Problem Set 2

Basil R. Yap

40.316 Game Theory - Term 8

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Question 1: [Declining Industry.] Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Every quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter, then the payoffs are (-1, 1) because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

(a) Write down this game in matrix form

(b) Is there any pure strategy that is dominated by some mixed strategy? Why?

(c) Find the pure strategy Nash equilibria.

(d) Find the unique mixed strategy Nash equilibrium (hint: you can use your answer to (b) to make things easier).

Solution

(a)

$$\begin{aligned} A:&\quad \text{Exit Immediately} \\ B:&\quad \text{Exit at the end of this quarter} \\ C:&\quad \text{Exit at the end of next quarter} \end{aligned}$$

	A	B	C
A	(0,0)	(0,2)	(0,4)
B	(2,0)	(-1,-1)	(-1,1)
C	(4,0)	(1,-1)	(-2,-2)

(b) Pure Strategy B is weakly dominated by a mixed strategy of A and C. This can be confirmed by comparing the expected pay-off, such that:

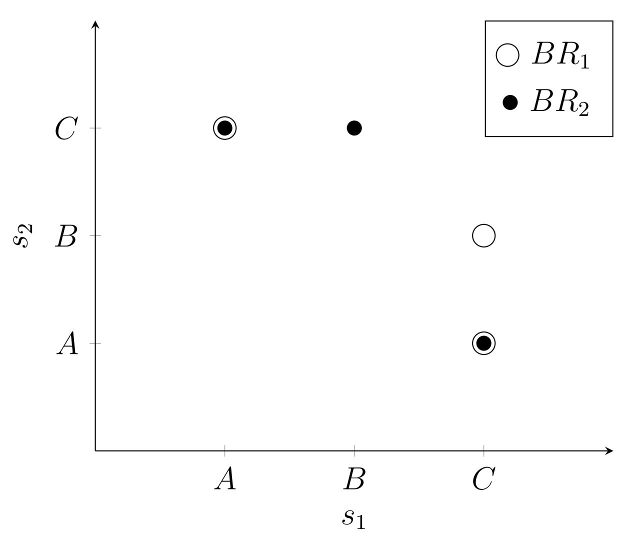
$$u_i([\frac{1}{2}, 0, \frac{1}{2}], s) \geq u_i(B, s) \quad \forall i \in \{1, 2\}\ \forall s \in \{A, B, C\}$$

$$\begin{aligned}
u_1([\frac{1}{2}, 0, \frac{1}{2}], A) &\geq u_1(B, A) \\
\frac{1}{2}(0) + \frac{1}{2}(4) &\geq 2 \\
2 &\geq 2 \text{ // True} \\
u_1([\frac{1}{2}, 0, \frac{1}{2}], B) &\geq u_1(B, B) \\
\frac{1}{2}(0) + \frac{1}{2}(1) &\geq -1 \\
\frac{1}{2} &\geq -1 \text{ // True} \\
u_1([\frac{1}{2}, 0, \frac{1}{2}], C) &\geq u_1(B, C) \\
\frac{1}{2}(0) + \frac{1}{2}(-2) &\geq -1 \\
-1 &\geq -1 \text{ // True}
\end{aligned}$$

Since the game is symmetrical, same applies for Column Player.

(c) The Best Response for Row/Column Player:

$$\begin{aligned}
BR_i(A) &= C \\
BR_i(B) &= C \\
BR_i(C) &= A
\end{aligned}$$



$\therefore$ Pure Strategy Nash Equilibria at (A, C) and (C, A)

(d) Based on the result in (b), we only consider the strategies A and C.

p : Probability row player chooses A over C
q : Probability column player chooses A over C

Expected pay-off for row player given row player's action:

$$\begin{aligned} u_1(A, [q, 1-q]) &= 0 \\ u_1(C, [q, 1-q]) &= 4q - 2(1-q) \\ \text{Solve } u_1(A, [q^*, 1-q^*]) &= u_1(C, [q^*, 1-q^*]) \\ 4q^* &= 2(1-q^*) \\ 6q^* &= 2 \\ q^* &= \frac{1}{3} \end{aligned}$$

Expected pay-off for column player given column player's action:

The pay-off matrix is symmetrical.
$\therefore p^* = \frac{1}{3}$

$\therefore$ Mixed Strategy Nash Equilibrium at $[(\frac{1}{3}, 0, \frac{2}{3}), (\frac{1}{3}, 0, \frac{2}{3})]$

Question 2: Consider the zero-sum game known as the "matching-pennies" game. Here and elsewhere we refer to the row player as player 1 and the column player as player 2.

