Problem Set 1

Basil R. Yap
40.316 Game Theory - Term 8

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Question 1: Each of two players has two possible actions, *Quiet* and *Fink*; each action pair results in the players' receiving amounts of money equal to the numbers corresponding to that action pair in the following game matrix:

	Quiet	*Fink*
Quiet	$2, 2$	$0, 3$
Fink	$3, 0$	$1, 1$

For example, if player 1 chooses *Quiet* and player 2 chooses *Fink*, then player 1 receives nothing, whereas player 2 receives \$3.

The players are not "selfish"; rather the preferences of each player i are represented by the payoff function $m_i(a) + \alpha m_j(a)$, where $m_i(a)$ is the amount of money received by player i when the action profile is a, j is the other player, and α is a given non-negative number. Player 1's payoff to the action pair *(Quiet,Quiet)*, for example, is $2 + 2\alpha$.

a) Formulate a strategic game that models this situation in the case $\alpha = 1$. Is this game the *Prisoners Dilemma*?

b) Find the range of values of α for which the resulting game is the *Prisoners Dilemma*.

Solution

a)
- Players: Row Player and Column Player
- Actions: Quiet or Fink

- Pay-off matrix:

	Quiet	*Fink*
Quiet	$(2+2\alpha, 2+2\alpha)$	$(3\alpha, 3)$
Fink	$(3, 3\alpha)$	$(1+\alpha, 1+\alpha)$

When $\alpha = 1$, the game is not a *Prisoner's Dilemma*.
$u(Quiet, \{Quiet, Fink\})$ is the dominant strategy for both players, regardless of the other player's decision.

b) For the game to be a *Prisoner's Dilemma*,
$u(Quiet, Quiet) > u(Fink, Fink) > u(Quiet, Fink)$ for both players.

$$2 + 2\alpha > 1 + \alpha > 3\alpha$$
$$2 + \alpha > 1 > 2\alpha$$
$$2 + \alpha > 1 \text{ is trivial, because } \alpha \geq 0$$
$$1 > 2\alpha$$
$$\alpha < \frac{1}{2}$$

The game is a *Prisoner's Dilemma*, when $0 \leq \alpha < \frac{1}{2}$

Question 2: Two players (player 1 and player 2) are bargaining over how to split one dollar. Both of them simultaneously name shares they would like to have, s_1 and s_2, where $0 \leq s_1, s_2 \leq 1$. If $s_1 + s_2 \leq 1$, then the players receive the shares they named; if $s_1 + s_2 > 1$, then both players receive zero. What are the pure strategy Nash equilibria of this game?

Solution

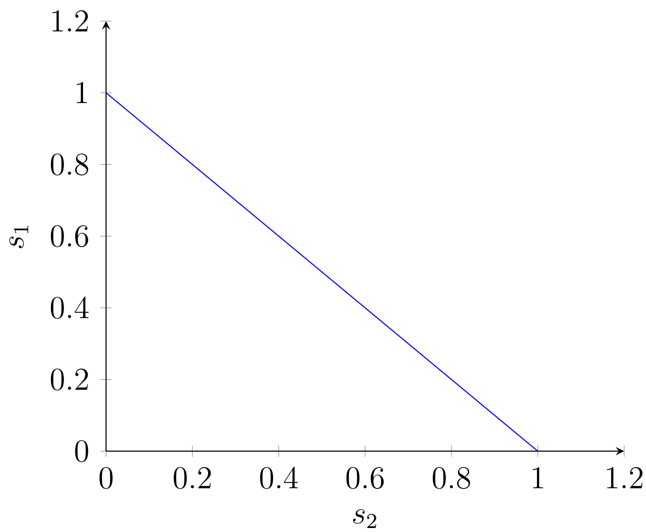
Pay-off Function:

$$u_i(s_i) = \begin{cases} s_i & \text{If } s_1 + s_2 \leq 1 \\ 0 & \text{Otherwise} \end{cases} \qquad \forall i \in \{1, 2\}$$

From the pay-off function, we can get the best response function:

$$BR_i(s_j) = 1 - s_j \quad \forall i, j \in \{1, 2\}$$

Best Response Graph:



Therefore, the Nash equilibria can be found at:

