TÉCNICO LISBOA

Quantum Pirates

A Quantum Game-Theory Approach to The Pirate's Game

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B

I can live with doubt and uncertainty and not knowing. I think it is much more interesting to live not knowing than to have answers that might be wrong. If we will only allow that, as we progress, we remain unsure, we will leave opportunities for alternatives. We will not become enthusiastic for the fact, the knowledge, the absolute truth of the day, but remain always uncertain... In order to make progress, one must leave the door to the unknown ajar.

Richard P. Feynman

Acknowledgments

I would like to thank the Academy and Teresa Guilherme!

Abstract

Keywords

Resumo

Palavras Chave

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Abbreviations

CLT - Central Limit Theorem

List of Symbols

1

Introduction

Contents

Adapt sub-sections at will

1.1 Motivation

Basically, what drives me is the need to complete my MSc.

1.2 Problem Description

My major problem is having to write an 80 page essay. Majorly.

1.3 Objectives

In this Master's thesis, I am going to do some awesome things and just going to be generally awesome.

1.4 Contributions

Everything plus a pair a boots. Also, I will lay an egg sometime along the way.

1.5 Publications

During the course of this work, my adviser told me he needed some extra publications for his career advancement. Therefore, I made these:

- Super paper

1.6 Thesis Outline

This is how the awesomeness will be presented to you.

2

Background

Contents

Present the chapter content.

2.1 Bayesian Probability

The probability theory has its roots in the 16th century with attempts to analyse games of chance by Cardano. It is not hard to understand why games of chance are in the foundations of the probability theory. Throughout History the concept of probability has fascinated the Human being. Luck, fate were words that reflect things we feel we have no control upon and are associated with those games, and almost paradoxically we evolved in a way that we do not feel comfortable around them. Uncertainty raises us fear and doubt, so the games of chance provide a controlled environment in which we can make our experiments with uncertainty. Also by studying these games we seek to minimize the uncertainty associated with them.

2.1.0.A Kolmogorov axioms

In spite of the fact that the problem of games of chance kept attracting numerous mathematicians (with some of the most influential ones being Fermat, Pascal and Laplace), it was not until the 20th century that the Russian mathematician Kolmogorov laid the foundations of the modern probability theory (first published in 1933) introducing three axioms [1]:

1. The probability of an event is a non-negative real number:

$$P(A) \in \mathbb{R} \ \wedge \ P(A) \geq 0 \tag{2.1}$$

   This number represents the likelihood of that event happening, the greater the probability the more certain is its associated outcome.

2. The sum of probabilities of all possible outcomes in a space is always 1 ($P(\Omega) = 1$). These first two axioms leave us with the corollary that probabilities are bounded:

$$0 \leq P(A) \leq 1 \tag{2.2}$$

3. The probability of a sequence of pairwise disjoint events is the sum these events. A corolary of this axiom is:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B) \tag{2.3}$$

2.1.0.B Conditional Probability

When some evidence is presented we have what we can conditional probability (or posterior probability). Conditional probability is represented as $P(A|B)$, that could be read as: the probability of A after evidence B is presented.

The product rule is used to calculate posterior probability2.4.

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \tag{2.4}$$

2.1.1 Markov Chains

Markov Chains define a system in terms of states and the probabilistic transitions from one state to another.
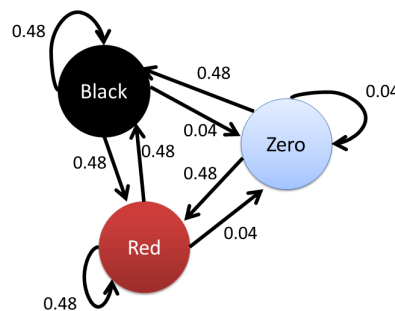
Figure 2.1: Markov Chain of a perspective on a roulette.

In 1913, at Monte Carlo Casino (Monaco), black came up twenty-six times in succession in roulette. Knowing that the probability of the ball landing on a red or on a black house is approximately 0.48 (the zero is a neutral house), many a gambler lost enormous sums while betting red as the they believed that the roulette was ripe with red, given the history. Players didn't want to believe that this and insisted that the roulette was biased; this became known as the Gamblers fallacy.

In the Markov Chain represented on Figure 2.1, we can see that in the state black the probability of transitioning to red is the same as to stay on the state black.