		2: L	R
1	T	$1, -1$	$-1, 1$
	B	$-1, 1$	$1, -1$

Table 1: Matching-pennies

This game has no equilibrium in pure strategies. It has a unique equilibrium in mixed strategies, where each player randomizes with equal probability yielding the equilibrium payoffs (0, 0). Can you design a correlated equilibrium to improve both players' payoffs? Hint: the mediator can design an event such that players assess with distinct subjective probabilities.

Solution

Consider the following states:

State 1: where State 1 occurs when Event D is True,
Tell Player 1 to play T and Player 2 to play L.

State 2: where State 2 occurs when Event D is False,
Tell Player 1 to play B and Player 2 to play L.

Where D has subjective probabilities,

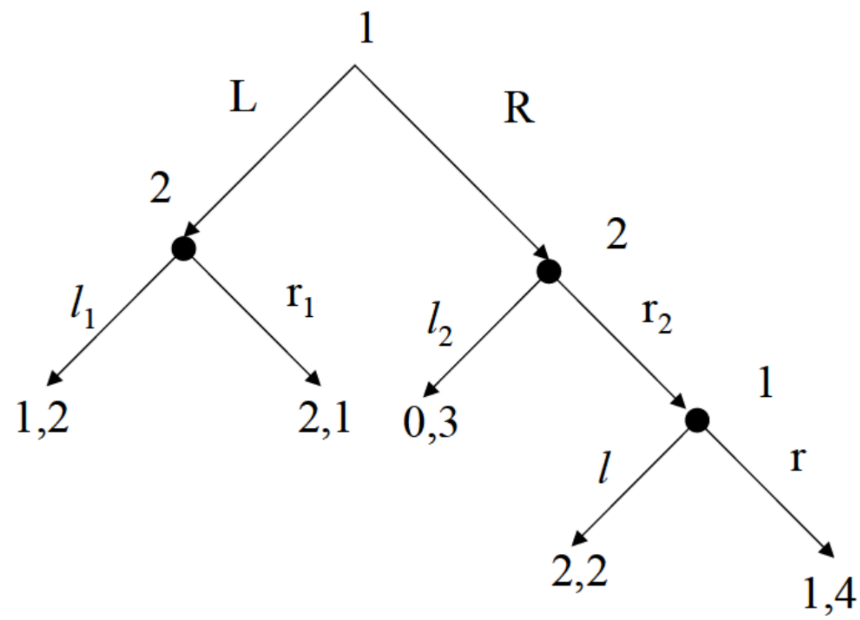
$$D = \begin{cases} \text{True} & \text{Where } \Pr(D; \text{Player 1}) = p \text{ and } \Pr(D; \text{Player 2}) = 1 - p \\ \text{False} & \text{Where } \Pr(D; \text{Player 1}) = 1 - p \text{ and } \Pr(D; \text{Player 2}) = p \end{cases}$$

The expected pay-off of both players are given by:

$$\begin{aligned} u_1(T| \text{ State 1}) &= p(1) + (1-p)(-1) \\ &= 2p - 1 \\ u_2(L| \text{ State 2}) &= p(1) + (1-p)(-1) \\ &= 2p - 1 \end{aligned}$$

When $0.5 \leq p < 1$, the expected pay-off of both players are improved while the value of the game remains at 0. As the expected pay-off for both player has improved, neither will have incentive to deviate. Therefore, the solution is a correlated equilibrium.

