数理方程 第二次小测

位势积分公式

推导位势积分公式：利用第二类Green公式，取v=1/r

第二类Green公式

设$\Omega\subset\mathbb{R}^3$是一有界区域，且$\partial\Omega$光滑，设$u$和$v$都在$C^2(\Omega)\cap C^1(\overline{\Omega})$中于是因为

$$u\Delta v=\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(u\frac{\partial v}{\partial x_j}\right)-\sum_{j=1}^{3}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j},$$

$$v\Delta u=\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(v\frac{\partial u}{\partial x_j}\right)-\sum_{j=1}^{3}\frac{\partial v}{\partial x_j}\frac{\partial u}{\partial x_j},$$

二者相减有$u\Delta v-v\Delta u=\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(u\frac{\partial v}{\partial x_j}-v\frac{\partial u}{\partial x_j}\right).$

两边在Ω上积分，利用数学分析中的高斯定理，即得以下重要的格林公式：

$$\iiint_{\Omega}(u\Delta v-v\Delta u)\,\mathrm{d}x=\iint_{\partial\Omega}\left[u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right]\mathrm{d}S,$$ n是$\partial\Omega$的单位外法向量。

令$v=\frac{1}{r}$，代入$Green$公式，设$B_\varepsilon(Q)$是以Q为圆心，ε为半径小球，$\Omega_\varepsilon=\Omega\backslash B_\varepsilon$，$P$是动点，$|PQ|=r$

则$v=\frac{1}{r}\in C^2(\Omega_\varepsilon)\cap C^1(\overline{\Omega}_\varepsilon)$，在$\Omega_\varepsilon$上使用$Green$公式，$u$仍为一般的$C^2(\Omega_\varepsilon)\cap C^1(\overline{\Omega}_\varepsilon)$

则 $$\iiint_{\Omega_\varepsilon}\left[u\Delta\left(\frac{1}{r}\right)-\frac{1}{r}\Delta u\right]\mathrm{d}x=\iint_{\partial\Omega_\varepsilon}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S$$

把这个空心小球的两部分拆开：$$\iiint_{\Omega_\varepsilon}\left[u\Delta\left(\frac{1}{r}\right)-\frac{1}{r}\Delta u\right]\mathrm{d}x=\iint_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S-\iint_{\partial B_\varepsilon(Q)}\left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial r}\right]\mathrm{d}S\quad(*)$$

其中因为$\frac{1}{r}$是基本解的主项，即$\Delta c_3\left(\frac{1}{r}\right)=\delta$，则在$\Omega_\varepsilon$中有$\Delta\left(\frac{1}{r}\right)=0\Rightarrow\iiint_{\Omega_\varepsilon}u\Delta\left(\frac{1}{r}\right)\mathrm{d}x_p=0$，则等式$(*)$左端$=-\iiint_{\Omega_\varepsilon}\frac{1}{r}\Delta u\mathrm{d}x$

下考虑$\iint_{\partial B_\varepsilon(Q)}\left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial r}\right]\mathrm{d}S$：$r=\varepsilon$，则为$-\frac{1}{\varepsilon^2}\iint_{\partial B_\varepsilon(Q)}u\mathrm{d}S-\frac{1}{\varepsilon}\iint_{\partial B_\varepsilon(Q)}\frac{\partial u}{\partial r}\mathrm{d}S$

由【积分中值定理】得，$\iint_{\partial B_\varepsilon(Q)}u\mathrm{d}S=u(Q^*)4\pi\varepsilon^2$，$\iint_{\partial B_\varepsilon(Q)}\frac{\partial u}{\partial r}\mathrm{d}S=\frac{\partial u}{\partial r}(\tilde{Q})4\pi\varepsilon^2$，其中$Q^*,\tilde{Q}\in\partial B_\varepsilon(Q)$

