波动方程的导出

$u(x,t)$表示t时刻x位置，弦的位移 $\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=\boxed{f(x,t)}$外力

波动算子的基本解

$\left(\frac{\partial^2}{\partial t^2}-a^2\Delta\right)E(x,t)=\delta(x,t)$两边关于$x$方向做$Fourier$变换 $\Rightarrow \frac{\partial^2}{\partial t^2}\hat{E}(\xi,t)+a^2|\xi|^2\hat{E}(\xi,t)=\delta(t)$

就是一个解ode方程的过程，k_2=0，代回去解k_1

(1)【求解$\widehat{E}(\xi,t)$】对应的齐次方程$\frac{\partial^2}{\partial t^2}\widehat{E}(\xi,t)+a^2|\xi|^2\widehat{E}(\xi,t)=0\Rightarrow$基本解组$\begin{cases}\widehat{E_1}(\xi,t)=\sin(a|\xi|t)\\ \widehat{E_2}(\xi,t)=\cos(a|\xi|t)\end{cases}$

常数变易法，令$\widehat{E}(\xi,t)=k_1(\xi,t)\sin(a|\xi|t)+k_2(\xi,t)\cos(a|\xi|t)$，待定$k_1(\xi,t),k_2(\xi,t)$

将$\widehat{E}(\xi,t)$代入，得$\frac{\partial}{\partial t}\widehat{E}(\xi,t)=\boxed{k_1'\sin(a|\xi|t)}+k_1(a|\xi|)\cos(a|\xi|t)+\boxed{k_2'\cos(a|\xi|t)}-k_2(a|\xi|)\sin(a|\xi|t)$

令$k_1'\sin(a|\xi|t)+k_2'\cos(a|\xi|t)=0$

$\Rightarrow\frac{\partial^2}{\partial t^2}\widehat{E}(\xi,t)=k_1'(a|\xi|)\cos(a|\xi|t)-k_1a^2|\xi|^2\sin(a|\xi|t)-k_2'(a|\xi|)\sin(a|\xi|t)-k_2a^2|\xi|^2\cos(a|\xi|t)$

$\Rightarrow\frac{\partial^2}{\partial t^2}\widehat{E}(\xi,t)+a^2|\xi|^2\widehat{E}(\xi,t)=k_1'(a|\xi|)\cos(a|\xi|t)-k_2'(a|\xi|)\sin(a|\xi|t)=\delta(t)$

$\Rightarrow\begin{cases}k_1'\sin(a|\xi|t)+k_2'\cos(a|\xi|t)=0 & ①\\ \frac{\partial^2}{\partial t^2}\widehat{E}(\xi,t)+a^2|\xi|^2\widehat{E}(\xi,t)=k_1'(a|\xi|)\cos(a|\xi|t)-k_2'(a|\xi|)\sin(a|\xi|t)=\delta(t) & ②\end{cases}$

$①\times a|\xi|\cos(a|\xi|t)-②\times\sin(a|\xi|t)\Rightarrow k_2'(a|\xi|)(\cos^2(a|\xi|t)+\sin^2(a|\xi|t))=\delta(t)\sin(a|\xi|t)$

$\Rightarrow k_2'(a|\xi|)=0\Rightarrow k_2'(x,t)=0$；由于此时只需要找到满足方程的$k_1,k_2$即可，故取$k_2$的最简形式$k_2(\xi,t)=0$

回代方程组$\Rightarrow\begin{cases}k_1'\sin(a|\xi|t)=0\\ k_1'(a|\xi|)\cos(a|\xi|t)=\delta(t)\end{cases}\Rightarrow k_1'(a|\xi|)\sin^2(a|\xi|t)+k_1'(a|\xi|)\cos^2(a|\xi|t)=\delta(t)\cos(a|\xi|t)$

$\Rightarrow k_1'(a|\xi|)=\delta(t)\Rightarrow k_1'=\frac{\delta(t)}{a|\xi|}\Rightarrow k_1(\xi,t)=\begin{cases}\frac{H(t)}{a|\xi|} & ,t>0\\ \frac{-H(-t)}{a|\xi|} & ,t<0\end{cases}\Rightarrow\widehat{E}(\xi,t)=\begin{cases}\widehat{E_+}(\xi,t)=\frac{H(t)}{a|\xi|}\sin(a|\xi|t) & ,t>0\\ \widehat{E_-}(\xi,t)=\frac{-H(-t)}{a|\xi|}\sin(a|\xi|t) & ,t<0\end{cases}$

类似热传导方程中凑成Gauss积分的，好求Fourier逆变换，此时是构造δ(r-|x|)的Fourier变换和F(基本解)很像

(2)法一

设曲面$S=\{x|p(x)=0,\mathrm{d}p(x)\neq0,p\in C^\infty\}$

定义：广义函数$\delta(p(x))\in\mathscr{D}'(\mathbb{R}^n):\forall\varphi\in C_0^\infty(\mathbb{R}^n),\langle\delta(p(x)),\varphi\rangle:=\int_{p(x)=0}\varphi(x)\mathrm{d}s$

从而有sing supp $\delta(p(x))=$ supp $\delta(p(x))=\{x|p(x)=0\}$

注：若$\{x|p(x)=0\}$为紧集，则$\delta(p(x))\in\mathscr{E}'(\mathbb{R}^n)\subset\mathbb{D}'\Rightarrow\widehat{\delta(p(x))}=\langle\delta(p(x)),e^{-ix\xi}\rangle=\int_{p(x)=0}e^{-ix\xi}\mathrm{d}s$

以$n=3$为例，设$p(x)=r-|x|\Rightarrow S=\{x||x|=r\}$，利用球坐标，以$\xi$方向为$x_3$方向，$x\xi=|x||\xi|\cos\theta$