Question 3: Consider the following game in extensive form:

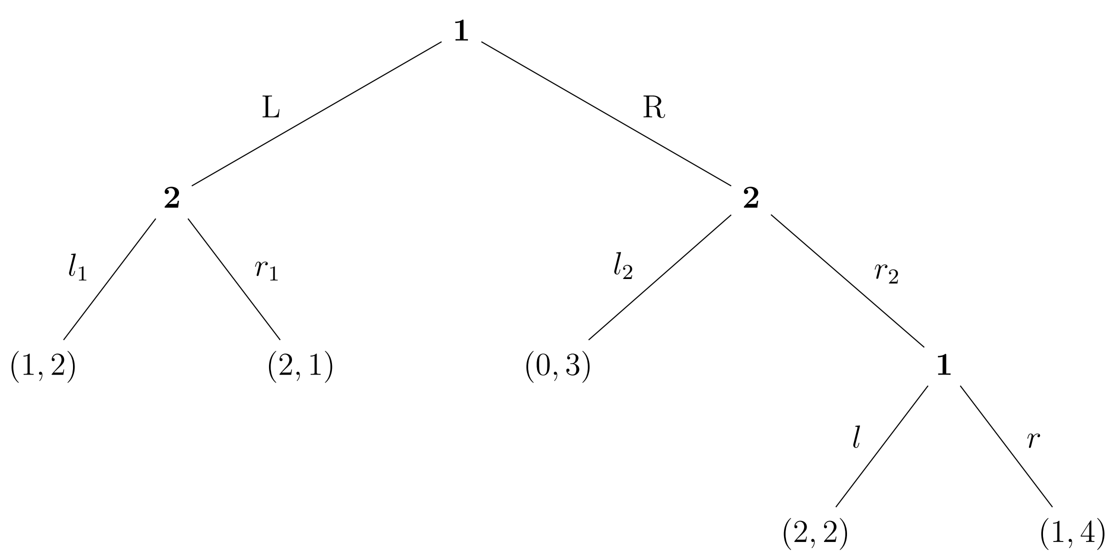


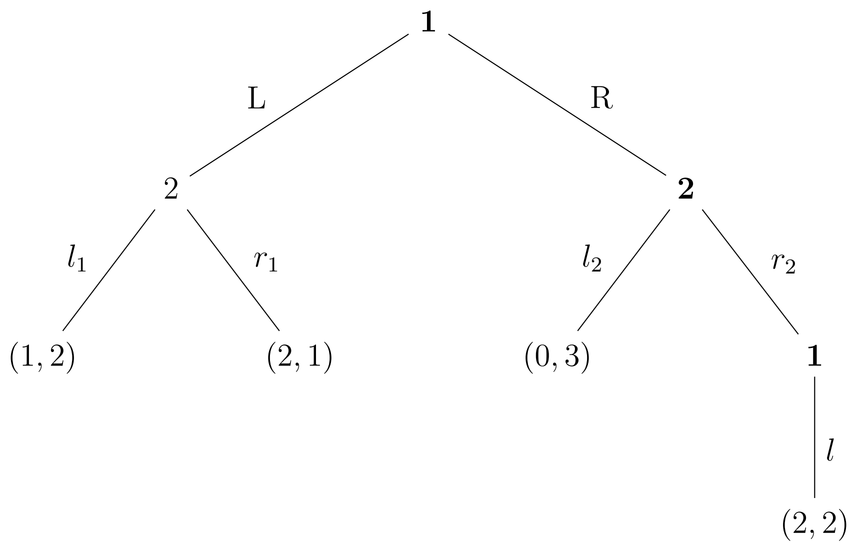
(a) Apply backwards induction in this game and find the subgame perfect equilibrium of the game

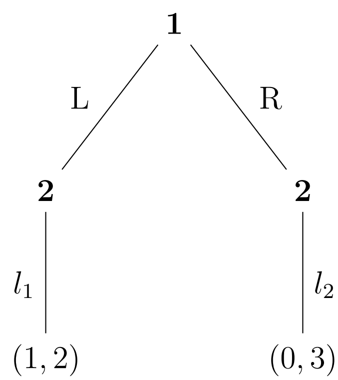
(b) Write the game in normal-form and find the set of pure strategy Nash equilibria.

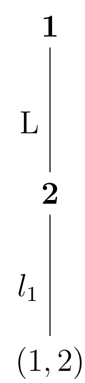
Solution

(a) Using backwards induction:









$\therefore$ The subgame perfect equilibrium is $(Ll, l_1 l_2)$.

