第七章 抛物线方程

热传导方程的基本解

以$n=3$为例，对$t>0$, $\frac{\partial}{\partial t}E(t)-a^2\Delta E(x,t)=\delta(x,t)$

对x做Fourier变换，ode方程求出F(E)，再做逆变换，凑成Gauss积分

等式两边对x做$Fourier$变换：$F_{x\to\xi}\left(\frac{\partial}{\partial t}E(x,t)-a^2\Delta E(x,t)\right)(\xi,t)=(F_{x\to\xi}\delta(x,t))(\xi,t)$

$\Rightarrow \frac{\partial}{\partial t}\hat{E}(\xi,t)+a^2|\xi|^2\hat{E}(\xi,t)=\delta(t)$即为$\hat{E}(\xi,t)$的一阶线性非齐次$ode$方程，解得为$\hat{E}(\xi,t)=H(t)e^{-a^2|\xi|^2t}$,

令$v(t)=e^{a^2|\xi|^2\hat{E}(\xi,t)}\hat{E}(\xi,t)$，这样方程化为$v'(t)=\delta(t)$, $v(t)=H(t)+C$，因为做$Fourier$变换要求$E(x,t)\in\mathscr{S}'(\mathbb{R}^{n+1}_{x,t})$,

则也要求$\hat{E}(x,t)\in\mathscr{S}'(\mathbb{R}^{n+1}_{x,t})$，所以要求$t\to-\infty$时$\hat{E}(\xi,t)=(H(t)+C)e^{a^2|\xi|^2\hat{E}(\xi,t)}$的任意解导数都趋于0，则$C=0$

则$E(x,t)=F^{-1}\left[H(t)e^{-a^2|\xi|^2t}\right]\xlongequal[\text{把}a\sqrt{2t}\xi\text{当成}\xi\text{,利用相似变换的性质}]{F^{-1}\left(e^{-\frac{|\xi|^2}{2}}\right)=(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}}H(t)\left(a\sqrt{2t}\right)^{-n}(2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2}\left|\frac{x}{a\sqrt{2t}}\right|^2}=H(t)(4\pi a^2t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4a^2t}}, t>0$

热传导方程的基本解$K(x,t)=(4\pi t)^{-\frac{n}{2}}H(t)e^{-\frac{|x|^2}{4t}}$

法一：叠加原理

Step 1：求解（P_1）

和基本解的求法基本一致，只是把H(t)换成了hatφ(ξ),再利用先卷积再变化=先变化再相乘的性质

证明当$u(x,t)$满足(P_1)时，$u(x,t)=E(x,t)\underset{(x)}{*}\varphi(x)$

$$(P_1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=0, & t>0\\ u|_{t=0}=\varphi(x)\end{cases}\text{关于}x\text{做}Fourier\text{变换}\Rightarrow\begin{cases}\frac{\partial}{\partial t}\hat{u}(t,\xi)+a^2|\xi|^2\hat{u}(t,\xi)=0, & t>0\\ \hat{u}|_{t=0}=\hat{\varphi}(\xi)\end{cases}$$

$\Rightarrow \hat{u}(t,\xi)=\hat{\varphi}(\xi)e^{-a^2|\xi|^2t}$, $u(t,x)=F^{-1}_{\xi\to x}\left(\hat{\varphi}(\xi)e^{-a^2|\xi|^2t}\right)$【和求基本解的做法一致】

由于$\widehat{f_1*f_2}=\hat{f}_1\cdot\hat{f}_2$，令$\begin{cases}f_1(t,x)=F^{-1}\left(\hat{e}^{-a^2|\xi|^2t}\right)=(4\pi a^2t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4a^2t}}\\ f_2(x)=\varphi(x)\end{cases}$, $\Rightarrow \hat{f}_1\cdot\hat{f}_2=u(t,x)=\widehat{f_1*f_2}=(4\pi a^2t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4a^2t}}\underset{(x)}{*}\varphi(x)$

就是基本解去掉$H(t)$，方法一样的，凑成$Gauss$积分 $=(4\pi a^2t)^{-\frac{n}{2}}\int\varphi(y)e^{-\frac{|x-y|^2}{4a^2t}}dy, t>0$

