第四章 曲面论基本定律

4.1 外微分式

线性空间的微分式

对偶空间，一次形式

设T是$\mathbb{R}$上的n维线性空间，令$T^*=\mathrm{Hom}(T,\mathbb{R})$为$T\to\mathbb{R}$所有的线性映射的集合，

即$\omega\in T^* \iff \begin{cases}\omega(X+Y)=\omega(X)+\omega(Y),\forall X,Y\in T\\ \omega(aX)=a\omega(X),\forall a\in\mathbb{R},X\in T\end{cases}$

在T^*中可定义加法、数乘，使T^*成为一个线性空间，称为T的对偶空间；

若$e_1,\dots,e_n$是T的一组基，我们可以定义T^*的一组基$e^1,\dots,e^n$,使满足：

$e^j(e_i)=\delta_{ij},1\leqslant i,j\leqslant n$; $e^1,\dots,e^n$称为$e_1,\dots,e_n$的对偶基

k次微分式

如果映射$\Omega:\underbrace{T\times\cdots\times T}_{k个}\to\mathbb{R}$满足：

(1)【反对称】$\Omega(X_1,\cdots,X_k)=\mathrm{sgn}\sigma\cdot\Omega\Big(X_{\sigma(1)},\cdots,\Omega(X_{\sigma(k)}\Big),\forall\sigma\in S_k$

(2)【线性】$\Omega(aX_1+bX',\cdots,X_k)=a\Omega(X_1,\cdots,X_k)+b\Omega(X',\cdots,X_k)$

则称Ω时T上的k次微分式，用$\wedge^k T$表示T上所有k次微分形式的集合

性质

- (i)对任一变量X_i,Ω都是线性的(k线性型)
- (ii)若$X_1,\cdots,X_k$线性相关，则$\Omega(X_1,\cdots,X_k)=0$
- $(iii)\ \wedge^m T=0,m>n$
- $(iv)\Omega(X_1,\cdots,X_k)=\dfrac{1}{k!}\sum\limits_{\sigma\in S_k}\mathrm{sgn}\sigma\Omega(X_{\sigma(1)},\cdots,X_{\sigma(k)})$

定理

在$\wedge^k T$中定义加法和数乘为$\begin{cases}(\Omega_1+\Omega_2)(X_1,\cdots,X_k)=\Omega_1(X_1,\cdots,X_k)+\Omega_2(X_1,\cdots,X_k)\\(a\Omega)(X_1,\cdots,X_k)=a\Omega(X_1,\cdots,X_k)\end{cases}$

则$\wedge^k T$是$\mathbb{R}$上的线性空间，且$\dim\wedge^k T=C_n^k$

外代数$\wedge T$

令$\wedge^0 T=\mathbb{R}$;我们在$\wedge T=\wedge^0 T\oplus\wedge^1 T\oplus\cdots\oplus\wedge^n T$中定义一种外乘法,设$\Omega\in\wedge^k T,\omega\in\wedge^l T$,定义

$\Omega\wedge\omega(X_1,\cdots,X_k,X_{k+1},\cdots,X_{k+l})=\dfrac{1}{(k+l)!}\sum\limits_{\sigma\in S_{k+l}}\mathrm{sgn}\sigma\cdot\Omega(X_{\sigma(1)},\cdots,X_{\sigma(k)})\cdot\omega(X_{\sigma(k+1)},\cdots,X_{\sigma(k+l)})$

定理

$\Omega,\Omega_1,\Omega_2\in\wedge^k T,\eta,\eta_1,\eta_2\in\wedge^l T,\theta\in\wedge^h T$,则有

(1)$(a_1\Omega_1+a_2\Omega_2)\wedge\eta=a_1(\Omega_1\wedge\eta)+a_2(\Omega_2\wedge\eta),\Omega\wedge(\eta_1+\eta_2)=\Omega\wedge\eta_1+\Omega\wedge\eta_2$

(2)$\Omega\wedge\eta=(-1)^{kl}\eta\wedge\Omega$

(3)$(\Omega\wedge\eta)\wedge\theta=\Omega\wedge(\eta\wedge\theta)$

推论

- $1\leqslant i_1<i_2<\cdots<i_k\leqslant n,e^1,\cdots,e^n\in T^*=\wedge^1 T$，则$\{e^{i_1}\wedge\cdots\wedge e^{i_k}\}$构成$\wedge^k T$的一组基
- $\sigma\in S_k$,则有$e^{i_{\sigma(1)}}\wedge\cdots\wedge e^{i_{\sigma(k)}}=\mathrm{sgn}\sigma\cdot(e^{i_1}\wedge\cdots\wedge e^{i_k})$

定理

设$\omega^i=\sum\limits_{j=1}^n a^i_j e^j$,即$\begin{pmatrix}\omega^1\\\omega^2\\\vdots\\\omega^k\end{pmatrix}=\begin{pmatrix}a^1_1&\cdots&a^1_n\\a^2_1&\cdots&a^2_n\\\vdots&\ddots&\vdots\\a^k_1&\cdots&a^k_n\end{pmatrix}\begin{pmatrix}e^1\\e^2\\\vdots\\e^k\end{pmatrix}$,则