则$$\lim_{\varepsilon\to0}\iint_{\partial B_\varepsilon(Q)}\left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial r}\right]\mathrm{d}S=-4\pi u(Q)$$，此时原方程化为$$-\iiint_{\Omega}\frac{1}{r}\Delta u\mathrm{d}x=\iint_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial u}{\partial n}\right]\mathrm{d}S+4\pi u(Q)$$

整理得到位势积分公式：$$u(Q)=\frac{1}{4\pi}\left[\iint_{\partial\Omega}\frac{1}{r}\frac{\partial u}{\partial n}\mathrm{d}S-\iint_{\partial\Omega}u\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\mathrm{d}S-\iiint_{\Omega}\frac{1}{r}\Delta u\mathrm{d}x\right]$$

$n>3$时的位势积分公式：$$u(Q)=\frac{1}{(n-2)|S^{n-1}|}\left[\int_{\partial\Omega}r^{2-n}\frac{\partial u}{\partial n}\mathrm{d}S_P-\int_{\partial\Omega}u\frac{\partial}{\partial n}(r^{2-n})\mathrm{d}S_P-\int_{\Omega}r^{2-n}\Delta u\mathrm{d}x_P\right]$$

可去奇点定理

证明调和函数的可去奇点定理：在以A为球心R为半径的小球上研究问题，构造补充定义后的函数u_1，用W衡量两个函数的差距，手段是再构造一个V比较V和W，先让Q→A，极值原理证明|W|≤V在小圆环上成立，再让ε→0，证明在去心圆盘上成立

当$n=2$时，若$u(Q)$在A点附近调和，且$u(Q)=o(1)\ln r(A,Q)$，则可补充定义，$s.t.\ \Delta u=0$，$x\in B_\delta(A)$

证明：取小球$B_R(A)\subset\Omega$作为研究的对象，$B_\delta(A)\subset B_R(A)$

设u_1满足$\begin{cases}\Delta u_1=0\\ u_1|_{\partial B_R}=u|_{\partial B_R}\end{cases}$，下证明$u_1$和$u$在除了$A$点点地方都离得很近

【手段：构造辅助函数】令$W=u_1-u\Rightarrow\begin{cases}\Delta W=0,B_R\backslash B_\delta\\ W|_{\partial B_R}=0\end{cases}$，

令$V_\varepsilon=\varepsilon\left(\ln\frac{1}{r(A,Q)}-\ln\frac{1}{R}\right)$，则$V_\varepsilon$满足$\begin{cases}\Delta V_\varepsilon=0\\ V_\varepsilon|_{\partial B_R}=0\end{cases}$，进而通过比较$V_\varepsilon$和$W$来说明$u_1$和$u$离得很近

由条件，$\lim_{Q\to A}\frac{u(Q)}{\ln r(A,Q)}=0$，则有$\lim_{Q\to A}\frac{u_1(Q)}{\ln r(A,Q)}=0$($Q\to A$时$\ln r(A,Q)\to\infty$，而$u_1(Q)$的大小能被边界所控制住)

则$\lim_{Q\to A}\frac{|W|}{\ln r(A,Q)}=\lim_{Q\to A}\frac{|u_1-u|}{\ln r(A,Q)}=0$，而显然$\forall\varepsilon>0$，$\lim_{Q\to A}\frac{|W|}{V_\varepsilon}=0$，则$\exists\delta_0$，$s.t.\ |W|\leqslant V_\varepsilon,\forall x\in\partial B_{\delta_0}$

$\Rightarrow\begin{cases}\Delta(W-V_\varepsilon)=0,B_R\backslash B_{\delta_0} & \text{，在去掉小圆盘上调和}\\ (W-V_\varepsilon)|_{\partial B_R}=0 & \text{，在大圆盘边界上是0}\\ (W-V_\varepsilon)|_{\partial B_{\delta_0}}\leqslant0\text{或}\geqslant0 & \text{，在小圆盘边界上保号}\end{cases}$ 由于极值原理$\Rightarrow W-V_\varepsilon$在$B_R\backslash B_{\delta_0}$上保号，令$\delta_0\to0^+$仍成立

