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$$\mathrm{Re}\left(\sum_{n=0}^{N-1} e^{int}\right)=\mathrm{Re}\left(\frac{1-e^{iNt}}{1-e^{it}}\right)=\frac{1-\cos t-\cos Nt+\cos(N-1)t}{2-2\cos t}$$

则 $(*)=\dfrac{\cos\frac{t}{2}(\sin t+\sin(N-1)t-\sin Nt)+\sin\frac{t}{2}(1-\cos t-\cos Nt+\cos(N-1)t)}{2-2\cos t}$

$$=\frac{\sin(t-\frac{t}{2})+\sin(N-1+\frac{1}{2})t-\sin(N+\frac{1}{2})t+\sin\frac{t}{2}}{2-2\cos t}$$

$$=\frac{2\sin\frac{t}{2}+\sin(N-\frac{1}{2})t-\sin(N+\frac{1}{2})t}{2-2\cos t}=\frac{2\sin\frac{t}{2}-2\cos Nt\sin\frac{t}{2}}{2-2\cos t}=\frac{\sin\frac{t}{2}(1-\cos Nt)}{1-\cos t}$$

$$=\frac{\sin\frac{t}{2}\cdot 2\sin^2\frac{Nt}{2}}{2\sin^2\frac{t}{2}}$$

$$=\frac{\sin^2\frac{Nt}{2}}{\sin\frac{t}{2}}$$

$(*)$得证. 因此 $F_N(t)=\frac{1}{N}\left(\dfrac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2$.

(ii) 由(i)知 $F_N(t)\geqslant 0$, 故 $\|F_N\|_1=\int_0^{2\pi}F_N(t)\frac{dt}{2\pi}$, 又对 $\forall n\in\mathbb{N}$. $\int_0^{2\pi}D_n(t)\frac{dt}{2\pi}=\int_0^{2\pi}1+2\cos t+\cdots+2\cos nt\,\frac{dt}{2\pi}$

$=1$

因此 $\int_0^{2\pi}F_N(t)\frac{dt}{2\pi}=\frac{1}{N}\int_0^{2\pi}\sum_{n=0}^{N-1}D_n(t)\frac{dt}{2\pi}=\frac{1}{N}\cdot N=1.$

(iii) 任取 $\delta>0$, $\int_{\delta\leq|t|\leq\pi}F_N(t)\frac{dt}{2\pi}=\frac{2}{N}\int_{\delta\leq t\leq\pi}\left(\dfrac{\sin\frac{Nt}{2}}{\sin\frac{t}{2}}\right)^2\frac{dt}{2\pi}$ $\cancel{\leq 2\int_{\delta\leq t\leq\pi}\frac{1}{(\sin\frac{t}{2})^2}\cdot\sin\frac{Nt}{2}}$

$$\leq\frac{2}{N}\int_{\delta\leq t\leq\pi}\left(\frac{1}{\sin\frac{t}{2}}\right)^2\frac{dt}{2\pi}\to 0\quad(N\to\infty).$$

(b)

Lemma: 设 $p,q,r\geqslant 1$ 且 $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. 若 $f\in L^p(\mathbb{R}), g\in L^q(\mathbb{R})$. 则 f 与 g 的卷积 $f*g$ 满足

$$\|f*g\|_r\leq\|f\|_p\cdot\|g\|_q.$$

证明: $(f*g)(x)=\int f(x-y)g(y)\,dy$,

$$|(f*g)(x)|\leq\int|f(x-y)|\cdot|g(y)|\,dy=\int|f(x-y)|^{1-\frac{p}{r}+\frac{p}{r}}\cdot|g(y)|^{1+\frac{q}{r}-\frac{q}{r}}dy$$

$$=\int|f(x-y)|^{\frac{p}{r}}\cdot|g(y)|^{\frac{q}{r}}\,|f(x-y)|^{1-\frac{p}{r}}\,|g(y)|^{1-\frac{q}{r}}dy$$

$$=\int(|f(x-y)|^p\cdot|g(y)|^q)^{\frac{1}{r}}\cdot|f(x-y)|^{\frac{r-p}{r}}\cdot|g(y)|^{\frac{r-q}{r}}dy$$

$$\leq\underbrace{\|(|f(x-y)|^p\cdot|g(y)|^q)^{\frac{1}{r}}\|_r}_{I_1}\cdot\underbrace{\||f(x-y)|^{\frac{r-p}{r}}\|_{\frac{pr}{r-p}}}_{I_2}\cdot\underbrace{\||g(y)|^{\frac{r-q}{r}}\|_{\frac{qr}{r-q}}}_{I_3}$$

其中最后一个不等式是由于 $\frac{1}{r}+\frac{r-p}{pr}+\frac{r-q}{qr}=\frac{1}{r}+\frac{1}{p}-\frac{1}{r}+\frac{1}{q}-\frac{1}{r}=1$, 用3个函数的Hölder不等式得的。

其中 I_1: $\|(|f(x-y)|^p\cdot|g(y)|^q)^{\frac{1}{r}}\|_r=\left(\int(|f(x-y)|^p\cdot|g(y)|^q)^{\frac{1}{r}\cdot r}dy\right)^{\frac{1}{r}}=\left(\int|f(x-y)|^p\cdot|g(y)|^q\,dy\right)^{\frac{1}{r}}$

I_2: $\||f(x-y)|^{\frac{r-p}{r}}\|_{\frac{pr}{r-p}}=\left(\int|f(x-y)|^{\frac{r-p}{r}\cdot\frac{pr}{r-p}}dy\right)^{\frac{r-p}{pr}}=\left(\int|f(x-y)|^p\,dy\right)^{\frac{1}{p}\cdot\frac{r-p}{r}}=\|f\|_p^{\frac{r-p}{r}}$

I_3: $\||g(y)|^{\frac{r-q}{r}}\|_{\frac{qr}{r-q}}=\left(\int|g(y)|^{\frac{r-q}{r}\cdot\frac{qr}{r-q}}dy\right)^{\frac{r-q}{qr}}=\cancel{\left(\int|g(y)|^{\frac{r-q}{r}\cdot\frac{qr}{r-q}}dy\right)^{\frac{r-q}{qr}}}=\left(\int|g(y)|^q dy\right)^{\frac{1}{q}\cdot\frac{r-q}{r}}$

$=\|g\|_q^{\frac{r-q}{r}}$

$\boxed{3}$