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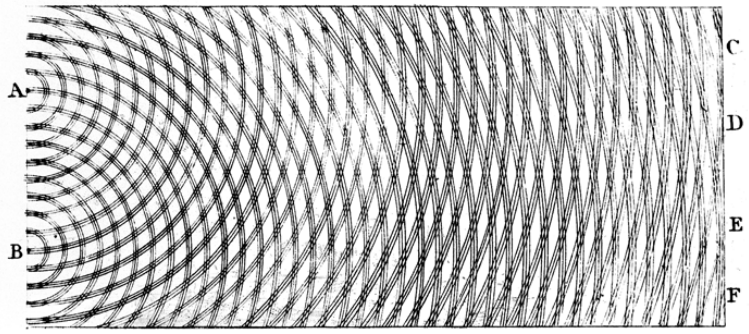

Quantum Pirates

A Quantum Game-Theory Approach to The Pirate's Game

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I can live with doubt and uncertainty and not knowing. I think it is much more interesting to live not knowing than to have answers that might be wrong. If we will only allow that, as we progress, we remain unsure, we will leave opportunities for alternatives. We will not become enthusiastic for the fact, the knowledge, the absolute truth of the day, but remain always uncertain... In order to make progress, one must leave the door to the unknown ajar.

Richard P. Feynman

Acknowledgments

[TO DO]

Abstract

Keywords

Quantum Game Theory

Pirate Game

Quantum Mechanics

Game Theory

Probability Theory

Resumo

Palavras Chave

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Abbreviations

CLT - Central Limit Theorem

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Background

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2.1 Bayesian Probability

The probability theory has its roots in the 16^{th} century with attempts to analyze games of chance by Cardano. It is not hard to understand why games of chance are in the foundations of the probability theory. Throughout History the concept of probability has fascinated the Human being. Luck, fate were words that reflect things we feel we have no control upon and are associated with those games, and almost paradoxically we evolved in a way that we do not feel comfortable around them.

2.1.0.A Kolmogorov axioms

In spite of the fact that the problem of games of chance kept attracting numerous mathematicians (with some of the most influential ones being Fermat, Pascal and Laplace), it was not until the 20^{th} century that the Russian mathematician Kolmogorov laid the foundations of the modern probability theory (first published in 1933) introducing three axioms [1]:

1. The probability of an event is a non-negative real number:

$$P(A) \in \mathbb{R} \ \wedge \ P(A) \geq 0 \tag{2.1}$$

   This number represents the likelihood of that event happening, the greater the probability the more certain is its associated outcome.

2. The sum of probabilities of all possible outcomes in a space is always 1 ($P(\Omega) = 1$). These first two axioms leave us with the corollary that probabilities are bounded:

$$0 \leq P(A) \leq 1 \tag{2.2}$$

3. The probability of a sequence of pairwise disjoint events is the sum these events. A corolary of this axiom is:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B) \tag{2.3}$$

2.1.0.B Conditional Probability

When some evidence is presented we have what we can conditional probability (or posterior probability). Conditional probability is represented as $P(A|B)$, that could be read as: the probability of A after evidence B is presented.

The product rule is used to calculate posterior probability2.4.

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \tag{2.4}$$

2.1.1 Markov Chains

Markov Chains define a system in terms of states and the probabilistic transitions from one state to another.

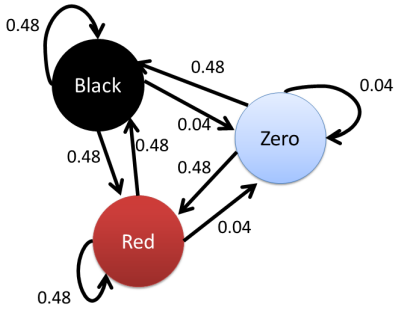

Figure 2.1: Markov Chain of a perspective on a roulette.

In 1913, at Monte Carlo Casino (Monaco), black came up twenty-six times in succession in roulette. Knowing that the probability of the ball landing on a red or on a black house is approximately 0.48 (the zero is a neutral house), many a gambler lost enormous sums while betting red as the they believed that the roulette was ripe with red, given the history. Players didn't want to believe that this and insisted that the roulette was biased; this became known as the Gamblers fallacy.

In the Markov Chain represented on Figure 2.1, we can see that in the state black the probability of transitioning to red is the same as to stay on the state black.

Given a system represented by the states $\{x_0, x_1, ..., x_i, ..., x_n\}$, and considering p_{ij} the probability of being in the state j and transitioning to the state i, the mixed state vector 2.5, which represents the probabilities of the system in the that i to transition to the other states. .

$$\overrightarrow{x_i} = \{p_{0i}x_0, p_{1i}x_1, ..., p_{ii}x_i, ..., p_{ni}x_n\} \tag{2.5}$$

The law of total probability is verified as $\sum_{j=0}^{n} p_{ji} = 1$, by specifying the every transition we get a stochastic matrix P, named the Markov matrix.

To illustrate how to construct a Markov Chain we will pick up on the example of Figure 2.1.

In this simplification of the Roulette we have 3 states:

- Black (B);
- Red (R);
- Zero (0).

We indifferently assign an index to each state, in order to construct the mixed state vector as in 2.5. Having the mixed state vectors defined the next step is to use them to create the stochactic matrix that has specified every transition 2.6.

$$R = \begin{bmatrix} p_{BB} & p_{BR} & p_{B0} \\ p_{RB} & p_{RR} & p_{R0} \\ p_{0B} & p_{0R} & p_{00} \end{bmatrix} = \begin{bmatrix} 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \end{bmatrix} \tag{2.6}$$

2.2 Von Neumann Probability

In the beginning on the 20^{th} century the nature of light was once again in the spotlight. The question whether light would be a particle (corpuscular theory), or a wave (undulatory theory), was posed throughout History. Newton, notoriously, considered light to be a particle and presented arguments such as the fact that light travels in a straight like, not bending when presented with obstacles, unlike waves, and gave an interpretation of the diffraction mechanism by resorting to a special medium (aether), where the light corpuscles could create a localized wave [2].