$=E(x,t)\underset{(x)}{*}\varphi(x)$【直接就能看出来】

Step 2：利用Duhamel原理求解(P_2)

构造U(x,t,τ)将(P_2)转化为(P_1)的形式

$$(P_2)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t), & t>0\\ u|_{t=0}=0\end{cases}$$

【齐次化原理($Duhamel$原理)】将(P_2)化为齐次方程非零初值问题【即将(P_2)形式的方程转化为(P_1)形式的方程】

构造辅助函数$U(x,t,\tau)$满足$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)U=0, & t>\tau\\ U(x,t,\tau)|_{t=\tau}=f(x,\tau)\end{cases}$，令$u(x,t)=\int_0^t U(x,t;\tau)\,d\tau\Rightarrow\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t), t>0\\ u|_{t=0}=0\end{cases}$

具体来说，$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}\int_0^t U(x,t,\tau)d\tau=U(x,t,t)+\int_0^t\frac{\partial U(x,t,\tau)}{\partial t}d\tau=f(x,t)+\int_0^t a^2\Delta U(x,t,\tau)d\tau=f(x,t)+a^2\Delta u(x,t)\\ u|_{t=0}=\int_0^0\cdots d\tau=0\end{cases}$

从而就可由u_1直接写出U(x,t,τ)的解，把积分写开变成n+1重积分就是u_2的解

利用(P_1)的解，写出$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)U=0, & t-\tau>0\\ U(x,t,\tau)|_{t-\tau=0}=f(x,\tau)\end{cases}$的解

$U(x,t,\tau)=E(x,t-\tau)\underset{(x)}{*}f(x,\tau)=\int_{\mathbb{R}^n}(4\pi a^2(t-\tau))^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4(t-\tau)}}f(y,\tau)dy=(4\pi a^2(t-\tau))^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(y,\tau)e^{-\frac{|x-y|^2}{4a^2(t-\tau)}}dy$

$\Rightarrow$把解出的$U(x,t,\tau)$回代$u(x,t)=\int_0^t U(x,t,\tau)d\tau$

$=\int_0^t(4\pi a^2(t-\tau))^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(y,\tau)\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)dyd\tau$

$=\int_0^t\int_{\mathbb{R}^n}f(y,\tau)\boxed{(4\pi a^2(t-\tau))^{-\frac{n}{2}}\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)}dyd\tau$

$\xlongequal{\text{后面要证}}E(x,t)*(H(t)f(x,t))=u_2(x,t)$ $Green$函数，$Heat\ Kernal$

验证$u_2(x,t)$为(P_2)的解：通过$Duhamel$原理的推导，已经保证了$u_2(x,t)(x,t)$必然满足非齐次方程(P_2)的所有要求

于是令$u(x,t)=u_1(x,t)+u_2(x,t)$，为$(P)\begin{cases}\frac{\partial u}{\partial t}-a^2\Delta u=f(x,t), t>0\\ u|_{t=0}=\varphi(x)\end{cases}$的解

$$(P)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t)\\ u|_{t=0}=\varphi(x)\end{cases}\quad(P_1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=0\\ u|_{t=0}=\varphi(x)\end{cases}\quad(P_2)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t)\\ u|_{t=0}=0\end{cases}$$

若u_1为P_1的解，u_2为P_2的解，则$u_1+u_2=u$为(P)的解（线性所以才能叠加）

法二：基本解$E(x,t)$直接求形式解

就是一步δ(t)u(x,t)=δ(t)φ(x)的化简，类似于证明u(x,t)的变量可以拆分，δ(t)作用在上面直接等价于作用t=0的情况

$\left(\frac{\partial}{\partial t}-a^2\Delta\right)\tilde{u}(x,t)=\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))\xlongequal{\triangle}F(x,t)\Rightarrow\tilde{u}(x,t)=E(x,t)*F(x,t)\xlongequal{t>0}u(x,t)$

$\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))=\delta(t)u(x,t)+\boxed{H(t)\frac{\partial}{\partial t}u(x,t)-a^2H(t)\Delta u(x,t)}=\delta(t)u(x,t)+H(t)f(x,t)$