则$|W|\leqslant V\varepsilon$在$B_R\backslash\{A\}$上成立，再让$\varepsilon\to0$，则$|W|=|u_1-u|=0,B_R\backslash\{A\}$，即可在$u(A)$处补充定义使之成为$u_1$

热传导算子的基本解

以$n=3$为例，对$t>0$，$\frac{\partial}{\partial t}E(t)-a^2\Delta E(x,t)=\delta(x,t)$

对x做Fourier变换，ode方程求出F(E)，再做逆变换，凑成Gauss积分

等式两边对x做$Fourier$变换：$F_{x\to\xi}\left(\frac{\partial}{\partial t}E(x,t)-a^2\Delta E(x,t)\right)(\xi,t)=(F_{x\to\xi}\delta(x,t))(\xi,t)$

$\Rightarrow\frac{\partial}{\partial t}\hat{E}(\xi,t)+a^2|\xi|^2\hat{E}(\xi,t)=\delta(t)$即为$\hat{E}(\xi,t)$的一阶线性非齐次$ode$方程，解得为$\hat{E}(\xi,t)=H(t)e^{-a^2|\xi|^2t}$，

令$v(t)=e^{a^2|\xi|^2\hat{E}(\xi,t)}\hat{E}(\xi,t)$，这样方程化为$v'(t)=\delta(t)$，$v(t)=H(t)+C$，因为做$Fourier$变换要求$E(x,t)\in\mathscr{S}'(\mathbb{R}^{n+1}_{x,t})$，

则也要求$\hat{E}(x,t)\in\mathscr{S}'(\mathbb{R}^{n+1}_{x,t})$，所以要求$t\to-\infty$时$\hat{E}(\xi,t)=(H(t)+C)e^{a^2|\xi|^2\hat{E}(\xi,t)}$的任意解导数都趋于0，则$C=0$

则$E(x,t)=F^{-1}\left[H(t)e^{-a^2|\xi|^2t}\right]=(2\pi)^{-3}\int e^{ix\xi}H(t)e^{-a^2|\xi|^2t}\mathrm{d}\xi$

$$\xlongequal{\text{凑成}Gauss\text{积分}}H(t)(2\pi)^{-3}\int_{\mathbb{R}^3}\exp\left[i\cdot\frac{x}{a\sqrt{2t}}\left(a\sqrt{2t}\xi\right)\right]\cdot\exp\left(-\frac{1}{2}|a\sqrt{2t}\xi|^2\right)\mathrm{d}\xi$$

$$\xlongequal{F_{x\to\xi}\left(e^{-\frac{|x|^2}{2}}\right)=(2\pi)^{\frac{n}{2}}e^{-\frac{|\xi|^2}{2}}\Rightarrow F^{-1}\left(e^{-\frac{|\xi|^2}{2}}\right)=(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}}H(t)\left(a\sqrt{2t}\right)^{-n}(2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2}\left|\frac{x}{a\sqrt{2t}}\right|^2}=H(t)\left(4\pi a^2t\right)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4a^2t}},t>0$$

热传导方程的基本解$K(x,t)=(4\pi t)^{-\frac{n}{2}}H(t)e^{-\frac{|x|^2}{4t}}$

Cauchy问题

$(P)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t),\quad t>0\\ u|_{t=0}=\varphi(x)\end{cases}$，$(P_2)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t),\quad t>0\\ u|_{t=0}=0\end{cases}$

$(P_1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=0,\quad t>0\\ u|_{t=0}=\varphi(x)\end{cases}$，若$u_1$为$P_1$的解，$u_2$为$P_2$的解，则$u_1+u_2=u$为$(P)$的解(线性所以才能叠加)

