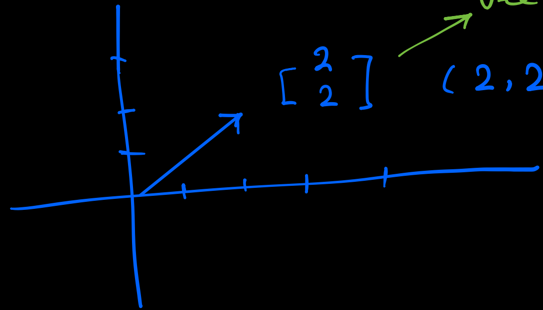
Some NoTes on linear algebra

Inspiration

3Blue1Brown



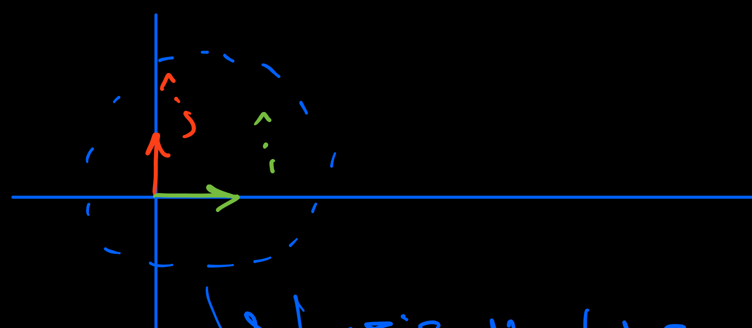
$\begin{bmatrix}2\\2\end{bmatrix}$ → vector

$(2,2)$ → Point

scalar "scales the vector".

$$2\cdot\begin{bmatrix}x\\4\end{bmatrix} = \begin{bmatrix}2x\\2\,4\end{bmatrix}$$

scaled each of it's components by that scalar.



basis vectors

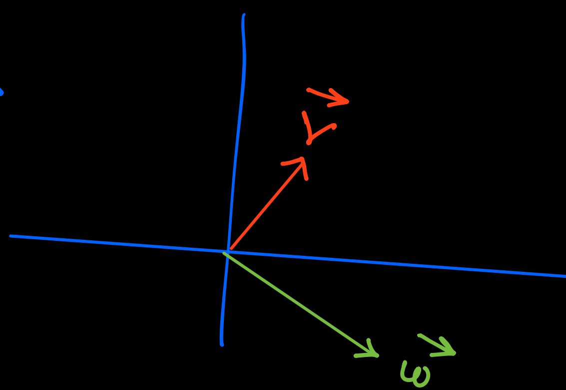
$$\begin{bmatrix}3\\1\end{bmatrix}$$

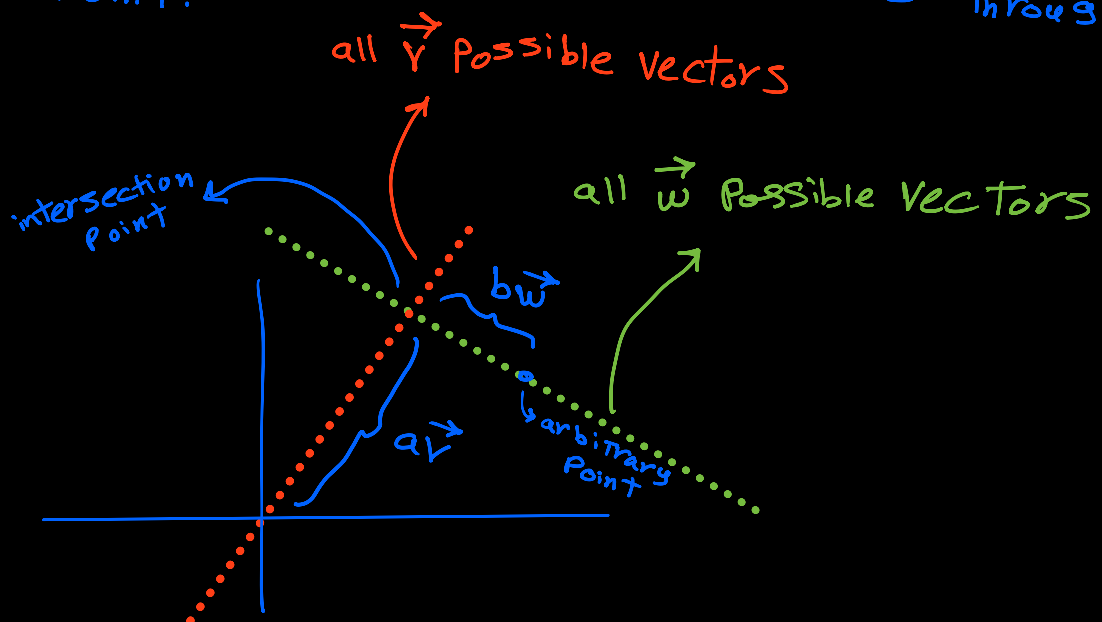
The scalar components in the directions of $\hat{i}$ & $\hat{j}$ vectors.