The idea of light as a particle stood up until the 18^{th}century as many scientists (Robert Hooke, Christian Huygens and Leonhard Euler to name a few) tried to explain contradictions found in corpuscular theory. This brought back the idea that light behaves like a wave.

One of the most famous experiments that corroborates the undulatory theory is the Young's experiments (19^{th} century), or the double-slit interferometer.

The apparatus for the double-slit experiment can be seen in Figure 2.2. A light source is placed in such a way that "two portions" of light arrive at same time at the slits. Behind the barrier is a "wall" placed to intercept the light. The light captured at a wall will sport an interference pattern similar to the pattern

[0]Source: Young, Thomas: Probability. `http://en.wikipedia.org/wiki/File:Young_Diffraction.png(1803)`

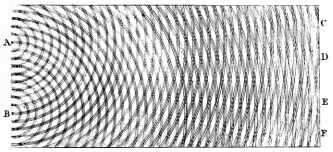

Figure 2.2: Thomas Young's sketch of two-slit diffraction of light.

when two waves interfere. The double-slit experiment was considered for a while the full stop on the discussion on the nature of light. However with experiments on the spectres of the light emitted by diverse substances and its relation with temperature, a new problem was posed.

The black body radiation problem was the theoretical problem where a body that absorbs light in all the electromagnetic spectrum, this makes the body acting as an natural vibrator, where each mode would have the same energy, according to the classic theory.

When a black body is at a determined temperature the frequency of the radiation it emits depends on the temperature. The classic theory predicted that most of the energy of the body would be in the high frequency part of the spectrum (violet part) where most modes would be found, this led to a prediction called the ultraviolet catastrophe. According to the classic theory the black body would emit radiation with an infinite power for temperatures above approximately 5000K. Max Plank(1901), provided an explanation where the light was exchanged in discrete amounts called quanta, so that each frequency would only have specific levels of energy. Plank also determined through experimentation the value of the energy of the quanta that became known as photons later, that value became the physical constant called Plank constant:

$$h = 6.62606957(29) \times 10^{-34} J.s \tag{2.7}$$

In 1905, Einstein used the concept of quanta (photons) to explain the photoelectric effect. De Broglie(1924), suggested that all the matter had a wave-particle duality. This prediction was confirmed by studying the interference patterns caused by electron diffraction.

2.2.1 Mathematical Foundations of Quantum Probability

As previously explained, Quantum Theory is a branch of physics that has arised from the need to explain certain phenomena that could not be explained with the current classical theory. In the beginning of the 20^{th} century Dirac and von Neumann helped to create the mathematical formalisms for this theory [3] [4].

Von Neumann's contributions were focused in the mathematical rigor, as is framework is strongly based

in Hilbert's theory of operators. Dirac's concerns were more of a practical nature. Their combined contributions were invaluable to establish this area.

From Dirac it is important to point the Dirac's notation (also known as Bra-ket notation or $\langle Bra|c|ket\rangle$) (introduced in 1939 [3]), that is widely used in literature based on quantum theory. This notation uses angle brackets and vertical bars to represent quantum states (or abstract vectors) as it can be seen in the formulas (2.8) and (2.9).

$$\langle z| = \left[\begin{array}{cccc} z_1^* & z_2^* & ... & z_n^* \end{array} \right] \tag{2.8}$$

$$|z\rangle = (\langle z|)^* = \left[\begin{array}{c} z_1 \\ z_2 \\ ... \\ z_n \end{array} \right] \tag{2.9}$$

This notation provides for an elegant representation of the inner product (2.10) and verifies the has linearity as it follows in equation (2.11). While the Bra-ket notation can be useful in terms of condensing information, using vectors and matrices to represent the states turns out to be a more approachable way to understand and manipulate data.

$$\langle z|z\rangle = \sum_{i=1}^{n} \bar{z_i} z_i \tag{2.10}$$

where $\bar{z_i}$ is the complex conjugate of z_i.

$$\langle z|(\alpha|x\rangle + \beta|y\rangle) = \alpha\langle z|x\rangle + \beta\langle z|y\rangle \tag{2.11}$$

2.2.1.A Born rule

The Born rule was formulated by Born in 1926. This law allows to predict the probability that a measurement on a quantum system will yield a certain result. This law provides a link between the mathematical foundation of Quantum Mechanics and the experimental evidence [5] [6].

A quantum system is represented by a n-dimensional Hilbert Space, a complex vector space in which the inner product is defined.

A two-dimensional Hilbert Space (corresponding to a qubit for example), can be represented by a Bloch

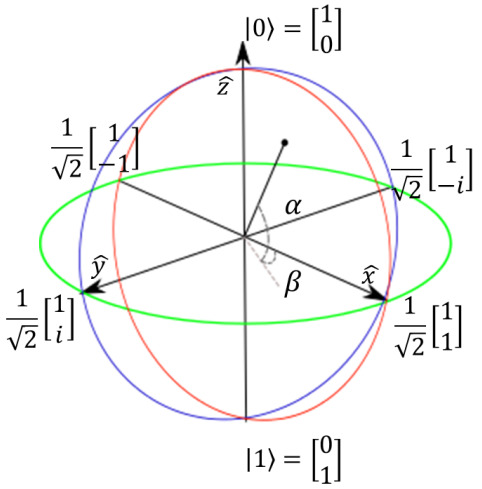

Figure 2.3: Representation of a two-dimensional Hilbert Space(H^2)

Sphere. In the Figure 2.3 we have a representation of quantum state $|v\rangle$ in a two-dimensional Hilbert Space.