法一：叠加原理

Step 1：求解（P_1）

和基本解的求法基本一致，只是把H(t)换成了hatφ(ξ),再利用先卷积再变化=先变化再相乘的性质

step 1:证明当$u(x,t)$满足(P_1)时，$u(x,t)=E(x,t)\underset{(x)}{*}\varphi(x)$

$(P_1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=0,\quad t>0\\ u|_{t=0}=\varphi(x)\end{cases}$ 关于x做$Fourier$变换$\Rightarrow\begin{cases}\frac{\partial}{\partial t}\hat{u}(t,\xi)+a^2|\xi|^2\hat{u}(t,\xi)=0,\quad t>0\\ \hat{u}|_{t=0}=\hat{\varphi}(\xi)\end{cases}$

$\Rightarrow\hat{u}(t,\xi)=\hat{\varphi}(\xi)e^{-a^2|\xi|^2t}$，$u(t,x)=F^{-1}_{\xi\to x}\left(\hat{\varphi}(\xi)e^{-a^2|\xi|^2t}\right)$【和求基本解的做法一致】

由于$\widehat{f_1*f_2}=\hat{f}_1\cdot\hat{f}_2$，令$\begin{cases}f_1(t,x)=F^{-1}\left(\hat{e}^{-a^2|\xi|^2t}\right)=\left(4\pi a^2t\right)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4a^2t}}\\ f_2(x)=\varphi(x)\end{cases}$，$\Rightarrow\hat{f}_1\cdot\hat{f}_2=u(t,x)=\widehat{f_1*f_2}=\left(4\pi a^2t\right)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4a^2t}}\underset{(x)}{*}\varphi(x)$

↑ 就是基本解去掉$H(t)$，方法一样的，凑成$Gauss$积分 $=\left(4\pi a^2t\right)^{-\frac{n}{2}}\int\varphi(y)e^{-\frac{|x-y|^2}{4a^2t}}\mathrm{d}y,t>0$

$=E(x,t)\underset{(x)}{*}\varphi(x)$【直接就能看出来】

要验证<δ(t,x),f(x)>=δ(t)f(0)，即等价于说明δ(t,x) 的“时间”和“空间”是可以分离的

接下来要验证：$u(x,t)=E(x,t)\underset{(x)}{*}\psi(x)$满足$(P_1)$：

① $\left(\frac{\partial}{\partial t}-a^2\Delta\right)(E(x,t)\underset{(x)}{*}\psi(x))=\left[\left(\frac{\partial}{\partial t}-a^2\Delta\right)E(x,t)\right]\underset{(x)}{*}\psi(x)=\delta(x,t)\underset{(x)}{*}\psi(x)=\langle\delta(y,t),\psi(x-y)\rangle_y\overset{?}{=\!=}\delta(t)\psi(x)=0$

$claim$：$\langle\delta(t,x),f(x)\rangle=\delta(t)f(0)$

证明：将$\delta(x,t)$磨光，取磨光核$\Phi_\varepsilon(x,t)$，令$\delta_\varepsilon(t,x)=\delta(t,x)*\Phi_\varepsilon(t,x)\in C_0^\infty(\mathbb{R}^{n+1})$，$\forall\psi(t)\in C_0^\infty(\mathbb{R})$，有

$\langle\langle\delta_\varepsilon(t,x),f(x)\rangle,\psi(t)\rangle=\int\psi(t)\mathrm{d}t\int\delta_\varepsilon(t,x)f(x)\mathrm{d}x=\iint\delta_\varepsilon(t,x)f(x)\psi(t)\mathrm{d}x\mathrm{d}t=\langle\delta_\varepsilon(t,x),f(x)\psi(t)\rangle$

令$\varepsilon\to0$得$\langle\langle\delta(t,x),f(x)\rangle,\psi(t)\rangle=\langle\delta(t,x),f(x)\psi(t)\rangle=f(0)\psi(0)=f(0)\langle\delta(t),\psi(t)\rangle=\langle f(0)\delta(t),\psi(t)\rangle\Rightarrow\langle\delta(t,x),f(x)\rangle=\delta(t)f(0)$