下证：$\delta(t)u(x,t)=\delta(t)\varphi(x)$，取$\forall\psi(t)\in C_0^\infty$, $\left\langle\delta(t)u(x,t),\psi(t)\right\rangle=\left\langle\delta(t),u(x,t)\psi(t)\right\rangle=u(x,0)\psi(0)$

$\left\langle\delta(t)\varphi(x),\psi(t)\right\rangle=\left\langle\delta(t),\varphi(x)\psi(t)\right\rangle=\varphi(x)\psi(0)$

即得$\delta(t)u(x,t)=\delta(t)\varphi(x)$

则设$\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))=\delta(t)\varphi(x)+H(t)f(x,t)\xlongequal{\triangle}F(x,t)$

由基本解性质，可知$\tilde{u}(t,x)=E(x,t)*F(x,t)=E(x,t)*(\delta(t)\varphi(x)+H(t)f(x,t))$

两项卷积的第一项化简，依旧是对δ磨光，通过卷积的定义转化形式再取极限

① $\delta(t)\varphi(x)*E(x,t)$，取$\Phi_\varepsilon(t)$为磨光核，令$\delta_\varepsilon(t)=\delta(t)*\Phi_\varepsilon(t)\in C_0^\infty$, $(\delta_\varepsilon(t)\varphi(x))*E(x,t)=\int_{\mathbb{R}}\int_{\mathbb{R}^n}\delta_\varepsilon(\tau)\varphi(y)E(x-y,t-\tau)d\tau dy$

$\Rightarrow$上式$=\int_{\mathbb{R}^n}\varphi(y)dy\int\delta_\varepsilon(\tau)E(x-y,t-\tau)d\tau=\left\langle\varphi(y),\left\langle\delta_\varepsilon(\tau),E(x-y,t-\tau)\right\rangle\right\rangle$

两边取极限$\Rightarrow(\delta(t)\cdot\varphi(x))*E(x,t)=\left\langle\varphi(y),E(x-y,t)\right\rangle=E(x,t)\underset{(x)}{*}\varphi(x)=H(t)(4\pi a^2t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(y)\exp\left(-\frac{|x-y|^2}{4a^2t}\right)dy$

注：$\delta(t)\varphi(x)*E(x,t)=\varphi\underset{(x)}{*}E(x,t)$与法一中$(P_1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=0, & t>0\\ u|_{t=0}=\varphi(x)\end{cases}$的结果$u_1$是一样的

第二项卷积基本没什么可以化简的，只是把H(τ)和H(t-τ)的地方化简了，形式还是复杂的积分

② $(H(t)f(x,t))*E(x,t)=\int_{\mathbb{R}^n}\int_{\mathbb{R}}H(\tau)f(y,\tau)\underbrace{H(t-\tau)(4\pi a^2(t-\tau))^{-\frac{n}{2}}\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)}_{E(x-y,t-\tau)}d\tau dy$

只有当$\tau\in(0,t)$时才同时满足$H(\tau)=1$且$H(t-\tau)=1$，所以上式$=\int_{\mathbb{R}^n}\int_0^t f(y,\tau)(4\pi a^2(t-\tau))^{-\frac{n}{2}}\exp\left(-\frac{|x-y|^2}{4a^2(t-\tau)}\right)d\tau dy$

注：$(H(t)f(x,t))*E(x,t)$与法一中$(P_2)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t), & t>0\\ u|_{t=0}=0\end{cases}$的解$u_2$相同

综上，$\tilde{u}(x,t)=\varphi(x)\underset{(x)}{*}E(x,t)\,(H(t)f(x,t))*E(x,t)+(H(t)f(x,t))*E(x,t)\xlongequal{t>0}u(x,t)$此时恰为形式解

注：对得到的形式解仍需验证满足方程和初值条件，验证在法一已写过，不再展开

对于线性算子P，基本解为$E(x)$，则$u=E*f$为$Pu=f$的$\mathscr{D}'$解

$(P)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\Delta\right)u=f(x,t)\\ u|_{t=0}=\varphi(x)\end{cases}$，令$\tilde{u}(x,t)=H(t)u(x,t)$,

