1

Quantum Pirate Game

	Player 2: C	Player 2: D
Player 1: C	(100, 0)	(100, 0)
Player 1: D	(100, 0)	(-200, 100.5)

Table 1.1: Representation of the 2 player sub-game in normal form.

In this chapter we describe the Pirate Game and the steps to model a quantum approach to the problem.

1.1 Pirate Game

*O*n selecting the problem...

1.1.1 Problem Description

The original Pirate Game is a multi-player version of the Ultimatum game that was first published as a mathematical problem in the Scientific American as a mathematical problem posed by Omohundro [**?**]. The main objective of the Pirate Game was to present a fully explainable problem with a non-obvious solution. The problem can be formulated as it follows:

> Suppose there are 5 rational pirates: A; B; C; D; E. The pirates have a loot of 100 indivisible gold coins to divide among themselves.
>
> As the pirates have a strict hierarchy, in which pirate A is the captain and E has the lowest rank, the highest ranking pirate alive will propose a division. Then each pirate will cast a vote on whether or not to accept the proposal.
>
> If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.
>
> We consider that each pirate privileges her survival, and then will want to maximize the number of coins received. When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.

1.1.2 Analysis

We can arrive at an equilibrium in this problem by using backward induction. At the end of the problem, supposing there are two pirates left, the equilibrium is very straight forward. This sub-game is represented in Table **??**, and its Nash Equilibrium is (C, D).

As the highest ranking pirate can pass the proposal in spite of the other decision, her self-interest dictates that she will get the 100 gold coins. Knowing this, pirate E knows that any bribe other higher ranking pirate offers her will leave her better than if the game arrives to the last proposal.

When applying this reasoning to the three pirate move, as pirate C knows she needs one more vote to pass her proposal and avoiding death, she will offer the minimum amount of coins that will make pirate E better off than if it comes to the last stage with two pirates. This means that pirate C will offer 1 gold coin to pirate E, and keep the remaining 99 coins.

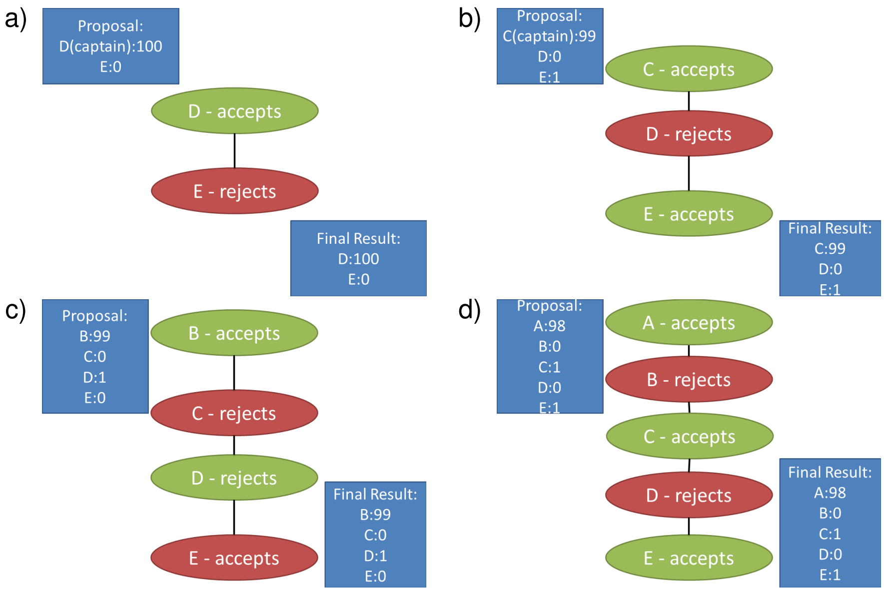


Table 1.2: The equilibrium for the Pirate Game can be found through backward induction. From a), where there's only two pirates left, to d), that corresponds to the initial problem we define the best response.

With 4 pirates, B would rather bribe pirate D with 1 gold coin, because E would rather like climb on the hierarchy and getting the same payoff. Finally, with 5 pirates the captain (A), will keep 98 gold coins and rely on pirate C and E to vote in favour of the proposal, by giving 1 gold coin each.

We can generalize this problem for N pirates. If we assign a number to each pirate, where the captain is number 1 and the lower the number the higher the rank. If the number of coins is superior to the number of pirates, the equilibrium will have the captain (highest ranking pirate), giving a gold coin to each odd pirate, in case the number of players alive is odd, while keeping the rest to herself. When we have a even number of players the captain will assign a gold piece to each pirate with a even number, and the the remaining coins to herself.