Given a system represented by the states $\{x_0, x_1, ..., x_i, ..., x_n\}$, and considering p_{ij} the probability of being in the state j and transitioning to the state i, the mixed state vector 2.5, which represents the probabilities of the system in the that i to transition to the other states. .

$$\vec{x_i} = \{p_{0i}x_0, p_{1i}x_1, ..., p_{ii}x_i, ..., p_{ni}x_n\} \tag{2.5}$$

The law of total probability is verified as $\sum_{j=0}^{n} p_{ji} = 1$, by specifying the every transition we get a stochastic matrix P, named the Markov matrix.

To illustrate how to construct a Markov Chain we will pick up on the example of Figure 2.1.

In this simplification of the Roulette we have 3 states:

- Black (B);

- Red (R);

- Zero (0).

We indifferently assign an index to each state, in order to construct the mixed state vector as in 2.5. Having the mixed state vectors defined the next step is to use them to create the stochactic matrix that has specified every transition 2.6.

$$R = \begin{bmatrix} p_{BB} & p_{BR} & p_{B0} \\ p_{RB} & p_{RR} & p_{R0} \\ p_{0B} & p_{0R} & p_{00} \end{bmatrix} = \begin{bmatrix} 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \end{bmatrix} \tag{2.6}$$

2.2 Von Neumann Probability

In the beginning on the 20th century the nature of light was once again in the spotlight. The question whether light would be a particle (corpuscular theory), or a wave (undulatory theory), was posed throughout History. Newton, notoriously, considered light to be a particle and presented arguments such as the fact that light travels in a straight like, not bending when presented with obstacles, unlike waves, and gave an interpretation of the diffraction mechanism by resorting to a special medium (aether), where the light corpuscles could create a localized wave [2].

The idea of light as a particle stood up until the 18th century as many scientists (Robert Hooke, Christian Huygens and Leonhard Euler to name a few) tried to explain contradictions found in corpuscular theory. This brought back the idea that light behaves like a wave.

One of the most famous experiments that corroborates the undulatory theory is the Young's experiments (19th century), or the double-slit interferometer.
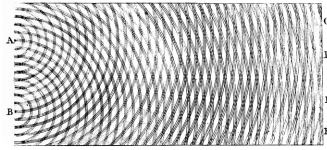

Figure 2.2: Thomas Young's sketch of two-slit diffraction of light.

The apparatus for the double-slit experiment can be seen in Figure 2.2. A light source is placed in such a way that "two portions" of light arrive at same time at the slits. Behind the barrier is a "wall" placed to intercept the light. The light captured at a wall will sport an interference pattern similar to the pattern when two waves interfere. The double-slit experiment was considered for a while the full stop on the discussion on the nature of light. However with experiments on the spectres of the light emitted by diverse substances and its relation with temperature, a new problem was posed.

The black body radiation problem was the theoretical problem where a body that absorbs light in all the electromagnetic spectrum, this makes the body acting as an natural vibrator, where each mode would have the same energy, according to the classic theory.

When a black body is at a determined temperature the frequency of the radiation it emits depends on the temperature. The classic theory predicted that most of the energy of the body would be in the high frequency part of the spectrum (violet part) where most modes would be found, this led to a prediction called the ultraviolet catastrophe. According to the classic theory the black body would emit radiation with an infinite power for temperatures above approximately 5000K. Max Plank(1901), provided an explanation where the light was exchanged in discrete amounts called quanta, so that each frequency would only have specific levels of energy. Plank also determined through experimentation the value of the energy of the quanta that became known as photons later, that value became the physical constant called Plank constant:

$$h = 6.62606957(29) \times 10^{-34} J.s \tag{2.7}$$

In 1905, Einstein used the concept of quanta (photons) to explain the photoelectric effect. De Broglie(1924), suggested that all the matter had a wave-particle duality. This prediction was confirmed by studying the interference patterns caused by electron diffraction.

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[0]Source: Young, Thomas: Probability. `http://en.wikipedia.org/wiki/File:Young_Diffraction.png(1803)`

2.2.1 Mathematical Foundations of Quantum Probability

As previously explained, Quantum Theory is a branch of physics that has arised from the need to explain certain phenomena that could not be explained with the current classical theory. In the beginning of the 20th century Dirac and von Neumann helped to create the mathematical formalisms for this theory [3] [4].