$\Rightarrow F(\delta(r-|x|))(\xi)=\int_{|x|=r}e^{-ix\xi}\mathrm{d}S_x=\int_{|x|=r}e^{-i|x||\xi|\cos\theta}\mathrm{d}S_x=\int_0^{2\pi}\int_0^{\pi}e^{-i|x||\xi|\cos\theta}r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$

$\overset{\cos\theta=y}{=\!=}-2\pi r^2\int_1^{-1}e^{-ir|\xi|y}\mathrm{d}y=\frac{2\pi r}{r|\xi|}\int_{-1}^1\cos(r|\xi|y)-i\sin(r|\xi|y)\mathrm{d}(r|\xi|y)$

$=\frac{2\pi r}{|\xi|}\sin(r|\xi|y)\Big|_{-1}^1=\frac{4\pi r}{|\xi|}\sin(r|\xi|)$

对$t>0$，令$r=at(a>0)$，$\widehat{E_+}(\xi,t)=\frac{H(t)4\pi at}{|\xi|}\sin(a|\xi|t)\frac{1}{4\pi a^2t}$

$\Rightarrow\begin{cases}E_+(x,t)=H(t)\frac{1}{4\pi a^2t}\delta(at-|x|)，对t<0，令r=-at(a>0)\\ E_-(x,t)=(-1)H(-t)\frac{1}{4\pi a^2t}\delta(-at-|x|)=(-1)H(-t)\frac{1}{4\pi a^2t}\delta(at+|x|)\end{cases}$

注：$\begin{cases}\text{supp }E_+(x.t)=\text{sing supp}E_+(x.t)=\{x||x|=at\}\\ \text{supp }E_-(x.t)=\text{sing supp}E_-(x.t)=\{x||x|=-at\}\end{cases}$

球坐标变换×ρ²sinθ，|ξ|=ρ，积化和差，极限情况写成δ

(3)法二：直接变换 以$n=3$为例

$E_+(x,t)=F^{-1}\widehat{E_+}(\xi,t)=(2\pi)^{-3}H(t)\int_{\mathbb{R}^3}\frac{\sin(a|\xi|t)}{a|\xi|}e^{ix\xi}\mathrm{d}\xi$

其中，$x\cdot\xi=|x|\cdot|\xi|\cdot\cos\theta$，令$\xi$轴过$x$点，记$|\xi|=\rho$

$\Rightarrow E_+(x,t)=(2\pi)^{-3}H(t)\int_0^{+\infty}\frac{\sin(a\rho t)}{a\rho}\rho^2\mathrm{d}\rho\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}e^{i|x|\rho\cos\theta}\sin\theta\mathrm{d}\theta$

$\overset{\cos\theta=y}{=\!=}(2\pi)^{-3}H(t)\frac{2\pi}{a}\int_0^{\infty}\sin(a\rho t)\rho\mathrm{d}\rho\int_1^{-1}\cos(|x|\rho y)+i\sin(|x|\rho y)(-1)\mathrm{d}y$

$=\frac{H(t)}{a|x|}(2\pi)^{-2}\int_0^{+\infty}\sin(a\rho t)2\cdot\sin(|x|\rho)\mathrm{d}\rho$

$=\frac{H(t)}{4\pi^2a|x|}\int_0^{+\infty}\cos(|x|-at)\rho-\cos(|x|+at)\rho\mathrm{d}\rho$

$=\frac{H(t)}{4\pi^2a|x|}\lim_{A\to+\infty}\frac{\sin[A(|x|-at)]}{|x|-at}-\frac{\sin[A(|x|+at)]}{|x|+at}$

$\overset{\lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin nx}{x}\mathrm{d}x=\delta(x)}{=\!=\!=}\frac{H(t)}{4\pi a|x|}(\delta(|x|-at)-\delta(|x|+at))$

$\overset{t>0时|x|+at>0所以\delta(|x|+at)=0}{=\!=\!=}\frac{H(t)}{4\pi a|x|}\delta(|x|-at),t>0$

由supp $\delta=\{|x|=at\}\Rightarrow E_+(x,t)=H(t)\frac{1}{4\pi a^2t}\delta(at-|x|),t>0$

$E_-(x,t)=-\frac{H(-t)}{4\pi a|x|}\delta(|x|+at),t<0$的情况完全类似

一维波动方程

初值问题

d'Alembert公式，仅限一维情形，还是叠加原理+Duhamel原理，坐标变换

$(P_0)\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t),t>0\\ u|_{t=0}=\varphi(x),\frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)\end{cases}$

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$d'Alembert$解 仅限一维情形

叠加原理，拆成$(P_1)\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=0,t>0\\ u|_{t=0}=\varphi(x),\frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)\end{cases}$；$(P_2)\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t),t>0\\ u|_{t=0}=0,\frac{\partial u}{\partial t}\Big|_{t=0}=0\end{cases}$

求解(P_1)：令$\xi=x-at;\eta=x+at$

计算得$\frac{\partial^2u}{\partial x^2}=\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)^2u;\frac{\partial^2u}{\partial t^2}=a^2\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right)^2u$

从而$\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=-4a^2\frac{\partial^2u}{\partial\xi\partial\eta}=0$，即$\frac{\partial^2u}{\partial\xi\partial\eta}=0,u(\xi,\eta)=F(\xi)+G(\eta)$

把$u|_{t=0}=\varphi(x),\frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)$代入$u(x,t)=F(x-at)+G(x+at)$得

方程组$\begin{cases}F(x)+G(x)=\varphi(x)\\ -a(F'(x)-G'(x))=\psi(x)\xrightarrow{两边积分}(-a)(F(x)-G(x))+c=\int_{x_0}^x\psi(y)\mathrm{d}y\end{cases}$

得$\begin{cases}F(x)=\frac{1}{2}\left[\varphi(x)-\frac{1}{a}\int_{x_0}^x\psi(y)\mathrm{d}y+\frac{c}{a}\right]\\ G(x)=\frac{1}{2}\left[\varphi(x)+\frac{1}{a}\int_{x_0}^x\psi(y)\mathrm{d}y-\frac{c}{a}\right]\end{cases}\Rightarrow u_1(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\mathrm{d}y$