If the number of pirates is greater than two times the amount of coins $N > 2C$, a new sittuation arises. If we have 100 coins and 201 pirates, the captain will not get any coin. By the same reasoning with 202 pirates the captain will still be able to survive by bribing the majority of the pirates and keeping no coins for herself. With 203 pirates the first captain will die. However with 204 pirates, the first captain will be able to survive even though he won't be able to bribe the majority, because her second in command knows that when she makes a proposal, she'll be thrown off board. In the game with 205 pirates, however the captain is not able to secure the vote from the second in command on the 204 pirate game, because the second captain is safe and she is able to make a have her proposal accepted and have the third pirate safe.

We can generalize this problem for $N > 2C$ as [**?**], as the games with a number of pirates equal to $2C$ plus a power of two will have an equilibrium in the first round, in the others every captain until a subgame with a number of pirates equal to $2C$ plus a power of two will be thrown off board.

1.2 Quantum Pirate Game

1.2.1 Hypothesis

The original Pirate Game is posed from the point of view of the captain. How should she allocate the treasure to the crew in order to maximize her payoff. We can find the a equilibrium to the original Pirates Game and, while the solution may seem unexpected at first sight, it is fully described using backwards induction.

When modelling this problem from a quantum theory perspective we are faced with some questions. The main difference from the original problem will rely on how the system is set up. Will the initial conditions provide different equilibria? Is there a condition where we are left with the classical problem? Is it possible for a captain, in a situation where we have more than two pirates left, to acquire all the coins?

We propose to study this problem for the 2 and 3 player games and trying to extrapolate for N players. We will analyse the role of entanglement and superposition in the game system. Another aspect worth studying is the variation in the coin distribution on the payoff functions for the players.

1.2.2 Quantum Model

In order to model the problem we will start by defining it using the definition of quantum game (Γ), referred in **??**, Section **??** [**?**].

We want to keep the problem as close to the original as possible in order to better compare the results. Thus we will analyse the game from the point of view of the captain. Will her best response change?

In terms of mechanics and steps, this problem could be described using 3 players and later extended to any number N of players.

We begin by assigning an offset to each pirate (in order to identify her), as in the Section . The captain is number 1 and the lower the number the higher the rank.

1.2.2.A Game system: Setting up the Initial State

A game Γ can be viewed as a system composed by qubits manipulated by players. In a 3 player game there will be 3 qubits, each manipulated by a different player.

With 3 players our system with be represented in a $\mathcal{H}^8$ using a state ψ. This means that to represent our system we will need vectors $2^3 \times 1$ vectors, our system grows exponentialy with the number of players/qubits. Each pure basis $\mathcal{B} = \{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ of $\mathcal{H}^8$ will represent a possible outcome in the game. We assign a pure basis as $|0\rangle = |C\rangle$ ("C" from "Cooperate"), and $|1\rangle = |D\rangle$ ("D" from "Defect").

For 200 players (for example), this game would be impractical to simulate in a classical computer. In this regard a quantum computer may enhance our power to simulate this kinds of experiments [**?**].

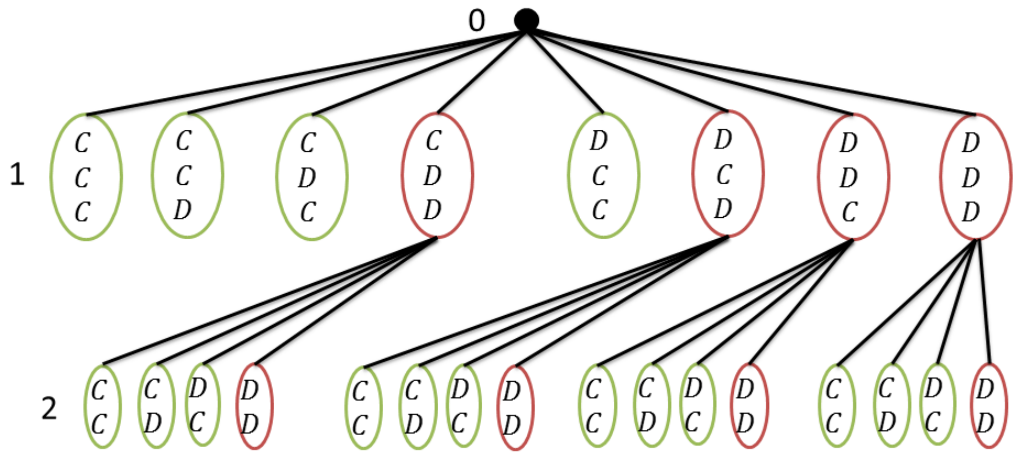


Figure 1.1: Game tree representation of the game (3 player). Red circles represent failed proposals, green represent accepted proposals.

The initial system ($|\psi_0(\gamma)\rangle$), will be set up by defining an entanglement coefficient γ, that affect the way the three qubits (belonging to the three pirate players), are related 1.2. We will entangle our state by applying the gate $\mathcal{J}$ [**?**]. The parameter γ becomes a way to measure the entanglement in the system [**?**].

