负电荷

$G(P,Q) = -\frac{1}{2\pi}\ln\frac{1}{r(P,Q)} + \frac{1}{2\pi}\ln\frac{R/\rho}{r_1(PQ_1)}$

其中，$r^2(P,Q) = \rho_P^2 + \rho^2 - 2\rho_P\rho\cos(\theta - \theta_P)$

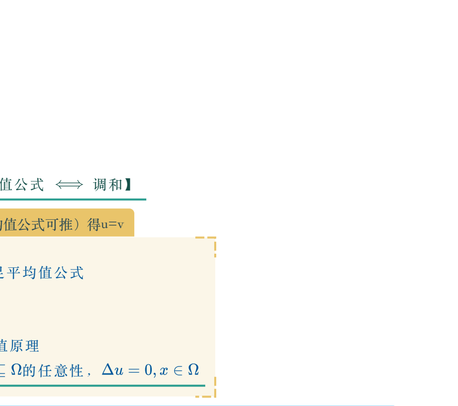
$r^2(P,Q_1) = \rho_P^2 + \rho_1^2 - 2\rho_P\rho_1\cos(\theta - \theta_P)$，且$\rho\rho_1 = R^2$

$\frac{\partial}{\partial n} = \frac{\partial}{\partial\rho_P} \Rightarrow \begin{cases} (1)\frac{\partial G(P,Q)}{\partial n} \\ (2)\text{当} P \in \partial\Omega\text{时} \Rightarrow \rho_P = R \end{cases}$

经一系列计算$\left.\frac{\partial G(P,Q)}{\partial n}\right|_{\partial\Omega} = \frac{1}{2\pi R}\frac{R^2 - \rho^2}{\rho^2 + R^2 - 2R\rho\cos\gamma}$

$u(Q) = \int_{\partial\Omega} \frac{\partial G(P,Q)}{\partial n} f(P) d\rho_P = \int_{\partial\Omega} \frac{1}{2\pi R}\frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta_P - \theta)} f(P) d\rho_P$

$\xlongequal{d\rho_P = Rd\theta_P} \frac{1}{2\pi R}\int_0^{2\pi} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta_P - \theta)} f(\theta_P) d\theta_P = u(Q) = u(\rho, \theta)$



Thm2.3.12的极坐标形式

证明若$F_\rho(\theta - \varphi)$是$\varphi \in [0, 2\pi]$周期为2π的连续函数(即圆周上的函数)，θ是一参数，则当适合以下两条件时必有$\lim_{\rho\to R^-} F_\rho(\theta - \varphi) = \delta(\theta - \varphi)$：

1.对任意$\varphi_1, \varphi_2 \in [0, 2\pi]$ $\left|\int_{\varphi_1}^{\varphi_2} F_\rho(\theta - \varphi) d\varphi\right| \leq C$.

2.在区间$[\varphi_1, \varphi_2]$上$\lim_{\rho\to R^-} \int_{\varphi_1}^{\varphi_2} F_\rho(\theta - \varphi) d\varphi = \begin{cases} 1, & \theta \in [\varphi_1, \varphi_2], \\ 0, & \theta \notin [\varphi_1, \varphi_2]. \end{cases}$

调和函数的另一些性质

平均值公式之逆

设$u(Q) \in C(\Omega)$满足平均值公式，则在Ω内是调和的【平均值公式$\iff$调和】

构造一个调和函数v，通过u-v在边界上是0，极值原理（只需平均值公式可推）得u=v

证明：取小球$B \subseteq \Omega$，令$\begin{cases} \Delta v = 0, x \in B \\ v|_{\partial B} = u|_{\partial B} \end{cases}$，$v$在$B$上调和，满足平均值公式

即$u(Q) = \frac{1}{|B|}\int_{\partial B} u(P) dx_P$，$v(Q) = \frac{1}{|B|}\int_{\partial B} v(P) dx_P$

则$u - v$在B上满足平均值公式，而考虑到平均值公式$\Rightarrow$极值原理

由于$u - v|_{\partial B} = 0$，则$u \equiv v, x \in \overline{B}$，则$u$在$B$中调和，由于$B \subseteq \Omega$的任意性，$\Delta u = 0, x \in \Omega$

可去奇点定理

证明调和函数的可去奇点定理：在以A为球心R为半径的小球上研究问题，构造补充定义后的函数u_1，用W衡量两个函数的差距，手段是再构造一个V比较V和W，先让Q→A，极值原理证明|W|＜V在小圆环上成立，再让ε→0，证明在去心圆盘上成立

u(Q)的奇性不能超过基本解

当$n = 2$时，若$u(Q)$在A点附近调和，且$u(Q) = o(1)\ln r(A,Q)$，则可补充定义，$s.t. \Delta u = 0$，$x \in B_\delta(A)$

证明：取小球$B_R(A) \subset \Omega$作为研究的对象，$B_\delta(A) \subset B_R(A)$

设u_1满足$\begin{cases} \Delta u_1 = 0 \\ u_1|_{\partial B_R} = u|_{\partial B_R} \end{cases}$，下证明$u_1$和$u$在除了$A$点点地方都离得很近