求解$(P_2)\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t),t>0\\ u|_{t=0}=0,\frac{\partial u}{\partial t}\Big|_{t=0}=0\end{cases}$

【$Duhamel$原理】令$W(x,t,\tau)$满足$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\Delta\right)W=0, & t>\tau\\ W(x,t,\tau)|_{t=\tau}=0\\ \frac{\partial W}{\partial t}\Big|_{t=\tau}=f(x,\tau)\end{cases}$，令$u(x,t)=\int_0^tW(x,t;\tau)\,d\tau$

得$W(x,t,\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(y,\tau)\mathrm{d}y$，则$u_2(x,t)=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(y,\tau)\mathrm{d}y\mathrm{d}\tau$

$\Rightarrow(P_0)$的解为$u(x)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\mathrm{d}y+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(y,\tau)\mathrm{d}y\mathrm{d}\tau$

初边值问题

齐次方程

初值函数做奇延拓，带回d'Alembert解，代边值条件得到等式. 讨论x-at的符号回代延拓函数解方程

例：$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=0,t>0,0<x<\infty\\ u|_{t=0}=\varphi(x),\frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)\\ u(0,t)=0\end{cases}$

设想$x<0$处仍有弦，只是在扰动过程中保持$x=0$，设$\Phi(x)$和$\Psi(x)$分别为$\varphi(x)$，$\psi(x)$奇延拓后的函数

$\Rightarrow\Phi(x)=\begin{cases}\varphi(x) & ,x\geqslant0\\ -\varphi(-x) & ,x<0\end{cases},\Psi(x)=\begin{cases}\psi(x) & ,x\geqslant0\\ -\psi(-x) & ,x<0\end{cases}$

延拓后代入$d'Alembert$公式$\Rightarrow\begin{cases}u(x,t)=\frac{1}{2}[\Phi(x+at)+\Phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(y)\mathrm{d}y\\ u(0,t)=0\end{cases}$

代入$u(0,t)=0$可得$\frac{1}{2}[\Phi(at)+\Phi(-at)]+\frac{1}{2a}\int_{-at}^{at}\Psi(y)\mathrm{d}y=0$，把$\varphi(x),\psi(x)$带回得【讨论】

① $x-at\geqslant0$，又有$x+at\geqslant0\Rightarrow u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\mathrm{d}y$

② $x-at<0$，仍有$x+at\geqslant0\Rightarrow\Phi(x-at)=-\varphi(at-x),\Psi(x-at)=-\Psi(at-x)$

$\Rightarrow u(x,t)=\frac{1}{2}[\Phi(x+at)+\Phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(y)\mathrm{d}y$

$=\frac{1}{2}[\varphi(x+at)+\varphi(at-x)]+\frac{1}{2a}\left(\int_0^{x+at}\psi(y)\mathrm{d}y+\int_{x-at}^0-\psi(-y)\mathrm{d}y\right)$

$=\frac{1}{2}[\varphi(x+at)+\varphi(at-x)]+\frac{1}{2a}\int_0^{x+at}\psi(y)\mathrm{d}y+\frac{1}{2a}\int_{at-x}^0\psi(-\tau)\mathrm{d}\tau$

$=\frac{1}{2}[\varphi(x+at)+\varphi(at-x)]+\frac{1}{2a}\int_{at-x}^{x+at}\varphi(y)\mathrm{d}y$

例：弦振动方程的$Cauchy$问题的$d'Alembert$解与初边值问题的$Fourier$方法解之间的关系

初边值问题中的初值条件$u|_{t=0}=\varphi(x),\frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)$可延拓为以$2l$为周期的奇函数

在此延拓思想下，两个解等价吗？

延拓.$\begin{cases}\varphi(x)=\sum_{k=1}^\infty a_k\sin\frac{k\pi x}{l}\\ \psi(x)=\sum_{k=1}^\infty b_k\sin\frac{k\pi x}{l}\end{cases}$，其中$\begin{cases}a_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi x}{l}\mathrm{d}x\\ b_k=\frac{2}{l}\int_0^l\psi(x)\sin\frac{k\pi x}{l}\mathrm{d}x\end{cases}$

$u(x,t)=\frac{1}{2}\left[\sum_{k=1}^\infty a_k\sin\frac{k\pi(x+at)}{l}+\sum_{k=1}^\infty a_k\sin\frac{k\pi(x-at)}{l}\right]+\frac{1}{2a}\int_{x-at}^{x+at}\sum_{k=1}^\infty b_k\sin\frac{k\pi y}{l}\mathrm{d}y$

$\overset{\sin\frac{k\pi(x+at)}{l}+\sin\frac{k\pi(x-at)}{l}=2\sin\frac{k\pi x}{l}\cos\frac{k\pi at}{l}}{=\!=\!=}\sum_{k=1}^\infty a_k\sin\frac{k\pi x}{l}\cos\frac{k\pi at}{l}+\sum_{k=1}^\infty\underbrace{\boxed{\frac{l}{ak\pi}}b_k}_{\widetilde{b_k}}\sin\frac{k\pi x}{l}\sin\frac{k\pi at}{l}$

$=\sum_{k=1}^\infty\left(a_k\cos\frac{k\pi at}{l}+\widetilde{b_k}\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}$

非齐次方程

2015考过

分离变量法（类似于热传导方程）

$(P)\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t),t>0\\ u|_{t=0}=\varphi(x),\frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)\\ u(0,t)=u(l,t)=0\end{cases}$

叠加原理$(P_1)\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=0,t>0\\ u|_{t=0}=\varphi(x),\frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x)\\ u(0,t)=u(l,t)=0\end{cases}$

