QML EX1

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1 Part A

We need to find the eigenvalues of the following matrix

$$A = \frac{1}{4}\begin{pmatrix} 15 & 9 & 5 & -3 \\ 9 & 15 & 3 & -5 \\ 5 & 3 & 15 & -9 \\ -3 & -5 & -9 & 15 \end{pmatrix}$$

notice that $|A - \lambda I|$ is a 4x4 symmetric matrix, so we will use the following formula:

$$\begin{vmatrix} a & b & c & d \\ b & a & e & f \\ c & e & a & g \\ d & f & g & a \end{vmatrix} = a^4 - a^2(b^2+c^2+d^2+e^2+f^2+g^2) + 2a(efg+cdg+bdf+bce) - 2(begd+bcfg+cdef) + (bg)^2 + (cf)^2 + (de)^2$$

Therefore, with our matrix we get that:

$$\begin{aligned} |A - \lambda I| = {} & (15-4\lambda)^4 - (15-4\lambda)^2 \cdot (9^2+5^2+(-3)^2+3^2+(-5)^2+(-9)^2) \\ & + 2 \cdot (15-4\lambda) \cdot (3 \cdot (-5) \cdot (-9) + 5 \cdot (-3) \cdot (-9) + 9 \cdot (-3) \cdot (-5) + 9 \cdot 5 \cdot 3) \\ & - 2 \cdot (9 \cdot 3 \cdot (-9) \cdot (-3) + 9 \cdot 5 \cdot (-5) \cdot (-9) + 5 \cdot (-5) \cdot 3 \cdot (-3)) + (9 \cdot (-9))^2 + (5 \cdot (-5))^2 + (3 \cdot (-3))^2 \end{aligned}$$

Which simplify to:

$$|A - \lambda I| = 256\lambda^4 - 3840\lambda^3 + 17920\lambda^2 - 30720\lambda + 16384 = (4-4\lambda)(8-4\lambda)(16-4\lambda)(32-4\lambda)$$

Therefore the eigenvalues are $\lambda = 1, 2, 4, 8$

2 Part B

For each eigenvalue λ_i, solve $(A - \lambda_i I)\vec{v} = 0$ to find the corresponding eigenvector $\vec{v_i}$

For $\lambda = 1$ we need to find $\vec{v_1}$ such that:

$$\begin{pmatrix} 14 & 9 & 5 & -3 \\ 9 & 14 & 3 & -5 \\ 5 & 3 & 14 & -9 \\ -3 & -5 & -9 & 14 \end{pmatrix} \vec{v_1} = \vec{0} \longrightarrow \vec{v_1} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 2$ we need to find $\vec{v_2}$ such that:

$$\begin{pmatrix} 13 & 9 & 5 & -3 \\ 9 & 13 & 3 & -5 \\ 5 & 3 & 13 & -9 \\ -3 & -5 & -9 & 13 \end{pmatrix} \vec{v_2} = \vec{0} \longrightarrow \vec{v_2} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 4$ we need to find $\vec{v_3}$ such that:

$$\begin{pmatrix} 11 & 9 & 5 & -3 \\ 9 & 11 & 3 & -5 \\ 5 & 3 & 11 & -9 \\ -3 & -5 & -9 & 11 \end{pmatrix} \vec{v_3} = \vec{0} \longrightarrow \vec{v_3} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = 8$ we need to find $\vec{v_4}$ such that:

$$\begin{pmatrix} 7 & 9 & 5 & -3 \\ 9 & 7 & 3 & -5 \\ 5 & 3 & 7 & -9 \\ -3 & -5 & -9 & 7 \end{pmatrix} \vec{v_4} = \vec{0} \longrightarrow \vec{v_4} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

3 Part C

Using the computer, let's find the vector $\vec{x}$ which solves the equation $A\vec{x} = \vec{b}$ for:

$$\begin{pmatrix} 15 & 9 & 5 & -3 \\ 9 & 15 & 3 & -5 \\ 5 & 3 & 15 & -9 \\ -3 & -5 & -9 & 15 \end{pmatrix} \vec{x} = \vec{b} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We got that:

$$\vec{x} = \frac{1}{32}\begin{pmatrix} -1 \\ 7 \\ 11 \\ 13 \end{pmatrix}$$

4 Part D

We will represent each eigenvector of matrix A with the computational basis states (for example $|01\rangle$):

$$\vec{v_1} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = -\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\vec{v_2} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\vec{v_3} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\vec{v_4} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

5 Part E

Now we will discuss the number of qubits needed to represent the eigenvalues and eigenvectors:
Notice the vectors are 4-dimensional therefore we need 2 qubits to represent each one.
The values range from 1 to 8 and all powers of 2. To represent them in binary we need 4 bits, and that also is true for qubits to acquire perfect precision.