【手段：构造辅助函数】令$W = u_1 - u \Rightarrow \begin{cases} \Delta W = 0, B_R \backslash B_\delta \\ W|_{\partial B_R} = 0 \end{cases}$，

令$V_\varepsilon = \varepsilon\left(\ln\frac{1}{r(A,Q)} - \ln\frac{1}{R}\right)$，则$V_\varepsilon$满足$\begin{cases} \Delta V_\varepsilon = 0 \\ V_\varepsilon|_{\partial B_R} = 0 \end{cases}$，进而通过比较$V_\varepsilon$和$W$来说明$u_1$和$u$离得很近

和证明G(P,Q)的取值范围时的思路相同

由条件，$\lim_{Q\to A} \frac{u(Q)}{\ln r(A,Q)} = 0$，则有$\lim_{Q\to A} \frac{u_1(Q)}{\ln r(A,Q)} = 0$，则$\lim_{Q\to A} \frac{|W|}{\ln r(A,Q)} = \lim_{Q\to A} \frac{|u_1 - u|}{\ln r(A,Q)} = 0$，则$\forall\varepsilon > 0$，$\lim_{Q\to A} \frac{|W|}{V_\varepsilon} = 0$

则在Q附近存在一个小邻域B_{δ_0}满足$\frac{|W|}{V_\varepsilon} < 1$，即$|W| \leqslant V_\varepsilon, \forall x \in \partial B_{\delta_0}$

$\Rightarrow \begin{cases} \Delta(W - V_\varepsilon) = 0, B_R \backslash B_{\delta_0} & \text{，在去掉小圆盘上调和} \\ (W - V_\varepsilon)|_{\partial B_R} = 0 & \text{，在大圆盘边界上是0} \\ (W - V_\varepsilon)|_{\partial B_{\delta_0}} \leqslant 0\text{或} \geqslant 0 & \text{，在小圆盘边界上保号} \end{cases}$ 由于极值原理$\Rightarrow W - V_\varepsilon$在$B_R \backslash B_{\delta_0}$上保号，令$\delta_0 \to 0^+$仍成立

则$|W| \leqslant V\varepsilon$在$B_R \backslash \{A\}$上成立，再让$\varepsilon \to 0$，则$|W| = |u_1 - u| = 0, B_R \backslash \{A\}$，即可在$u(A)$处补充定义使之成为$u_1$

调和函数的解析性

若$\Delta u = 0$，则$u \in C^\infty$且可以展成收敛幂级数(正则性很高)

弱：具有此形式，但正则性较低

若$\forall\varphi \in C_0^\infty(\Omega), \langle\Delta v, \varphi\rangle = \langle 0, \varphi\rangle = 0 \Rightarrow \langle v, \Delta\varphi\rangle = 0$，则称$v$是$\Delta u = 0$的$\mathscr{D}'$解(弱解，分布定义解)

$Liouville$定理

在$\mathbb{R}^n$上有界调和函数必为常函数

思路是证明u(x)-u(0)=0，方法是展开成平均值公式，对积分区域做放缩，证明R→∞时|u(x)-u(0)|→0

证明：$\forall x \in \mathbb{R}^n$，取$B_R(0), B_R(x)$，由于$u$调和，有平均值公式$u(0) = \frac{1}{|B_R(0)|}\int_{B_R(0)} u(P) dx_P$，$u(x) = \frac{1}{|B_R(x)|}\int_{B_R(x)} u(P) dx_P$

$$|u(x) - u(0)| = \frac{1}{|B_R|}\left|\int_{B_R(0)} u(y) dy - \int_{B_R(x)} u(y) dy\right| \leqslant \frac{1}{|B_R|}\left(\int_{B_R(0)\backslash B_R(x)} u(y) dy + \int_{B_R(x)\backslash B_R(0)} u(y) dy\right)$$

$$\leqslant \frac{M}{|B_R|}\left(\int_{B_R(0)\backslash B_R(x)} 1 dy + \int_{B_R(x)\backslash B_R(0)} 1 dy\right)$$

$|y - x| > R, |y| \leqslant R$ $\quad$ $|y - x| \leqslant R, |y| > R$

$R - |x| < |y| \leqslant R$ $\quad$ $R < |y| \leqslant |x| + R$

$$\leqslant \frac{M}{|B_R|}\int_{R-|x|}^{R+|x|} 1 dy$$

$$= \frac{M}{c_n \cdot R^n}c_n((R + |x|)^n - (R - |x|^n))$$

$$= O\left(\frac{1}{R}\right) \xrightarrow{R\to\infty} 0$$

即得$u(x) \equiv u(0)$，即$u(x)$恒为常数【对n维成立】

$Harnack$定理

调和函数列+一致收敛→极限函数也调和

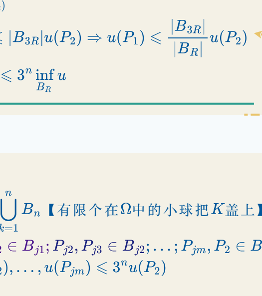
设$\{u_n(Q)\}$是Ω中的调和函数列，且在Ω中一致收敛于$u(Q)$，则$u(Q)$也是Ω中的调和函数