The Born rule states that if there is a system is in a state $|v\rangle$ (in a given n-dimensional Hilbert Space H), and an Hermitian operator A is applied then the probability of measuring a specific eigenvalue λ_i associated with the i-th eigenvector of A (ψ_i), will be given by [5]:

$$P_v(\lambda_i) = \langle v|Proj_i|v\rangle \tag{2.12}$$

where $Proj_i$ is a projection matrix corresponding to ψ_i :

$$Proj_i = |\psi_i\rangle\langle\psi_i| \tag{2.13}$$

Given the properties of A, the set of eigenvectors $\{\psi_1, \psi_2, ..., \psi_i, ..., \psi_n\}$ forms a orthogonal basis of the n-dimensional Hilbert Space considered. Thus the state $|v\rangle$ can be written as a linear combination of the eigenvectors of A:

$$|v\rangle = \alpha_1\psi_1 + \alpha_2\psi_2 + ... + \alpha_i\psi_i + ... + \alpha_n\psi_n \tag{2.14}$$

The coefficients α_i are complex numbers called probability amplitudes, and their squared sum is equal to 1:

$$\sum_{i=0}^{n} |\alpha_i|^2 = 1 \tag{2.15}$$

this brings us to:

$$P_v(\lambda_i) = \langle v|\psi_i\rangle\langle\psi_i|v\rangle = |\langle v|\psi_i\rangle|^2 = |\alpha_i^*\alpha_i|^2 \tag{2.16}$$

So the determination of the probability of an event ($P(A)$), is made by projecting the quantum state on the eigenvectors corresponding to the operator A of the Hilbert Space and measuring the squared length of the projection. [7]

$$P(A) = (Proj_A|z\rangle)^2 \tag{2.17}$$

Applying an operator can be seen as applying a rotation matrix on the system ($|v\rangle$) and measuring the projection of $|v\rangle$ onto the imaginary axis and the real axis (Figure 2.3), or considering that we have a determined state vector and rotating the orthogonal basis of the Hilbert Space according to an operator and then to do a projection on the new chosen orthogonal basis.

According to Leiffer [8] “quantum theory can be thought of as a non-commutative, operator-valued, generalization of classical probability theory”.

As in the classical probability theory where from a random variable it is possible to establish a probability distribution, also known as density function, in the Hilbert space there is a equivalent density operator. The density operator (ρ) is a Hermitian operator that has the particularity of having its trace equal to 1 [5].

$$\rho = \sum_{i=0}^{n} \alpha_i|\psi_i\rangle\langle\psi_i| \tag{2.18}$$

$$tr(\rho) = 1 \tag{2.19}$$

2.2.1.B Lüders Rule

Lüders Rule defines the state of a quantum system after a partial measurement. We can establish a parallel between this selective measurement and conditional probability2.1.0.B [9].

To measure the state of a system we first use a projection operator on the quantum state. This is also the first stage of applying Lüders Rule2.20.

$$A = Proj_A|s\rangle \tag{2.20}$$

After the measurement the resulting state is normalized2.21.

$$|s_A\rangle = \frac{A}{|A|} \tag{2.21}$$

2.2.1.C Example of the double-slit experiment with electrons

Like the Young's Experiment with light created an interference pattern similar to a wave, firing electrons one at the time produces a similar pattern. The unobserved fired electron behaved like a wave and after passing the slits the wavelets interfered with one another to create a interference pattern. However if a measuring device was active while the electron was fired the interference pattern wasn't registered.

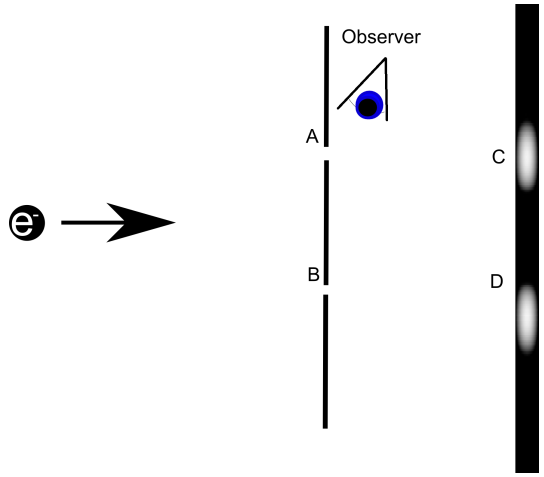

Figure 2.4: Double-slit experiment where there is a measuring device that allows to know through which slit the electron passed.

The fact that the electron was measured while passing through a slit produced a particle behaviour, explained by the classical theory (Figure 2.4).

In this experiment a single electron is shoot at a time. So in the start of the experiment (S), we know the initial position of the electron.

A final measurement (F) is made when the electron hits the wall behind the slits, where we know the final position of the electron. [1]

[1] Mohrhoff, U.: Two Slits. http://thisquantumworld.com/wp/the-mystique-of-quantum-mechanics/two-slit-experiment/#fn1back

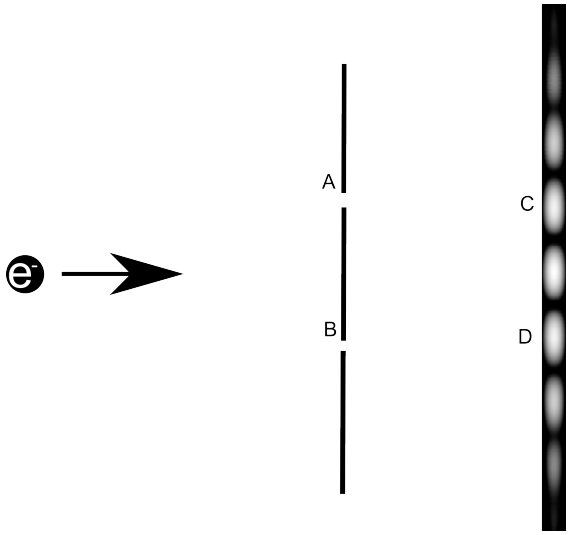

Figure 2.5: Double-slit experiment, where electrons exhibit the interference pattern characteristic in waves

If this experiment is observed, there is an intermediate measure that tells us whether the electron went throught the slit A or B. The corresponding probability amplitudes related to this measurement are ω_A and ω_B, and:

$$\omega_A = \langle F|A\rangle\langle A|S\rangle \tag{2.22}$$

$$\omega_B = \langle F|B\rangle\langle B|S\rangle \tag{2.23}$$

If we consider the intermediate measurement the probability $P(F|S)$ will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle|^2 + |\langle F|B\rangle\langle B|S\rangle|^2 \tag{2.24}$$

But if we only measure the position of the electron at the end of the experiment that probability will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle + \langle F|B\rangle\langle B|S\rangle|^2 \tag{2.25}$$

The latter equation will be dependent on a interference coefficient that will be responsible the interference pattern observed in the unobserved experiment.