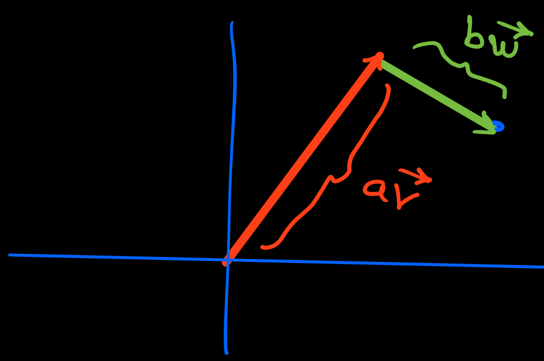
any Two non-Parrallel, non-zero vectors Can span the space.

Proof: any two non-parrallel lines at the same Plane meet at a Point.

Therefore, any point in space can be constructed as The intersection of two non-parrallel, non zero lines, where one of them passes through The origin, and The other passes through The Point.







Since The Two lines must meet $\longrightarrow$ The vector can be constructed as The vector from The origin To The intersection point "$a\vec{r}$" + The vector from The intersection point To The arbitrary Point "$b\vec{w}$"

Linear combination of $\vec{v}$ and $\vec{w}$

$$a\vec{v} + b\vec{w}$$

The span of $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations.

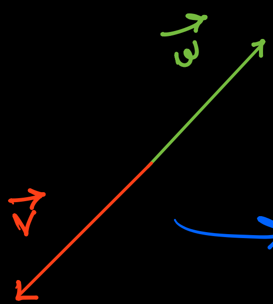
$$a\vec{v} + b\vec{w}$$

Let a and b vary over all real numbers

The span of two non-parrallel, non-zero vectors is the entire 2-D space

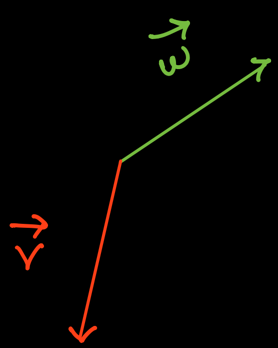
- The span of two parrallel vectors is a line they are parallel to.
- The span of a vector and a zero vector is the line parallel to the non-zero vector
- The span of two zero vectors is a point.

$$\boxed{\begin{array}{l}\text{Linear Combination of } \vec{v}, \vec{w}, \text{ and } \vec{u} \\[1em] \quad a\vec{v} + b\vec{w} + c\vec{u}\end{array}}$$

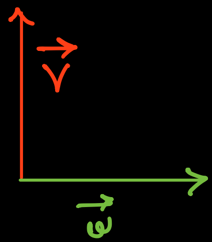


$\rightarrow$ $\vec{v}$ and $\vec{w}$ are $\boxed{\text{linearly dependent}}$

one can be expressed in terms of the other
"as a linear combination of the other"



$\rightarrow$ $\vec{v}$ and $\vec{w}$ are $\boxed{\text{Linearly independent}}$



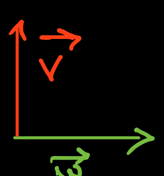
Let $\vec{v}$ and $\vec{w}$ be perpendicular,

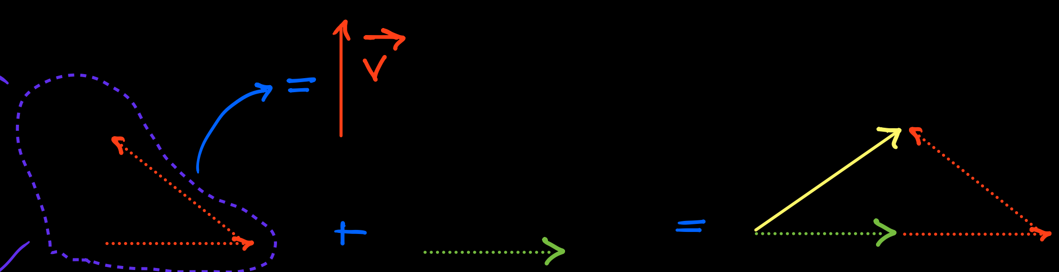
Then neither $\vec{v}$ contribute to $\vec{w}$, nor $\vec{w}$ contribute to $\vec{v}$, in the convention of horizontal & vertical axis.

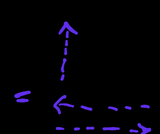
Note from future

a counter example is that; they still can contribute to each other.

I didn't consider this vector which is a sum of 2 vectors as a whole







still has no contribution

$\vec{v}$ contributed to $\vec{w}$.