The index 0 represents the depth of the game tree which can be examined in Figure 1.1.

Due to the nature of quantum mechanics we have to pay attention to some details whe setting up our architecture. For example we cannot copy or clone unkown quantum states [**?**].

The concept of entanglement is crucial to explain some phenomena in Quantum Mechanics (Section **??**). Analysing the role of entanglement in the system is also considered to be a major source of different results from the traditional Game Theory approach [**?**] [**?**] [**?**] [**?**] [**?**].

$$\mathcal{J} = exp\left\{ i\frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{1.1}$$

$$\begin{aligned} |\psi_{in}(\gamma)\rangle &= exp\left\{ i\frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} |000\rangle \\ &= cos(\frac{\gamma}{2})|000\rangle + isin(\frac{\gamma}{2})|111\rangle, \gamma \in (0, \pi) \end{aligned} \tag{1.2}$$

1.2.2.B Strategic Space

In Equation **??** there is the notion of a subset of unitary operators that the players can use to manipulate their assigned qubits.

Each player will be able to manipulate a qubit in the system, in this case $|\varphi_1\rangle$, $|\varphi_2\rangle$, and $|\varphi_3\rangle$, with one of two operators shown in Equation 1.3.

An operator is an unitary 2×2 matrix that is used to manipulate a qubit in the system. This restriction of the strategic space is relevant to keep the problem as close to the classical version as possible. The two operators will correspond to the action of voting "Yes" or to Cooperate, and voting "No", meaning that they will not accept the proposal.

The cooperation operator will be represented by the Identity operator (o_{i0}, where i identifies the qubit upon which player i will act). When assigned to a qubit this operator will leave it unchanged.

The defection operator (D), will be represented by one of Pauli's Operators - the Bit-flip operator. This operator was chosen because it performs the classical operation NOT on a qubit.

These operators are also permutation matrices, so our players are in fact permutating the state of their qubit as in the roullete quantum model (Section **??**). It is also notewhorty that this operators correspond to pure-strategies.

$$\mathcal{U}_i = \begin{cases} C = o_{i0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D = o_{i1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases} , i \in \{1, 2, 3\} \tag{1.3}$$

1.2.2.C Final State

We can play the Pirate Game by considering a succession of steps or voting rounds. In each step we have a simultaneous move, however considering the potential rounds the game has we have a sequential game.

With three players, the first move will correspond to the player 1 (or the captain), if the proposal fails we will proceed to the second step in the game, where the remaining two players will vote on a new proposal made by player 2 (who will be the new captain). The first captain is indifferent to the outcome of the second game so he uses a coin matrix to vote.

After a move we have a "final state" that can be identified by an index which points to the depth k of the game (Figure 1.1). This state is calculated by constructing a super-operator, by performing the tensor product of each player chosen strategy1.3. The super-operator, containing each player strategy, will then be applied to the initial state, this will correspond to the players making a simultaneous move1.4.

In the Figure 1.2 we have a step in the game.We start by building our initial state $k-1$, then the players will select their strategy, a super operator is constructed by performing a tensor product of the selected operators.

In order to calculate the expected payoff functions we need to disentangle the system, before measuring. The act of measuring, in quantum computing, gives an expected value that can be understood as the probability of the system collapsing into that state.

We can disentagle the our $\mathcal{H}^8$ system by applying $\mathcal{J}^\dagger$ (Equation 1.5), this will produce a final state that we will be able to measure.

$$|\psi_k\rangle = \otimes_{i=1}^{N} \mathcal{U}_i |\psi_{k-1}\rangle \tag{1.4}$$

$$|\psi_{fin}\rangle = \mathcal{J}^\dagger |\psi_k\rangle \tag{1.5}$$

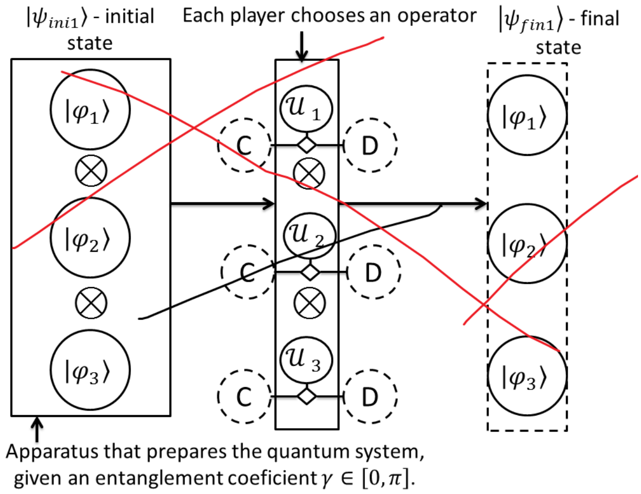


Figure 1.2: Playing the first round of the Pirate Game with 3 players.