$Fourier$方法求解(P_1)：$u(x,t)=X(x)T(t)\Rightarrow X(x)T''(t)-a^2X''(x)T(t)=0$

$\Rightarrow\frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T'(t)}\overset{\triangle}{=\!=}-\mu\Longrightarrow X''(x)+\mu X(x)=0$

满足边界条件：$X(0)=X(l)=0\Rightarrow X_k(x)=C_k\sin\frac{k\pi}{l}x,k=1,2,\ldots$

代入$T_k''(t)+a\mu_kT_k(t)=0\Rightarrow T_k(t)=\overline{A_k}\cos\frac{ak\pi}{l}t+\overline{B_k}\sin\frac{ak\pi}{l}t$

$u_k(x,t)=X_k(x)T_k(t)\Rightarrow u_1(x,t)=\sum_{k=1}^\infty\left(A_k\cos\frac{ak\pi}{l}t+B_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x$

$\Rightarrow$代入边值条件求得$A_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi}{l}x\mathrm{d}x;B_k=\frac{2}{ak\pi}\int_0^l\psi(x)\sin\frac{k\pi}{l}x\mathrm{d}x$

$Duhamel$原理$(P_2)\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t),t>0\\ u|_{t=0}=\frac{\partial u}{\partial t}\Big|_{t=0}=0\\ u(0,t)=u(l,t)=0\end{cases}$

令$W(x,t,\tau)=\sum_{k=1}^\infty B_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)\sin\frac{k\pi}{l}x,B_k(\tau)=\frac{2}{k\pi a}\int_0^lf(x,\tau)\sin\frac{k\pi}{l}x\mathrm{d}x$

$\Rightarrow u_2(x,t)=\int_0^tW(x,t,\tau)\mathrm{d}\tau=\sum_{k=1}^\infty\left[\int_0^tB_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)\mathrm{d}\tau\right]\sin\frac{k\pi}{l}x$

则$u(x,t)=u_1(x,t)+u_2(x,t)$为初边值问题(P)的解

高维波动方程

坐标变换的思路一致

$Kirchhof$公式（三维）

$Cauchy$问题$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t),t>0,x\in\mathbb{R}^3，其中f\in\mathbb{C}^3,g_0\in\mathbb{C}^3,g_1\in\mathbb{C}^2\\ u|_{t=0}=g_0(x),\frac{\partial u}{\partial t}\Big|_{t=0}=g_1(x)\end{cases}$

回顾热传导方程高维的$Cauchy$问题$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,t),t>0\\ u|_{t=0}=\varphi(x)\end{cases}$

用基本解法直接求，化简出F(x,t)的形式

$\tilde{u}(x,t)=H(t)u(x,t)\Rightarrow\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)\tilde{u}(x,t)=F(x,t)$，计算得$=H(t)f(x,t)+\delta'(t)u(x,t)+2\delta(t)\frac{\partial u}{\partial t}$

$\forall\varphi\in C_0^\infty,\langle\delta'(t)u,\varphi\rangle=\langle\delta'(t),u\cdot\varphi\rangle=-\langle\delta'(t),\partial_t(u\varphi)\rangle=-\langle\delta'(t),\partial_tu\varphi+u\partial_tv\rangle=\langle-(\partial_tu)\delta(t)+\partial_t(\delta(t)u),\varphi\rangle$

$\Rightarrow\delta'(t)u=\partial_t(\delta(t)u)-\delta(t)\frac{\partial u}{\partial t}\Rightarrow F(x,t)=H(t)f(x,t)+\partial_t(\delta(t)u)+\delta(t)\frac{\partial u}{\partial t}$

$\langle\partial_t(\delta(t)u),\varphi\rangle=-\langle\delta(t),u\partial_t\varphi\rangle=-u(x,0)\partial_t\varphi|_{t=0}=-g_0(x)(\partial_t\varphi|_{t=0})=-g_0(x)\langle\delta(t),\partial_t\varphi\rangle=\left\langle\partial_t(g_0(x)\delta(t)),\varphi\right\rangle$

同理，$\left\langle\delta(t)\frac{\partial u}{\partial},\varphi\right\rangle=\langle g_1(x)\delta(t),\varphi\rangle\Rightarrow F(x,t)=H(t)f(x,t)+\partial_t(g_0(x)\delta(t))+g_1(x)\delta'(t)$

已知$E_+(x,t)$为$\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)$在$t>0$的基本解

$\Rightarrow H(t)u(x,t)=E_+(x,t)*[H(t)f(x,t)+\partial_t(g_0(x)\delta(t))+g_1(x)\delta'(t)]$

$=E_+(x,t)*[H(t)f(x,t)]+E_+(x,t)*[\partial_t(g_0(x)\delta(t))]+E_+(x,t)*[g_1(x)\delta(t)]$

其中，② $\langle E_+(x,t)*[\partial_t(g_0(x)\delta(t))],\varphi(x)\rangle=\langle\partial_tE_+(x,t)*(g_0(x)\delta(t)),\varphi(x)\rangle$

$=\langle\partial_tE_+(x,t),\langle\delta(\tau)g_0(y),\varphi(x+y,t+\tau)\rangle\rangle$

$=\langle\partial_tE_+(x,t),\langle g_0(y),\varphi(x+y,t)\rangle\rangle$

$=\partial_tE_+(x,t)\underset{(x)}{*}g_0(x)$

$=\partial_t\left(E_+(x,t)\underset{(x)}{*}g_0(x)\right)$

同理，$E_+(x,t)*(\delta(t)g_1(x))=E_+(x.t)\underset{(x)}{*}g_1(x)$

于是有$\tilde{u}(x,t)=E_+(x,t)*(H(t)f(x,t))+\partial_t\left(E_+(x,t)\underset{(x)}{*}g_0(x)\right)+E_+(x.t)\underset{(x)}{*}g_1(x)$