$\omega^1\wedge\cdots\wedge\omega^k=\sum\limits_{i_1<\cdots<i_k}A\begin{pmatrix}i_1&\cdots&i_k\\j_1&\cdots&j_k\end{pmatrix}e^{i_1}\wedge\cdots\wedge e^{i_k}$

设$e_1,\cdots,e_n$为T的基，$e^1,\cdots,e^n$是T^*的对偶基，$f_1,\cdots,f_n$是T的另一组基，

$f^1,\cdots,f^n$是其对偶基，若$\begin{pmatrix}f_1\\\vdots\\f_n\end{pmatrix}=A\begin{pmatrix}e_1\\\vdots\\e_n\end{pmatrix},\begin{pmatrix}f^1\\\vdots\\f^n\end{pmatrix}=B\begin{pmatrix}e^1\\\vdots\\e^n\end{pmatrix}$，则$BA^T=I$

曲面上的外微分式

$\mathscr{F}(S)$

设S是一个k次连续可微的正则曲面片，映射$f:S\to\mathbb{R}$称为S上的k次连续可微函数，

如果S有正则参数表示$P(i!,u^2),s.t.f(P(u^1,u^2))$是$u^1,u^2$的$k$次连续可微函数.

用$\mathscr{F}(S)$表示S上所有k次连续可微函数的集合.$\mathscr{F}(S)$是交换环

$\mathscr{D}(S)$

设$D:\mathscr{F}(S)\to\mathscr{F}(S)$，满足：$\begin{cases}(1)D(af+bg)=aD(f)+bD(g),\forall a,b\in\mathbb{R},f,g\in\mathscr{F}(S)\\(2)D(fg)=(Df)\cdot g+f\cdot(Dg),\forall f,g\in\mathscr{F}(S)\end{cases}$

则称D为$\mathscr{F}(S)$的一个导子，所有导子的集合记为$\mathrm{Der}\mathscr{F}(S)=\mathscr{D}^1(S)$,显然$\dfrac{\partial}{\partial u^i}\in\mathscr{D}^1(S)$

定理

- 设$D\in\mathscr{D}^1(S)$,则$D=Du^i\dfrac{\partial}{\partial u^i}$
- $p_0\in S$,则$T_{p_0}S\cong\{X_{p_0}|X\in\mathscr{D}^1(S)\}$，而且与参数选取无关

  这样，我们可以将$X_{p_0}=a\dfrac{\partial}{\partial u^1}+b\dfrac{\partial}{\partial u^2}$视为$T_{p_0}S$中的元素，故$X\in\mathscr{D}^1(S)$是$S$上的切向量场

$\mathscr{A}_k(S)$

如果映射$\Omega:\underbrace{\mathscr{D}^1(S)\times\cdots\times\mathscr{D}^1(S)}_{k个}\to\mathscr{F}(S)$满足：

(1)【反对称】$\forall X_1,\cdots,X_k\in\mathscr{D}^1(S),\forall\sigma\in S_k$，有

$\Omega(X_1,\cdots,X_k)=\mathrm{sgn}\sigma\cdot\Omega\Big(X_{\sigma(1)},\cdots,\Omega(X_{\sigma(k)}\Big)$,

(2)【线性】$\forall X_1,\cdots,X_k\in\mathscr{D}^1(S),\forall f,g\in\mathscr{F}(S)$,有

$\Omega(fX_1+bX',\cdots,X_k)=f\Omega(X_1,\cdots,X_k)+g\Omega(X',\cdots,X_k)$.

则称Ω时S上的k次微分式，用$\mathscr{A}_k(S)T$表示S上所有k次微分形式的集合

令$\mathscr{A}_0(S)=\mathscr{F}(S)$;我们在$\mathscr{A}(S)=\oplus_k\mathscr{A}_k(S)$中定义外积$\wedge$,设$\omega\in\mathscr{A}_k(S),\theta\in\mathscr{A}_l(S),\omega\wedge\theta\in\mathscr{A}_{k+l}(S)$,

$\omega\wedge\theta(X_1,\cdots,X_{k+l})=\dfrac{1}{(k+l)!}\sum\limits_{\sigma\in S_{k+l}}\mathrm{sgn}\cdot\omega(X_{\sigma(1)},\cdots,X_{\sigma(k)})\cdot\theta(X_{\sigma(k+1)},\cdots,X_{\sigma(k+l)})$

$\mathscr{A}(S)$称为S上的$Grassmann$代数

T_pS中元素表示为$x=x^1P_1+x^2P_2,x^1,x^2\in\mathbb{R}$; 映射$\mathrm{d}u^i:x\to x^i$是$T_pS$上的线性映射，

且$\mathrm{d}u^i\left(\dfrac{\partial P}{\partial u^j}\right)=\delta_{ij}$,故$\mathrm{d}u^1,\mathrm{d}u^2$是$P_1,P_2$的对偶基,则$\mathrm{d}u^i\left(\dfrac{\partial}{\partial u^j}\right)=\dfrac{\partial u^i}{\partial u^j}=\delta_{ij}$