Von Neumann's contributions were focused in the mathematical rigor, as is framework is strongly based in Hilbert's theory of operators. Dirac's concerns were more of a practical nature. Their combined contributions were invaluable to establish this area.

From Dirac it is important to point the Dirac's notation (also known as Bra-ket notation or $\langle Bra|c|ket \rangle$) (introduced in 1939 [3]), that is widely used in literature based on quantum theory. This notation uses angle brackets and vertical bars to represent quantum states (or abstract vectors) as it can be seen in the formulas (2.8) and (2.9).

$$\langle z| = \left[\begin{array}{cccc} z_1^* & z_2^* & ... & z_n^* \end{array} \right] \tag{2.8}$$

$$|z\rangle = (\langle z|)^* = \left[\begin{array}{c} z_1 \\ z_2 \\ ... \\ z_n \end{array} \right] \tag{2.9}$$

This notation provides for an elegant representation of the inner product (2.10) and verifies the has linearity as it follows in equation (2.11). While the Bra-ket notation can be useful in terms of condensing information, using vectors and matrices to represent the states turns out to be a more approachable way to understand and manipulate data.

$$\langle z|z\rangle = \sum_{i=1}^{n} \bar{z}_i z_i \tag{2.10}$$

where $\bar{z}_i$ is the complex conjugate of z_i.

$$\langle z|(\alpha|x\rangle + \beta|y\rangle) = \alpha\langle z|x\rangle + \beta\langle z|y\rangle \tag{2.11}$$

2.2.1.A Born rule

The Born rule was formulated by Born in 1926. This law allows to predict the probability that a measurement on a quantum system will yield a certain result. This law provides a link between the mathematical foundation of Quantum Mechanics and the experimental evidence [5] [6].

A quantum system is represented by a n-dimensional Hilbert Space, a complex vector space in which the inner product is defined.
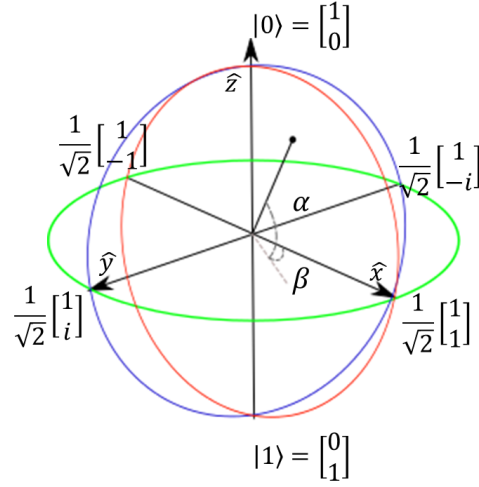
Figure 2.3: Representation of a two-dimensional Hilbert Space(H^2)

A two-dimensional Hilbert Space (corresponding to a qubit for example), can be represented by a Bloch Sphere. In the Figure 2.3 we have a representation of quantum state $|v\rangle$ in a two-dimensional Hilbert Space.

The Born rule states that if there is a system is in a state $|v\rangle$ (in a given n-dimensional Hilbert Space H), and an Hermitian operator A is applied then the probability of measuring a specific eigenvalue λ_i associated with the i-th eigenvector of A (ψ_i), will be given by [5]:

$$P_v(\lambda_i) = \langle v|Proj_i|v\rangle \tag{2.12}$$

where $Proj_i$ is a projection matrix corresponding to ψ_i :

$$Proj_i = |\psi_i\rangle\langle\psi_i| \tag{2.13}$$

Given the properties of A, the set of eigenvectors $\{\psi_1, \psi_2, ..., \psi_i, ..., \psi_n\}$ forms a orthogonal basis of the

n-dimensional Hilbert Space considered. Thus the state $|v\rangle$ can be written as a linear combination of the eigenvectors of A:

$$|v\rangle = \alpha_1\psi_1 + \alpha_2\psi_2 + ... + \alpha_i\psi_i + ... + \alpha_n\psi_n \tag{2.14}$$

The coefficients α_i are complex numbers called probability amplitudes, and their squared sum is equal to 1:

$$\sum_{i=0}^{n} |\alpha_i|^2 = 1 \tag{2.15}$$

this brings us to:

$$P_v(\lambda_i) = \langle v|\psi_i\rangle\langle\psi_i|v\rangle = |\langle v|\psi_i\rangle|^2 = |\alpha_i^*\alpha_i|^2 \tag{2.16}$$

So the determination of the probability of an event ($P(A)$), is made by projecting the quantum state on the eigenvectors corresponding to the operator A of the Hilbert Space and measuring the squared length of the projection. [7]

$$P(A) = (Proj_A|z\rangle)^2 \tag{2.17}$$

Applying an operator can be seen as applying a rotation matrix on the system ($|v\rangle$) and measuring the projection of $|v\rangle$ onto the imaginary axis and the real axis (Figure 2.3), or considering that we have a determined state vector and rotating the orthogonal basis of the Hilbert Space according to an operator and then to do a projection on the new chosen orthogonal basis.