先证明基本解载全空间上的积分是1，再利用该引理将绝对值分成两段，小球内积分自然可任意小，小球外积分要将基本解展开写，换元后易证趋于0

② 验证$\lim_{t\to0^+}u(t,x)=\lim_{t\to0^+}E(t,x)\underset{(x)}{*}\varphi(x)=\varphi(x)$，即证$\lim_{t\to0^+}E(t,x)=\delta(x)$，即证$\forall\varphi(x)\in C_0^\infty(\mathbb{R}^n)$，$\langle E(t,x),\varphi(x)\rangle\xrightarrow{t\to0^+}\varphi(0)$

$claim$：$\int_{\mathbb{R}^n}E(x,t)\mathrm{d}x=1,t>0$，$proof$：$\int_{\mathbb{R}^n}E(x,t)\mathrm{d}x=\int_{\mathbb{R}^n}(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}\mathrm{d}x\xlongequal{\text{令}2\sqrt{t}z=x}(\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-|z|^2}\mathrm{d}z=1$

则$E(x,t)\in L_{loc}(\mathbb{R}^n)$，$\forall\varphi\in C_0^\infty(\mathbb{R}^n)$，由连续性可知$\forall\varepsilon>0,\exists\,\delta>0$，s.t.$|x|<\delta$时，$|\varphi(x)-\varphi(0)|<\varepsilon$

则$\left|\left\langle E(x,t),\varphi(x)\right\rangle-\varphi(0)\right|=\left|\int_{\mathbb{R}^n}E(x,t)\varphi(x)\mathrm{d}x-\varphi(0)\right|\xlongequal{\varphi(0)=\varphi(0)\int E(x,t)\mathrm{d}x}\left|\int_{\mathbb{R}^n}E(x,t)\left[\varphi(x)-\varphi(0)\right]\mathrm{d}x\right|$

$=\left|\int_{B_\delta(0)}E(x,t)\left[\varphi(x)-\varphi(0)\right]\mathrm{d}x\right|+\left|\int_{\mathbb{R}^n\backslash B_\delta(0)}E(x,t)\left[\varphi(x)-\varphi(0)\right]\mathrm{d}x\right|$

$\left|\int_{B_\delta(0)}E(x,t)\left[\varphi(x)-\varphi(0)\right]\mathrm{d}x\right|\leqslant\varepsilon\left|\int_{B_\delta(0)}E(x,t)\mathrm{d}x\right|\leqslant\varepsilon$

$\left|\int_{\mathbb{R}^n\backslash B_\delta(0)}E(x,t)\left[\varphi(x)-\varphi(0)\right]\mathrm{d}x\right|\leqslant2\|\varphi\|_\infty\int_{\mathbb{R}^n\backslash B_\delta(0)}E(x,t)\mathrm{d}x\xlongequal{\text{令}2\sqrt{t}z=x}(\pi)^{\frac{n}{2}}\|\varphi\|_\infty\int_{\mathbb{R}^n\backslash B_{\frac{\delta}{2\sqrt{t}}}(0)}e^{-|z|^2}\mathrm{d}x\xrightarrow{t\to0^+}0$

综上，即得$\forall\varphi(x)\in C_0^\infty(\mathbb{R}^n)$，$\langle E(t,x),\varphi(x)\rangle\xrightarrow{t\to0^+}\varphi(0)$

综上，得到了$t>0$，$u_1(x,t)=E(t,x)\underset{(x)}{*}\varphi(x)=H(t)(4\pi a^2t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(y)e^{-\frac{|x-y|^2}{4a^2t}}\mathrm{d}y$为$(P_1)$的解

Step 2：利用Duhamel原理求解(P_2)

构造U(x,t,τ)将(P_2)转化为(P_1)的形式

$(P_2)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t),\quad t>0\\ u|_{t=0}=0\end{cases}$