2.2.1.D Example of the Polarization of Light

The photons in a beam of light don't vibrate all the same direction in most of the natural sources of light. To filter the light polaroids are used. A polaroid only allows the passage of light in a well-defined direction and thus reducing the intensity of the light. In the Figure 2.6 we can observe that the introduction of the oblique polaroid in the third situation led to a passage of light. Although there is a classical explanation to this phenomenon if we consider waves when we are considering a beam of light, if our light source emits one photon at the time a quantum mechanical explanation is needed [10].

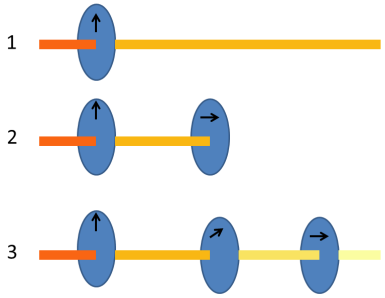

Figure 2.6: 1. With one vertical polaroid the unpolarized light is attenuated by a half. 2. Vertical polarization followed by a horizontal polarization will block all the passing light. 3. Inserting a oblique polaroid between the vertical and horizontal polaroids will allow light to pass.

To model the polarization of the photon in a quantum setting, we will use a vector $|v\rangle$ in a two-dimensional Hilbert Space:

$$|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.26}$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ would represent the vertical direction (could also be represented by the state vector $|\uparrow\rangle$), and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the horizontal one (another possible representation to this basis could be $|\rightarrow\rangle$). We can consider the Figure 2.3 as a graphical representation for this system.

In the first situation if $a = \frac{1}{\sqrt{2}}$, that would mean that the probability of passing the vertical polaroid would be $a = (\frac{1}{\sqrt{2}})^2 = 0.5$, that would light to the expected reduction of a half of the intensity of light. After passing through the vertical polaroid the photon will have a polarization of $|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Considering now the second situation after we have our photon polarized vertically (like in the end on the first situation), the probability of being vertically polarized is 1, thus making the probability of passing through the horizontal polaroid 0. In the third situation, after being vertically polarized the photon will pass through an oblique polaroid that makes its direction

$$|v\rangle = i.sen(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + cos(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.27}$$

θ being the angle of the polaroid. The photon filtered by the vertical polaroid will pass this second polaroid with a probability of $(i.sen(\theta))^2$, becoming polarized according to the filter, as we can observe depending on the value of θ we will now have a horizontal component in the vector that describes the state of the photon. This will make the photon pass the horizontal polaroid with a probability of $cos(\theta)^2$.

2.3 Quantum Computing

The quantum equivalent of a bit (the basic unit of information in computers), is a qubit (or quantum bit). A qubit is a two-state quantum system that can be interpreted as normalized vectors in a two-dimensional Hilbert space. This Hilbert Space(H^2), has two basis (also known as pure states). In quantum computing, the basis used are $|0\rangle, |1\rangle$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The qubit can be described in terms of linear transformations of pure states as in 2.41, ω_0 and ω_1 are complex numbers called probability amplitudes. When a system is in a mixture of this pure states (as the general representation in 2.41 might point), it is in a phenomenon known as superposition. Measuring forces the system to collapse and assume one of the pure states with a certain probability.

$$|\psi\rangle = \omega_0|0\rangle + \omega_1|1\rangle = \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix} \tag{2.28}$$

In the example stated in the representation 2.41 the probability of the system falling into the state $|0\rangle$ would be $|\omega_0|^2$. The probability of the system falling into $|1\rangle$ would be $|\omega_1|^2$. The second axiom of Kolmogorov (law of total probability) is verified (2.42).

$$|\omega_0|^2 + |\omega_1|^2 = 1 \tag{2.29}$$

ω_0 and ω_1 (2.41) are complex numbers, the so-called probability amplitudes. When squared, the probability amplitude represents a probability.

2.3.1 Compound Systems

The representation of a system comprising multiple qubits grows exponentially. If to represent a single qubit system there is a two-dimensional Hilbert space, to represent a system with m qubits a 2^m-dimension space would be required. To represent a higher dimension multiple-qubit system one can perform a tensor product of single-qubit and/or lower dimension systems.

The tensor product is an operation denoted by the symbol $\otimes$. Given two vector spaces V and W with basis 2.43 and 2.44 respectively, their tensor product would be the mn-dimensional vector space with a basis with elements of the form 2.45 [10].

$$A = \{|\alpha_1\rangle, |\alpha_2\rangle, ..., |\alpha_m\rangle\} \tag{2.30}$$

$$B = \{|\beta_1\rangle, |\beta_2\rangle, ..., |\beta_n\rangle\} \tag{2.31}$$

$$|\alpha_i\rangle \otimes |\beta_j\rangle \tag{2.32}$$

For example, if we consider two Hilbert spaces H^2 with basis $A = \{|0\rangle, |1\rangle\}$ and $B = \{|-\rangle, |+\rangle\}$, their tensor product would be a H^4 with basis 2.46.

$$AB = \{|0-\rangle, |0+\rangle, |1-\rangle, |1+\rangle\} \tag{2.33}$$

Now taking the former Hilbert spaces, supposing we have the qubits 2.47 and 2.48.

$$|v\rangle = a_0|0\rangle + a_1|1\rangle \tag{2.34}$$

$$|w\rangle = b_0|-\rangle + b_1|+\rangle \tag{2.35}$$

$$|v\rangle \otimes |w\rangle = a_0b_0|0-\rangle + a_0b_1|0+\rangle + a_1b_0|1-\rangle + a_1b_1|1+\rangle \tag{2.36}$$

The Bra-ket notation provides a way to prevent the escalation of the basis notation. When specified the vector space the basis can be specified in base 10 for simplicity sake. According to this the basis of the last example would be $AB = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ and a system with 3-qubit could be represented in a Hilbert space with basis $H = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}$.