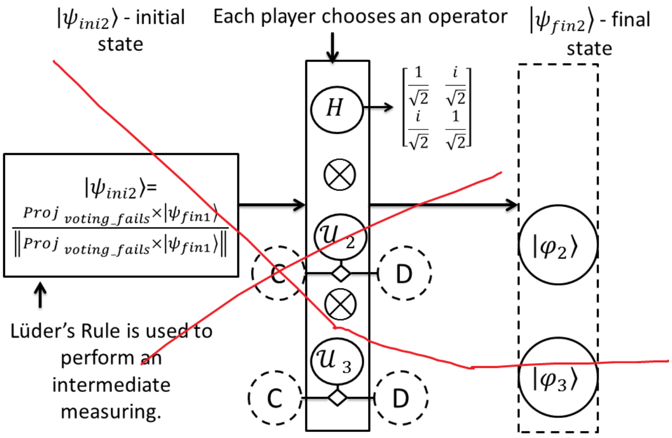


Figure 1.3: Playing the second round of the Pirate Game with 3 players by performing an intermediate measuring.

The second stage in the game can be played in one of two ways. Either we perform an intermediate measuring step on the final state from the first round of voting as in Figure 1.3, or we "ask" what are the two player's strategy for this round without informing them of the results in the last round, like in the Figure 1.4. In the original game the voting results are displayed in between rounds:

If a majority or a tie is reached the goods will be allocated according to the proposal.

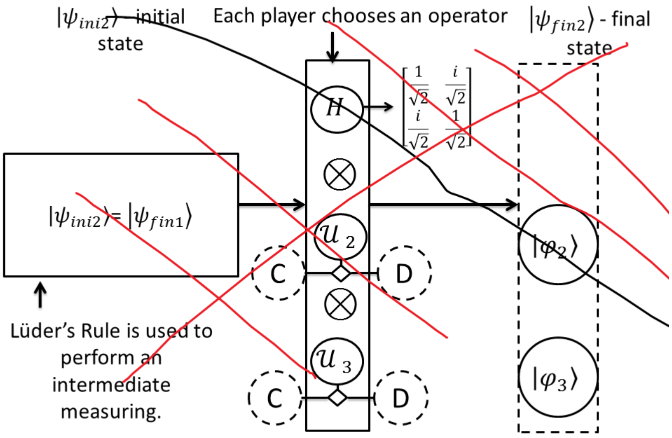


Figure 1.4: Playing the second round of the Pirate Game with 3 players without performing an intermediate measurement.

> Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.

When modelling this problem we want to study if withholding the results in the intermediate step will give way to new strategies.

1.2.2.D Utility

To build the expected payoff functionals for the three player situation we must take into account the subgames created when the proposal fail. In Figure 1.1 we can see an extensive form representation of the game.

As defined on Equation **??**,for each player we must specify a utility functional that attributed a real number to the measurement of the projection of a basis in the quantum state that we get after the game.

This measurement can be understood as a probability of the system collapsing into that state (that derives from the Born Rule, Section **??**).

These utility functions will represent the degree of satisfaction for each pirate after game by atributiong a real number to a measurement performed to the system (as in Equation **??**, Section **??**). The real numbers used convey the logical relations of utility posed by the original problem description. Those numbers will represent the utility associated with the number of coins that a pirate gets, a death penalty, and a small incentive to climb the hierarchy. As each pirate wants to maximize her utility, the Nash equilibrium will be thoroughly used to find the strategies that the pirates will adopt [**?**] [**?**].

The number of coins will translate directly the utility associated with getting those coins. For example if a pirate receives 5 gold coins and the proposal is accepted he will get a utility of 5.

The highest ranking pirate in the hierarchy will be responsible to make a proposal to divide the 100 gold coins. This proposal is modelled as choosing some parameters for the payoff functionals for every player, according to some rules. For the initial step in the game with three pirates these parameters will