According to Leiffer [8] "quantum theory can be thought of as a non-commutative, operator-valued, generalization of classical probability theory".

As in the classical probability theory where from a random variable it is possible to establish a probability distribution, also known as density function, in the Hilbert space there is a equivalent density operator. The density operator (ρ) is a Hermitian operator that has the particularity of having its trace equal to 1 [5].

$$\rho = \sum_{i=0}^{n} \alpha_i|\psi_i\rangle\langle\psi_i| \tag{2.18}$$

$$tr(\rho) = 1 \tag{2.19}$$

2.2.1.B Lüders Rule

Lüders Rule defines the state of a quantum system after a partial measurement. We can establish a parallel between this selective measurement and conditional probability2.1.0.B [9].

To measure the state of a system we first use a projection operator on the quantum state. This is also the first stage of applying Lüders Rule2.20.

$$A = Proj_A|s\rangle \tag{2.20}$$

After the measurement the resulting state is normalized2.21.

$$|s_A\rangle = \frac{A}{|A|} \tag{2.21}$$

2.2.1.C Example of the double-slit experiment with electrons

Like the Young's Experiment with light created an interference pattern similar to a wave, firing electrons one at the time produces a similar pattern. The unobserved fired electron behaved like a wave and after passing the slits the wavelets interfered with one another to create a interference pattern. However if a measuring device was active while the electron was fired the interference pattern wasn't registered.
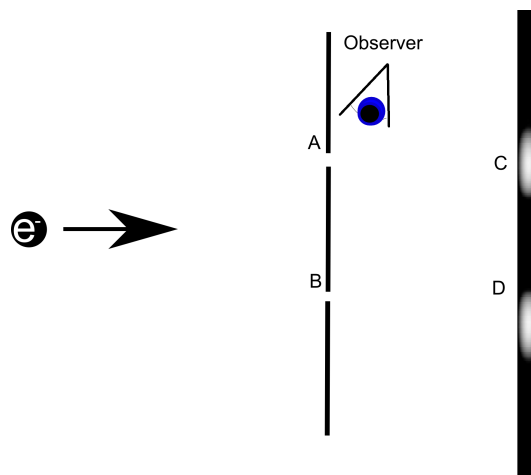

Figure 2.4: Double-slit experiment where there is a measuring device that allows to know through which slit the electron passed.

The fact that the electron was measured while passing through a slit produced a particle behaviour, explained by the classical theory (Figure 2.4).
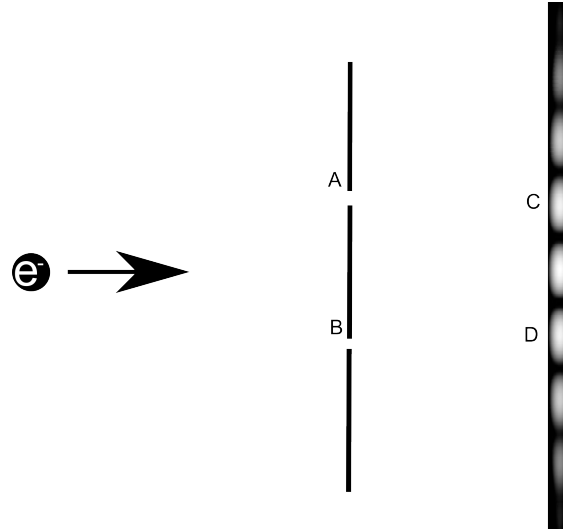
Figure 2.5: Double-slit experiment, where electrons exhibit the interference pattern characteristic in waves

In this experiment a single electron is shoot at a time. So in the start of the experiment (S), we know the initial position of the electron.