2.3.2 Entanglement

In multiple qubit systems, qubits can interfere with each other, thus making impossible to determine the state of part of the system without “disturbing” the whole. In other words, there are states in a multi-qubit system that can’t be described as a tensor product of single-qubit systems; this superposition is called quantum entanglement. This property is not local as transformations that act separately in different

parts of an entangled system cannot break the entanglement. Quantum entanglement is one of the core aspects when we are trying to explore the full potential of quantum systems [10].

For example, supposing we consider the following quantum states, known as Bell states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.37}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.38}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.39}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.40}$$

These states form a particular basis in H^4 as they are all entangled states and they are maximally entangled. If we have a system of two qubits in a Hilbert space in the mixed state $|\Phi^+\rangle$ and we measure the qubit A (by deciding the outcome 0 or 1 with a probability of 0.5 for each),we are automatically uncovering the value of the qubit B. In this state we know the second qubit will always yield the same value of the first measured qubit. Se second value is correlated with the first one.

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$$AB = \{|0-\rangle, |0+\rangle, |1-\rangle, |1+\rangle\} \tag{2.46}$$

Now taking the former Hilbert spaces, supposing we have the qubits 2.47 and 2.48.

$$|v\rangle = a_0|0\rangle + a_1|1\rangle \tag{2.47}$$

$$|w\rangle = b_0|-\rangle + b_1|+\rangle \tag{2.48}$$

$$|v\rangle \otimes |w\rangle = a_0b_0|0-\rangle + a_0b_1|0+\rangle + a_1b_0|1-\rangle + a_1b_1|1+\rangle \tag{2.49}$$

The Bra-ket notation provides a way to prevent the escalation of the basis notation. When specified the vector space the basis can be specified in base 10 for simplicity sake. According to this the basis of the last example would be $AB = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ and a system with 3-qubit could be represented in a Hilbert space with basis $H = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}$.

2.4.2 Entanglement

In multiple qubit systems, qubits can interfere with each other, thus making impossible to determine the state of part of the system without "disturbing" the whole. In other words, there are states in a multi-qubit system that can't be described as a tensor product of single-qubit systems; this superposition is called quantum entanglement. This property is not local as transformations that act separately in different parts of an entangled system cannot break the entanglement. Quantum entanglement is one of the core aspects when we are trying to explore the full potential of quantum systems [10].

For example, supposing we consider the following quantum states, known as Bell states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.50}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.51}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.52}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.53}$$

These states form a particular basis in H^4 as they are all entangled states and they are maximally entangled. If we have a system of two qubits in a Hilbert space in the mixed state $|\Phi^+\rangle$ and we measure the qubit A (by deciding the outcome 0 or 1 with a probability of 0.5 for each),we are automatically uncovering the value of the qubit B. In this state we know the second qubit will always yield the same value of the first measured qubit. Se second value is correlated with the first one.

2.5 Game Theory

2.5.1 Zero-Sum Game

A zero-sum game is a mathematical representation of a system where the gains are completely evened out by the losses; this means that the sum of the utilities of all players will always be zero.

2.6 Quantum Game Theory

[Intro SAUCEs]

The definition of Piotr Frackiewicz in Quantum information approach to normal representation of extensive games proposes a representation for finite extensive form games. The representation assumes that there are only two available actions, which can be represented by a qubit. Those actions could be a yes/no decision or a cooperate/defeat as those found in many classical game theory problems. A game in this form is represented by a six-tuple2.54, where:

$$\Gamma = (\mathcal{H}^a,\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\}) \tag{2.54}$$

- a is the number of actions (qubits), in the game;
- $\mathcal{H}^a$ is a a-dimensional Hilbert space constructed as $\otimes_{j=1}^{a}\mathbb{C}^2$, with basis $\mathcal{B}$;
- $|\psi_{in}\rangle$ is the initial state of the compound-system composed by a qubits: $|\varphi_1\rangle,\ |\varphi_2\rangle, ..., |\varphi_j\rangle, ..., |\varphi_a\rangle$;
- ξ is a mapping function that assigns each action to a player;
- For each qubit j $\mathcal{U}_j$ is a subset of unitary operators that can be used by the player to manipulate her qubit;

- Finally, for each player i E_i is a utility functional that specifies her payoff. This is done by attributing a real numbered utility to the measurent of the projection of a basis in the final state2.55.

$$E_i = \sum_{b \in \mathcal{B}} u_i(b) |\langle b|\psi_{fin}\rangle|^2, u_i(b) \in \mathbb{R} \tag{2.55}$$

The strategy of a player i is a map τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j \in \xi^{-1}(i)$). The simultaneous move is represented in 2.56.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{N} \otimes_{j \in \xi^{-1}(i)} U_j |\psi_{in}\rangle \tag{2.56}$$

3

Related Work

Contents

Foreword.

3.1 Quantum Walks

3.1.1 Quantum Walk on a Line

Quantum Walk is the quantum version of random walks, which are mathematical formalisms that describe a path composed of random steps. A Markov Chain might be used to describe these processes.

We can define the Discrete Quantum Walk on a Line as a series of Left/Right decisions. Understanding this algorithm is important towards being able to define and design more complex algorithms that make use of quantum properties.

We followed an approach suggested by [11] towards simulating n-steps of a Quantum Walk on a Line. The algorithm can be consulted on A. In a discrete quantum walk in a line we want to preserve the fact that the probability of turning left is equal to the probability of turning right. To represent a state in this algorithm we will need the number of the node and a direction (identified as L,R) 3.1.

$$|\psi\rangle = |n, L\rangle \tag{3.1}$$

With two equally possible direction choices in each step, we can use a coin metaphor [11] [12] to approach the decision. We toss a coin and go either Left or Right depending on the result.