Net sum of

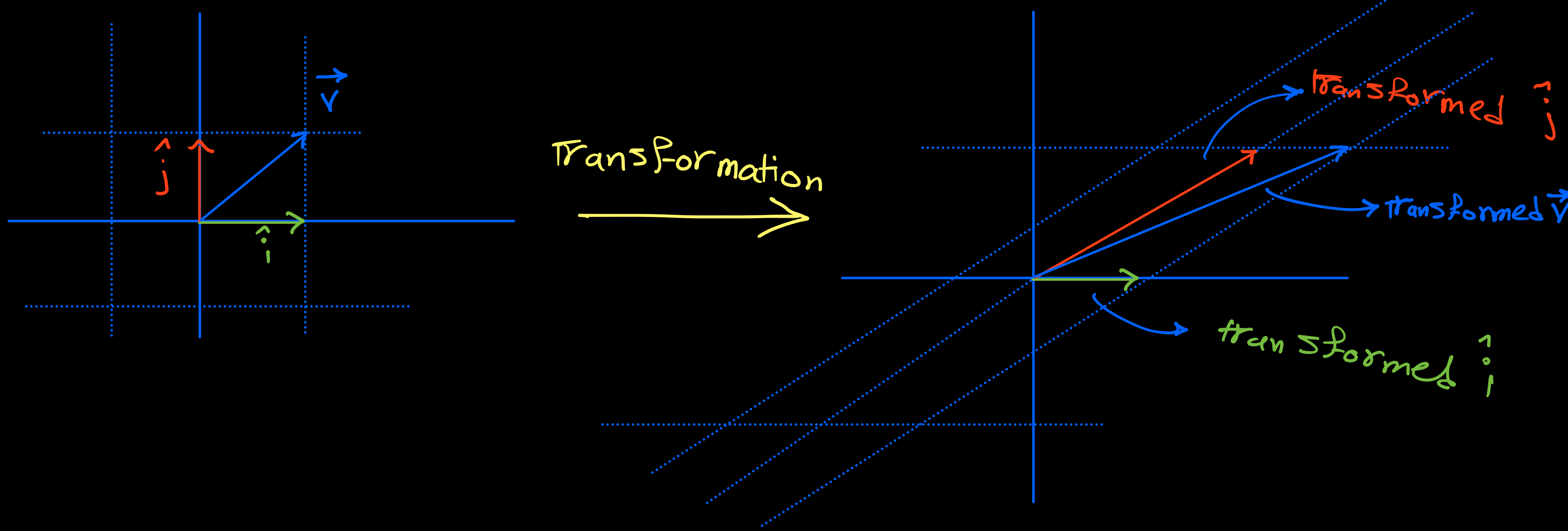
using that logic it contributes then cancels that out = no contribution but Choosing the basis vectors To be perpendicular has a benefit of

1- symmetry between quarters
2- each vector has no projection on the other. "useful in dot product"

yet non-orthogonal basis vectors are equally capable in theory.

$$\vec{V} = a\hat{i} + b\hat{j}$$

$$\text{Transformed } \vec{V} = a(\text{Transformed } \hat{i}) + b(\text{Transformed } \hat{j})$$



$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{Transformation}} x\underset{\substack{\text{Transformed} \\ \hat{i}}}{\begin{bmatrix} a \\ c \end{bmatrix}} + y\underset{\substack{\text{Transformed} \\ \hat{j}}}{\begin{bmatrix} b \\ d \end{bmatrix}} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

any vector can be transformed linearly using only the the transformation of the basis vectors.

2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

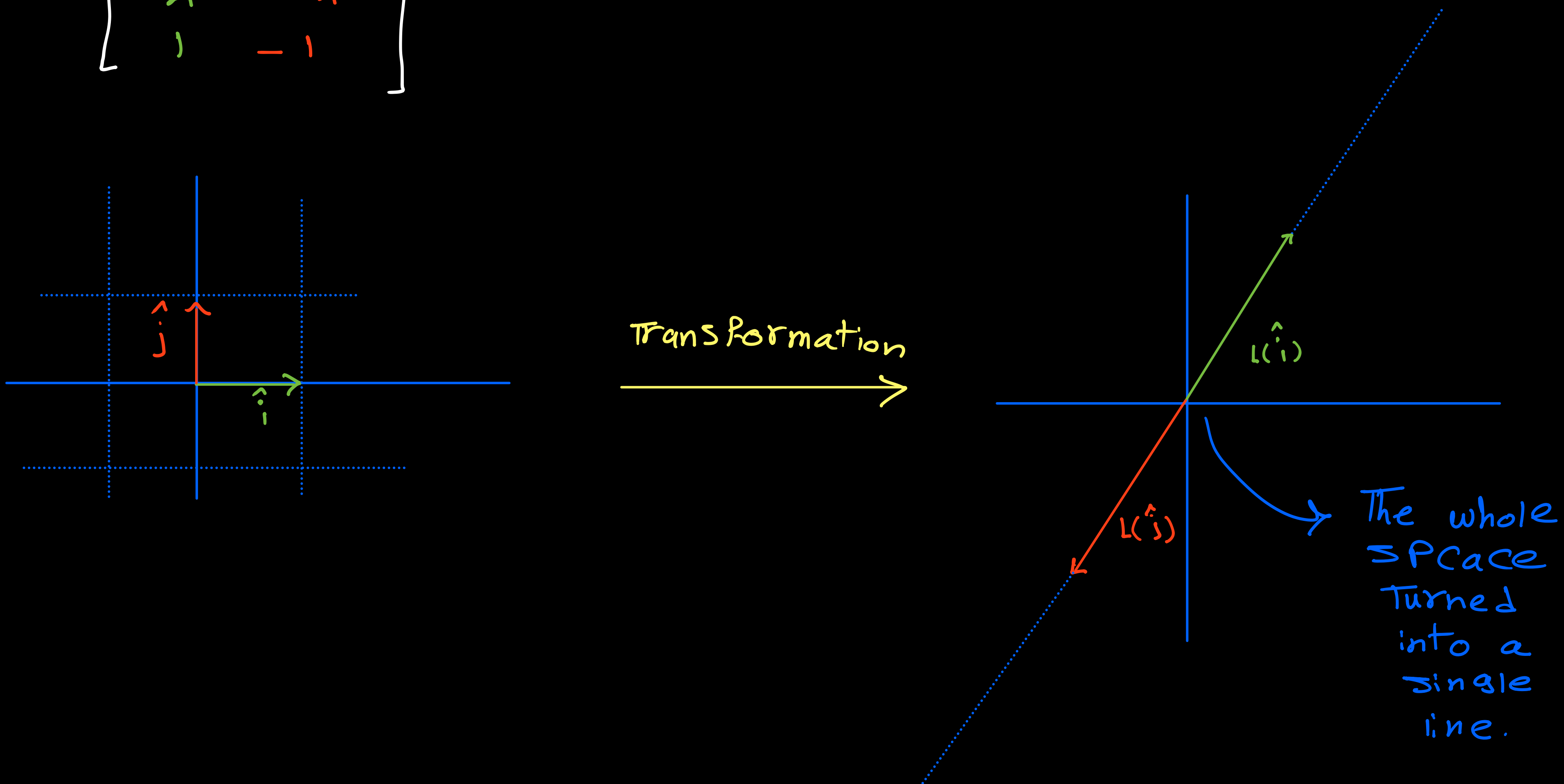
$$\underbrace{\begin{array}{cc} \text{where } \hat{i} \text{ lands} & \text{where } \hat{j} \text{ lands} \end{array}}_{\text{a convection}}$$