A final measurement (F) is made when the electron hits the wall behind the slits, where we know the final position of the electron. [1]

If this experiment is observed, there is an intermediate measure that tells us whether the electron went throught the slit A or B. The corresponding probability amplitudes related to this measurement are ω_A and ω_B, and:

$$\omega_A = \langle F|A\rangle\langle A|S\rangle \tag{2.22}$$

$$\omega_B = \langle F|B\rangle\langle B|S\rangle \tag{2.23}$$

If we consider the intermediate measurement the probability $P(F|S)$ will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle|^2 + |\langle F|B\rangle\langle B|S\rangle|^2 \tag{2.24}$$

But if we only measure the position of the electron at the end of the experiment that probability will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle + \langle F|B\rangle\langle B|S\rangle|^2 \tag{2.25}$$

The latter equation will be dependent on a interference coefficient that will be responsible the interference pattern observed in the unobserved experiment.

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[1]Mohrhoff, U.: Two Slits. `http://thisquantumworld.com/wp/the-mystique-of-quantum-mechanics/two-slit-experiment/#fn1back`

2.2.1.D   Example of the Polarization of Light

The photons in a beam of light don't vibrate all the same direction in most of the natural sources of light. To filter the light polaroids are used. A polaroid only allows the passage of light in a well-defined direction and thus reducing the intensity of the light. In the Figure 2.6 we can observe that the introduction of the oblique polaroid in the third situation led to a passage of light. Although there is a classical explanation to this phenomenon if we consider waves when we are considering a beam of light, if our light source emits one photon at the time a quantum mechanical explanation is needed [10].
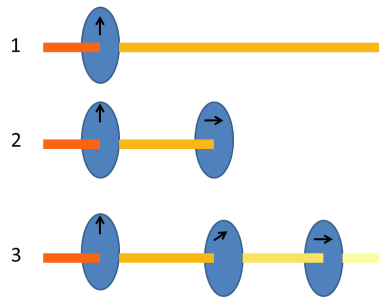

Figure 2.6: 1. With one vertical polaroid the unpolarized light is attenuated by a half. 2. Vertical polarization followed by a horizontal polarization will block all the passing light. 3. Inserting a oblique polaroid between the vertical and horizontal polaroids will allow light to pass.

To model the polarization of the photon in a quantum setting, we will use a vector $|v\rangle$ in a two-dimensional Hilbert Space:

$$|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.26}$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ would represent the vertical direction (could also be represented by the state vector $|\uparrow\rangle$), and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the horizontal one (another possible representation to this basis could be $|\rightarrow\rangle$). We can consider the Figure 2.3 as a graphical representation for this system.

In the first situation if $a = \frac{1}{\sqrt{2}}$, that would mean that the probability of passing the vertical polaroid would be $a = (\frac{1}{\sqrt{2}})^2 = 0.5$, that would light to the expected reduction of a half of the intensity of light. After passing through the vertical polaroid the photon will have a polarization of $|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Considering now the second situation after we have our photon polarized vertically (like in the end on the first situation), the probability of being vertically polarized is 1, thus making the probability of passing through the horizontal polaroid 0. In the third situation, after being vertically polarized the photon will pass through an oblique polaroid that makes its direction

$$|v\rangle = i.sen(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + cos(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.27}$$

θ being the angle of the polaroid. The photon filtered by the vertical polaroid will pass this second polaroid with a probability of $(i.sen(\theta))^2$, becoming polarized according to the filter, as we can observe depending on the value of θ we will now have a horizontal component in the vector that describes the state of the photon. This will make the photon pass the horizontal polaroid with a probability of $cos(\theta)^2$.

3

Related Work

Contents

Present the chapter content.

3.1 Quantum Walks

3.1.1 Quantum Walk on a Line

A Quantum Walk is the quantum version of random walks, which are a mathematical formalism to describe a path constructed by random steps. This processes can be described by a Markov Chain.

We can define the Discrete Quantum Walk on a Line as a series of Left/Right decisions. Understanding this algorithm is important towards being able to define and design more complex algorithms.