【齐次化原理($Duhamel$原理)】将(P_2)化为齐次方程非零初值问题【即将(P_2)形式的方程转化为(P_1)形式的方程】

构造辅助函数$U(x,t,\tau)$满足$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)U=0,\quad t>\tau\\ U(x,t,\tau)|_{t=\tau}=f(x,\tau)\end{cases}$，令$u(x,t)=\int_0^tU(x,t;\tau)\,d\tau\Rightarrow\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t),t>0\\ u|_{t=0}=0\end{cases}$

具体来说，$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}\int_0^tU(x,t,\tau)\mathrm{d}\tau=U(x,t,t)+\int_0^t\frac{\partial U(x,t,\tau)}{\partial t}\mathrm{d}\tau=f(x,t)+\int_0^ta^2\Delta U(x,t,\tau)\mathrm{d}\tau=f(x,t)+a^2\Delta u(x,t)\\ u|_{t=0}=\int_0^0\cdots\mathrm{d}\tau=0\end{cases}$

从而就可由u_1直接写出U(x,t,τ)的解，把积分写开变成n+1重积分就是u_2的解

利用(P_1)的解，写出$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)U=0,\quad t-\tau>0\\ U(x,t,\tau)|_{t-\tau=0}=f(x,\tau)\end{cases}$的解

$U(x,t,\tau)=E(x,t-\tau)\underset{(x)}{*}f(x,\tau)=\int_{\mathbb{R}^n}\left(4\pi a^2(t-\tau)\right)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)\mathrm{d}y=\left(4\pi a^2(t-\tau)\right)^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(y,\tau)e^{-\frac{|x-y|^2}{4a^2(t-\tau)}}\mathrm{d}y$

$\Rightarrow$把解出的$U(x,t,\tau)$回代$u(x,t)=\int_0^tU(x,t,\tau)\mathrm{d}\tau$

$=\int_0^t\left(4\pi a^2(t-\tau)\right)^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(y,\tau)\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)\mathrm{d}y\mathrm{d}\tau$

$=\int_0^t\int_{\mathbb{R}^n}f(y,\tau)\boxed{\left(4\pi a^2(t-\tau)\right)^{-\frac{n}{2}}\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)}\mathrm{d}y\mathrm{d}\tau$

$\xlongequal{\text{后面要证}}E(x,t)*(H(t)f(x,t))=u_2(x,t)$ $Green$函数，$Heat\ Kernal$

验证$u_2(x,t)$为(P_2)的解：通过$Duhamel$原理的推导，已经保证了$u_2(x,t)(x,t)$必然满足非齐次方程(P_2)的所有要求

于是令$u(x,t)=u_1(x,t)+u_2(x,t)$，为$(P)\begin{cases}\frac{\partial u}{\partial t}-a^2\Delta u=f(x,t),t>0\\ u|_{t=0}=\varphi(x)\end{cases}$的解

法二：利用基本解$E(x,t)$直接求形式解

对于线性算子P，基本解为$E(x)$，则$u=E*f$为$Pu=f$的$\mathscr{D}'$解

思路：$(P)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t),\quad t>0\\ u|_{t=0}=\varphi(x)\end{cases}$，令$\tilde{u}(x,t)=H(t)u(x,t)$，其中$u(x,t)$为$P$的解$\Rightarrow\tilde{u}(t,x)\xlongequal{t>0}u(x,t)$

$\left(\frac{\partial}{\partial t}-a^2\Delta\right)\tilde{u}(x,t)=\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))\overset{\triangle}{=\!=}F(x,t)\Rightarrow\tilde{u}(x,t)=E(x,t)*F(x,t)\xlongequal{t>0}u(x,t)$