matrix - vector multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

if the transformation of the basis vectors are linearly dependent, then they squish the whole space into a single line.

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$



right to left

$$\underset{\text{2nd Transformation}}{\begin{bmatrix} & \\ & \end{bmatrix}}\underset{\text{1st transformation}}{\begin{bmatrix} & \\ & \end{bmatrix}} = \underset{\text{Product}}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

From function notation

$$\underbrace{f(\overbrace{g(x)}^{\text{1st function}})}_{\text{2nd function}}$$

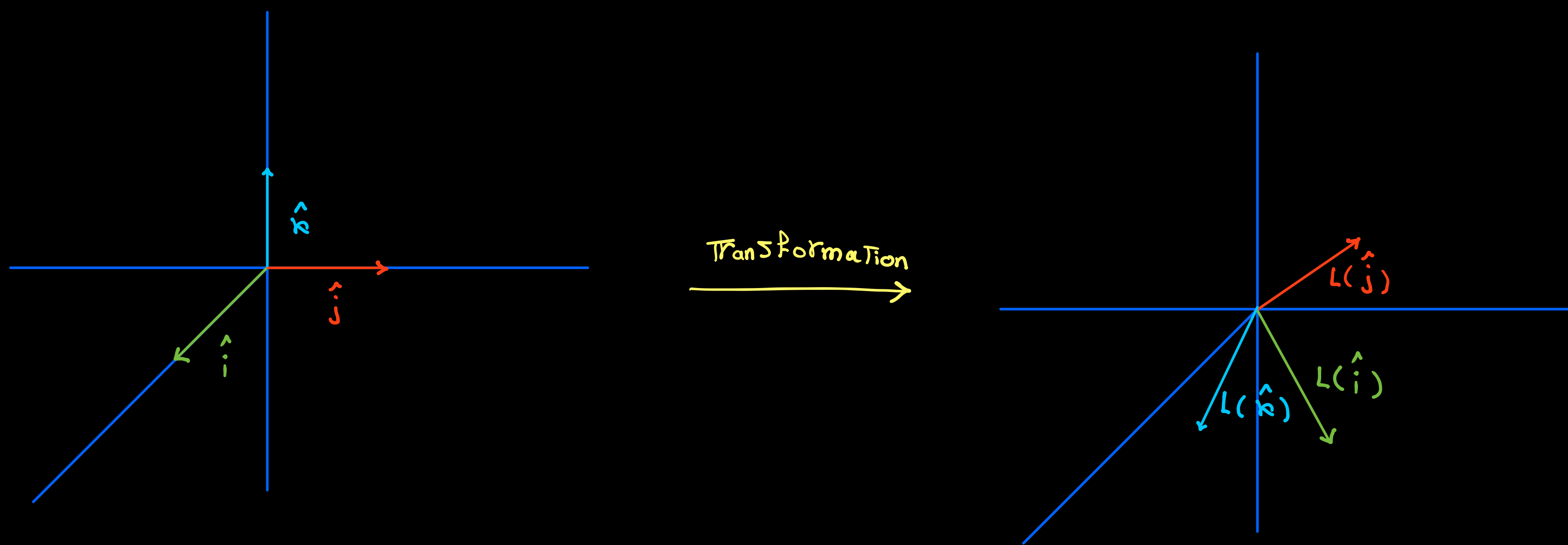
> The Product of Two matrices has The geometric meaning of doing one Transformation Then The other.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \; & \; \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\overbrace{\begin{bmatrix} e \\ g \end{bmatrix}}^{\text{where } \hat{i} \text{ lands after the 1st Transformation}}}_{\text{where } \hat{i} \text{ lands after both Transformations}} = \begin{bmatrix} \vdots \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\overbrace{\begin{bmatrix} f \\ h \end{bmatrix}}^{\text{where } \hat{j} \text{ lands after The 1st Transformation}}}_{\text{where } \hat{j} \text{ lands after both Transformations}} = \begin{bmatrix} \vdots \end{bmatrix}$$