In a quantum version we need to define a Coin Operator (Coin Matrix), which is responsible to imprint a direction to the current state. This operator is a unitary matrix in a 2-dimension Hilbert space. Some examples of Coin Operators are the Hadamard matrix3.2 and a symmetric unitary matrix 3.3.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3.2}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{3.3}$$

Taking the Hadamard matrix as an example3.2, the coin matrix will operate on the state in the following way 3.43.5 [11].

$$C|n, L\rangle = \frac{1}{\sqrt{2}}|n, L\rangle + \frac{1}{\sqrt{2}}|n, R\rangle \tag{3.4}$$

$$C|n, R\rangle = \frac{1}{\sqrt{2}}|n, L\rangle - \frac{1}{\sqrt{2}}|n, R\rangle \tag{3.5}$$

The Coin Matrix obtains its name by being the quantum equivalent of flipping a classic coin. After tossing a coin comes an operator that will move the node in the direction assigned. The operator responsible for this modification is commonly referred as Shift Operator 3.63.7.

$$S|n, L\rangle = \frac{1}{\sqrt{2}}|n-1, L\rangle \tag{3.6}$$

$$S|n, R\rangle = \frac{1}{\sqrt{2}}|n+1, R\rangle \tag{3.7}$$

These matrices (Coin Matrix and Shift Operator) are referred conceptually ubiquitously throughout the literature [**?**], therefore it is important to be familiar with them. A single step of the algorithm A is illustrated in Figure 3.1.

[needs more explaining]

Depending on the Coin Matrix we can get different distributions. In Figures 3.3 and 3.2

Starting on the middle of a line we can shift one unit left or right. If we took the classical approach in which we tossed a fair coin, and after n-steps we measured the final node repeatedly, by the Central Limit Theorem (CLT) the final distribution would converge to a normal distribution.

3.1.2 Quantum Walk on a Graph

3.2 Quantum Models

The rationale behind building a quantum version of a Game Theory and/or Statistics problem lays in bringing phenomena like quantum superposition, and entanglement into known frameworks allowing different results, that seemingly perform at least as good as their classical versions.

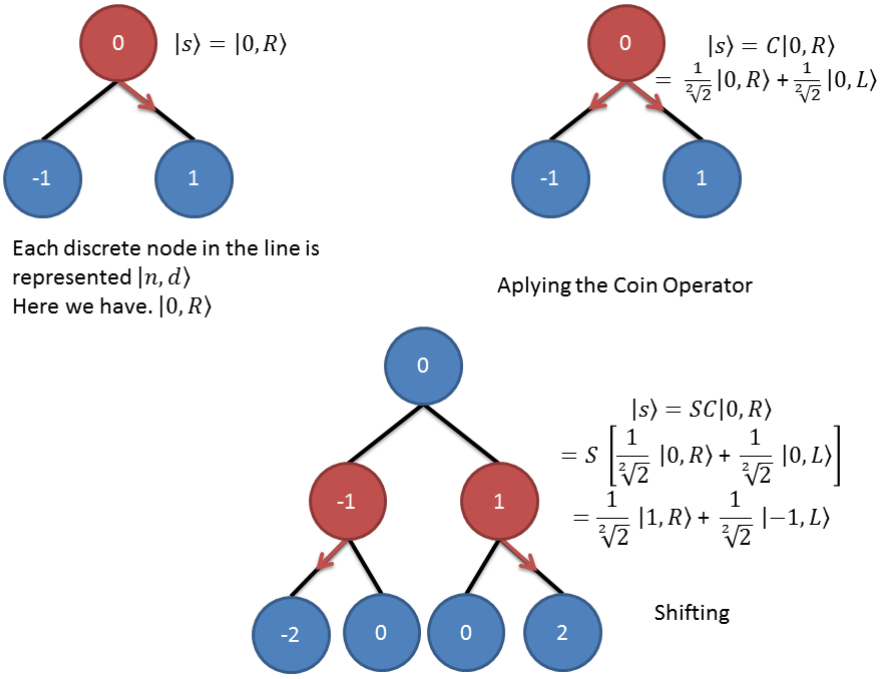

Figure 3.1: Simulating a step of a discrete quantum walk on a line. In the beginning we have a state characterized by the position (0) and a direction (either Left or Right).

[To Introduce more SOURCES]

The effort put in converting known classical problems also enables the familiarization with the potential differences these models bring.

3.2.1 Quantum Roulette

In the arbitrary N-State quantum roulette, [] presented a N-State roulette model using permutation matrices. To verify this model with two players we developed a Matlab simulation **??**, as proposed by the authors.

The game in represented in a N-Dimensional Hilbert Space. There is a basis in the space that represents each of the equally probable entries as shown in 3.8. In a sense this is a generalization of a quantum coin flip that is also used in 3.1.1.

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, |N\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{3.8}$$

Each state transition is obtained using a permutation matrix denoted by P^i. There are $N!$ permutation

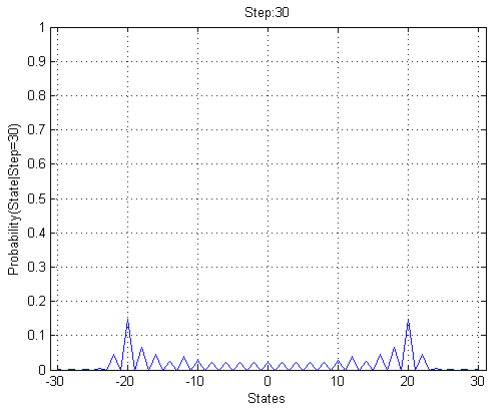

Figure 3.2: 30 Step of the Simulation A using Matrix 3.2 as a Coin Operator.