$\Rightarrow\tilde{u}(t,x)\xlongequal{t>0}u(x,t)$为$P$的解

$\left(\frac{\partial}{\partial t}-a^2\Delta\right)\tilde{u}(x,t)=\left(\frac{\partial}{\partial t}-a^2\Delta\right)(H(t)u(x,t))\xlongequal{\triangle}F(x,t)$

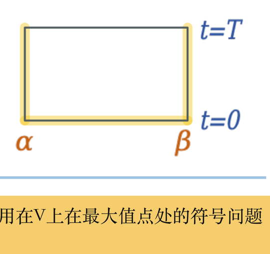
$\Rightarrow\tilde{u}(x,t)=E(x,t)*F(x,t)\xlongequal{t>0}u(x,t)$

初边值问题：

$$\begin{cases}\partial_t u-a^2\Delta u=0\\ \quad t>0, x\in\Omega\\ u|_{t=0}=f(x)\\ u|_{\partial\Omega}=g(x,t)\end{cases}$$

极值原理

初边值问题的极值原理



设$u(x,t)$在$R_T=\{\alpha\leqslant x\leqslant\beta, 0\leqslant t\leqslant T\}$，且在$R_T$内满足$\partial_t u-a^2\partial_x^2u=0$,
则在R_T的抛物边界上取最大值和最小值，记Γ_T为R_T的抛物边界，则有
$\max\limits_{R_T}u(x,t)=\max\limits_{\Gamma_T}u(x,t)$; $\min\limits_{R_T}u(x,t)=\min\limits_{\Gamma_T}u(x,t)$【极值在抛物边界上取到】

经典反证法，但是不是直接用u在内部取到的最大值点，而是构造辅助函数V，热传导算子作用在V上在最大值点处的符号问题 构造V的方式很巧妙，应该需要背下来

证明：只需证：$\max\limits_{R_T}u=\max\limits_{\Gamma_T}u$，因为将$-u$代替$u$，最小值的情形将变为最大值的情形

反证：假设$u(x,t)$在内部(x^*,t^*)取到最大值$M>$边界最大值m, $x^*\in(\alpha,\beta)$, $t^*\in(0,T]$

作函数$V(x,t)=u(x,t)+\frac{M-m}{4l^2}(x-x^*)^2$，其中$l=\beta-\alpha$

$\Rightarrow V(x,t)|_{\Gamma_T}=u(x,t)|_{\Gamma_T}+\frac{M-m}{4l^2}(x-x^*)\Big|_{\Gamma_T}\xrightarrow{\frac{(x-x^*)^2}{l^2}<1}V(x,t)|_{\Gamma_T}<m+\frac{M-m}{4}=\frac{M+3m}{4}=\theta M, 0<\theta<1$;

而$V(x^*,t^*)=u(x^*,t^*)+0=M\Rightarrow$设$V(x,t)$在$R_T$内部$(x_1,t_1)$取到最大值(不一定在$(x^*,t^*)$点)

$$\Rightarrow\begin{cases}\left.\frac{\partial^2V}{\partial x^2}\right|_{(x_1,t_1)}\leqslant 0\\ \left.\frac{\partial V}{\partial t}\right|_{(x_1,t_1)}\geqslant 0\text{，因为}\begin{cases}\text{当}0<t_1<T\text{时，有}\left.\frac{\partial V}{\partial t}\right|_{(x_1,t_1)}=0\\ \text{当}t_1=T\text{时，有}\left.\frac{\partial V}{\partial t}\right|_{(x_1,t_1)}\geqslant 0\end{cases}\end{cases}\Longrightarrow\text{于是有}\left.\frac{\partial V}{\partial t}-a^2\frac{\partial^2V}{\partial x^2}\right|_{(x_1,t_1)}\geqslant 0$$

但是，由$V(x,t)=u(x,t)+\frac{M-m}{4l^2}(x-x^*)^2$代入$(\partial_t-a^2\partial_x^2)V=\boxed{(\partial_t-a^2\partial_x^2)u(x,t)}+(\partial_t-a^2\partial_x^2)\frac{M-m}{4l^2}(x-x^*)^2$