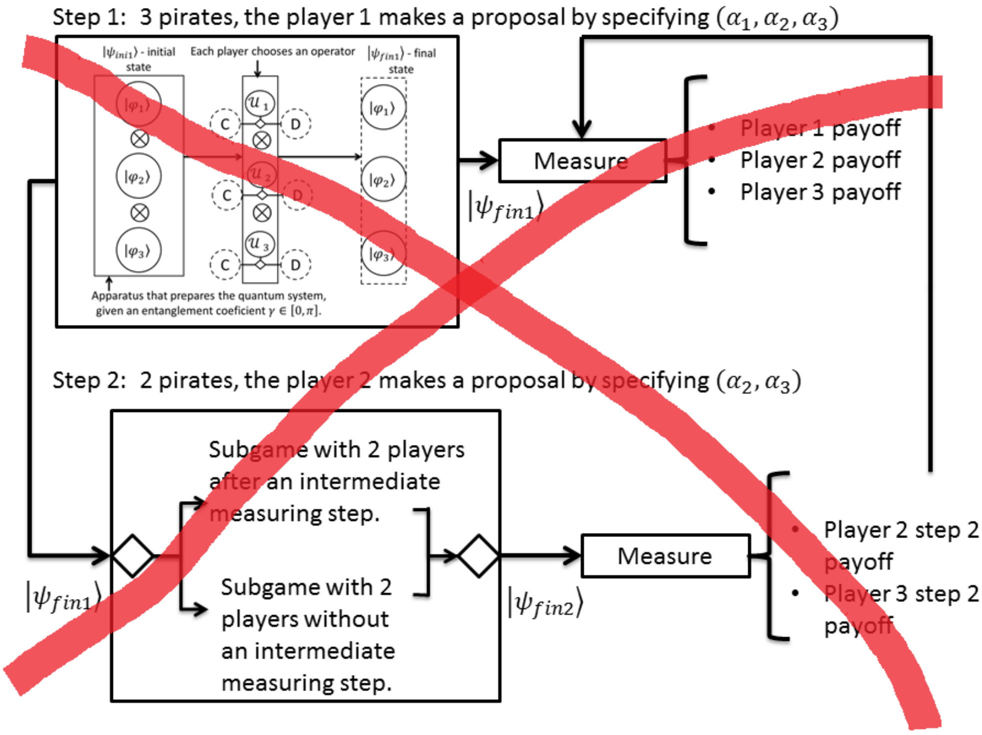


Figure 1.5: Overal view on the Quantum Pirate Game architecture.

be $\alpha_1, \alpha_2, \alpha_3$, and they will obey to the Equation 1.6, where k is the offest of the current captain, and N the number of pirates in the game.

$$\sum_{i=k}^{N} \alpha_i = 100, \forall i : \alpha_i \in \mathbb{N}_0 \tag{1.6}$$

The most interesting values for $(\alpha_1, \alpha_2, \alpha_3)$ will be the allocation that results in a Nash equilibrium in the original Pirate Game $(99, 0, 1)$, and the case where the captain maximizes the number of coins he can get $(100, 0, 0)$. Will the game modelled as a quantum system allow the captain to acquire all the coins?

The proposed goods allocation will be executed if there is a majority (or a tie), in the voting step. A step in the game consists on the highest ranking pirate defining a proposal and the subsequent vote, where all players choose simultaneously an operator.

If the proposal is rejected the captain will be thrown off board, to account for the fact that this situation is very undesirable for the captain he will receive a negative payoff of -200. This value was chosen to be much less than the highest number of coins a pirate could get.

> "When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy."

This means that the pirates have a small incentive to climb the hierarchy. For example in the three player classical game, the third player, who has the lowest rank, will prefer to defect the initial proposal if the player 1 doesn't give her a coin, even knowing that in the second round the player 2 will be able to keep the 100 coins. We will account for this preference by assigning an expected value of half a coin (0.5), to the payoff of the players that will climb on the hierarchy if the voting fails.

We can observe that in Equations 1.7 and 1.8 that we have two separate groups of outcomes:

- Outcomes where the proposal is passed:
  - $|CCC\rangle$ or $|000\rangle$ (which we measure with $|\langle 000|\psi_{fin}\rangle|^2$);
  - $|DCC\rangle$ or $|100\rangle$;
  - $|CDC\rangle$ or $|010\rangle$;
  - $|CCD\rangle$ or $|001\rangle$.
- outcomes where the captain will be eliminated and the remaining players will keep playing:
  - $|DDD\rangle$ or $|111\rangle$;
  - $|DDC\rangle$ or $|110\rangle$;
  - $|CDD\rangle$ or $|011\rangle$;
  - $|DCD\rangle$ or $|101\rangle$.

So with 3 pirates we start with step 1 (Figure 1.5), if the voting is rejected we will get to step 2, where 1 vote is enough to pass a proposal. The final payoff function (for example Equation 1.7 for a 3 player game), will be calculated recursively, the base case being the 2 player sub-game in a 3 player system will be Equation 1.8.