4.2 幺正活动标价

我们希望在研究曲面时也使用幺正活动标架$\{P;e_1,e_2,e-3\}$，而且时$e_3=n$是单位法向量

我们可以将$\dfrac{\partial P}{\partial u},\dfrac{\partial P}{\partial v},n$视为$\wedge^0(S)\times\wedge^0(S)\times\wedge^0(S)=\wedge^0(S)\oplus\mathbb{R}^3$中的元素，

于是e_1,e_2,e_3也可以视为$\wedge^0(S)\oplus\mathbb{R}^3$中的元素

此时,$\mathrm{d}P=\left(\dfrac{\partial f}{\partial u}\mathrm{d}u+\dfrac{\partial f}{\partial v}\mathrm{d}v,\dfrac{\partial g}{\partial u}\mathrm{d}u+\dfrac{\partial g}{\partial v}\mathrm{d}v,\dfrac{\partial h}{\partial u}\mathrm{d}u+\dfrac{\partial h}{\partial v}\mathrm{d}v\right)\in\wedge^1(S)\oplus\mathbb{R}^3$

同样，可以认$\mathrm{d}P,\mathrm{d}e_j\in\wedge^1(S)\oplus\mathbb{R}^3$

引理

设$\{P;e_1,e_2,e_3\}$是S上的幺正标架场，则对$\forall e\in\wedge^k(S)\oplus\mathbb{R}^3$,

$\exists$唯一的一组$\omega_1,\omega_2,\omega_3\in\wedge^k(S)$,使得$e=(\omega_1,\omega_2,\omega_3)\wedge\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$

则有$\mathrm{d}P=(\omega_1,\omega_2,\omega_3)\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix},\mathrm{d}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$

$\omega_3=0,\omega_{ij}=-\omega_{ji}$即

$\mathrm{d}P=(\omega_1,\omega_2,0)\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix},\mathrm{d}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$

结构方程：$\mathrm{d}(\omega_1,\omega_2,0)=(\omega_1,\omega_2,0)\wedge\Omega$; $\mathrm{d}(\Omega)=\Omega\wedge\Omega$

即$\mathrm{d}(\omega_1,\omega_2,0)=(\omega_1,\omega_2,0)\wedge\begin{pmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{pmatrix}$

$\mathrm{d}\begin{pmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{pmatrix}=\begin{pmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{pmatrix}\wedge\begin{pmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{pmatrix}$

用分量形式写出：$\mathrm{d}\omega_1=\omega_{12}\wedge\omega_1$

$\mathrm{d}\omega_2=\omega_1\wedge\omega_{12}$

$\omega_1\wedge\omega_{13}+\omega_2\wedge\omega_{23}=0$

$\mathrm{d}\omega_{12}=\omega_{23}\wedge\omega_{13}$【$Gauss$方程】

$\mathrm{d}\omega_{13}=\omega_{12}\wedge\omega_{23}$

$\mathrm{d}\omega_{23}=\omega_{13}\wedge\omega_{12}$【$Codazzi$方程】

$(i)\forall\omega\in\wedge^1(S),\exists$唯一的$f,g\in\wedge^0(S)$，$s.t.\omega=f\omega_1+g\omega_2$

$(ii)\forall\omega\in\wedge^2(S),\exists$唯一的$h\in\wedge^0(S)$，$s.t.\omega=h\omega_1\wedge\omega_2$

【$Cartan$引理】设$\begin{cases}\omega_{12}=h^1_{12}\omega_1+h^2_{12}\omega_2\\\omega_{13}=h^1_{13}\omega_1+h^2_{13}\omega_2\\\omega_{23}=h^1_{23}\omega_1+h^2_{23}\omega_2\end{cases}$,则$\begin{cases}h^1_{12}\omega_1\wedge\omega_2=\mathrm{d}\omega_1\\h^2_{12}\omega_1\wedge\omega_2=\mathrm{d}\omega_2\end{cases},h^2_{13}=h^1_{23}$

方便起见，设$\begin{cases}\omega_{12}=f\omega_1+g\omega_2\\\omega_{13}=a\omega_1+b\omega_2\\\omega_{23}=b\omega_1+c\omega_2\end{cases}$,则$\begin{cases}f\omega_1\wedge\omega_2=\mathrm{d}\omega_1\\g\omega_1\wedge\omega_2=\mathrm{d}\omega_2\end{cases}$,

4.3 基本形式与Gauss曲率

设$\{P;e_1,e_2,e_3\}$为幺正标架场，又$\mathrm{d}P=(\omega_1,\omega_2,0)\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix},\mathrm{d}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\Omega\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$为基本方程

且$\omega_{13}=a\omega_1+b\omega_2$; $\omega_{23}=b\omega_1+c\omega_2$，则$S$的第一基本形式、第二基本形式与总曲率$K$分别为

$\mathrm{I}=\omega_1^2+\omega_2^2$; $\mathrm{II}=a\omega_1^2+2b\omega_1\omega_2+c\omega_2^2$; $K=ac-b^2$

$\mathrm{d}\omega_{12}=-K\omega_1\wedge\omega_2$