Social Choice Theory

Basic Terminology

- ***Social choice problem***: a decision problem faced by a **group**,
    - in which individual is willing to state at least ***ordinal*** preferences over outcomes.
        - *Individual preference orderings*
        - must satisfy axioms of completeness, asymmetry and transitivity.
- ***Social preference ordering*** is formed by combine the individual preference ordering.
    - It reflects the preferences of the group.
- ***Social state***: the state of the world that includes **everything** that individuals are **care** about.
    - Example, choosing a tax level among high, moderate and low,
    - each tax level corresponds to a social state.
- ***Social welfare function*** refers to any decision rule that
    - **aggregates** a **set** of individual preference orderings over ***social states*** into
    - a ***social preference ordering*** over those states.

Mathematic Notation

- An individual preference ordering is a vector / a list of ordered objects.
    - $I = [a, b, c, d, \{e, f\}, g, h, \dots]$.
    - If some objects are equi-preferred, put them into a set and include that set into the vector.
- All individual preference orderings in society is a set of vectors.
    - $G = \{I, K, L, \dots\}$.
- The aim of social choice theory is to analyze if and how G can be aggregated in a systematic manner into a social preference ordering S.
    - Given an arbitrary set G, which ***SWF*** would produce the best S.

- S is another vector that lists the group's preference ordering over the objects its individuals hold preferences over.
- ***SWF***: $G \rightarrow S$

Voting Paradox

- **Majority rule** might end up with a cyclic preference ordering.
    - A: x > y > z
    - B: y > z > x
    - C: z > x > y
    - A and C => x > y
    - A and B => y > z
    - B and C => z > x
    - In summary, x > y > z > x.

Four Axioms

- Every normatively reasonable **SWF** should be *nondictatorial*.
    - It must not be the case that S *always* coincides with the preference orderings of a particular individual.
    - No individual should be allowed to be a dictator.

Decisiveness

- The ability to make decisions quickly and confidently.
- A group of people D (which may be a single-member group), which is part of the group of all individuals G,
    - $D \in G$
- is **decisive with respect to** the ordered pair of social states (a, b)
- **if and only if** state a is socially preferred to b whenever **everyone in D** prefers to a to b
- A group that is decisive with respect to all pairs of social states is simply **decisive**.

- **Nondictatorship** (**Condition D**): No single individual (no single-member group D) of the group G is decisive.
    - Majority rule meets this condition.
    - No individual will be a dictator as long as the majority rule is accepted.
- **Ordering** (**Condition U**): For every possible combination of individual preference orderings,
    - the social preference ordering must be complete, asymmetric, and transitive.
    - Majority rule is ruled out by this condition because of cyclic.
    - ***unrestricted domain***
- **Pareto** (**Condition P**): The **group** of **all individuals** in society is decisive.
    - Remark: it's the group being decisive but not all individuals.
    - i.e, if everyone in the group prefers a to b, then the group should prefer a to b.
- **Independence of irrelevant alternatives** (**Condition I**)
    - If all individuals have the same preference between a and b
    - in two different set of individual preference orderings G and G'
    - ,then society's preference between a and b must be same in G and G'
- The problem is that it effectively excludes all **SWF**s
    - that are sensitive to relational properties of the individual preference orderings.

Example of Independence of Irrelevant Alternatives

- a and b should depend only on individual preferences over **that** pair of social states (a, b).
- The social ranking of *a* and *b* must **not** depend on how some third (**irrelevant**) social state *c* is ranked by the individuals.
- In the old society, society preferred a to b, simply because
    - Old A - a > b > c
    - Old B - c > a > b
- In the new society, things are different, but the only difference that object c is ranked differently:
    - New A - c > a > b
    - New B - a > c > b
- Since New A and New B still agree that *a* is better than *b*, the new society must also prefer *a* to *b*.

- How *c* is ranked is ***irrelevant*** when it comes to determining the social preference between *a* and *b*.

Violation of Four Axioms

- An aggregation procedure in which the group preference ordering **always mimics**
    - the preference ordering of a certain individual or that of a certain subgroup violates the **condition D**.
- The pairwise comparisons method (often referred to as ***majority rule***) violates the **condition U**.
- **The Borda count** method violates the **condition I**.