$$\begin{cases} E_{11}(|\psi_{fin}\rangle,\alpha_1) = \alpha_1 \times (|\langle 000|\psi_{fin}\rangle|^2 + |\langle 100|\psi_{fin}\rangle|^2 + |\langle 010|\psi_{fin}\rangle|^2 + |\langle 001|\psi_{fin}\rangle|^2) - \\ \qquad -200 \times (|\langle 111|\psi_{fin}\rangle|^2 + |\langle 110|\psi_{fin}\rangle|^2 + |\langle 101|\psi_{fin}\rangle|^2 + |\langle 011|\psi_{fin}\rangle|^2) \\ E_{12}(|\psi_{fin}\rangle,\alpha_2) = \alpha_2 \times (|\langle 000|\psi_{fin}\rangle|^2 + |\langle 100|\psi_{fin}\rangle|^2 + |\langle 010|\psi_{fin}\rangle|^2 + |\langle 001|\psi_{fin}\rangle|^2) - \\ \qquad +(0.5+E_{22}) \times (|\langle 111|\psi_{fin}\rangle|^2 + |\langle 110|\psi_{fin}\rangle|^2 + |\langle 101|\psi_{fin}\rangle|^2 + |\langle 011|\psi_{fin}\rangle|^2) \\ E_{13}(|\psi_{fin}\rangle,\alpha_3) = \alpha_3 \times (|\langle 000|\psi_{fin}\rangle|^2 + |\langle 100|\psi_{fin}\rangle|^2 + |\langle 010|\psi_{fin}\rangle|^2 + |\langle 001|\psi_{fin}\rangle|^2) - \\ \qquad +(0.5+E_{23}) \times (|\langle 111|\psi_{fin}\rangle|^2 + |\langle 110|\psi_{fin}\rangle|^2 + |\langle 101|\psi_{fin}\rangle|^2 + |\langle 011|\psi_{fin}\rangle|^2) \end{cases} \tag{1.7}$$

$$\begin{cases} E_{22}(|\psi_{fin2}\rangle,\alpha_2) = \alpha_2 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \qquad +0.5 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \\ E_{23}(|\psi_{fin2}\rangle,\alpha_3) = \alpha_3 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \qquad +100.5 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \end{cases} \tag{1.8}$$

1.3 Analysis and Results

1.3.1 3 Player Game

1.3.1.A The captain proposes: $(99, 0, 1)$

The outcome CDC with a proposal of $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$ would represent the Nash Equilibrium of the classic Pirate Game (for 3 players).

b) b1)

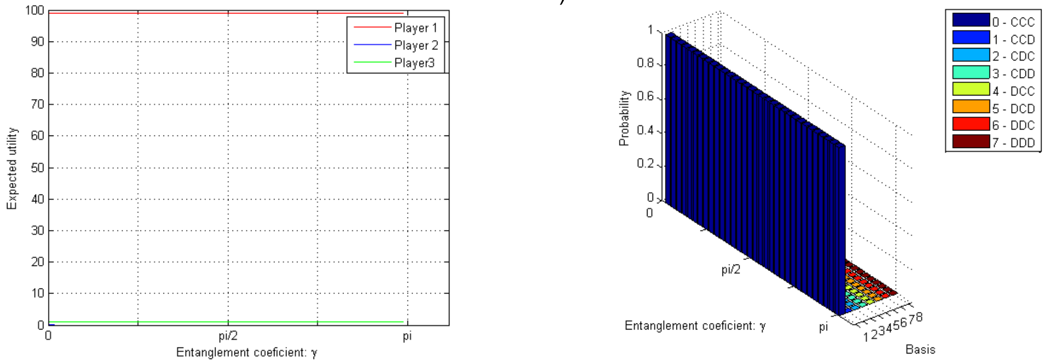


Table 1.3: a) Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Cooperate)$ operators. The initial proposal is $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$. a1) Probability distribution of the final state depending on the entanglement coefitient γ.

c) c1)

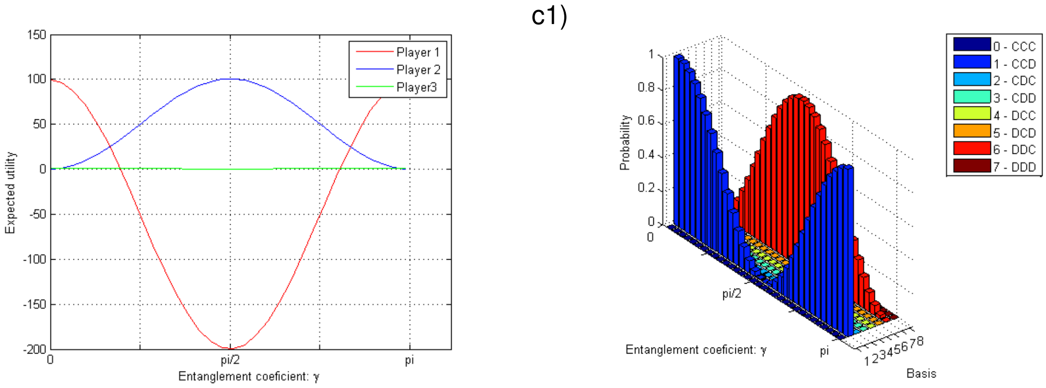


Table 1.4: b) Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Defect)$ operators. b1) Probability distruibution of the final state depending on the entanglement coefitient γ.

When the players chose at least 2 operators $Cooperate$ on the initial proposal the game ends right away, the disentangle operator $\mathcal{J}^\dagger$ is applied, and the payoff functionals are calculated given the final state. The final state will be calculated as in Equation 1.9.