思路：$u(Q) = \int_{\partial\Omega} \frac{\partial G(P,Q)}{\partial n} f(P) dS_P$是$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = f \end{cases}$的形式解

取$B \subseteq \Omega$，u_n在∂B上一致收敛于u，$f_n(P) = u_n|_{\partial B} \rightrightarrows f(P) = u|_{\partial B}(n \to \infty)$

$\Rightarrow \underset{\downarrow}{u_n(Q)} = \int_{\partial B} \frac{\partial G(P,Q)}{\partial n} f_n(P) dS_P \to \int_{\partial B} \frac{\partial G(P,Q)}{\partial n} f(P) dS_P$

$u(Q)$ $\quad \Rightarrow \Delta u = 0, x \in B$，由$B$的任意性得在$V$中调和



u非负很重要，在大的区域上积分大，2023伯苓考过

一般的$Harnack$不等式：对紧集$K \subseteq \Omega$，有【非负】调和函数u，有$\sup_K u \leqslant c^n \inf_K u$

证明了在小球中的结论

证明：取$P_1, P_2 \in B_R$，$u(P_1) = \frac{1}{|B_R(P_1)|}\int_{B_R(P_1)} u(y) dy$，$u(P_2) = \frac{1}{|B_{3R}|}\int_{B_{3R}(P_2)} u(y) dy$

$\Rightarrow B_R(P_1) \subset B_{3R}(P_2) \xrightarrow{u\text{非负}} \boxed{\int_{B_R(P_1)} u(y) dy \leqslant \int_{B_{3R}(P_2)} u(y) dy} \Rightarrow |B_R|u(P_1) \leqslant |B_{3R}|u(P_2) \Rightarrow u(P_1) \leqslant \frac{|B_{3R}|}{|B_R|}u(P_2)$

$|B_R| = c_nR^n, |B_{3R}| = c_n(3R)^n \Rightarrow u(P_1) \leqslant 3^nu(P_2)$，由$P_1, P_2$的任意性，$\sup_{B_R} u \leqslant 3^n \inf_{B_R} u$

在紧集K中，添加若干个辅助点，每两点都在一个小球中，有限个小球覆盖K

对连通紧子集$K \to$可由有限个球覆盖住$\to$球的半径如何取

取$R = \frac{1}{4}\text{dist}(K, \partial\Omega)$(因$\Omega$无非负调和)，$\exists$以$R$为半径的球$B_1, \ldots, B_n, s.t. K \subseteq \bigcup_{k=1}^n B_n$【有限个在$\Omega$中的小球把$K$盖上】

$\forall P_1, P_2 \in K$，设$P_1 \in B_{i1}, P_2 \in B_{i2}$，则$\exists P_{j1}, \ldots, P_{jm}$，s.t. $P_1, P_{j1} \in B_{i1}$；$P_{j1}, P_{j2} \in B_{j1}$；$P_{j2}, P_{j3} \in B_{j2}$；$\ldots$；$P_{jm}, P_2 \in B_{i2}$

【保证每两点都在同一个小球中】$\Rightarrow u(P_1) \leqslant 3^nu(P_{j1}), \ldots, u(P_{j1}) \leqslant 3^nu(P_{j2}), \ldots, u(P_{jm}) \leqslant 3^nu(P_2)$

$\Rightarrow u(P_1) \leqslant c\,u(P_2) \Rightarrow \sup_K u \leqslant c \inf_K u$

例题

习题6.2(6)：设u在以Q为圆心R为半径的圆B内非负调和，而在$\overline{B}$上连续，则$\frac{R - \rho}{R + \rho}u(Q) \leqslant u(P) \leqslant \frac{R + \rho}{R - \rho}u(Q)$

证明：$u(P) = u(\rho, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)} f(P) d\varphi$

$-1 \leqslant \cos(\theta - \varphi) \leqslant 1 \Rightarrow \begin{cases} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)} \leqslant \frac{R^2 - \rho^2}{(R - \rho)^2} = \frac{R + \rho}{R - \rho} \\ \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)} \geqslant \frac{R^2 - \rho^2}{(R + \rho)^2} = \frac{R - \rho}{R + \rho} \end{cases}$

由u是非负的得$\frac{R - \rho}{R + \rho}\frac{1}{2\pi}\int_0^{2\pi} f(P) d\varphi \leqslant u(P) \leqslant \frac{R + \rho}{R - \rho}\frac{1}{2\pi}\int_0^{2\pi} f(P) d\varphi$

$claim$：$u(Q) = \frac{1}{2\pi}\int_0^{2\pi} f d\varphi$，实际上，由平均值公式或$u(P) \xlongequal{P=Q} u(Q) \xlongequal{P=0} \frac{1}{2\pi}\int_0^{2\pi} f d\varphi$