$(s_1, s_2) \ \ \forall s_1, s_2$ where $s_1 + s_2 = 1$

Question 3: Two candidates are competing in a political race. Each candidate i can spend $s_i \geq 0$ on ads which affect the probability of winning the election. Given a strategy profile (s_1, s_2), candidate 1 wins the election with probability

$$\frac{s_1}{s_1 + s_2}$$

and candidate 2 wins with probability

$$\frac{s_2}{s_1 + s_2}.$$

If both candidates spend zero, then each wins with probability one half. The payoff to candidate i is the probability of winning minus the amount spent on ads.

a) Write an expression for the payoff function of candidate 1.

b) Solve for the best response function of each candidate.

c) Find the pure strategy Nash equilibrium.

Solution

a) Pay-off function for Candidate 1:

$$u_1(s_1) = \frac{s_1}{s_1 + s_2} - s_1$$

b) Pay-off function for both Candidates:

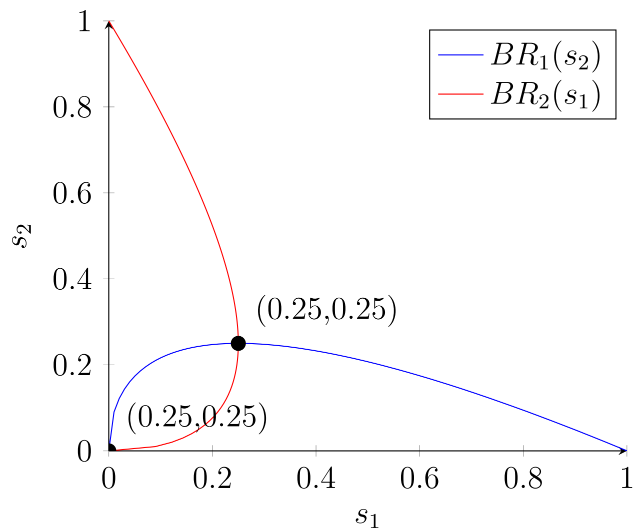
$$\begin{aligned} u_i(s_i) &= \frac{s_i}{s_1 + s_2} - s_i \\ &= \frac{s_i - s_i(s_1 + s_2)}{s_1 + s_2} \qquad \forall i \in \{1, 2\} \end{aligned}$$

Best Response for both Candidates:

$$\begin{aligned} \nabla u_i(s_i) &= \frac{d}{ds_i} \left(\frac{s_i - s_i^2 - s_i s_j}{s_i + s_j} \right) \\ &= \frac{(s_i + s_j)(1 - 2s_i - s_j) - (s_i - s_i^2 - s_i s_j)}{(s_i + s_j)^2} \\ &= \frac{s_i - 2s_i^2 - s_i s_j + s_j - 2s_i s_j - s_j^2 - s_i + s_i^2 + s_i s_j}{s_i^2 + 2s_i s_j + s_j^2} \\ &= \frac{-s_i^2 + s_j - 2s_i s_j - s_j^2}{s_i^2 + 2s_i s_j + s_j^2} \\ &= \frac{s_j}{(s_i + s_j)^2} - 1 \qquad \forall i, j \in \{1, 2\} \end{aligned}$$

$$\begin{aligned} \frac{s_j}{(s_i + s_j)^2} - 1 &= 0 \\ s_j &= (s_i + s_j)^2 \\ \pm\sqrt{s_j} &= s_i + s_j \\ s_i &= -s_j \pm \sqrt{s_j} \\ BR_i(s_j) &= -s_j \pm \sqrt{s_j} \\ &= \sqrt{s_j} - s_j \qquad \because s_i \geq 0 \end{aligned}$$

c) Best Response Graph:



Therefore, the Nash equilibria can be found at:

(0, 0) and (0.25, 0.25)

Question 4: An employer hires an employee and promises him wage w. The employee can work (W) or shirk (S). Working is associated with cost of effort, e, where $w > e > 0$. If the employee works, the employer obtains revenue r and otherwise $\frac{r}{2}$.
Employer cannot tell whether the employee is working or shirking unless she chooses to run inspection (I) at a cost of $c > 0$. Inspection reveals whether the employee is working or not and in the latter case the wage is withheld. The decisions of the two players (W/S and I/NI) are taken simultaneously. The payoff matrix is as follows:

	I	NI
W	w-e, r-c-w	w-e,r-w
S	0,$r/2$-c	w,$r/2$-w

Employee is choosing a row (row player) and the employer is choosing a column (column player).

a) Under what condition(s) will this game have a pure strategy Nash equilibrium?

b) Assume the aforementioned condition(s) is (are) not satisfied. Derive the formulas for probability of working, p and probability of inspection, q in the mixed strategy Nash equilibrium.