(b) Normal-form pay-off matrix:

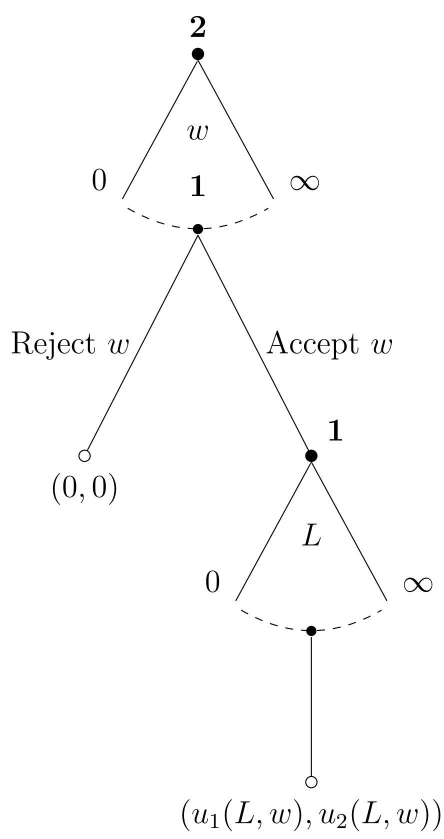
	$l_1 l_2$	$l_1 r_2$	$r_1 l_2$	$r_1 r_2$
Ll	$(\mathbf{1}, \mathbf{2})$	$(1, 2)$	$(2, 1)$	$(2, 1)$
Lr	$(\mathbf{1}, \mathbf{2})$	$(1, 2)$	$(2, 1)$	$(2, 1)$
Rl	$(0, 3)$	$(2, 2)$	$(0, 3)$	$(2, 2)$
Rr	$(0, 3)$	$(1, 4)$	$(0, 3)$	$(1, 4)$

$$\therefore \text{ the pure strategy equilibria is } (Ll, l_1 l_2) \text{ and } (Lr, l_1 l_2)$$

Question 4: A firm's output is $L(100 - L)$ when it uses $L \leq 50$ units of labour, and 2500 when it uses $L > 50$ units of labour. The price of output is 1. A union that represents workers presents a wage demand (a nonnegative number w), which the firm either accepts or rejects. If the firm accepts the demand, it chooses the number L of workers to employ (which you should take to be a continuous variable, not an integer); if it rejects the demand, no production takes place ($L = 0$). The firm's preferences are represented by its profit; the union's preferences are represented by the value of wL. Formulate this situation as an extensive form game and find the subgame perfect equilibria of the game. Is there an outcome of the game that both parties prefer to any subgame perfect equilibrium outcome?

Solution

- Players: Firm (*Player 1*), Union (*Player 2*)
- Extensive form tree:

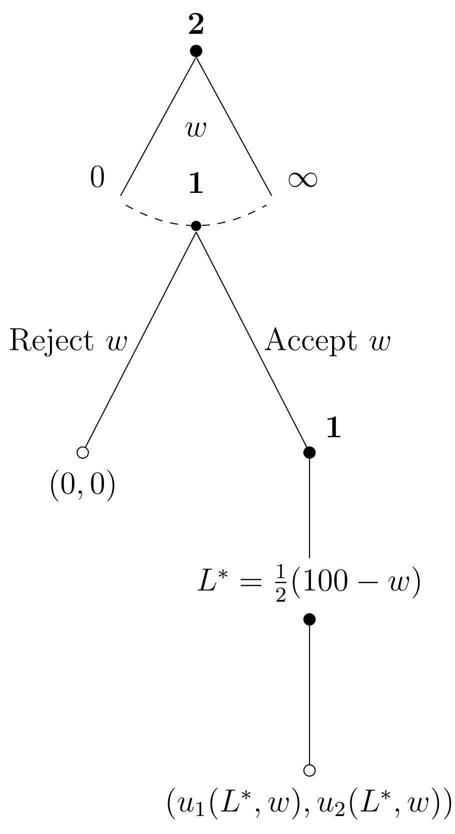


$$u_1(L, w) = \begin{cases} L(100 - L) - wL & L \leq 50 \\ 2500 - wL & L > 50 \end{cases}$$