就是一步δ(t)u(x,t)=δ(t)φ(x)的化简

$\left(\frac{\partial}{\partial t}-a^2\Delta\right)\tilde{u}(x,t)=\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))\overset{\triangle}{=\!=}F(x,t)\Rightarrow\tilde{u}(x,t)=E(x,t)*F(x,t)\xlongequal{t>0}u(x,t)$

$\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))=\delta(t)u(x,t)+H(t)\frac{\partial}{\partial t}u(x,t)-a^2H(t)\Delta u(x,t)=\delta(t)u(x,t)+H(t)f(x,t)$

下证：$\delta(t)u(x,t)=\delta(t)\varphi(x)$，取$\forall\psi(t)\in C_0^\infty$，$\left\langle\delta(t)u(x,t),\psi(t)\right\rangle=\left\langle\delta(t)u(x,t),\psi(t)\right\rangle=\left\langle\delta(t),u(x,t)\psi(t)\right\rangle=u(x,0)\psi(0)$

$||$

$\left\langle\delta(t)\varphi(x),\psi(t)\right\rangle=\left\langle\delta(t),\varphi(x)\psi(t)\right\rangle=\varphi(x)\psi(0)$

即得$\delta(t)u(x,t)=\delta(t)\varphi(x)$

则设$\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))=\delta(t)\varphi(x)+H(t)f(x,t)\overset{\triangle}{=\!=}F(x,t)$

由基本解性质，可知$\tilde{u}(t,x)=E(x,t)*F(x,t)=E(x,t)*(\delta(t)\varphi(x)+H(t)f(x,t))$

两项卷积的第一项化简，依旧是对δ磨光，通过卷积的定义转化形式再取极限

① $\delta(t)\varphi(x)*E(x,t)$，取$\Phi_\varepsilon(t)$为磨光核，令$\delta_\varepsilon(t)=\delta(t)*\Phi_\varepsilon(t)\in C_0^\infty$，$(\delta_\varepsilon(t)\varphi(x))*E(x,t)=\int_{\mathbb{R}}\int_{\mathbb{R}^n}\delta_\varepsilon(\tau)\varphi(y)E(x-y,t-\tau)\mathrm{d}\tau\mathrm{d}y$

由$\delta_\varepsilon(t)\in C_0^\infty(\mathbb{R})$，$\varphi(x)$有界连续，$E(x,t)$在空间方向$x$为急减的，从而以上积分有意义

$\Rightarrow$上式$=\int_{\mathbb{R}^n}\varphi(y)\mathrm{d}y\int\delta_\varepsilon(\tau)E(x-y,t-\tau)\mathrm{d}\tau=\left\langle\varphi(y),\left\langle\delta_\varepsilon(\tau),E(x-y,t-\tau)\right\rangle\right\rangle$

两边取极限$\Rightarrow(\delta(t)\cdot\varphi(x))*E(x,t)=\left\langle\varphi(y),E(x-y,t)\right\rangle=E(x,t)\underset{(x)}{*}\varphi(x)=H(t)\left(4\pi a^2t\right)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(y)\exp\left(-\frac{|x-y|^2}{4a^2t}\right)\mathrm{d}y$

注：$\delta(t)\varphi(x)*E(x,t)=\varphi\underset{(x)}{*}E(x,t)$与法一中$(P_1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=0,\quad t>0\\ u|_{t=0}=\varphi(x)\end{cases}$的结果$u_1$是一样的

第二项卷积基本没什么可以化简的，只是把H(τ)和H(t-τ)的地方化简了，形式还是复杂的积分

② $(H(t)f(x,t))*E(x,t)=\int_{\mathbb{R}^n}\int_{\mathbb{R}}H(\tau)f(y,\tau)\underbrace{H(t-\tau)\left(4\pi a^2(t-\tau)\right)^{-\frac{n}{2}}\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)}_{E(x-y,t-\tau)}\mathrm{d}\tau\mathrm{d}y$