We followed an approach suggested by [11] towards simulating n-steps of a Quantum Walk on a Line, the algorithm can be consulted on A. In a discrete quantum walk in a line we want to preserve some properties, namely: the probability of turning left should be equal to the probability of turning right. To represent a state in this algorithm we will need the number of the node and a direction (identified as L,R) 3.1.

$$|\psi\rangle = |n, L\rangle \tag{3.1}$$

Having two equally possible choices we can resort to a coin metaphor [11] [12]. We toss a coin and go either Left or Right depending on the result. In a quantum version we need to define a Coin Operator (Coin Matrix) which is responsible to imprint a direction to our state with equal probability. This operator is a unitary matrix in a 2-dimension Hilbert space. Some examples of Coin Operators are the Hadamard matrix3.2 and a symmetric unitary matrix 3.3.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3.2}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{3.3}$$

Taking the Hadamard matrix as an example3.2, the coin matrix will operate on the state in the following way 3.43.5 [11].

$$C|n, L\rangle = \frac{1}{\sqrt{2}}|n, L\rangle + \frac{1}{\sqrt{2}}|n, R\rangle \tag{3.4}$$

$$C|n, R\rangle = \frac{1}{\sqrt{2}}|n, L\rangle - \frac{1}{\sqrt{2}}|n, R\rangle \tag{3.5}$$

The Coin Matrix obtains its name by being the quantum equivalent of flipping a classic coin. After tossing a coin comes an operator that will move the node in the direction assigned. The operator responsible for this modification is commonly referred as Shift Operator 3.63.7.

$$S|n, L\rangle = \frac{1}{\sqrt{2}}|n - 1, L\rangle \tag{3.6}$$

$$S|n, R\rangle = \frac{1}{\sqrt{2}}|n + 1, R\rangle \tag{3.7}$$

These matrices (Coin Matrix and Shift Operator) are referred conceptually ubiquitously throughout the literature [sources], therefore it is important to be familiar with them. A single step of the algorithm A is illustrated in Figure 3.1.

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Depending on the Coin Matrix we can get different distributions. In Figures 3.3 and 3.2

Starting on the middle of a line we can shift one unit left or right. If we took the classical approach in which we tossed a fair coin, and after n-steps we measured the final node repeatedly, by the Central Limit Theorem (CLT) the final distribution would converge to a normal distribution.

3.2   Quantum Models

The rationale behind building a quantum version of a Game Theory and/or Statistics problem lays in bringing phenomena like quantum superposition, and entanglement into known frameworks allowing different results, that seemingly perform at least as good as their classical versions.
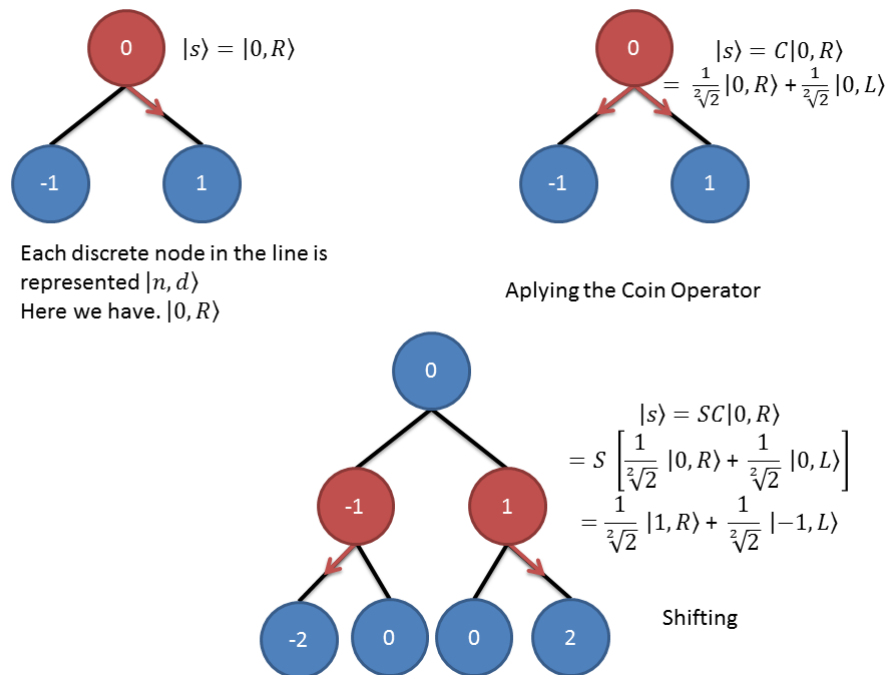
Figure 3.1: Simulating a step of a discrete quantum walk on a line. In the beginning we have a state characterized by the position (0) and a direction (either Left or Right).

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The effort put in converting known classical problems also enables the familiarization with the potential differences these models bring.