$=0$ 齐次方程

$=-a^2\frac{M-m}{2l^2}<0$，矛盾！$\Rightarrow m=M$

初边值问题的唯一性、稳定性

$$(P)\begin{cases}\partial_t u-a^2\partial_x^2u=f(x,t), t>0\\ u|_{t=0}=\varphi(x)\\ u(\alpha,t)=\mu_1(t); u(\beta,t)=\mu_2(t)\end{cases}$$

唯一性

设$u_1(x,t)$, $u_2(x,t)$均满足(P)，令$u(x,t)=u_1-u_2\Rightarrow\begin{cases}(\partial_t-a^2\partial_x^2)u=0, t>0\\ u|_{\Gamma_T}=0\text{【整个抛物边界都是0】}\end{cases}$

极值原理$\max\limits_{R_T}u(x,t)=\max\limits_{\Gamma_T}u(x,t)$; $\min\limits_{R_T}u(x,t)=\min\limits_{\Gamma_T}u(x,t)$; $\Rightarrow u\equiv 0\Rightarrow u_1\equiv u_2$

稳定性

u_1, u_2分别满足$(P_i)\begin{cases}(\partial_t-a^2\partial_x^2)u=f(x,t), & t>0\\ u|_{\Gamma_T}=g_i(x), i=1,2\end{cases}$，且$\max\limits_{\Gamma_T}|g_1-g_2|\leqslant\varepsilon$

$u=u_1-u_2$满足$\begin{cases}(\partial_t-a^2\partial_x^2)u=0, & t>0\\ u|_{\Gamma_T}=g_1-g_2\end{cases}\xrightarrow{\text{极值原理}}|u_1-u_2|\leqslant\max\limits_{\Gamma_T}|g_1-g_2|\leqslant\varepsilon$

$Cauchy$问题（初值问题）在有界函数类中的解是唯一的且连续依赖于初值条件

考虑初值问题$(P_{初})\begin{cases}(\partial_t-a^2\partial_x^2)u=0, & t>0\\ u|_{t=0}=\varphi(x)\end{cases}$【$n=1$时带形区域】

由初值问题的极值原理证明初值问题的唯一性和稳定性

注：① 利用热传导方程初边值问题的极值原理来证明$Cauchy$问题的唯一性和稳定性
② $Cauchy$问题的唯一性和稳定性在有界函数中研究，
【加条件：$u(x,t)$一致有界】∃正常数B，对$\forall t\geqslant 0$, $\forall x$，成立$|u(x,t)|\leqslant B$，其中$u(x,t)$表示物体温度

已知|u|≤B，要证明|u|≡0，构造辅助函数v,热传导算子作用=0且在抛物边界上v≥u，利用初边值问题的极值原理证明

① 唯一性：设u_1, u_2均满足$(P_{初})$，令$u=u_1-u_2$，则$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=0\\ u|_{t=0}=0\end{cases}$

取$\forall$固定(x_0,t_0), $t_0>0$，考虑$R_0=\{(x,t)||x-x_0|\leqslant L, 0\leqslant t\leqslant t_0\}$

作辅助函数$v(x,t)=\frac{4B}{L^2}\left(\frac{(x-x_0)^2}{2}+a^2t\right)\in C(R_0)$，且有在$R_0$内部$\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)v=\frac{4Ba^2}{L^2}-a^2\frac{4B}{L^2}=0$满足方程

$\begin{cases}\text{底边：}v(x,t)|_{t=0}=\frac{2B}{L^2}(x-x_0)^2\geqslant 0=u|_{t=0}\\ \text{侧边：}v(x_0\pm L,t)=2B+a^2\frac{4B}{L^2}t\geqslant 2B\geqslant u(x_0\pm L,t)\end{cases}\Rightarrow$在$R_0$的抛物边界上成立$v\geqslant u\Rightarrow\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)v=0, t\geqslant 0\\ v|_{\partial R_0}\geqslant u|_{\partial R_0}\end{cases}$

由初边值问题的极值原理可知在R_0上成立$v(x,t)\geqslant u(x,t)$，即$u(x,t)\leqslant\frac{4B}{L^2}\left(\frac{(x-x_0)^2}{2}+a^2t\right)$

同理可证$-\frac{4B}{L^2}\left(\frac{(x-x_0)^2}{2}+a^2t\right)\leqslant u(x,t)\Rightarrow|u(x,t)|\leqslant\left|\frac{4B}{L^2}\left(\frac{(x-x_0)^2}{2}+a^2t\right)\right|$