对得到的u(x,t)的卷积形式进行展开、平均值公式

$I_1=E_+(x,t)*(H(t)f(x,t))=\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}H(\tau)f(y,\tau)H(t-\tau)\delta(a(t-\tau)-|x-y|)\mathrm{d}y\mathrm{d}\tau$

当$\tau>0,H(\tau)=1,t-\tau>0,H(t-\tau)=1\iff\tau<t$

$\Rightarrow I_1=\int_0^t\mathrm{d}\tau\int_{|x-y|=a(t-\tau)}\frac{f(y,\tau)}{4\pi a^2(t-\tau)}\mathrm{d}y=\frac{1}{4\pi a}\int_0^t\mathrm{d}\tau\int_{|x-y|=a(t-\tau)}\frac{f(y,\tau)}{|x-y|}\mathrm{d}S_y$

$=\frac{1}{4\pi a^2}\int_{|x-y|\leqslant at}\frac{f\left(y,t-\frac{|x-y|}{a}\right)}{|x-y|}\mathrm{d}y$

$I_3=E_+(x,t)\underset{(x)}{*}g_1(x)=\frac{H(t)}{4\pi a^2t}\int_{\mathbb{R}^3}g_1(y)\delta(at-|x-y|)\mathrm{d}y=\frac{1}{4\pi a^2t}\int_{|x-y|=at}g_1(y)\mathrm{d}S_y=tM\{g_1\}$

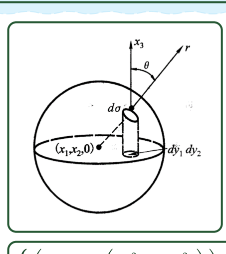
同理，$I_2=\partial_t(tM\{g_1\})$

最终结果

$\tilde{u}(x,t)=\frac{1}{4\pi a^2}\int_{|x-y|\leqslant at}\frac{f\left(y,t-\frac{|x-y|}{a}\right)}{|x-y|}\mathrm{d}y+\partial_t(tM\{g_1\})+tM\{g_1\}\overset{t>0}{=\!=}u(x,t)$

降维法

二维



$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_1^2}\right)\right)u=0\\ u|_{t=0}=g_0(x),\frac{\partial u}{\partial t}\Big|_{t=0}=g_1(x)\end{cases}$

降维法，若有初值g_0,g_1不含有x_3分量，$u(x,t)=\partial_t(tM(g_0))+tM(g_1)$

考虑$tM(g_1)=\frac{t}{4\pi(at)^2}\int_{|x-y|=at}g_1(y)\mathrm{d}S_y,\cos\theta\mathrm{d}S_y=\mathrm{d}y_1\mathrm{d}y_2$

$\cos\theta=\frac{\sqrt{(at)^2-((x_1-y_1)^2+(x_2-y_2)^2)}}{at}$

$\Rightarrow tM(g_1)=2\frac{t}{4\pi(at)^2}\int_{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\leqslant at}\frac{g_1(y_1,y_2)}{\cos\theta}\mathrm{d}y_1\mathrm{d}y_2$

$=\frac{1}{2\pi a}\int_{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\leqslant at}\frac{g_1(y_1,y_2)}{\sqrt{(at)^2-((x_1-y_1)^2+(x_2-y_2)^2)}}\mathrm{d}y_1\mathrm{d}y_2$

同理得$\partial_t(tM(g_0))$

一维

$\begin{cases}\left(\frac{\partial}{\partial t}-a^2\frac{\partial^2}{\partial x^2}\right)u=0\\ u|_{t=0}=g_0(x),\frac{\partial u}{\partial t}\Big|_{t=0}=g_1(x)\end{cases}$

柱面面积$2\pi r\mathrm{d}y_1$

$\frac{2\pi r\mathrm{d}y_1}{\mathrm{d}S}=\cos\theta=\frac{r}{at}\Rightarrow\mathrm{d}S=\frac{at2\pi r\mathrm{d}y_1}{r}=2\pi at\mathrm{d}y_1$

$\Rightarrow tM(g_1)=\frac{1}{4\pi a^2t}\int_{|x-y|=at}g_1(y)\mathrm{d}S_y$

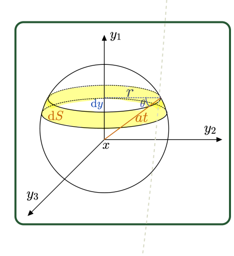
$=\frac{1}{4\pi a^2t}\int_{x_1-at}^{x_1+at}g_1(y_1)2\pi at\mathrm{d}y_1$

$=\frac{1}{2a}\int_{x_1-at}^{x_1+at}g_1(y_1)\mathrm{d}y_1$

$\Rightarrow\partial_t(tM(g_0))=\partial_t\left(\frac{1}{2a}\int_{x_1-at}^{x_1+at}g_1(y_1)\mathrm{d}y_1\right)$

$=\frac{1}{2a}(g_0(x_1+at)a-g_0(x_1-at)(-a))$

$=\frac{1}{2}(g_0(x_1+at)+g_0(x_1-at))$



例题：波动方程的经典解形式

例：证明函数$E(x,t)=\begin{cases}\frac{1}{2a}, & |x|\leqslant at,t\geqslant0\\ 0, & \text{if not}\end{cases}$是弦振动方程的基本解

证明：要证$\left\langle\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)E(x,t),\varphi(x,t)\right\rangle=\langle\delta(x,t),\varphi(x,t)\rangle=\varphi(0,0),\forall\varphi(x,t)\in C_0^\infty$

$\left\langle\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)E(x,t),\varphi(x,t)\right\rangle=\left\langle E(x,t),\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)\varphi(x,t)\right\rangle=\int_{x^2\leqslant(at)^2,t\geqslant0}\frac{1}{2a}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)\varphi\mathrm{d}x\mathrm{d}t$