Solution

a) The game will have a pure strategy Nash Equilibrium when one or more of the following conditions are met:

(i) (W,I)

$$w - e \geq 0 \text{ // True}$$
$$r - c - w \geq r - w \text{ // Not Possible}$$

(ii) (W,NI)

$$w - e \geq w \text{ // Not Possible}$$
$$r - w \geq r - c - w \text{ // True}$$

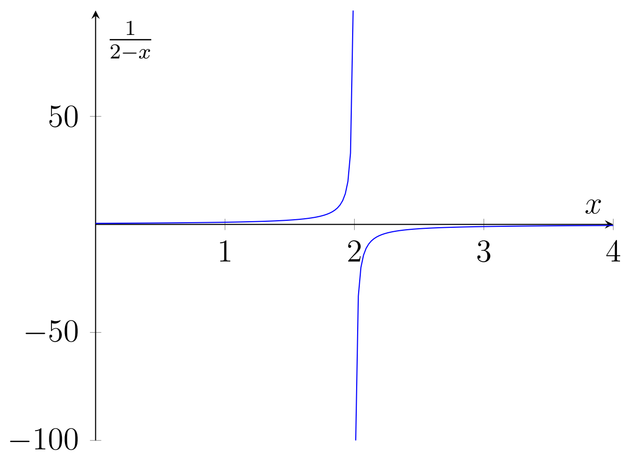
(iii) (S,I)

$$0 \geq w - e \text{ // Not Possible}$$
$$\frac{r}{2-c} \geq \frac{r}{2-w} \text{ // Possible}$$

(iv) (S,NI)

$$w \geq w - e \text{ // True}$$
$$\frac{r}{2-w} \geq \frac{r}{2-c} \text{ // Possible}$$

(S,NI) is the only pure strategy that can be at Nash Equilibrium.
It can only be a Nash Equilibrium when:



$$\begin{aligned}\frac{r}{2-w} &\geq \frac{r}{2-c}\\ 2-w &\leq 2-c\\ w &\geq c\end{aligned}$$

Where $w, c \neq 2$ and both $w, c < 2$ or $w, c > 2$

OR $w \in [0, 2)$ and $c \in (2, \infty)$

b) Expected pay-off for row player given row player's action:

$$\begin{aligned}u_1(W, [q, 1-q]) &= q(w-e) + (1-q)(w-e)\\ u_1(S, [q, 1-q]) &= (1-q)w\\ \text{Solve } u_1(W, [q^*, 1-q^*]) &= u_1(S, [q^*, 1-q^*])\\ q^*(w-e) + (1-q^*)(w-e) &= (1-q^*)w\\ q^*w - q^*e + q^*e - e &= 0\\ q^*w &= e\\ q^* &= \frac{e}{w}\end{aligned}$$

Expected pay-off for column player given column player's action:

$$
\begin{aligned}
u_2(I,[p,1-p]) &= p(r-c-w)+(1-p)(\frac{r}{2-c}) \\
u_2(NI,[p,1-p]) &= p(r-w)+(1-p)(\frac{r}{2-w}) \\
\text{Solve } u_2(I,[p^*,1-p^*]) &= u_2(NI,[p^*,1-p^*]) \\
p^*(r-c-w)+(1-p^*)(\frac{r}{2-c}) &= p^*(r-w)+(1-p^*)(\frac{r}{2-w}) \\
-p^*c &= (1-p^*)\left(\frac{r}{2-w}-\frac{r}{2-c}\right) \\
\frac{p^*}{p^*-1} &= \frac{r}{2c-cw}-\frac{r}{2c-c^2} \\
&= \frac{2rc-rc^2-2rc+rcw}{4c^2-2c^3-2c^2w+c^3w} \\
&= \frac{rw-rc}{4c-2c^2-2cw+c^2w} \\
1-p^* &= \frac{4c-2c^2-2cw+c^2w}{rw-rc} \\
p^* &= 1-\frac{c}{r}\frac{4-2c-2w+cw}{w-c} \\
&= 1-\frac{c}{r}\frac{(2-c)(2-w)}{w-c}
\end{aligned}
$$

