5. 设H是Hilbert空间，(A_n)是H中递减的非空闭子集列. 任取$x\in H$，令$d_n(x)=d(x,A_n)$
且$d(x)=\lim\limits_{n\to\infty} d_n(x)$.

(a)证明：若对某一个$x\in H$，有$d(x)<\infty$，则对所有的$x\in H$，$d(x)<\infty$. 我们在下面假设该命题成立，并且$A(x,\varepsilon,n)$表示中心在x，半径为$d(x)+\varepsilon$的闭球与A_n的交集，即

$$A(x,\varepsilon,n)=A_n\cap\overline{B}(x,d(x)+\varepsilon).$$

(b)证明

$$\lim_{\varepsilon\to0,n\to\infty}\operatorname{diam}(A(x,\varepsilon,n))=0$$

(c)证明所有A_n的交集A非空且$d(x)=d(x,A)$.

证明：(a) 设$x\in H$. s.t. $d(x)<\infty$ 下证对$\forall y\in H$，$d(y)<\infty$.

考虑子集A_n, $n\ge1$. 对$\forall z\in A_n$，由于$d(y,z)\le d(y,x)+d(x,z)\Rightarrow$

$$\inf_{z\in A_n} d(y,z)\le d(y,x)+\inf_{z\in A_n}d(x,z)$$

即 $d_n(y)\le d(y,x)+d_n(x)\le d(y,x)+d(x)$，故 $\lim\limits_n d_n(y)<\infty$.

(b) 设$y,z\in A_n\cap\overline{B}(x,d(x)+\varepsilon)$. 则

$$d_n(x)\le d(x,y)\le d(x)+\varepsilon$$
$$d_n(x)\le d(x,z)\le d(x)+\varepsilon$$

且由于A_n为凸集知$\frac{y+z}{2}\in A_n$. 则

$$\|y-z\|^2=2(\|x-y\|^2+\|x-z\|^2)-\|2(x-\tfrac{y+z}{2})\|^2 \text{（平行四边形法则）}$$
$$\le 2((d(x)+\varepsilon)^2+(d(x)+\varepsilon)^2)-4\|x-\tfrac{y+z}{2}\|^2$$
$$\le 4(d(x)+\varepsilon)^2-4d_n(x)$$
$$\le 4(d(x)+\varepsilon)^2-4(d(x)-\varepsilon)^2 \text{（当n充分大）}$$
$$=16d(x)\cdot\varepsilon$$

故 $\operatorname{diam}(A(x,\varepsilon,n))\le16d(x)\varepsilon$（当n充分大）$\Rightarrow\lim\limits_{\varepsilon\to0,n\to\infty}\operatorname{diam}(A(x,\varepsilon,n))=0$.

(c) 由(b)的证明过程，取$\varepsilon=\frac1n$知 $\operatorname{diam}(A(x,\frac1n,n))\le16d(x)\cdot\frac1n$ 故

$$\lim_{n\to\infty}\operatorname{diam}(A(x,\tfrac1n,n))=0$$

令$B_n=A(x,\frac1n,n)$，则(B_n)是H中递减的非空闭子集列且$\lim\limits_n \operatorname{diam}(B_n)=0$. 则由定理2.2.6知$\bigcap\limits_{n\ge1}B_n$是单点集，设$\bigcap\limits_{n\ge1}B_n=\{x_0\}$. 则由$\bigcap\limits_{n\ge1}B_n\subset\bigcap\limits_{n\ge1}A_n=A$知$A$非空且$d(x,x_0)\ge d(x,A)$.
故为证明$d(x)\ge d(x,A)$只需证$d(x)\ge d(x,x_0)$.

对$\forall n\ge1$，取$y_n\in A_n$ s.t. $d_n(x)-\frac1n>d(x,y_n)$ 则 $d(x)-\frac1n\ge d_n(x)-\frac1n>d(x,y_n)$ (*)

设$1\le n<m$，由于$y_m\in B_m\subset B_n$ 则 $\limsup\limits_{n\to\infty,m\to\infty}d(y_n,y_m)\le\lim\limits_{n\to\infty}\operatorname{diam}(A_n)=0$