3.2.1 Monty Hall Problem

The Monty Hall problem became popularized in 1990 in a column, in the magazine Parade [13]. It is loosely based on a television game show hosted by Monty Hall, but it was first attributed to a statistician, named Selvin. The problem is posed as it follows:

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Most people, when faced with this problem, will be indifferent about whether to switch or to stay with the initially picked door.
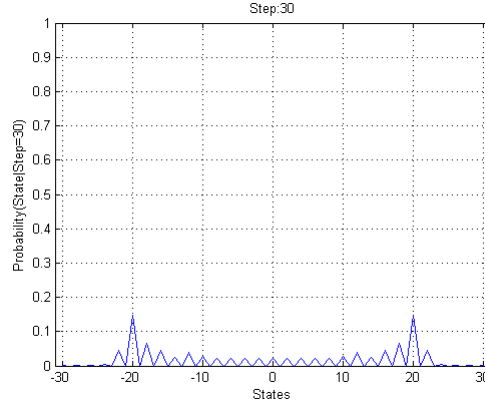
Figure 3.2: 30 Step of the Simulation A using Matrix 3.2 as a Coin Operator.
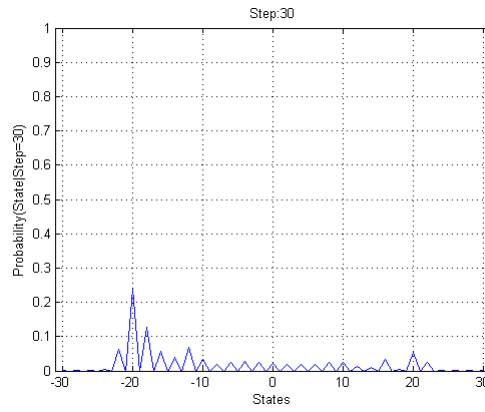

Figure 3.3: 30 Step of the Simulation A using a Hadamard Matrix 3.2 as a Coin Operator.

It is also verified that they will tend to stick with their first choice. According to Granberg and Brow [14], only 13% of 228 subjects decided to switch their initial choice. However, by using probability theory, one can arrive at the conclusion that, in fact, it is advantageous to switch given the previous formulation.

The action of the Host implies a belief update on the probabilities of the variables in the system. This poses a violation of rational decision making; subjects do not seem to follow the best strategy which would maximize their chances of winning the prize.

To understand this exercise, one can look at the decision tree in Figure 3.4. We assumed indifferently that the player chose the door No. 1. The situation is symmetric whichever door she chooses.

Assuming we call C_1 to the variable that describes whether or not the car in behind door No. 1. The variables C_2 and C_3 will respectively describe the probability associated with the car being (or not), behind doors 2 and 3 ($P(C_2) = 1$ or $P(C_2) = 0$, for example).

After the player has had her choice, the host will perform an operation on the remaining two doors. The host of the show has complete information of the game, unlike the player.
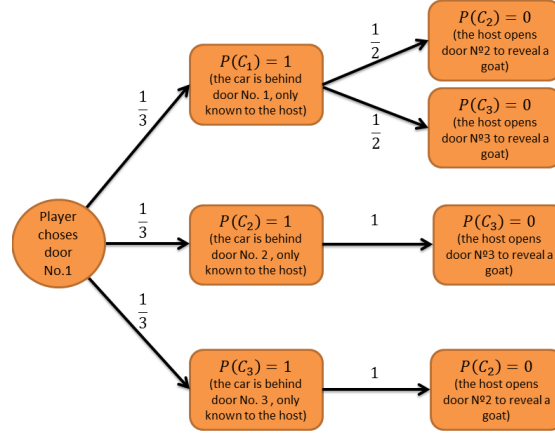
Figure 3.4: Decision tree modelling the Monty Hall problem.

$$P(C_1) = P(C_1|\neg C_2) + P(C_1|\neg C_3) \tag{3.8}$$

$$P(C_1) = (\frac{1}{3} \times \frac{1}{2}) + (\frac{1}{3} \times \frac{1}{2}) = 1 \tag{3.9}$$

$$P(\neg C_1) = P(C_2|\neg C_3) + P(C_3|\neg C_2) = (\frac{1}{3} \times 1) + (\frac{1}{3} \times 1) = 2/3 \tag{3.10}$$

The probability of switching and getting the car is twice 3.10 as likely of staying with the first choice and getting the prize3.9.