令$y=x+at,z=at-x,\varphi(x,t)=\varphi\left(\frac{y-z}{2},\frac{y+z}{2a}\right)=\varphi(y,z)\in C_0^\infty$

被积函数变换：$\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)\varphi=4a^2\frac{\partial^2\varphi}{\partial y\partial z},J(y,z)=\begin{vmatrix}\frac{\partial x}{\partial y} & \frac{\partial x}{\partial z}\\ \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z}\end{vmatrix}=\frac{1}{2a}$，被积区域变换：$\left.\begin{matrix}x^2\leqslant(at)^2\Rightarrow y,z\geqslant0\\ t\geqslant0\Rightarrow y+z\geqslant0\end{matrix}\right\}\Rightarrow y\geqslant0,z\geqslant0$

上式$=\int_{y=0}^{+\infty}\int_{z=0}^{+\infty}\frac{1}{2a}4a^2\frac{\partial^2\varphi}{\partial y\partial z}\frac{1}{2a}\mathrm{d}y\mathrm{d}z=\int_{y=0}^{+\infty}\int_{z=0}^{+\infty}\frac{\partial^2\varphi}{\partial y\partial z}\mathrm{d}z\mathrm{d}y\overset{\varphi\in C_0^\infty}{=\!=}\int_{y=0}^{+\infty}\int_{z=0}^{+\infty}\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial y}\right)\mathrm{d}z\mathrm{d}y\overset{分部积分}{=\!=\!=}(-1)(-1)\varphi(0,0)=\varphi(0,0)$

$\Rightarrow E(x,t)$为基本解$\Rightarrow\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=f(x,y,t),(x,y)\in\Omega\\ u|_{t=0}=g_0\\ \frac{\partial u}{\partial t}\Big|_{t=0}=g_1\end{cases}\Rightarrow u(x,t)=\partial_t\left(E(x,t)\underset{(x)}{*}g_0(x)\right)+E(x,t)\underset{(x)}{*}g_1(x)$

能量方法

初边值问题

$n=2,\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)u=f(x,y,t),(x,y)\in\Omega\\ u|_{t=0}=\varphi(x,y)\\ \frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x,y)\\ u|_{\partial\Omega}=\mu(x,y,t)\end{cases}$

$\begin{cases}动能U=\frac{1}{2}\int_\Omega\rho u_t^2\mathrm{d}x\mathrm{d}y\\ 位能V=\frac{1}{2}\int_\Omega T\left(u_x^2+u_y^2\right)\mathrm{d}x\mathrm{d}y\end{cases}$其中$\rho$为密度，$T$为张力

$\Rightarrow$总能量$E(t)=\int_\Omega u_t^2+a^2\left(u_x^2+u_y^2\right)\mathrm{d}x\mathrm{d}y,a^2=\frac{T}{\rho}$(不计一个常数因子)

对于有限薄膜振动，在无外力作用即$f=0$时，应有能量守恒$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0$

求导，分部积分，把u_t提出来，齐次方程=0，第一类Gauss公式，把u_t提出来=0，因为u在边界上恒为0

从数学考虑，齐次方程齐次边值条件$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-a^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right)\right)u=0\\ u|_{t=0}=\varphi(x,y)\\ \frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x,y)\\ u|_{\partial\Omega}=0\end{cases}$，证明：$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0$

证明：已知$E(t)=\int_\Omega u_t^2+a^2(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y\Rightarrow\frac{\mathrm{d}E}{\mathrm{d}t}=\int_\Omega2u_tu_{tt}+a^2(2u_xu_{xt}+2u_yu_{yt})\mathrm{d}x\mathrm{d}y$

分部积分：$u_xu_{xt}=\partial_x(u_xu_t)-u_{xx}u_t;u_yu_{yt}=\partial_y(u_yu_t)-u_{yy}u_t$

即$\frac{\mathrm{d}E}{\mathrm{d}t}=2\int_\Omega u_tu_{tt}+a^2(\partial_x(u_xu_t)-u_{xx}u_t+\partial_y(u_yu_t)-u_{yy}u_t)\mathrm{d}x\mathrm{d}y$

$=2\int_\Omega u_t(u_{tt}-a^2(u_{xx}+u_{yy}))\mathrm{d}x\mathrm{d}y+2a^2\int_\Omega\partial_x(u_xu_t)+\partial_y(u_yu_t)\mathrm{d}x\mathrm{d}y$【把带$u_t$的分开写】

$\overset{\iint_\Omega\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\mathrm{d}x\mathrm{d}y=\int_{\partial\Omega}(P\,\mathrm{d}y-Q\,\mathrm{d}x)}{=\!=\!=\!=}2a^2\int_{\partial\Omega}u_xu_t\cos(\vec{n},x)+u_yu_t\cos(\vec{n},y)\mathrm{d}S$【再把$u_t$提出来】

$=2a^2\int_{\partial\Omega}u_t\left(u_x\cos(\vec{n},x)+u_yu\cos(\vec{n},y)\right)\mathrm{d}S$

$=0$，因为$u|_{\partial\Omega}=0\Rightarrow u_t|_{\partial\Omega}=0$

于是得证：$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0$，即能量守恒

2023、2015、2015伯苓考过

唯一性

根据能量守恒可得$(P)\begin{cases}u_{tt}-a^2(u_{xx}+u_{yy})=f(x,y,t),(x,y)\in\Omega\\ u|_{t=0}=\varphi(x,y)\\ \partial_tu|_{t=0}=\psi(x,y)\\ u|_{\partial\Omega}=0\end{cases}$的唯一性

满足能量守恒，E(t)=E(0)=0(初值条件得)，即得u为常数，u=0

证明：设u_2,u_2均为(P)的解，$u=u_1-u_2$满足$(P_u)\begin{cases}u_{tt}-a^2(u_{xx}+u_{yy})=0,(x,y)\in\Omega\\ u|_{t=0}=0\\ \partial_tu|_{t=0}=0\\ u|_{\partial\Omega}=0\end{cases}$