取点(x_0,t_0)处$\Rightarrow|u(x_0,t_0)|\leqslant\frac{4B}{L^2}a^2t_0$，令$L\to\infty\Rightarrow|u(x_0,t_0)|=0$

又由(x_0,t_0)是上半平面的任一点$\Rightarrow u(x,t)\equiv 0, t>0\Rightarrow u_1\equiv u_2$

好像思路差不多

② 稳定性：设u_1, u_2分别满足$(P_i)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=0\\ u|_{t=0}=\varphi_i(x), i=1,2\end{cases}$，且$|\varphi_1(x)-\varphi_2(x)|\leqslant\eta$

令$u=u_1-u_2$，则$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=0\\ |u|\Big|_{t=0}=|\varphi_1-\varphi_2|\Big|_{t=0}\leqslant\eta\end{cases}$

令辅助函数$V(x,t)=\frac{4B}{L^2}\left(\frac{(x-x_0)^2}{2}+a^2t\right)+\eta\Rightarrow\left(\frac{(x-x_0)^2}{2}+a^2t\right)v=0$

$\forall$固定(x_0,t_0)有，同理可证$|u(x_0,t_0)|\leqslant\eta$

$Fourier$方法

齐次方程齐次边值

首先看最简单的情况$(P_1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=0, & t>0\\ u(0,t)=u(l,t)=0\\ u|_{t=0}=\varphi(x)\end{cases}$

一个解ode初值问题的套路

思路：试求$u(x,t)=X(x)T(t)$变量分离的形式解，将$u(x,t)=X(x)T(t)$代入(P_1)，试求$X(x)$和$T(t)$
$\Rightarrow u(x,t)=X(x)T(t)$为形式解，再验证$u(x,t)$满足方程，初边值条件
$\Rightarrow u(x,t)$为(P_1)的解，又由极值原理解的唯一性结论$\Rightarrow u(x,t)=X(x)T(t)$为$(P_1)$的唯一解

(1)代入方程$X(x)T'(t)-a^2X''(x)T(t)=0$

$\Rightarrow\frac{X''(x)}{X(x)}=\frac{T'(t)}{a^2T'(t)}\xlongequal{\triangle}-\mu\Longrightarrow X''(x)+\mu X(x)=0$

(2)满足边界条件：$\begin{cases}u(0,t)=X(0)T(t)=0\\ u(l,t)=X(l)T(t)=0\end{cases}\Longrightarrow X(0)=X(l)=0$

$\Longrightarrow$ 从而得到形式解应满足ode方程$\begin{cases}X''(x)+\mu X(x)=0\\ X(0)=X(l)=0\end{cases}$

①：$\mu<0$，设$\mu=-\lambda^2\Rightarrow\lambda=\pm\sqrt{-\mu}$

$\Rightarrow X(x)=C_1e^{\sqrt{\mu}x}+C_2e^{-\sqrt{\mu}x}\Rightarrow$代入边界条件：$\begin{cases}X(0)=C_1+C_2=0\\ X(l)=C_1e^{\sqrt{\mu}l}+C_2e^{-\sqrt{\mu}l}=0\end{cases}$

有唯一解需满足系数行列式不为0 $\Rightarrow\begin{vmatrix}1 & 1\\ e^{\sqrt{\mu}l} & e^{-\sqrt{\mu}l}\end{vmatrix}\neq 0\Rightarrow C_1=C_2=0\Rightarrow u$解为平凡解，舍去

②：$\mu=0$, $X(x)=C_1x+C_2\Rightarrow$边界条件：$\begin{cases}X(0)=C_2=0,\\ X(l)=C_1l=0\end{cases}\Rightarrow C_1=0$

$\Rightarrow C_1=C_2=0\Rightarrow$解为平凡解：$X(x)\equiv 0$，舍去

③：$\mu>0$，$X(x)=C_1\cos(\sqrt{\mu}x)+C_2\sin(\sqrt{\mu}x)$，边界条件：$\begin{cases}X(0)=C_1=0\\ X(l)=C_2\sin(\sqrt{\mu}l)=0\end{cases}$

