Mecânica Clássica II

André Del Bianco Giuffrida

Fetter - Exercício *4.1*

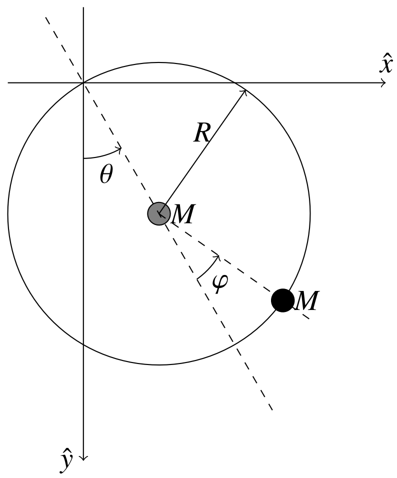
A thin hoop of radius R and mass M oscillates in its own plane with one point of the hoop fixed. Attached to the hoop is a point mass M constrained to move without friction along the hoop. The system is in a uniform gravitational field g. Consider only small oscillations.

(a) Show that the normal-mode frequencies are

$$\omega_1 = \frac{1}{2}\sqrt{\frac{2g}{R}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{2g}{R}}$$

(b) Find the normal-mode eigenvectors. Sketch the motion.
(c) Construct the modal matrix.
(d) Find the normal coordinates and show that they diagonalize the lagrangian.

Podemos partir da lagrangiana usando o centro de massa para o aro.



$$\mathcal{L} = \frac{1}{2}2MR^2\dot{\theta}^2 + \frac{1}{2}MR^2((-\sin(\theta)\dot{\theta} - \sin(\phi)\dot{\phi})^2 + (\cos(\theta)\dot{\theta} + \cos(\phi)\dot{\phi})^2) + V(\theta, \phi)$$

$$\mathcal{L} = \frac{1}{2}2MR^2\dot{\theta}^2 + \frac{1}{2}MR^2(2\dot{\theta}^2 + 2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\phi)) + V(\theta, \phi)$$

$$\mathcal{L} = \frac{1}{2}2MR^2\dot{\theta}^2 + \frac{1}{2}MR^2(2\dot{\theta}^2 + 2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)) + V(\theta, \phi)$$

$$V(\theta, \phi) = MgR\cos(\theta) + MgR(\cos(\theta) + \cos(\phi)$$

$$\mathcal{L} = \frac{1}{2}2MR^2\dot{\theta}^2 + \frac{1}{2}MR^2(2\dot{\theta}^2 + 2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)) + MgR(2\cos(\theta) + \cos(\phi))$$

Fazendo a aproximação para pequenos angulos, ou seja fazendo a lagrangeana quadratica:

$$\cos(\alpha) = (1 - \alpha^2/2) \quad , \quad \sin(\alpha) = \alpha$$

$$\mathcal{L} = \frac{1}{2}2MR^2\dot{\theta}^2 + \frac{1}{2}MR^2((2\dot{\theta}^2 + 2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) + MgR\left(2\left(1 - \frac{\theta^2}{2}\right) + \left(1 - \frac{\phi^2}{2}\right)\right)$$

$$\mathcal{L} = \frac{1}{2}2MR^2\dot{\theta}^2 + \frac{1}{2}MR^2(2\dot{\theta}^2 + 2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) - MgR\left(\theta^2 + \frac{\phi^2}{2}\right) + 3MgR$$

$$\mathcal{L} = \frac{1}{2}2MR^2\dot{\theta}^2 + \frac{1}{2}MR^2(2\dot{\theta}^2 + 2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) - MgR\left(\theta^2 + \frac{\phi^2}{2}\right) + V_0$$

$$\mathcal{L} = \frac{1}{2}MR^2\left[4\dot{\theta}^2 + 2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right] - MgR\left(\theta^2 + \frac{\phi^2}{2}\right) + V_0$$

Podemos escrever na forma geral, onde $R\theta = \eta_1$ e a derivada $R^2\dot{\theta}^2 = \dot{\eta}_1^2$ analogamente, $R\phi = \eta_2$ e a derivada $R^2\dot{\phi}^2 = \dot{\eta}_2^2$

$$\mathcal{L} = \frac{1}{2}M\left[4\dot{\eta}_1^2 + 2\dot{\eta}_2^2 + 2\dot{\eta}_1\dot{\eta}_2\right] - \frac{Mg}{R}\left(\eta_1^2 + \frac{\eta_2^2}{2}\right) + V_0$$

Usando a notação de *bra-kets* $\langle\eta|,|\eta\rangle$

$$\mathcal{L} = \frac{1}{2}\langle\dot{\eta}|\ \mathcal{M}\ |\dot{\eta}\rangle - \frac{1}{2}\langle\eta|\ \mathcal{V}\ |\eta\rangle$$

Onde podemos encontrar as matrizes $\mathcal{M}$ e $\mathcal{V}$.

$$\mathcal{M} = M\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{e} \quad \mathcal{V} = \frac{Mg}{R}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Desenvolvendo por Euler-Lagrange.

$$\mathcal{M}|\,\ddot{\eta}\,\rangle + \mathcal{V}|\,\eta\,\rangle = 0$$

$$|\,\eta\,\rangle = \cos(\omega t + \phi)\,|\,\rho\,\rangle$$

Chegamos a equação de Auto-Valores:

$$\left(\mathcal{V} - \mathcal{M}\omega^2\right)|\,\rho\,\rangle = 0$$

Ou seja resolvendo, $\det(\mathcal{V} - \mathcal{M}\omega^2) = 0$ podemos encontrar os valores de ω^2 ou seja, os Modos Normais.

Assim:

$$\det\begin{pmatrix} \frac{Mg}{R} - 4M\omega^2 & -M\omega^2 \\ -M\omega^2 & \frac{Mg}{2R} - 2M\omega^2 \end{pmatrix} = 0$$

$$\left(\frac{Mg}{R} - 4M\omega^2\right)\left(\frac{Mg}{2R} - 2M\omega^2\right) - M^2\omega^4 = 0$$

$$\frac{M^2g^2}{2R^2} - \frac{4M^2g\omega^2}{R} - \frac{2M^2g\omega^2}{R} + 7M^2\omega^4 = 0$$

$$\frac{g^2}{2R} - 6g\omega^2 + 7\omega^4R = 0$$