由外力$f=0,u|_{\partial\Omega}=0\Rightarrow$能量守恒$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0\Rightarrow E(t)=E(0)=\int_\Omega u_t^2+a^2(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y\Big|_{t=0}$

由(P_u)初值条件，$u_t|_{t=0}=0$，由$u|_{t=0}=u(x,y,t)|_{t=0}=0,(x,y)\in\Omega$

$\Rightarrow u_x|_{t=0}=0,u_y|_{t=0}=0\Rightarrow u_t^2+a^2(u_x^2+u_y^2)|_{t=0}=0\Rightarrow E(0)=0$

$\Rightarrow\int_\Omega u_t^2+a^2(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y=0\Rightarrow u_t^2+a^2(u_x^2+u_y^2)=0\Rightarrow u_t=u_x=u_y=0\Rightarrow u\equiv const$

又$u|_{t=0}=0,u_1\equiv u_2\Rightarrow$唯一性

2020, 2023伯苓考过

稳定性

考虑$\begin{cases}u_{tt}-a^2(u_{xx}+u_{yy})=f(x,y,t) & \text{in }\Omega,\\ u|_{t=0}=\varphi, & \partial_tu|_{t=0}=\psi,\\ u|_{\partial\Omega}=0.\end{cases}$，对$\forall\varepsilon>0$，$\exists\eta(\varepsilon,T)>0,s.t.$：当$\begin{cases}\|\varphi_1-\varphi_2\|_{L^2(\Omega)}<\eta\\ \|\varphi_{1x}-\varphi_{2x}\|_{L^2(\Omega)}<\eta\\ \|\varphi_{1y}-\varphi_{2y}\|_{L^2(\Omega)}<\eta\\ \|\psi_1-\psi_2\|_{L^2(\Omega)}<\eta\\ \|f_1-f_2\|_{L^2((0,T)\times\Omega)}<\eta\end{cases}$

则对应φ_1,ψ_1,f_1的解u_1和对应φ_2,ψ_2,f_2的解u_2在$[0,T]$上成立以下稳定性，即$\begin{cases}\|u_1-u_2\|_{L^2(\Omega)}<\varepsilon\\ \|u_{1x}-u_{2x}\|_{L^2(\Omega)}<\varepsilon\\ \|u_{1y}-u_{2y}\|_{L^2(\Omega)}<\varepsilon\\ \|u_{1t}-u_{2t}\|_{L^2(\Omega)}<\varepsilon.\end{cases}$

step 1:考虑有外力的情况：$(P_f)\begin{cases}u_{tt}-a^2(u_{xx}+u_{yy})=f(x,y,t),(x,y)\in\Omega\\ u|_{t=0}=\varphi(x,y)\\ \partial_tu|_{t=0}=\psi(x,y)\\ u|_{\partial\Omega}=0\end{cases}$，估计总能量$E(t)$

$\frac{\mathrm{d}E}{\mathrm{d}t}=2\int_\Omega u_tu_{tt}+a^2(u_xu_{xt}+u_yu_{yt})\mathrm{d}x\mathrm{d}y=2\int_\Omega u_t\cdot f\mathrm{d}x\mathrm{d}y\leqslant\int_\Omega u_t^2+f^2\mathrm{d}x\mathrm{d}y\leqslant E(t)+\int_\Omega f^2\mathrm{d}x\mathrm{d}y$

两边同乘$e^{-t}\Rightarrow e^{-t}\frac{\mathrm{d}E}{\mathrm{d}t}-e^{-t}E(t)\leqslant e^{-t}\int_\Omega f^2\mathrm{d}x\mathrm{d}y\Rightarrow\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}E(t))\leqslant e^{-t}\int_\Omega f^2\mathrm{d}x\mathrm{d}y$从0到$t$积分

$\Rightarrow e^{-t}E(t)-E(0)\leqslant\int_0^te^{-\tau}\int_\Omega f(x,y,\tau)\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\Rightarrow E(t)\leqslant e^t\left(E(0)+\int_0^t\int_\Omega e^{-\tau}f^2\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\right)$

考虑有限的时间$0\leqslant t\leqslant T\Rightarrow\int_\Omega u_t^2+a^2(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y\Rightarrow\frac{\mathrm{d}E}{\mathrm{d}t}=E(t)\leqslant C_0\left(E(0)+\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\right)$

step 2:为了得到稳定性，即u_0,u_t,u_x,u_y;关于初值和外力f的依赖，需要估计$E_0(t)=\int_\Omega u^2\mathrm{d}x\mathrm{d}y$

$\frac{dE_0(t)}{dt}=\frac{d}{dt}\int_\Omega u^2\mathrm{d}x\mathrm{d}y=\int_\Omega2uu_t\mathrm{d}x\mathrm{d}y\leqslant\int_\Omega u_t^2+u^2\mathrm{d}x\mathrm{d}y\leqslant E_0(t)+E(t)$

两边同乘e^{-t}：$e^{-t}\frac{\mathrm{d}E_0(t)}{\mathrm{d}t}-e^{-t}E_0(t)\leqslant e^{-t}E(t)\Rightarrow\frac{d}{dt}(e^{-t}E_0(t))\leqslant e^{-t}E(t)$

从0到t积分，得$e^{-t}E_0(t)-E_0(0)\leqslant\int_0^te^{-\tau}E(\tau)\mathrm{d}\tau\Rightarrow E_0(t)\leqslant e^t\left(E_0(0)+\int_0^te^{-\tau}E(\tau)\mathrm{d}\tau\right)$

代入$E(t)$得$E_0(t)\leqslant e^t\left(E_0(0)+\widetilde{C_0}T\left(E(0)+\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\right)\right)$