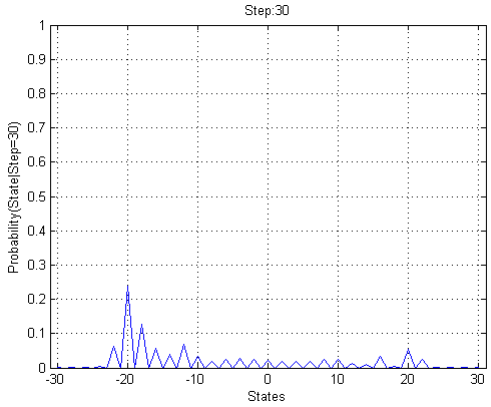

Figure 3.3: 30 Step of the Simulation A using a Hadamard Matrix 3.2 as a Coin Operator.

matrices, so in the particular case of having a 3-State roulette, there are 6 possible transition choices. The classical strategy considered will rely on choosing an arbitrary probability distribution, that verifies3.10, and that maps the usage of the permutation matrices. This step will not affect the density matrix (ρ) of the roulette3.9.

$$\rho = \frac{1}{N!} \sum_{i=0}^{N!-1} P^i \tag{3.9}$$

$$\sum_{i=0}^{N!-1} p_i = 1 \tag{3.10}$$

The density matrix is diagonalizable by a Discrete Fourier Transform because it is a kind of circulant matrix [13], as we can see in 3.11. In 3.11 λ_k are eigenvalues of ρ. $\lambda_1 = 1$ while $\lambda_2 = \lambda_3 = \lambda_k = \lambda_{N-1} = 0$. Each column i of the Fourier matrix will represent a eigenvector $|\lambda_i\rangle$. If we construct the diagonilizing matrix by rotating the columns of the Fourier Matrix we can obtain the projection states as in 3.12.

$$F^{\dagger}\rho F = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N-1} \end{bmatrix} \tag{3.11}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = F^{\dagger}\rho F \tag{3.12}$$

The quantum strategy advantage in this case is that the first player will not alter the density matrix3.13.

$$\rho = \sum_{i=0}^{N!-1} p_i P^i \rho P^{i\dagger}, \ \sum_{i=0}^{N!-1} p_i = 1 \tag{3.13}$$

This means that if the second player knows the initial state and the first player plays with a classical strategy, thus never modifing the system density matrix, the first player will be able to manipulade the game under optimal conditions.

We extended this work for three players in order to answer if adding a third player with access to a quantum strategy would affect the strategy of player two. We introduced a third character that will try to manipulate the game independently from the previous characters using also quantum strategies.

3.2.2 Prisioner's Dillema

The Prisioner's Dillema is one example of a game that can be represented in normal form. This problem has received a great deal of attention because, in its simple form, rational individuals will seem to deviate from solutions that would represent the best interest overall solution, the Pareto Optinal solution. In this game the pareto optimal solution is not a Nash Equilibrium. The Prisioneers Dillema can be formulated as it follows:

> Two suspects of being partners in a crime are arrested. The police needs more evidence in order to prossecute the prisioners. So each prisioner is locked in solitary confinement and has no means of communicating with the other suspect. The police will then try to extort a confession from the prisioners. A bargain will be proposed to the suspects:
>
> - If the suspect testifies against the other suspect (Defeats), he will go free and the second will get three years sentence.

	Player 2: C	Player 2: D
Player 1: C	(1,1)	(0,2)
Player 1: D	(2,0)	(0,0)

Table 3.1: One possible normal form representation of Prisioner's Dillema.

- If they both testify against one another (both Defeat), both will be convicted and they will get two years.
- In case both suspects deny the involvement (both Cooperate) of the other, they will be set free because there's no conclusive evidence.

A matrix representation of the problem is in Table 3.1.

3.2.2.A Quantum Prisioner's Dillema

This game has been modelled several times. However in order to exemplify the use of the definition described in Section 2.6, it poses a good consolidation exercise.

Each player i in the quantum version of Prisioner's Dillema will be able to manipulate one qubit (φ_1 and φ_2)3.143.15, with two possible operators (shown in 3.16) corresponding to the classical actions: Cooperate (C), and Defeat (D).

$$\varphi_1 = a_0|0\rangle + a_1|1\rangle, \sum_{i=0}^{1} |a_i|^2 = 1 \tag{3.14}$$

$$\varphi_2 = b_0|0\rangle + b_1|1\rangle, \sum_{j=0}^{1} |b_j|^2 = 1 \tag{3.15}$$

$$\begin{cases} O_{i0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ O_{i1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}, i \in \{1, 2\} \tag{3.16}$$

The system that holds the game is represented in a $\mathcal{H}^4$. Each basis ($|1\rangle, |2\rangle, |3\rangle, |4\rangle$), represents a final outcome as Table 3.2 suggests.

$\bigotimes$	C-$\|0\rangle$	D-$\|1\rangle$
C-$\|0\rangle$	$\|0,0\rangle$	$\|0,1\rangle$
D-$\|1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$

Table 3.2: Construction of the basis for the game space; $\mathcal{H}^4$.

The utility functions for each player is calculated by projecting the final state in each base and atributing a real number to each measurement, as in equation. In order to compare a classical version with this quantum model, the real numbers assigned are those in the classical example in Table 3.1, and the basis are presented in Table 3.2.

$$E_0 = 1 \times |\langle 00|\psi_{fin}\rangle|^2 + 2 \times |\langle 10|\psi_{fin}\rangle|^2 \tag{3.17}$$

$$E_1 = 1 \times |\langle 00|\psi_{fin}\rangle|^2 + 2 \times |\langle 01|\psi_{fin}\rangle|^2 \tag{3.18}$$

The fundamental difference from the classical version lies in the way the initial state is formulated3.19. The way this state is constructed will alow to entangle the moves made by the players. The entanglement in quantum game theory can be viewed as an intrinsic unbreakable contract.[sources]

$$|\psi_{in}(\gamma)\rangle = cos(\gamma)|00\rangle + isen(\gamma)|11\rangle, \gamma \in (0,\pi) \tag{3.19}$$

3.2.3 Monty Hall Problem

The Monty Hall problem became popularized in 1990 in a column, in the magazine Parade [14]. The reason for its notoriety rests mainly in its counter-intuitive nature. Although the Monty Hall problem can be modeled as a Bayesian probability problem, human beings have difficulty in grasping the probabilities involved.

3.2.3.A Problem Description

The Monty Hall problem is loosely based on a television game show hosted by its namesake - Monty Hall. Furthermore it was first attributed to a statistician, named Selvin. The problem is posed as it follows:

Suppose you're on a game show, and you're given the choice of three doors: Behind one

door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Most people, when faced with this problem, will be indifferent about whether to switch or to stay with the initially picked door.