3-D Linear Transformation



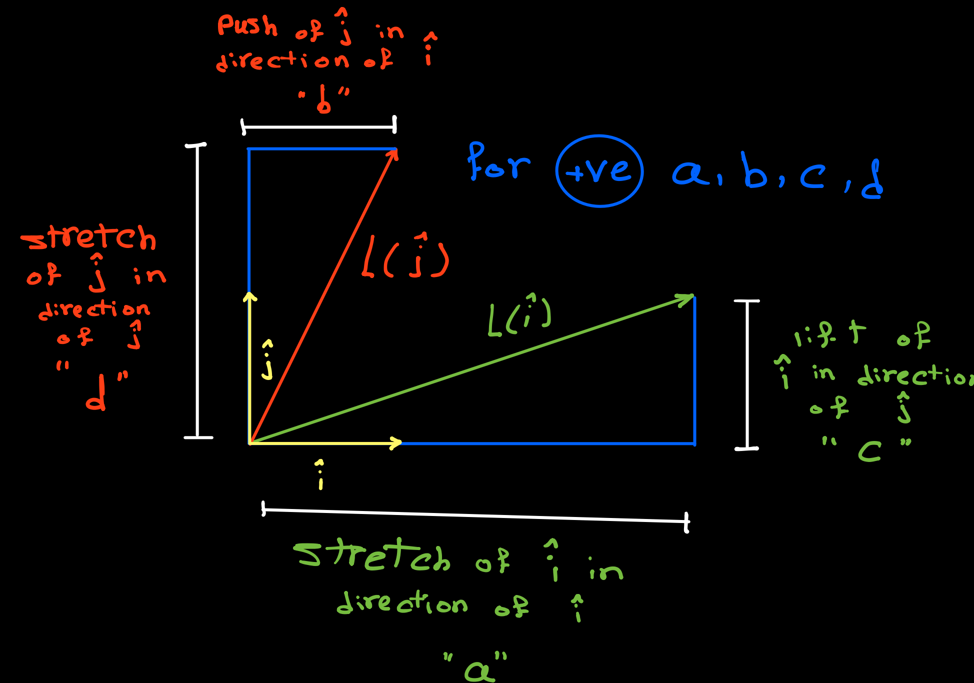
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

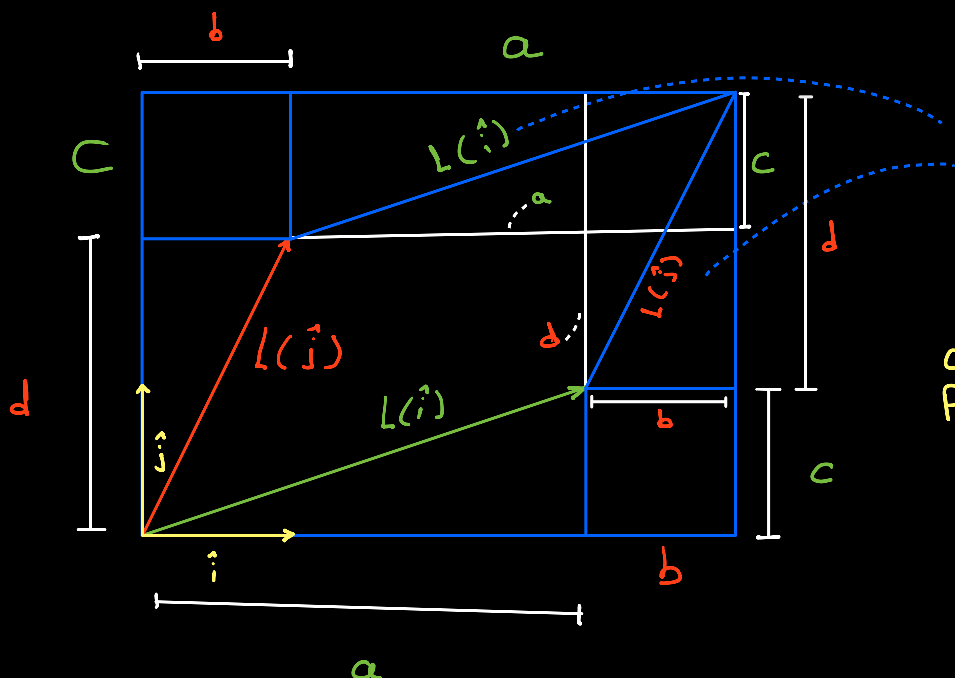
where $\hat{i}$ lands | where $\hat{j}$ lands | where $\hat{k}$ lands $\longrightarrow$ a Convention

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $\hat{i}$ lands   where $\hat{j}$ lands

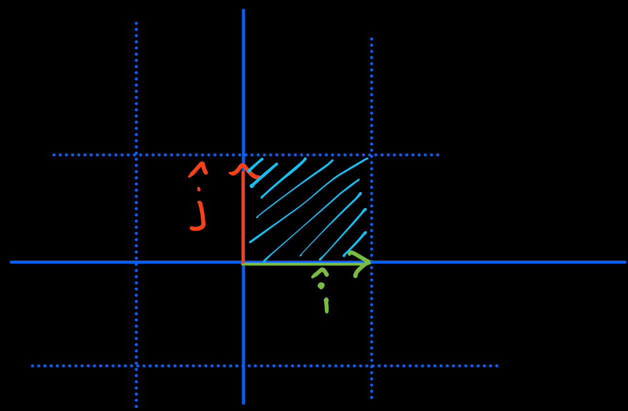
for (+ve) a, b, c, d





Two non-zero, non parrallel vectors can be veiewed as the adjacent sides of a parrallelogram

