Mecânica Clássica II
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Exemplo visto em aula
Um Pêndulo duplo! para pequenas oscilações:

$$\mathcal{L} = \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m(\dot{\eta}_1 + \dot{\eta}_2)^2 - V(\eta_1, \eta_2)$$

$$V(\eta_1, \eta_2) = \frac{mg}{l}\left(3 + \frac{2\eta_1^2 + \eta_2^2}{2}\right)$$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{\eta}_1^2 + \left(\dot{\eta}_1 + \dot{\eta}_2\right)^2\right) - \frac{1}{2}\frac{mg}{l}\left(2\eta_1^2 + \eta_2^2\right) + V_0$$

Usando a notação mais genérica:

$$\mathcal{L} = \langle \dot{\eta} \,|\, M \,|\, \dot{\eta} \rangle - \langle \eta \,|\, V \,|\, \eta \rangle$$

Assim podemos chegar as matrizes:

$$M = m\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{e} \quad V = \frac{mg}{l}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

E assim chegamos a Equação de Auto-Valores:

$$(V - M\omega^2)\,|\,\eta\rangle = 0 \quad \text{Fugindo da solução trivial} \quad \det(V - M\omega^2) = 0$$

$$\det\begin{pmatrix} 2\frac{mg}{l} - 2m\omega^2 & -m\omega^2 \\ -m\omega^2 & \frac{mg}{l} - m\omega^2 \end{pmatrix} = 0$$

$$2\left(\frac{mg}{l} - m\omega^2\right)^2 - m^2\omega^4 = 0$$

$$2\frac{g^2}{l^2} + 2\omega^4 - 4\frac{g}{l}\omega^2 - \omega^4 = 0 \quad \text{usando,} \quad \omega_0^2 = \frac{g}{l}$$

$$2\omega_0^4 + \omega^4 - 4\omega_0^2\omega^2 = 0$$

$$\omega^2 = 2\omega_0^2 \pm \sqrt{4\omega_0^4 - 2\omega_0^4}$$

$$\omega^2 = \omega_0^2\left(2 \pm \sqrt{2}\right)$$

Obtemos assim dois valores para ω^2, substituindo novamente na equação de Auto-Valores.

$$\left(V - M\omega_0^2(2 + \sqrt{2})\right)|\,\eta^{(1)}\rangle = 0$$

$$\begin{pmatrix} 2m\omega_0^2 - 2m\omega_1^2 & -m\omega_1^2 \\ -m\omega_1^2 & m\omega_0^2 - m\omega_1^2 \end{pmatrix}\begin{pmatrix} \eta_1^{(1)} \\ \eta_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(1) \quad 2\omega_0^2\,\eta_1^{(1)} - 2\omega_0^2\left(2 + \sqrt{2}\right)\eta_1^{(1)} - \omega_0^2\left(2 + \sqrt{2}\right)\eta_2^{(1)} = 0$$

$$(2) \quad -\omega_0^2\left(2 + \sqrt{2}\right)\eta_1^{(1)} + \omega_0^2\,\eta_2^{(1)} - \omega_0^2\left(2 + \sqrt{2}\right)\eta_2^{(1)} = 0$$

$$(1) \quad 2\eta_1^{(1)} - 2\left(2+\sqrt{2}\right)\eta_1^{(1)} - \left(2+\sqrt{2}\right)\eta_2^{(1)} = 0$$

$$(2) \quad -\left(2+\sqrt{2}\right)\eta_1^{(1)} + \eta_2^{(1)} - \left(2+\sqrt{2}\right)\eta_2^{(1)} = 0$$

$$\frac{-2(1+\sqrt{2})}{\left(2+\sqrt{2}\right)} = \eta_2^{(1)}$$

$$|\eta^{(1)}\rangle = \begin{pmatrix} 1 \\ \frac{-2\left(1+\sqrt{2}\right)}{\left(2+\sqrt{2}\right)} \end{pmatrix}$$

Analogamente para a segunda solução de ω:

$$\left(V - M\omega_0^2(2-\sqrt{2})\right)|\eta^{(2)}\rangle = 0$$

$$|\eta^{(2)}\rangle = \begin{pmatrix} 1 \\ \frac{-2\left(1-\sqrt{2}\right)}{\left(2-\sqrt{2}\right)} \end{pmatrix}$$