6 Part F

We need to find β such that $|b\rangle = \sum_j \beta_j |u_j\rangle$ therefore $\beta_j = \langle u_j|b\rangle$:

$$\beta_1 = \frac{1}{4}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}^\dagger \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2}$$

$$\beta_2 = \frac{1}{4}\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}^\dagger \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2}$$

$$\beta_3 = \frac{1}{4}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}^\dagger \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2}$$

$$\beta_4 = \frac{1}{4}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}^\dagger \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2}$$

7 Part G

We choose t to be $\frac{2\pi}{16}$ so our eigenvalues will become:

$\lambda_1 = 1 \rightarrow 0.0625 = |0001\rangle$, $\lambda_2 = 2 \rightarrow 0.125 = |0010\rangle$, $\lambda_3 = 4 \rightarrow 0.25 = |0100\rangle$, $\lambda_4 = 8 \rightarrow 0.5 = |1000\rangle$

8 Part H

after the quantum phase estimation, the quantum state in registers B and C will be the following:

$$\frac{1}{2}|0001\rangle|v_1\rangle + \frac{1}{2}|0010\rangle|v_2\rangle + \frac{1}{2}|0100\rangle|v_3\rangle + \frac{1}{2}|1000\rangle|v_4\rangle$$

9 Part I

The constant C needs to be chosen such that it is less than the smallest eigenvalue $\frac{1}{16}$, but as large as possible so that when the auxiliary qubit is measured, the probability of it being in the state $|1\rangle$ is large. therefore we will choose $C = \frac{1}{32}$.

$$\begin{aligned}
&\frac{1}{2}|0001\rangle|v1\rangle \cdot (\sqrt{1 - \frac{(\frac{1}{32})^2}{(\frac{1}{16})^2}}|0\rangle + \frac{\frac{1}{32}}{\frac{1}{16}}|1\rangle) + \frac{1}{2}|0010\rangle|v2\rangle \cdot (\sqrt{1 - \frac{(\frac{1}{32})^2}{(\frac{1}{8})^2}}|0\rangle + \frac{\frac{1}{32}}{\frac{1}{8}}|1\rangle) + \\
&\frac{1}{2}|0100\rangle|v3\rangle \cdot (\sqrt{1 - \frac{(\frac{1}{32})^2}{(\frac{1}{4})^2}}|0\rangle + \frac{\frac{1}{32}}{\frac{1}{4}}|1\rangle) + \frac{1}{2}|1000\rangle|v4\rangle \cdot (\sqrt{1 - \frac{(\frac{1}{32})^2}{(\frac{1}{2})^2}}|0\rangle + \frac{\frac{1}{32}}{\frac{1}{2}}|1\rangle) \\
&= \frac{1}{2}|0001\rangle|v1\rangle \cdot (\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle) + \frac{1}{2}|0010\rangle|v2\rangle \cdot (\frac{\sqrt{15}}{4}|0\rangle + \frac{1}{4}|1\rangle) \\
&+ \frac{1}{2}|0100\rangle|v3\rangle \cdot (\frac{\sqrt{63}}{8}|0\rangle + \frac{1}{8}|1\rangle) + \frac{1}{2}|1000\rangle|v4\rangle \cdot (\frac{\sqrt{255}}{16}|0\rangle + \frac{1}{16}|1\rangle)
\end{aligned}$$

10 Part J

After applying the inverse quantum phase we will get

$$\begin{aligned}
&\frac{1}{2}|0000\rangle|v1\rangle \cdot (\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle) + \frac{1}{2}|0000\rangle|v2\rangle \cdot (\frac{\sqrt{15}}{4}|0\rangle + \frac{1}{4}|1\rangle) \\
&+ \frac{1}{2}|0000\rangle|v3\rangle \cdot (\frac{\sqrt{63}}{8}|0\rangle + \frac{1}{8}|1\rangle) + \frac{1}{2}|0000\rangle|v4\rangle \cdot (\frac{\sqrt{255}}{16}|0\rangle + \frac{1}{16}|1\rangle)
\end{aligned}$$

Which is the quantum state but the qubits are set to zeros.

11 Part K

The next step of the HHL algorithm is to measure the auxiliary qubit, and the probability of success for each of the eigenvalues is

$$P(|1\rangle) = (\frac{1}{4})^2 + (\frac{1}{8})^2 + (\frac{1}{16})^2 + (\frac{1}{32})^2 = \frac{85}{1024}$$

12 Part L

On outcome 1 when measuring the auxiliary qubit, the state is

$$\frac{\frac{1}{4}|0000\rangle|v1\rangle|1\rangle + \frac{1}{8}|0000\rangle|v2\rangle|1\rangle + \frac{1}{16}|0000\rangle|v3\rangle|1\rangle + \frac{1}{32}|0000\rangle|v4\rangle|1\rangle}{\sqrt{\frac{85}{1024}}}$$

$$= \frac{8|0000\rangle|v1\rangle|1\rangle + 4|0000\rangle|v2\rangle|1\rangle + 2|0000\rangle|v3\rangle|1\rangle + |0000\rangle|v4\rangle|1\rangle}{\sqrt{85}}$$

13 Part M

To get x we need to substitute $v1, v2, v3, v4$

$$\frac{|0000\rangle(8|v1\rangle + 4|v2\rangle + 2|v3\rangle + |v4\rangle)|1\rangle}{\sqrt{85}}$$

$$= \frac{|0000\rangle((-8+4+2+1)|00\rangle + (8-4+2+1)|01\rangle + (8+4-2+1)|10\rangle + (8+4+2-1)|11\rangle)|1\rangle}{2\sqrt{85}}$$

$$= \frac{|0000\rangle(-1|00\rangle + 7|01\rangle + 11|10\rangle + 13|11\rangle)|1\rangle}{2\sqrt{85}}$$

The answer x is almost correct, we got a normalized x.

14 Part N

We would expect to get the same result for x but with a probability close to $\frac{85}{1024}$ (considering noise), and other quantum states when the auxiliary bit is 0 which we don't care about.