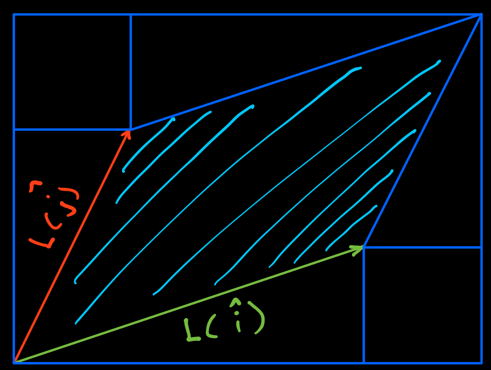
$$\begin{aligned}
\text{area of Parrallegram} &= (a+b)(d+c) - 2(bc) \\
&\quad - 2\left(\tfrac{1}{2}ac\right) - 2\left(\tfrac{1}{2}bd\right) \\
&= ad + \cancel{ac} + \cancel{bd} + bc - 2bc \\
&\quad - \cancel{ac} - \cancel{bd} \\
&= ad - bc
\end{aligned}$$



area of the parallelogram "square" before transformation $= 1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $\hat{i}$ lands   where $\hat{j}$ lands



area of the parallelogram after transformation $= ad - bc$

The area of any shape in the space after transformation is resized by a factor of $\underbrace{ad - bc}_{\det(\text{matrix})}$.

↳ linear transformation

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det\begin{bmatrix} e & f \\ h & i \end{bmatrix}$$

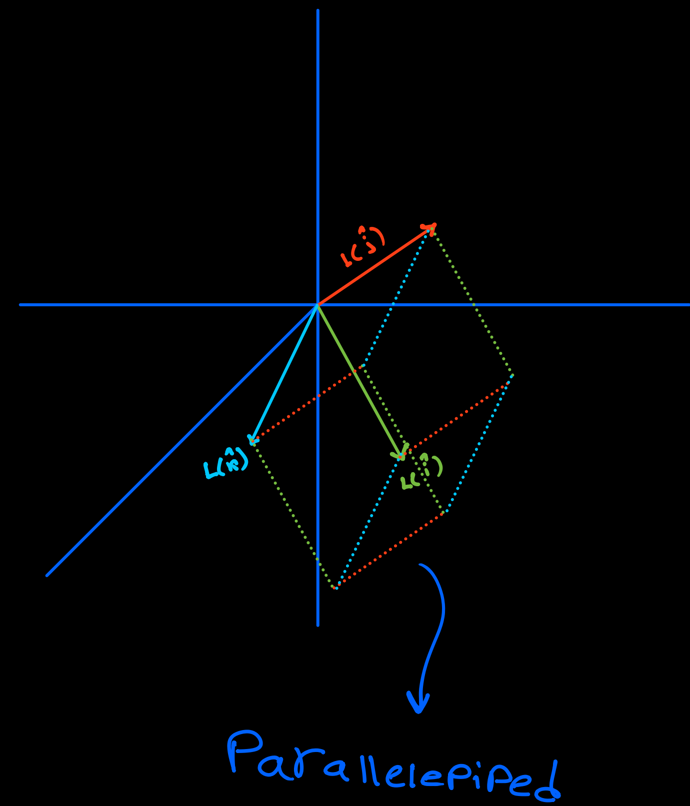
$$- b \det\begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

$$+ c \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\det(M_1 M_2) = \det(M_1)\det(M_2)$$

$$\underbrace{\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}$$

geometrically represents the scale of volume after the transformation.



The system of equations:

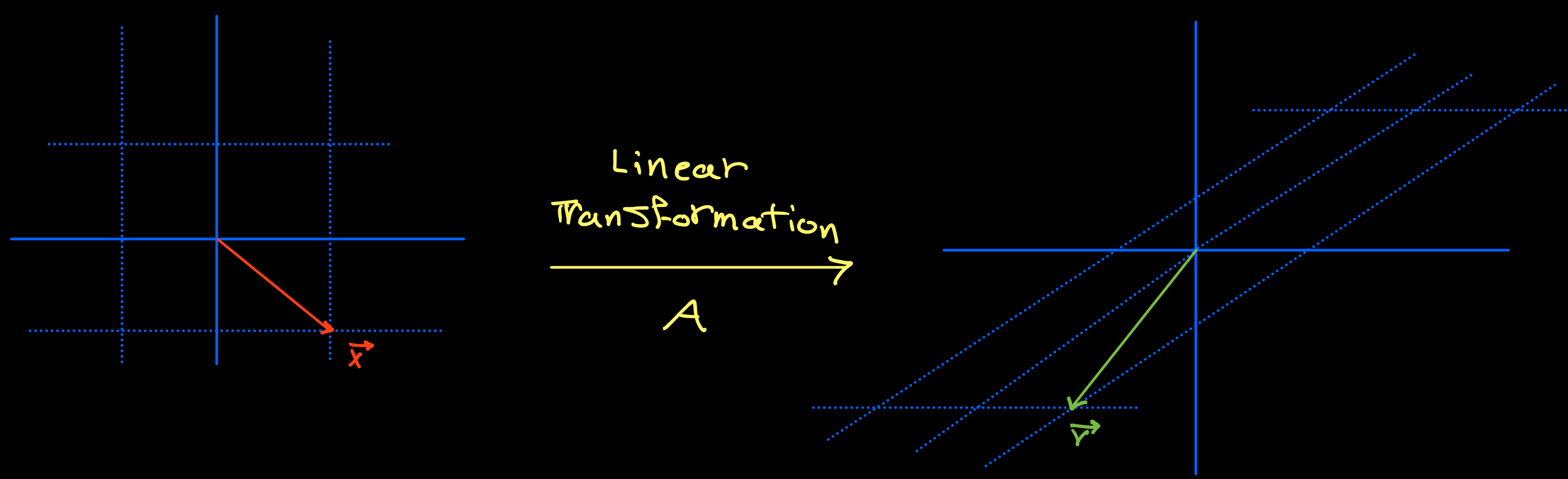
$$2x + 5y + 3z = -3$$
$$4x + 0y + 8z = 0$$
$$1x + 3y + 0z = 2$$