$\Rightarrow\sin(\sqrt{\mu}l)=0\Rightarrow\mu_k=\left(\frac{k\pi}{l}\right)^2, k=1,2,\cdots\Rightarrow X_k(x)=C_k\sin\frac{k\pi}{l}x$, $k=1,2,\ldots$

通过正交性和完备性，可以将u(x,t) 表示成一系列特征函数和时间因子的乘积形式，这给出了偏微分方程解的分离变量形式

此时$\mu_k=\left(\frac{k\pi}{l}\right)^2, k=1,2,\ldots, X_k(x)=C_k\cdot\sin\frac{k\pi}{l}x$

$\Rightarrow T_k'(t)+a^2\mu_kT_k(t)=0\Rightarrow T_k'(t)=B_k\exp\left[-a^2\left(\frac{k\pi}{l}\right)^2t\right]$

$\Rightarrow\mu_k(x,t)=X_k(x)T_k(t)=A_ke^{-a^2\left(\frac{k\pi}{l}\right)^2t}\sin\frac{k\pi}{l}x, k=1,\ldots$

$\Rightarrow u(x,t)=\sum_{k=1}^\infty u_k(x,t)=\sum_{k=1}^\infty A_ke^{-a^2\left(\frac{k\pi}{l}\right)^2t}\sin\frac{k\pi}{l}x$

求A_k：$u|_{t=0}=\varphi(x)$，那$u|_{t=0}=\sum_{k=1}^\infty A_k\sin\frac{k\pi}{l}x=\varphi(x)$

固定m，等式两边同$\times\sin\frac{m\pi}{l}x$并积分$\Rightarrow\int_0^l\sum_{k=1}^\infty A_k\sin\frac{k\pi}{l}x\sin\frac{m\pi}{l}xdx=\int_0^l\varphi(x)\sin\frac{m\pi}{l}xdx$

$\int_0^l A_m\left(\sin\frac{k\pi}{l}x\right)^2dx=A_m\frac{l}{2}=\int_0^l\varphi(x)\sin\frac{m\pi}{l}xdx\Rightarrow A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi}{l}xdx$

下验证满足(P_1)？$u(x,t)$是否有意义，是否收敛？

其中，$u(x,t)$是$[0,l]$上的连续函数，由$\forall\delta>0$，当$t\geqslant\delta$时，对$\forall p>0$, $\sum_{k=1}^\infty\mu_k^pe^{-a^2\frac{k\pi}{l}\pi t}$一致收敛$\Rightarrow$有意义

$Sturm-Liouville$问题

注：对$\begin{cases}X''(x)+\mu X(x)=0\\ X(0)=X(l)=0\end{cases}$确定$\mu$的值，使得上述线性$ode$问题有(非平凡)解，称为$Sturm-Liouville$问题，
所得$\{\mu_k\}_{k=1,2}$称为$S-L$问题的特征值；$\{X_k(x)\}$为$S-L$问题的特征函数

$S-L$问题的特征函数$X_k(x)_{k=1,\ldots}$构成$L^2([0,l])$中的完全正交系。

(1)正交：对$m\neq n$时，成立$\int_0^l X_m(x)X_n(x)dx=0$

证明：$\begin{cases}X_m''(x)+\mu_mX_m(x)=0\\ X_m(0)=X_m(l)=0\end{cases}$ ① 和 $\begin{cases}X_n''(x)+\mu_nX_n(x)=0\\ X_n(0)=X_n(l)=0\end{cases}$ ②

令①$X_m(x)-$②$X_n(x)=X_m''(x)X_n(x)-X_n''(x)X_m(x)=\mu_nX_nX_m-\mu_mX_mX_n$

$(\mu_m-\mu_n)\int_0^l X_m(x)X_n(x)dx=\int_0^l X_m''X_n-X_n''X_mdx=X_m'X_n|_0^l-X_n'X_m|_0^l+\int_0^l X_m'X_n'-X_n'X_m'dx=0$

由于$\mu_m\neq\mu_n$, $\Rightarrow\int_0^l X_m(x)X_n(x)dx=0$

证明非常长，可参考中科大陈祖墀《偏微分方程》

(2)完全系：完备性是指特征函数$X_k(x)$的集合可以用来展开$L^2([0,l])$空间中的任何函数。
即若函数$f(x)\in L^2([0,l])$，且$f(x)$与每个$X_k(x), k=1,\ldots$均正交，则$f\equiv 0$