1	**2**	**2**
A	A	B
B	C	C
C	B	A

1	**2**	**2**
D	A	B
A	C	C
B	B	D
C	D	A

Arrow's Impossibility Theorem

- No ***social welfare function*** satisfies the four conditions, namely,
    - **non-dictatorship**,
    - **ordering**,
    - **Pareto**,
    - **independence of irrelevant alternatives**,
- unless the group has **just one member** or the **number of social states is fewer than three**.
- Some proposals to avoid the theorem's implications:

- To defend the **majority rule** by rejecting the **condition U**
- To defend the **Borda count** method by rejecting the **condition I**

Sen on Liberalism and the Pareto Principle

- Sen argued that the Pareto principle is incompatible with the basic ideals of liberalism.
- **Minimal Liberalism**: There are **at least two individuals** in **society** such that
    - for each of them there is **at least one pair** of **alternatives** with respect to which she is ***decisive***,
    - that is, there is a pair a and b, such that if she prefers a to b,
    - then society prefers a to b
    - (and society prefers b to a if she prefers b to a).
- What Sen proved is that no **SWF** satisfies minimal liberalism, Pareto and the ordering condition.
    - ***The paradox of the Paretian Liberal***.

Robert Nozick's View

- One of the most well-known proponents of liberalism in recent years,
- His main point is that Sen is **wrong** in constructing liberalism as property of an SWF.
- He claimed that it's better to better to think of liberalism as,
- a ***constraint*** on the set of alternatives that society should be allowed to make decision about.
- Nozick denies a part of the ordering condition known as "unrestricted domain".
- According to him, it's simply **false** that
    - an ***SWF*** should be a function from **all possible** individual preference orderings
    - to a social preference ordering over the same set of objects.

Harsanyi's Utilitarian Theorems

Individual Rationality

- Harsanyi rejects Arrow's view that individual preference orderings **carry nothing** but **ordinal information**.

- On his view, it is reasonable to assume that
    - rational individual preference orderings can be represented in an ***interval scale***,
    - which satisfy the von Neumann and Morgenstern axioms for preferences over lotters.

- This directly implies that rational individual can represent their utility of a social state on an interval state.
    - preferences can be represented by a utility function that measures your utility on an interval scale

- ***Individual rationality***: All individual preference orderings satisfy the von Neumann and Morgenstern axioms of preferences over lotteries.

The Chairperson

- Imagine an individual (who may or may not be a fellow citizen) who
    - evaluates all social states from a moral point of view.
    - The ***Chairperson***.
- If the **Chairperson** is a fellow citizen, then he has two separate prefernce orderings
    - one personal preference ordering over all states
        - that reflects his **personal opinion**.
    - a separate preference ordering over the **same set** of social states
        - that reflects the **social preference ordering**.
- The social preference ordering is the preferences the **Chairperson**,
    - exhibits in those - possibly quite rare - moments

        - when he forces a special impartial and impersonal attitude,
            - a moral attitude
        - upon himself.
- The conditions imposed upon the Chairperson's preference orderings:
    - The rationality condition
    - The moral condition

- What can be conclude about the Chairperson's social preference ordering?
    - given that it fulfills certain structural conditions
- **Rationality of a social preferences**:
    - The **Chairperson**'s social preference ordering satisfies the von Neumann and Morgenstern axioms for preferences over lotteries.

Pareto

- It's the moral condition imposed on the Chairperson
- If a is preferred to b in at least one individual preference ordering,
    - and that there is **no** individual preference ordering in which
        - b is preferred to a
    - then, a is preferred to b in the Chairperson's social preference ordering.
- Furthermore, if all individuals are different, then so is the Chairperson in her social preference ordering.

Harsanyi's First Theorem

- The three conditions imply that the Chairpersons' social preference must be
    - a **weighted** sum of the individual preference orderings,
    - in which the **weight**
        - represents its moral importance relative to the others
- From *individual rationality*, it follows that
    - individual preference orderings can be represented by utility functions that measure utility on interval scale.
- From *rationality of a social preferences*, it follows that
    - the same holds true of social preference ordering.

- Let $u_i(a)$ denote individual i's utility of state a
- Let $u_s(a)$ denote the utility of a as reflected in the Chairperson's social preference ordering.
- Let α be a real number between 0 and 1.
- Individual rationality, rationality of social preferences and Pareto together entail that:
    - $u_s(a) = \sum_{i=1}^{n} \alpha_i \cdot u_i(a)$ with α_i > 0 for i = 1, ..., n.
    - society's utility of state a is weighted sum of all individuals' utility of that state.

Remark

- The theorem doesn't guarantee that every individual preference ordering will be assigned the same weight.
    - Another moral constraint is needed.
- Within this theorem it is possible that different weights are assigned to different individuals.
    - Because α is greater than 0, each individual's preference ordering is assigned some weight
    - but the weights can be unequal.

Equal Treatment of All Individuals

- If all individual's utility functions $u_1, \ldots, u_n$ are expressed in **equal utility units**,
    - then the Chairperson's social utility function u_c must assign the **same weight** to **all individual** utility functions.

Harsanyi's Second Theorem

- Given equal treatment of all individuals, the coefficients in ***Harsanyi's first theorem*** will be equal:
    - $\alpha_1 = \ldots = \alpha_n$
- $u_s(a) = u_1(a) + u_2(a) + u_3(a) + \ldots u_n(a)$