can be written as $\longrightarrow$

$$\overbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{\vec{x}} = \overbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}^{\vec{v}}$$

$$A\vec{x} = \vec{v}$$

Can be viewed as which vector $\vec{x}$ under the linear transformation A gives vector $\vec{v}$.



if A doesn't Turn The space To a lower dimension (Transformation A maintains The dimensionality of The space), Then a Transformation A^{-1} can Turn back The space to it's original form.

Linear Transformation A doesn't lower The dimension of space iff $\det(A) \neq 0$.

$$\boxed{\begin{aligned} A^{-1}(A\vec{x}) &= A^{-1}(\vec{v}) \\ \vec{x} &= A^{-1}\vec{v} \end{aligned}}$$

if The output of all The vectors after The Transformation is a line, Then it's Said That The Transformation has a Rank of 1.

if The output is a Plane, Then The Transformation has a Rank of 2.

Rank $\longleftrightarrow$ Number of dimensions in The output

Set of all possible outputs $A\vec{X}$ $\longleftrightarrow$ Column space of A

$\vec{X} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

where $x, y, z \in \mathbb{R}$

The span of the columns vectors "where each basis vector land" is the set of all possible vectors after the transformation.

Column

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{\text{Span of columns}}$$

$$\Updownarrow$$

Column space

Rank $\longleftrightarrow$ Number of dimensions in column space

When The Rank is as high as it can be "equals The number of columns" we call The matrix full Rank.

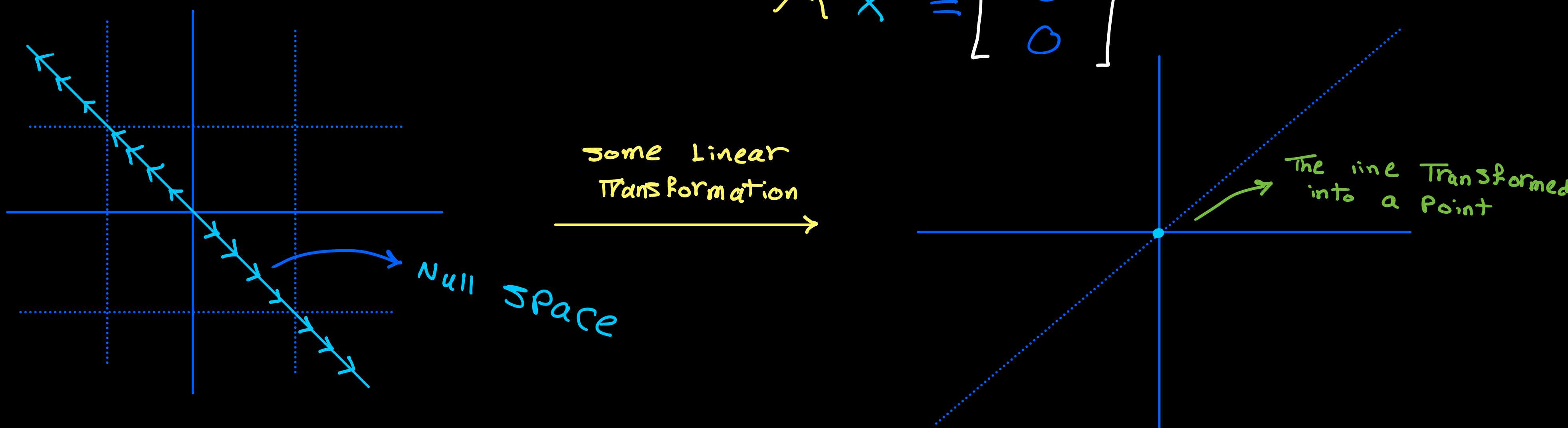
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always in The column space.

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ always lands on $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For matrices which are not full rank (The space Turned into a lower dimension), many vectors Can land on a Point.

The Set of Vectors That land on The Zero vector after some linear Transformation are Called The null space or The kernel.

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Non square matrices

3x2 matrix

$$\begin{array}{cc} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} & \\ \text{where } \hat{i} \text{ lands} \quad \text{where } \hat{j} \text{ lands} & \end{array}$$

→ Span of columns $\Updownarrow$ Column space

→ maps 2D space To 3D space but with the span of 2 columns.

The matrix has a full rank since the number of dimensions in the Column space is equal To The number of dimensions in the input space "2".