非齐次问题

$$(P_0)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t), & t>0\\ u(0,t)=g_1(t), u(l,t)=g_2(t)\\ u|_{t=0}=\varphi(x)\end{cases}$$

法一：叠加原理 + $Duhamel$原理

(1)化成零边值问题
构造函数$v(x,t)$满足边值条件，$s.t. W=u-v$满足零边值条件
$v(x,t)=g_1(t)+\frac{x}{l}(g_2(t)-g_1(t))$

则$(P_w)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)W=f_W(x,t), & t>0\\ W(0,t)=W(l,t)=g_2(t)=0\\ W|_{t=0}=\varphi(x)-v|_{t=0}=\varphi_W(x)\end{cases}$，即化成了零边值条件

(2)对(P_W)用叠加原理

$\boxed{(P_w-1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)W=0, & t>0\\ W(0,t)=W(l,t)=g_2(t)=0\\ W|_{t=0}=\varphi(x)-v|_{t=0}=\varphi_W(x)\end{cases}}\longrightarrow$齐次方程零边值问题已经求过

$(P_w-2)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)W=f_W(x,t), & t>0\\ W(0,t)=W(l,t)=g_2(t)=0\\ W|_{t=0}=0\end{cases}$

(3)用$Duhamel$原理求解(P_W-2)

(4)$W(x,t)=W_1(x,t)+W_2(x.t)$即为(P_W)的解
$u(x,t)=W(x,t)+v(x,t)$

法二：常数变易法

(1)化成零边值问题$(P_w)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)W=f_W(x,t), & t>0\\ W(0,t)=W(l,t)=g_2(t)=0\\ W|_{t=0}=\varphi(x)-v|_{t=0}=\varphi_W(x)\end{cases}$

(2)先求齐次方程(P_W-1)
$(P_w-1)\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)W=0, & t>0\\ W(0,t)=W(l,t)=g_2(t)=0\\ W|_{t=0}=\varphi(x)-v|_{t=0}=\varphi_W(x)\end{cases}$

$\Rightarrow Fourier$方法$\Rightarrow S-L$问题$\Rightarrow$特征函数系$\left\{X_k(x)=\sin\frac{k\pi}{l}x\right\}$

(3)求解(P_W)
由(2)中的齐次方程问题得到了特征函数系，用其将$W(x,t)$, $f_W(x,t)$, $\varphi_W(x,t)$进行展开

$W(x,t)=\sum_{k=1}^\infty W_k(t)\sin\frac{k\pi}{l}x$，其中$W_k(t)$位置；$f_W(x,t)=\sum_{k=1}^\infty f_k(t)\sin\frac{k\pi}{l}t$，其中$f_k(t)$已知

$\varphi_W(x)=\sum_{k=1}^\infty\varphi_k\sin\frac{k\pi}{l}x$，其中$\varphi_k$已知

要求求$W_k(t)$，将$W(x,t)$, $f_W(x,t)\varphi_W(x)$的展开式代入(P_W)中

$\Rightarrow\left(W_k'(t)+\left(\frac{k\pi}{l}\right)^2W_k(t)\right)\sin\frac{k\pi}{l}x=f_k(t)\sin\frac{k\pi}{l}x$

$W_k(0)\sin\frac{k\pi}{l}x=\varphi_k\cdot\sin\frac{k\pi}{l}x$

$\Rightarrow\begin{cases}W_k'(t)+\left(\frac{k\pi}{l}\right)^2W_k(t)=f_k(t)\\ W_k(0)=\varphi_k\end{cases}$

$\Rightarrow W_k(t)=\left(\varphi+\int_0^t f_k(t)e^{a^2\left(\frac{k\pi}{l}\right)^2s}ds\right)e^{a^2\left(\frac{k\pi}{l}\right)^2t}$

$\Rightarrow W(x,t)=\sum_{k=1}^\infty W_k(t)\sin\frac{k\pi}{l}x$

(4)$u(x,t)=W(x,t)+v(x,t)$