从而得到：$\int_\Omega u^2\mathrm{d}x\mathrm{d}y=E_0(t)\leqslant C_1\left(E_0(0)+E(0)+\int_0^T\int_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\right)$进一步可得稳定性定理

step 3:设$u=u_1-u_2$，则u满足：$\begin{cases}u_{tt}-a^2(u_{xx}+u_{yy})=f_1-f_2 & \text{in }\Omega,\\ u|_{t=0}=\varphi_1-\varphi_2\\ \partial_tu|_{t=0}=\psi_1-\psi_2,\\ u|_{\partial\Omega}=0.\end{cases}$

$\|u_1-u_2\|^2_{L^2(\Omega)}=\int_\Omega u^2\mathrm{d}x\mathrm{d}y=E_0(t)$

$\leqslant C_1\left(E_0(0)+E(0)+\int_0^T\int_\Omega(f_1-f_2)^2\mathrm{d}x\mathrm{d}y\mathrm{d}t\right)$

$=C_1\left(\int_\Omega u^2\mathrm{d}x\mathrm{d}y\Big|_{t=0}+\int_\Omega u_t^2+a^2(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y\Big|_{t=0}\right)+C_1\|f_1-f_2\|^2_{L^2([0,T]\times\Omega)}$

$\leqslant C_1\left(\|\varphi_1-\varphi_2\|^2_{L^2(\Omega)}+\|\psi_1-\psi_2\|^2_{L^2(\Omega)}+a^2\left(\|\varphi_{1x}-\varphi_{2x}\|^2_{L^2(\Omega)}+\|\varphi_{1y}-\varphi_{2y}\|^2_{L^2(\Omega)}\right)+\|f_1-f_2\|^2_{L^2([0,T]\times\Omega)}\right).$

$\leqslant(3+2a^2)C_1\eta^2$

$\leqslant\widetilde{C}\cdot\eta^2<\varepsilon^2$

而其余三项$\begin{cases}\|u_{1t}-u_{2t}\|^2_{L^2(\Omega)}<\varepsilon\\ \|u_{1x}-u_{2x}\|^2_{L^2(\Omega)}<\varepsilon\\ \|u_{1y}-u_{2y}\|^2_{L^2(\Omega)}<\varepsilon\end{cases}$的估计只需应用$E(t)$的能量不等式即可得到

例：对受摩擦力作用的固定端点的有界弦振动方程

$(P_1)\begin{cases}\frac{\partial^2u}{\partial t^2}-a^2\frac{\partial^2u}{\partial x^2}+c\frac{\partial u}{\partial t}=0,x\in(0,l),c<0\\ u|_{x=0}=0,u|_{x=l}=0\\ u|_{t=0}=\varphi,\frac{\partial u}{\partial t}\Big|_{t=0}=\psi\end{cases}$，证明总能量$E(t)$随时间$t$减少，且$(P_1)$解唯一

证明：$E(t)=\int_a^bu_t^2+a^2u_x^2\mathrm{d}x$

$\frac{\mathrm{d}}{\mathrm{d}t}E=\int_a^b2u_tu_{tt}+2a^2u_{xt}u_x\mathrm{d}x$

$\overset{分部积分:u_xu_{xt}=\partial_x(u_xu_t)-u_{xx}u_t}{=\!=\!=\!=}\int_a^b2u_tu_{tt}+2a^2(\partial_x(u_xu_t)-u_{xx}u_t)\mathrm{d}x$

$=2c\int_a^b(u_t)^2\mathrm{d}x<0$，即$E(t)$减少

唯一性：令$u=u_1-u_2$满足$(P_0)\begin{cases}u_{tt}-a^2u_{xx}+cu_t=0\\ u|_{x=a}=u|_{x=b}=0\\ u|_{t=0}=\partial_tu|_{t=0}=0\end{cases}$

则$E(t)\leqslant E(0)\overset{(P_0)}{=\!=}\int_a^bu_t^2(x,0)+a^2u_x(x,0)\mathrm{d}x=0$

$\Rightarrow0\leqslant E(t)\leqslant0\Rightarrow E(t)=0\Rightarrow u_x=u_t=0\Rightarrow u\equiv c$

又由$u|_{t=0}=0\Rightarrow u_1\equiv u_2$

例：利用能量积分函数$E(t)=\frac{1}{2}\int_0^lk(x)u_x^2+\rho(x)u_t^2+q(x)u^2\mathrm{d}x$

证明初值问题：$\begin{cases}\rho(x)u_{tt}-(k(x)u_x)_x+q(x)u=0, & 0<x<l,t>0\\ u(x,0)=0,\ u_t(x,0)=0, & x\in[0,l],\\ u(0,t)=u(l,t)=0, & t\geqslant0,\end{cases}$的解$u\equiv0$

其中$k(x)\geqslant k_0>0,\rho(x)\geqslant\rho_0>0,q(x)\geqslant q_0>0.$

证明：$\frac{dE(t)}{dt}=\frac{1}{2}\int_0^l2k(x)u_xu_{xt}+2\rho(x)u_tu_{tt}+2q(x)uu_t\mathrm{d}x$

$=\int_0^l\partial_x(k(x)u_xu_t)-\partial_x(k(x)u_x)u_t+\rho(x)u_tu_{tt}+q(x)uu_t\mathrm{d}x$

$=\int_0^lu_t(\rho(x)u_{tt}-\partial_x(k(x)u_x)+q(x)u)\mathrm{d}x+\int_0^l\partial_x(k(x)u_xu_t)\mathrm{d}x$

$=k(x)u_xu_t|_0^l\overset{两端固定}{=\!=\!=}0\Rightarrow\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0\Rightarrow E(t)=0$

$E(t)=E(0)=\frac{1}{2}\int_0^lk(x)u_x^2+\rho(x)u_t^2+q(x)u^2\mathrm{d}x\Big|_{t=0}=0$

$\Rightarrow u_x=u_t=u=0\Rightarrow u\equiv0$