2 x 3 matrix

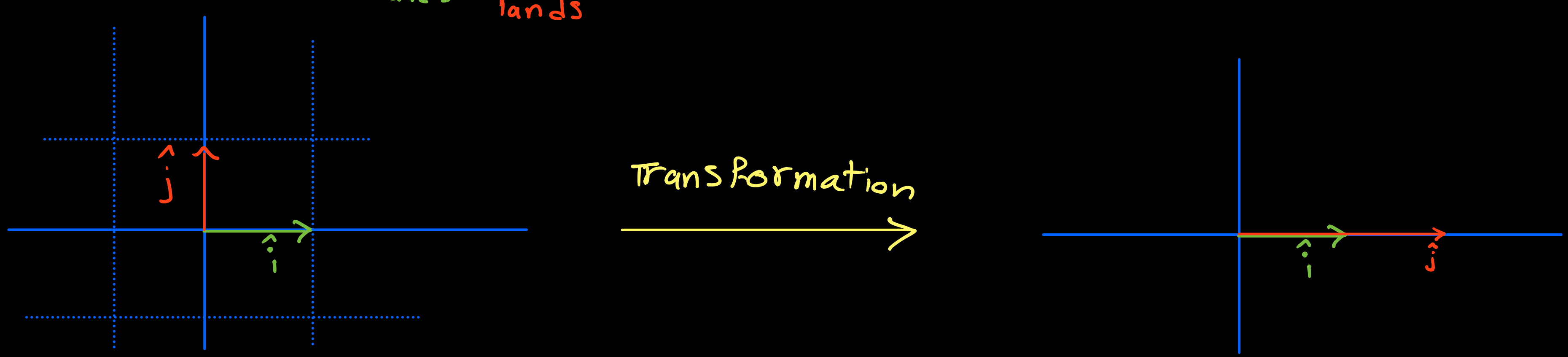
$$\begin{array}{ccc} \multicolumn{3}{c}{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}} \\ \text{where } \hat{i} \text{ lands} & \text{where } \hat{j} \text{ lands} & \text{where } \hat{k} \text{ lands} \end{array} \longrightarrow$$

maps 3D space To 2D space.

1 x 2 matrix

$$\begin{array}{cc} \multicolumn{2}{c}{\begin{bmatrix} 1 & 2 \end{bmatrix}} \\ \text{where } \hat{i} \text{ lands} & \text{where } \hat{j} \text{ lands} \end{array} \longrightarrow$$

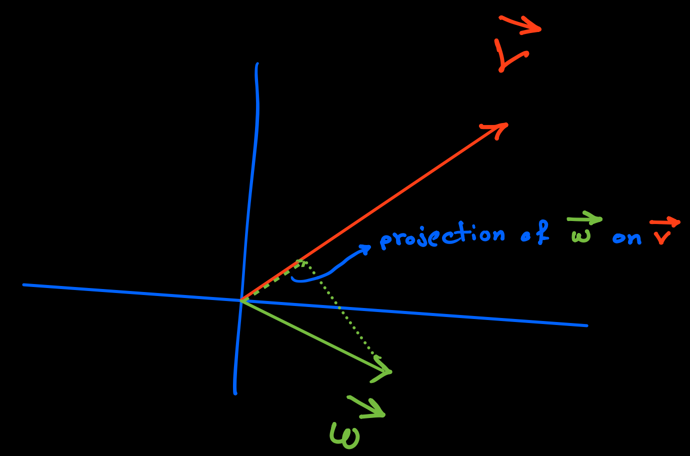
maps 2D space To 1D space.



Dot product

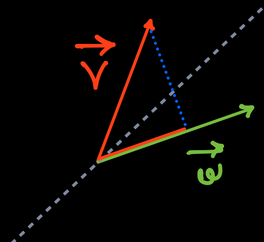
$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = a_1 a_2 + b_1 b_2$$

$$\vec{V} \cdot \vec{w} = |V| \, |w_{proj}|$$



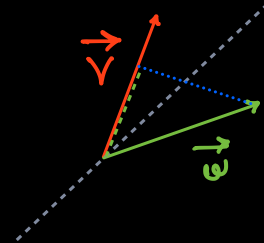
$$\begin{aligned} \vec{V} \cdot \vec{w} &= |V| \, |w_{proj}| \\ &= |V_{proj}| \, |w| \end{aligned}$$

if $\vec{V}$ & $\vec{w}$ were equal in magnitude:



projecting $\vec{V}$ on $\vec{w}$

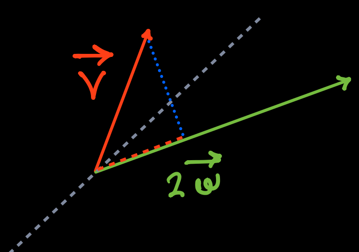
$\xleftrightarrow{\text{symmetry}}$



projecting $\vec{w}$ on $\vec{V}$

There is symmetry between the two $\xrightarrow{\text{implies}}$ $|V||W_{proj}| = |V_{proj}||W|$

if $\vec{V}$ & $\vec{w}$ were not equal in magnitude:



$\longrightarrow$ $|V_{proj}||2w| = 2\underbrace{|V_{proj}||w|}_{|V||W_{proj}|} = 2|V||w_{proj}| = |V||2w_{proj}|$

To be answered $\{$ is $|2w_{proj}|$ the projection of $2\vec{w}$?

The symmetry is broken but $\vec{V}\cdot 2\vec{w} = V\cdot(\vec{w}+\vec{w})$