$$u_2(L, w) = wL$$

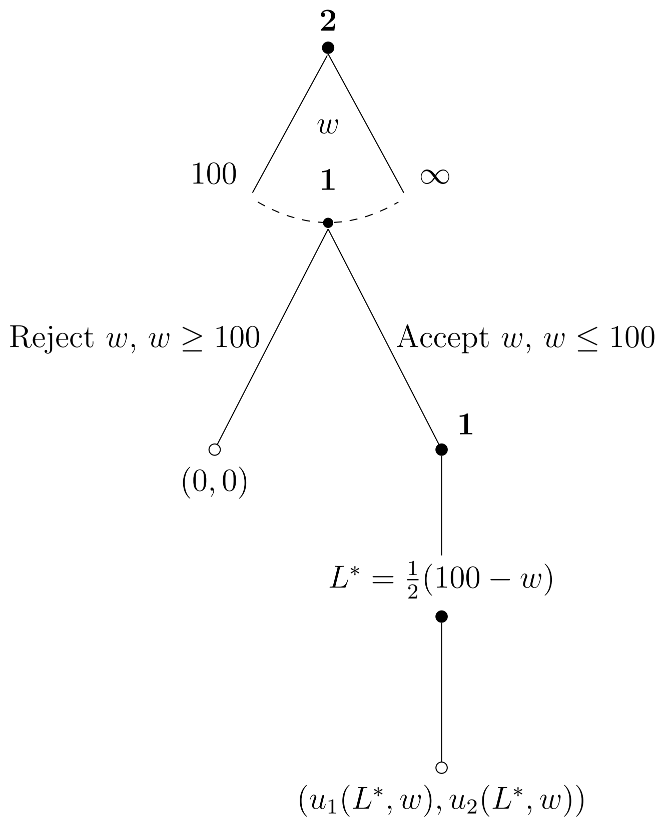
Row player does have increased pay-off when $L \leq 50$, $\therefore$ consider u_1 only when $L \leq 50$

$$\begin{aligned} u_1(L, w) &= L(100 - L) - wL \\ &= 100L - L^2 - wL \\ \nabla u_1(L, w) &= \frac{d}{dL}(100L - L^2 - wL) \\ &= 100 - 2L - w \\ \text{Solve } \nabla u_1(L^*, w) &= 0 \\ 100 - 2L^* - w &= 0 \\ 2L^* &= 100 - w \\ L^* &= \frac{1}{2}(100 - w) \end{aligned}$$



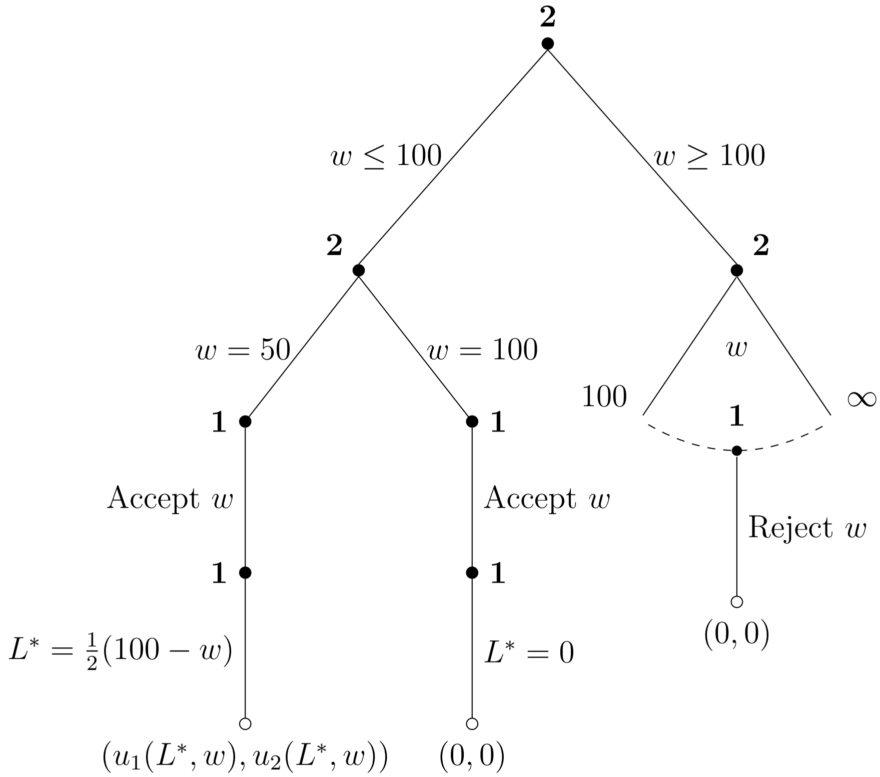
Consider the cases where $u_1(L^*, w) < 0$ and $u_1(L^*, w) > 0$,
When $u_1(L^*, w) \leq 0$, *Player 1* would reject the w proposed by *Player 2*.
When $u_1(L^*, w) \geq 0$, *Player 1* would accept the w proposed by *Player 2*.