只有当$\tau\in(0,t)$时才同时满足$H(\tau)=1$且$H(t-\tau)=1$，所以上式$=\int_{\mathbb{R}^n}\int_0^tf(y,\tau)\left(4\pi a^2(t-\tau)\right)^{-\frac{n}{2}}\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)\mathrm{d}\tau\mathrm{d}y$

注：$(H(t)f(x,t))*E(x,t)$与法一中$(P_2)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t),\quad t>0\\ u|_{t=0}=0\end{cases}$的解$u_2$相同

综上，$\tilde{u}(x,t)=\varphi(x)\underset{(x)}{*}E(x,t)\,(H(t)f(x,t))*E(x,t)+(H(t)f(x,t))*E(x,t)\xlongequal{t>0}u(x,t)$此时恰为形式解

注：对得到的形式解仍需验证满足方程和初值条件，验证在法一已写过，不再展开

$Fourier$方法求齐次方程齐次边值(P_1) $\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=0,t>0\\ u(0,t)=u(l,t)=0\\ u|_{t=0}=\varphi(x)\end{cases}$ 的解

一个解ode初值问题的套路

思路：试求$u(x,t)=X(x)T(t)$变量分离的形式解，将$u(x,t)=X(x)T(t)$代入(P_1)，试求$X(x)$和$T(t)$

$\Rightarrow u(x,t)=X(x)T(t)$为形式解，再验证$u(x,t)$满足方程，初边值条件

$\Rightarrow u(x,t)$为(P_1)的解，又由极值原理解的唯一性结论$\Rightarrow u(x,t)=X(x)T(t)$为$(P_1)$的唯一解

(1)代入方程$X(x)T'(t)-a^2X''(x)T(t)=0$

$\Rightarrow\frac{X''(x)}{X(x)}=\frac{T'(t)}{a^2T'(t)}\overset{\triangle}{=\!=}-\mu\implies X''(x)+\mu X(x)=0$

(2)满足边界条件：$\begin{cases}u(0,t)=X(0)T(t)=0\\ u(l,t)=X(l)T(t)=0\end{cases}\implies X(0)=X(l)=0$

$\implies$从而得到形式解应满足ode方程$\begin{cases}X''(x)+\mu X(x)=0\\ X(0)=X(l)=0\end{cases}$

①：$\mu<0$，设$\mu=-\lambda^2\Rightarrow\lambda=\pm\sqrt{-\mu}$

$\Rightarrow X(x)=C_1e^{\sqrt{\mu}x}+C_2e^{-\sqrt{\mu}x}\Rightarrow$代入边界条件：$\begin{cases}X(0)=C_1+C_2=0\\ X(l)=C_1e^{\sqrt{\mu}l}+C_2e^{-\sqrt{\mu}l}=0\end{cases}$

有唯一解需满足系数行列式不为0$\Rightarrow\begin{vmatrix}1&1\\ e^{\sqrt{\mu}l}&e^{-\sqrt{\mu}l}\end{vmatrix}\neq0\Rightarrow C_1=C_2=0\Rightarrow u$解为平凡解，舍去

②：$\mu=0$，$X(x)=C_1x+C_2\Rightarrow$边界条件：$\begin{cases}X(0)=C_2=0,\\ X(l)=C_1l=0\end{cases}\Rightarrow C_1=0$

$\Rightarrow C_1=C_2=0\Rightarrow$解为平凡解：$X(x)\equiv0$，舍去

③：$\mu>0$，$X(x)=C_1\cos(\sqrt{\mu}x)+C_2\sin(\sqrt{\mu}x)$，边界条件：$\begin{cases}X(0)=C_1=0\\ X(l)=C_2\sin(\sqrt{\mu}l)=0\end{cases}$

$\Rightarrow\sin(\sqrt{\mu}l)=0\Rightarrow\mu_k=\left(\frac{k\pi}{l}\right)^2,k=1,2,\cdots\Rightarrow X_k(x)=C_k\sin\frac{k\pi}{l}x,\quad k=1,2,\ldots$