Question 5: Consider the Cournot's game with n number of firms. Assume that the inverse demand function is given by:

$$P(Q) = \max\{\alpha - Q, 0\},$$

where $\alpha > 0$. The cost function of each firm i is $C(q_i) = cq_i$ for all q_i, with $0 \leq c \leq \alpha$. Assuming that all firms produce the same output in Nash equilibrium, solve for the equilibrium output for each firm. Find the market price at which output is sold. Does the equilibrium price increase or decrease as n increases?

Solution
Assuming that all firms produce the same output in Nash Equilibrium.
In Nash Equilibrium,

$$
\begin{aligned}
&Q = nq \text{ where } q = q_i \quad \forall i \in [1,n] \\
&u_i = q \times \max\{\alpha - nq, 0\} - cq
\end{aligned}
$$

$$
u_i = \begin{cases} q(\alpha - nq) - cq & \text{If } \alpha - nq > 0 \\ -cq & \text{Otherwise} \end{cases}
$$

Equilibrium output for all firms:

$$\begin{aligned}
\frac{du_i}{dq} &= \frac{d}{dq}(q(\alpha - nq) - cq) \\
&= \alpha - 2nq - c \\
\text{Solve for } q^* \text{ when } \frac{du_i}{dq} &= 0 \\
0 &= \alpha - 2nq^* - c \\
2nq^* &= \alpha - c \\
q^* &= \frac{\alpha - c}{2n}
\end{aligned}$$

Market price at Nash Equilibrium:

$$\begin{aligned}
P &= \max\{\alpha - nq^*, 0\} \\
&= \max\{\alpha - n(\frac{\alpha - c}{2n}), 0\} \\
&= \max\{\alpha - \frac{\alpha - c}{2}, 0\}
\end{aligned}$$

As n increases, there is no change to the equilibrium price.
But the optimal production quantity q^* decreases as n increases.

Question 6: Two people can perform a task if, and only if, they both exert effort. They are both better off if they both exert effort and perform the task than if neither exerts effort (and nothing is accomplished); the worst outcome for each person is that she exerts effort and the other person does not (in which case again nothing is accomplished). Specifically, the players' preferences are represented by the expected value of the payoff functions in the following figure (where c is a positive number less than 1 that can be interpreted as the cost of exerting effort):

	No effort	Effort
No effort	$0, 0$	$0, -c$
Effort	$-c, 0$	$1-c, 1-c$

Find all the mixed strategy Nash equilibria of the game. How do the equilibria change as c increases?

Solution
Expected pay-off for row player given row player's action:
Let q be the probability of column player picking *No Effort*.

$$\begin{aligned} u_1(\textit{No Effort}, [q, 1-q]) &= 0 \\ u_1(\textit{Effort}, [q, 1-q]) &= -cq + (1-c)(1-q) \\ \text{Solve } u_1(\textit{No Effort}, [q^*, 1-q^*]) &= u_1(\textit{Effort}, [q^*, 1-q^*]) \\ 0 &= -cq^* + (1-c)(1-q^*) \\ cq^* &= 1 - q^* - c + cq^* \\ q^* &= 1 - c \end{aligned}$$

Expected pay-off for column player given column player's action:
Let p be the probability of the row player picking *No Effort*.

$$p^* = 1 - c$$

$$\because \text{ pay-off matrix is symmetric.}$$

Nash Equilibrium for the game: $[(1-c,c),(1-c,c)]$ for $c \in (0,1)$

When $c \in \{0, 1\}$, there are two pure strategy Nash Equilibrium, (*No Effort*,*No Effort*) and (*Effort*,*Effort*).

When $c \in (0,1)$, the mix strategy Nash Equilibria exists and tends towards (*Effort*,*Effort*) as c increases.