㉑ 证明：(a) $1\le p<\infty$ 时，$L_p^*(0,2\pi)=L_q(0,2\pi)\subset L_1(0,2\pi)$（有限测度空间）.

对 $\forall f\in L_q(0,2\pi)$，$\int_0^{2\pi}e_n f\,dt=\overline{\int_0^{2\pi}e^{-int}\overline{f(t)}\,dt}=2\pi\hat{f}(n)\to 0$

（定理5.2.13），故 $e_n\xrightarrow{w}0$.

但 $\|e_n\|=\int_0^{2\pi}|e_n(t)|^2dt=2\pi\not\to 0$，故 e_n 不依范数收敛到0.

(b) $L_2^*(0,2\pi)=L_2(0,2\pi)$.

$\forall f\in L_2(0,2\pi)$，$\left|\int_0^{2\pi}f(t)f_n(t)dt\right|=\left|\frac{1}{n}\sum_{k=1}^{n^2}\int_0^{2\pi}f(t)e_k(t)dt\right|$

$$=\left|\frac{1}{n}\sum_{k=1}^{N}\overline{\int_0^{2\pi}\overline{f(t)}e^{-ikt}dt}+\frac{1}{n}\sum_{k=N+1}^{n^2}\overline{\int_0^{2\pi}\overline{f(t)}e^{-ikt}dt}\right|$$

$$=\left|\frac{1}{n}\sum_{k=1}^{N}\hat{f}(k)+\frac{1}{n}\sum_{k=N+1}^{n^2}\hat{f}(k)\right|\le\frac{1}{n}\sum_{k=1}^{N}|\hat{f}(k)|+\frac{1}{n}\sum_{k=N+1}^{n^2}|\hat{f}(k)|$$

$\forall\varepsilon>0$，取N使得 $\left(\sum_{k=N+1}^{\infty}|\hat{f}(k)|^2\right)^{\frac12}<\frac{\varepsilon}{2}$（由Parseval恒等式这是可以做到的），$\frac{1}{n}\sum_{k=N+1}^{n^2}|\hat{f}(k)|\le\frac{1}{n}\left(\sum_{k=N+1}^{\infty}|\hat{f}(k)|^2\right)^{\frac12}\cdot(n^2)^{\frac12}=\left(\sum_{k=N+1}^{\infty}|\hat{f}(k)|^2\right)^{\frac12}<\frac{\varepsilon}{2}$，

又由 $\sum_{k\ge1}|\hat{f}(k)|^2<\infty$ 知 $\hat{f}(k)\to 0$ 知 $(|\hat{f}(k)|)_{k\ge1}$ 有界，设 $K=\sup_{k\ge1}|\hat{f}(k)|$，则

$\frac{1}{n}\sum_{k=1}^{N}|\hat{f}(k)|\le\frac{N}{n}K\le\frac{\varepsilon}{2}$（n充分大），故n充分大时，$\left|\int_0^{2\pi}f(t)f_n(t)dt\right|<\varepsilon$.

所以 f_n 在 $L_2(0,2\pi)$ 中弱收敛到0.

又 $\|f_n\|=\frac{1}{n}(\underbrace{1+1\cdots+1}_{n^2个})^{\frac12}=1\not\to 0$，故 f_n 不依范数收敛到0.