$$
\begin{aligned}
\text{Solve } u_1(L^*, w) &= 0 \\
100(\frac{1}{2}(100-w)) - (\frac{1}{2}(100-w))^2 - w(\frac{1}{2}(100-w)) &= 0 \\
200(100-w) - (100-w)^2 - 2w(100-w) &= 0 \\
20000 - 200w - 10000 + 200w - w^2 - 200w + 2w^2 &= 0 \\
w^2 - 200w + 10000 &= 0 \\
(w-100)^2 &= 0 \\
w &= 0
\end{aligned}
$$



Consider the case when $w \leq 100$ and $L = L^*$

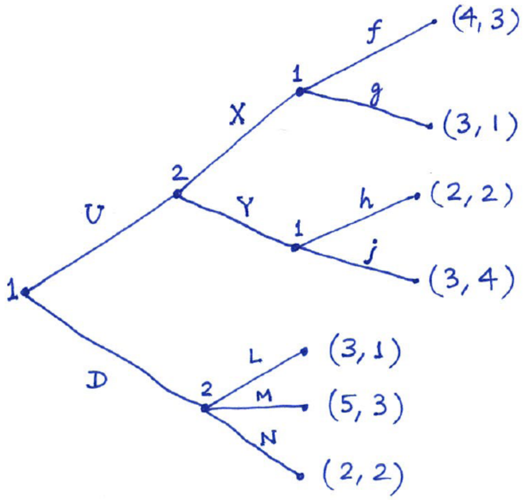
$$
\begin{aligned}
u_2(L^*, w) &= \frac{1}{2}w(100-w) \\
\text{Solve } \nabla u_2(L^*, w*) &= 0 \\
\frac{d}{dw^*}(\frac{1}{2}w^*(100 - w*)) &= 0 \\
50 - w* &= 0 \\
w^* &= 50
\end{aligned}
$$



The subgame perfect equilibria are
$(Accept/L = 25, w = 50)$, $(Accept/L = 0, w = 100)$
and $(Reject/L = 0, w \in [100, \infty])$

There is one subgame perfect equilibrium where both players benefit, that is at
$(Accept/L = 25, w = 50)$.
The expected pay-off would be $(625, 1250)$.

Question 5: Consider the following game in extensive form:



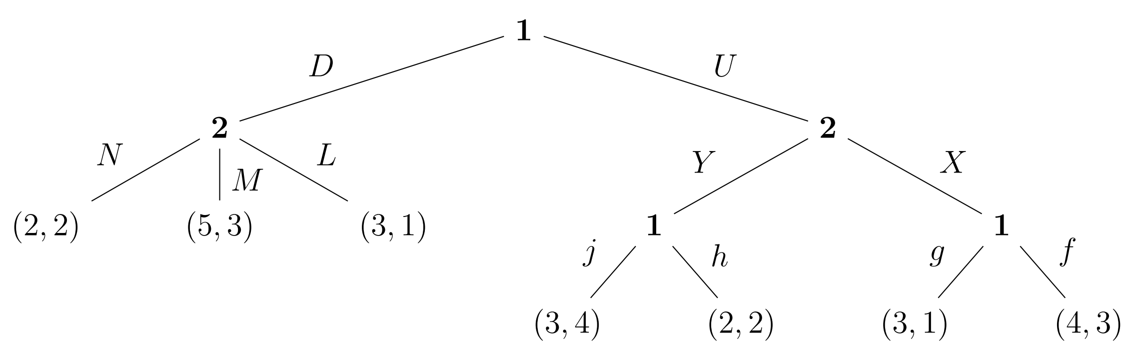
The nodes are labeled with a 1 or a 2 depending on which player moves at that decision node.