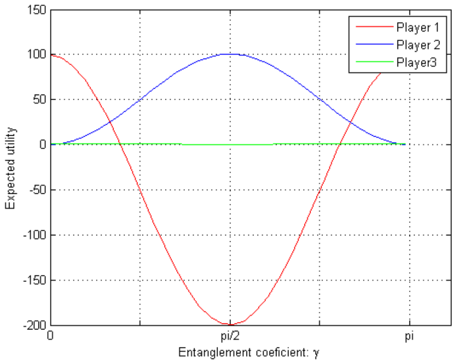
Tables 1.3, 1.4, 1.5, and 1.6present the results for the sittuation described above.

When the first proposal is rejected (more than 1 player chooses to $Defect$), the second round ensues. We calculate the final state for these states with Equation 1.10.

$$|\psi_{fin}\rangle = \mathcal{J}^\dagger |\psi_1\rangle \tag{1.9}$$

$$|\psi_{fin}\rangle = \mathcal{J}^\dagger |\psi_2\rangle \tag{1.10}$$

a)



a1)

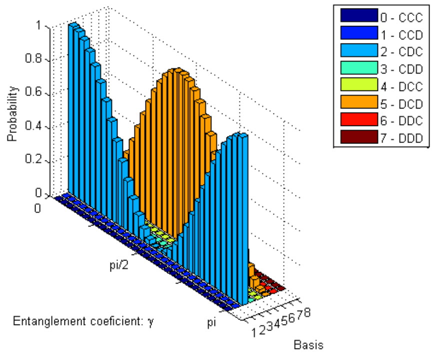
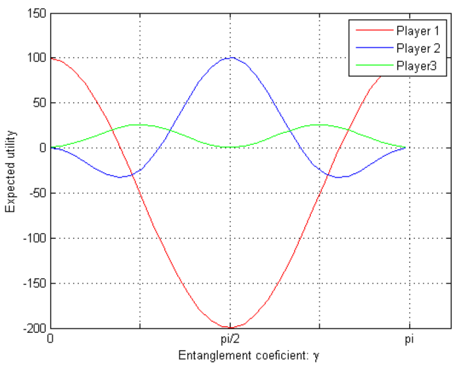


Table 1.5: c) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Cooperate)$ operators. c1) Probability distribution of the final state depending on the entanglement coefitient γ.

a)



a1)

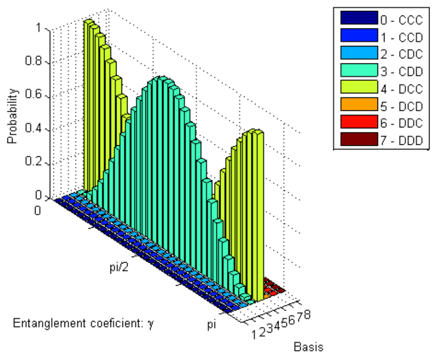


Table 1.6: b) Expected utility for 3 players, where the players will use the $(Defect, Cooperate, Cooperate)$ operators. b1) Probability distrubution of the final state depending on the entanglement coefitient γ.

a)

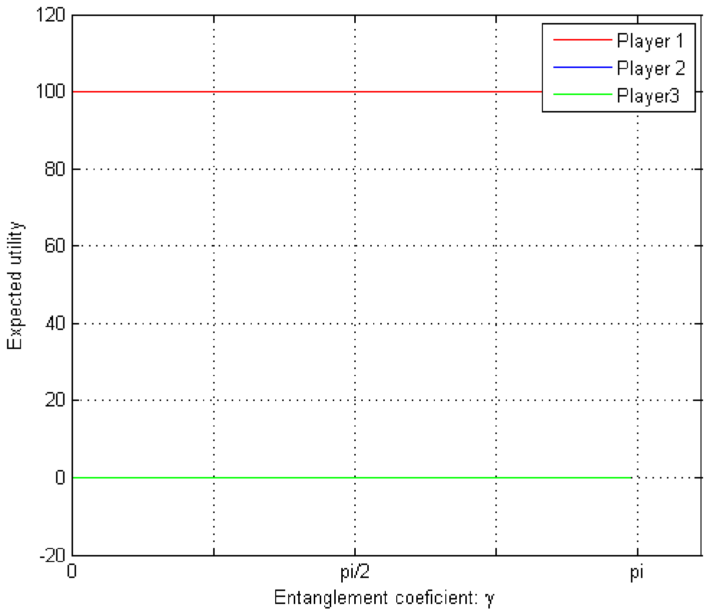


Table 1.7: a) Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Cooperate)$ operators. The initial proposal is $(\alpha_1, \alpha_2, \alpha_3) = (100, 0, 0)$.