3.2.1.A   Quantum Model

Various Models have been proposed to describe a quantum version of the Monty Hall problem.

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The Monty Hall problem has an interesting property; the host reveals information on the system. Despite being a counter-intuitive problem, a quantum approach to this problem allows a in-depth comparison between the classical measurement and the quantum measurement. The classic Monty Hall problem is modelled using conditional probability and Baye's rule. In a quantum game the result is determined measuring the outcome of the final state, rather than the intermediate of players [15].

4

Your Solution here

Contents

Present the chapter content.

4.1 Section A

4.1.1 Subsection A

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As seen in [**?**]. *Enfatizar*

4.1.2 Subsection B

Figure 4.1: Dummy Figure Caption.

Remember you can change the reference style. Another dummy citation [**?**].

4.2 Section B

4.2.1 Subsection A

The model described can also be represented as

$$\dot{\mathbf{x}}(t) = \mathbf{T}\mathbf{z}(y), \ \mathbf{y}(0) = \mathbf{y}_0, \ z \geq 0 \tag{4.1}$$

where

$$\mathbf{A} = \begin{bmatrix} -(a_{12} + a_{10}) & a_{21} \\ a_{12} & -(a_{21} + a_{20}) \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.2}$$

4.2.2 Subsection B

Table 4.1: Dummy Table.

Vendor Name	Short Name	Commercial Name	Manufacturer
Text in Multiple Row	ABC	ABC®	ABC SA
	DEF	DEF®	DEF SA
	GHF	GHF®	GHF SA
Text in Single Row	IJK	IJK®	IJK SA
Frescos SA	LMN	LMN®	LMN SA
Carros Lda.	Text in Multiple Column		

5

Evaluation

Contents

Adapt sub-sections at will

5.1   Section 1

5.2   Section 2

6

Conclusions and Future Work

Cap your work!

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A

Matlab Simulation: Discrete Quantum Walk on a Line

```matlab
1
2
3  %—— number of steps in the simulation
4  steps= 30;
5
6  %—— hadamard matrix
7  H = [1/sqrt(2) 1/sqrt(2); 1/sqrt(2) −1/sqrt(2) ];
8  %—— symetric matrix
9  M = [1/sqrt(2) 1i/sqrt(2); 1i/sqrt(2) 1/sqrt(2) ];
10 %—— coin flip unitary operator
11 C = H;
12 C=M;
13
14 %—— coin flip matrix
15 CM= zeros((steps*2+1),2);
16 %—— shift matrix
17 SM= zeros(steps*2+1,2);
18
19 %—— middle index
20 i0= steps+1;
21
22 %—— initialize flip probability amplitudes (coin has 1/2 chance of going +1 or −1)
23 CM(i0,1)=1/sqrt(2);
24 CM(i0,2)=1/sqrt(2);
25
26 for i=1:steps
27     %—— clean SM
28     SM= zeros(steps*2+1,2);
29
30     for j=1:(steps*2+1)
31         if CM(j, 1)≠0
32             SM(j−1,1)=CM(j, 1);
33         end
34         if CM(j, 2)≠0
35             SM(j+1,2)=CM(j, 2);
36         end
37     end
38     SM;
39     %disp('——————');
40     %—— clean CM
41     CM= zeros(steps*2+1,2);
42     for j=1:(steps*2+1)
43         if SM(j, 1)≠0
44             CM(j, 1)= CM(j, 1)+C(1,1)*SM(j, 1);
45             CM(j, 2)= CM(j, 2)+C(1,2)*SM(j, 1);
46         end
47         if SM(j, 2)≠0
48             CM(j, 1)= CM(j, 1)+C(2,1)*SM(j, 2);
49             CM(j, 2)= CM(j, 2)+C(2,2)*SM(j, 2);
50         end
51     end
52     CM;
53
54     %Display
55
56     figure
57
58     probability=zeros(steps*2+1);
59
60     for j=1:steps*2+1
61         probability(j)=abs(SM(j,1)).^2+ abs(SM(j,2)).^2;
62     end
63     axisP = −steps : steps;
64     plot(axisP,probability)
65     title(strcat('Step: ' ,num2str(i)))
66     axis([−steps−1 , steps+1, 0, 1]);
67     grid on
68     ylabel(strcat('Probability(State|Step=',num2str(i), ')'));
69     xlabel('States');
70
71     %Display
72 end
```