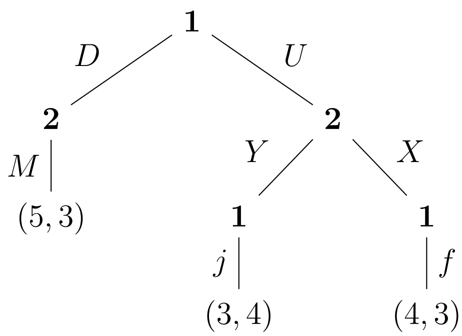
(a) Find a subgame perfect equilibrium (in pure strategies).

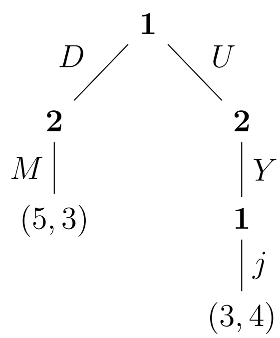
(b) Is there a Nash equilibrium for the game that is not SPE? If yes, write down one such equilibrium.

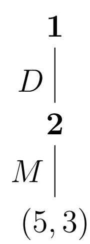
Solution

(a) Using backwards induction:









The subgame perfect equilibrium exists at (Djf, MY)

(b) Normal-form pay-off matrix:

	NY	NX	MY	MX	LY	LX
Djg	$(2,2)$	$(2,2)$	$(\mathbf{5},\mathbf{3})$	$(\mathbf{5},\mathbf{3})$	$(3,1)$	$(3,1)$
Djf	$(2,2)$	$(2,2)$	$(\mathbf{5},\mathbf{3})$	$(\mathbf{5},\mathbf{3})$	$(3,1)$	$(3,1)$
Dhg	$(2,2)$	$(2,2)$	$(\mathbf{5},\mathbf{3})$	$(\mathbf{5},\mathbf{3})$	$(3,1)$	$(3,1)$
Dhf	$(2,2)$	$(2,2)$	$(\mathbf{5},\mathbf{3})$	$(\mathbf{5},\mathbf{3})$	$(3,1)$	$(3,1)$
Ujg	$(\mathbf{3},\mathbf{4})$	$(3,1)$	$(3,4)$	$(3,1)$	$(\mathbf{3},\mathbf{4})$	$(3,1)$
Ujf	$(\mathbf{3},\mathbf{4})$	$(4,2)$	$(3,4)$	$(4,2)$	$(\mathbf{3},\mathbf{4})$	$(4,2)$
Uhg	$(2,2)$	$(3,1)$	$(2,2)$	$(3,1)$	$(2,2)$	$(3,1)$
Uhf	$(2,2)$	$(\mathbf{4},\mathbf{2})$	$(2,2)$	$(4,2)$	$(2,2)$	$(\mathbf{4},\mathbf{2})$

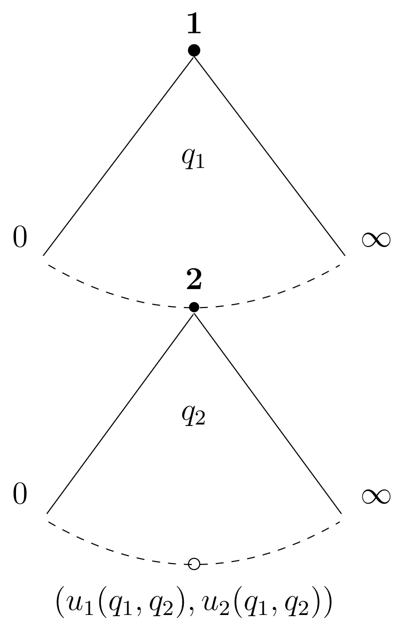
An example of a pure strategy Nash Equilibrium
which is not a sub-game perfect equilibrium is (Ujg, NY).