1.3.1.B The captain proposes: $(100, 0, 0)$

Suppose captain is greedy and proposes to get the 100 coins. In the classical Pirate Game this would pose a conflict with his self-preserving needs. A pertinent question would be if this Quantum Model of the Pirate Game would allow the first captain to approve an allocation proposal. The initial proposal will be accepted if there is at least 2 players play $\mathcal{U}_i = C$ in a round. In order to answer our question we must analyse the expected utilities for all players when the initial proposal is accepted.

Our final state will be calculated as in Equation 1.9.

b)

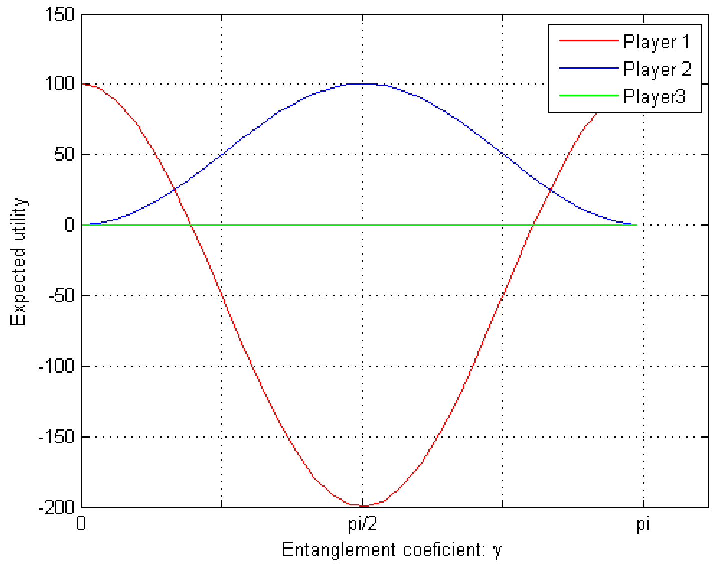


Table 1.8: b) Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Defect)$ operators.

c)

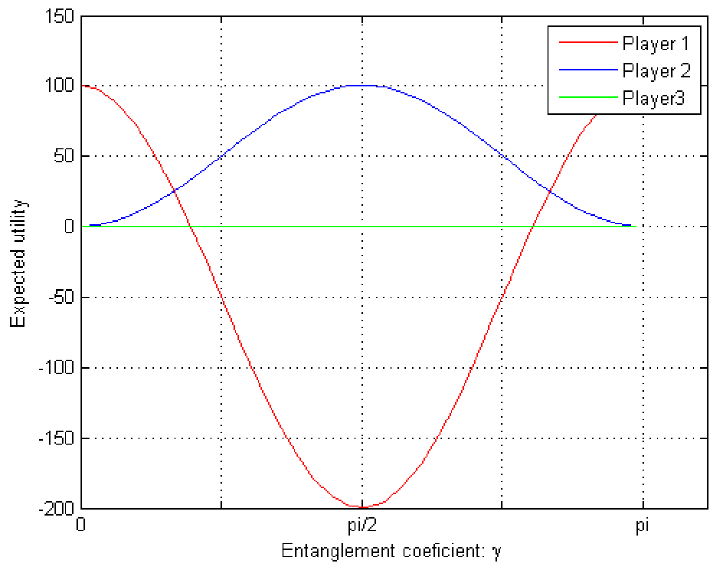


Table 1.9: c) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Cooperate)$ operators.

d)

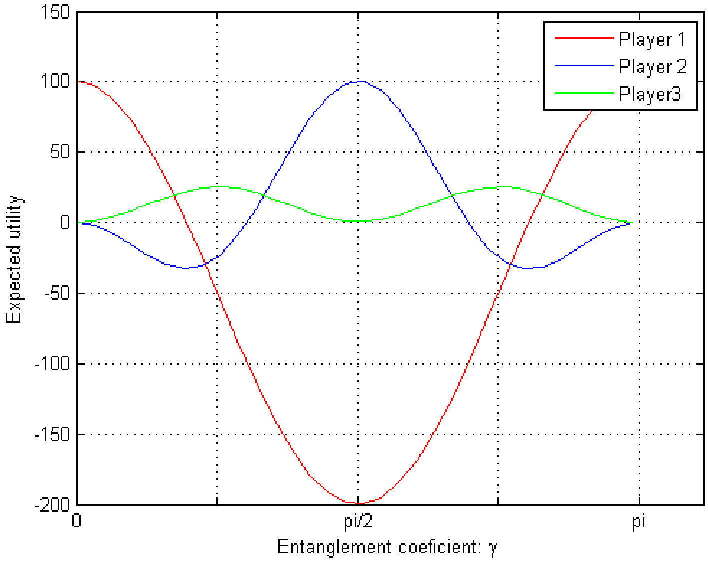


Table 1.10: d) Expected utility for 3 players, where the players will use the $(Defect, Cooperate, Cooperate)$ operators.