Question 6: Consider the Stackelberg game of sequential quantity choice where firm 1 moves first and firm 2 is the follower. After firm 1 sets a quantity, firm 2 chooses it's quantity after observing firm 1's choice. The inverse demand function in the market is given by:

$$p = 1000 - 2q_1 - 2q_2$$

Each firm incurs a cost of 200 for each unit of output it produces. Find the subgame perfect equilibrium of the game. What is the market price and profit for each firm in the SPE?

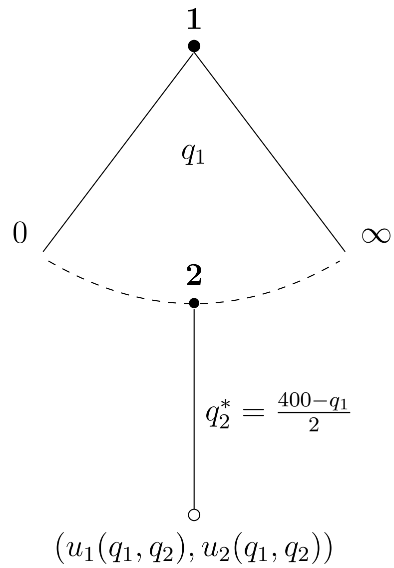
Solution



$$\begin{aligned} u_1(q_1, q_2) &= q_1(1000 - 2q_1 - 2q_2) - 200q_1 \\ u_2(q_1, q_2) &= q_2(1000 - 2q_1 - 2q_2) - 200q_2 \end{aligned}$$

$$\begin{aligned} \nabla u_2(q_1, q_2) &= \frac{d}{dq_2}\left(q_2(1000 - 2q_1 - 2q_2) - 200q_2\right) \\ &= 1000 - 2q_1 - 4q_2 - 200 \\ &= 800 - 2q_1 - 4q_2 \end{aligned}$$

$$
\begin{aligned}
\text{Solve } \nabla u_2(q_1, q_2^*) &= 0 \\
800 - 2q_1 - 4q_2^* &= 0 \\
4q_2^* &= 800 - 2q_1 \\
q_2^* &= \frac{400 - q_1}{2}
\end{aligned}
$$

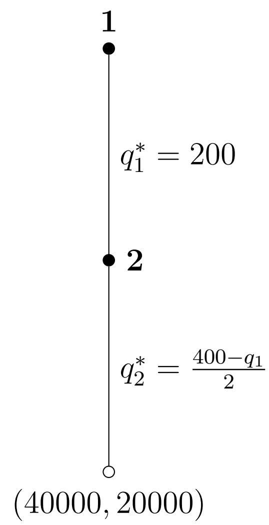


$$
\begin{aligned}
u_1(q_1, q_2^*) &= q_1 \left(1000 - 2q_1 - 2\left(\frac{400 - q_1}{2}\right)\right) - 200q_1 \\
&= 1000q_1 - 2q_1^2 - 400q_1 + q_1^2 - 200q_1 \\
&= 400q_1 - q_1^2
\end{aligned}
$$

$$
\begin{aligned}
\nabla u_1(q_1, q_2^*) &= \frac{d}{dq_1}(400q_1 - q_1^2) \\
&= 400 - 2q_1
\end{aligned}
$$

$$
\begin{aligned}
\text{Solve } \nabla u_1(q_1^*, q_2^*) &= 0 \\
400 - 2q_1^* &= 0 \\
2q_1^* &= 400 \\
q_1^* &= 200
\end{aligned}
$$

$$
\begin{aligned}
q_2^* &= 100 \\
u_1(200, 100) &= 40000 \\
u_2(200, 100) &= 20000
\end{aligned}
$$



$$
\begin{aligned}
p &= 1000 - 2q_1 - 2q_2 \\
&= 1000 - 2(200) - 2(100) \\
&= 400
\end{aligned}
$$

The sub-game perfect equilibrium is at $(q_1 = 200, q_2 = 100)$.
The market price for both firms is 400.
The pay-off of each firm is $(40000, 20000)$.