It is also verified that they will tend to stick with their first choice. According to Granberg and Brow [15], only 13% of 228 subjects decided to switch their initial choice. However, by using probability theory, one can arrive at the conclusion that, in fact, it is advantageous to switch given the previous formulation.

The action of the Host implies a belief update on the probabilities of the variables in the system. This poses a violation of rational decision making; subjects do not seem to follow the best strategy which would maximize their chances of winning the prize.

To understand this exercise, one can look at the decision tree in Figure 3.4. We assumed indifferently that the player chose the door No. 1. The situation is symmetric whichever door she chooses.

Assuming we call C_1 to the variable that describes whether or not the car in behind door No. 1. The variables C_2 and C_3 will respectively describe the probability associated with the car being (or not), behind doors 2 and 3 ($P(C_2) = 1$ or $P(C_2) = 0$, for example).

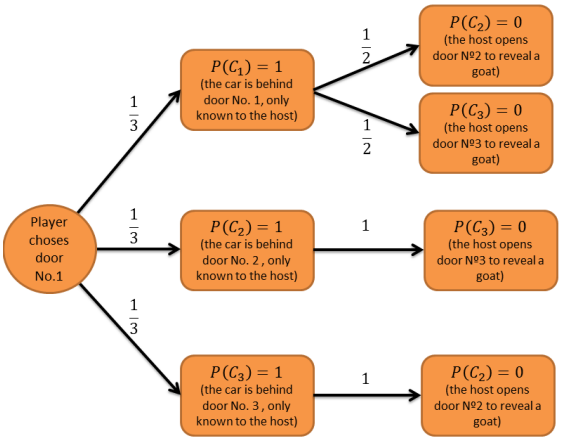

Figure 3.4: Decision tree modelling the Monty Hall problem.

After the player has had her choice, the host will perform an operation on the remaining two doors. The host of the show has complete information of the game, unlike the player.

$$P(C_1) = P(C_1|\neg C_2) + P(C_1|\neg C_3) \tag{3.20}$$

$$P(C_1) = (\frac{1}{3}\times\frac{1}{2}) + (\frac{1}{3}\times\frac{1}{2}) = 1 \tag{3.21}$$

$$P(\neg C_1) = P(C_2|\neg C_3) + P(C_3|\neg C_2) = (\frac{1}{3}\times 1) + (\frac{1}{3}\times 1) = 2/3 \tag{3.22}$$

The probability of switching and getting the car is twice 3.22 as likely of staying with the first choice and getting the prize3.21.

3.2.3.B Quantum Model

Various Models have been proposed to describe a quantum version of the Monty Hall problem.

[To Introduce more SOURCES]

As the host reveals information, the initial setup is modified. This is an interesting property. Despite being a counter-intuitive problem, a quantum approach to this problem allows an in-depth comparison between the classical measurement and the quantum measurement. The classic Monty Hall problem is modeled using conditional probability and Bayes Rule. In the quantum version, measuring the outcome of the final state yields the result, instead of taking into account the intermediate actions [16].

Moreover it is important to realize that there is not a unique way to model a classical problem [17]. Therefore, when modelling a classical problem, we need to select properties that could potentially benefit from a quantum approch. In [17] we can observe the attempt to stick as closely to the classical formulation as possible, the host has a system that is correlated to the game system.

4

Quantum Pirate's Game

Contents

In this chapter we describe the Pirate's Game and the steps to model a quantum approach to the problem.

4.1 Pirate Game

4.1.1 Problem Description

The Pirate Game is a multiplayer version of the Ultimatum game that is usually stated as follows:

> Suppose there are 5 rational pirates: A; B; C; D; E. The pirates have a loot of 100 gold coins to divide among themselves.
>
> As the pirates have a strict hierarchy, in which pirate A is the captain and E has the lowest rank, the highest ranking pirate alive will propose a division. Then each pirate will cast a vote on whether or not to accept the proposal.
>
> If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.
>
> We consider that each pirate privileges her survival, and then will want to maximize the number of coins received. When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.

We can arrive at an equilibrium in this problem by using backward induction.

5

Conclusions and Future Work

[TO DO]

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Matlab Simulation: Discrete Quantum Walk on a Line

```
1
2
3  %-- number of steps in the simulation
4  steps= 30;
5
6  %-- hadamard matrix
7  H = [1/sqrt(2) 1/sqrt(2); 1/sqrt(2) -1/sqrt(2) ];
8  %-- symetric matrix
9  M = [1/sqrt(2) 1i/sqrt(2); 1i/sqrt(2) 1/sqrt(2) ];
10 %-- coin flip unitary operator
11 C = H;
12 C=M;
13
14 %-- coin flip matrix
15 CM= zeros((steps*2+1),2);
16 %-- shift matrix
17 SM= zeros(steps*2+1,2);
18
19 %-- middle index
20 i0= steps+1;
21
22 %-- initialize flip probability amplitudes (coin has 1/2 chance of going +1 or -1)
23 CM(i0,1)=1/sqrt(2);
24 CM(i0,2)=1/sqrt(2);
25
26 for i=1:steps
27    %-- clean SM
28    SM= zeros(steps*2+1,2);
29
30    for j=1:(steps*2+1)
31        if CM(j, 1)≠0
32            SM(j-1,1)=CM(j, 1);
33        end
34        if CM(j, 2)≠0
35            SM(j+1,2)=CM(j, 2);
36        end
37    end
38    SM;
39    %disp('-------');
40    %-- clean CM
41    CM= zeros(steps*2+1,2);
42    for j=1:(steps*2+1)
43        if SM(j, 1)≠0
44            CM(j, 1)= CM(j, 1)+C(1,1)*SM(j, 1);
45            CM(j, 2)= CM(j, 2)+C(1,2)*SM(j, 1);
46        end
47        if SM(j, 2)≠0
48            CM(j, 1)= CM(j, 1)+C(2,1)*SM(j, 2);
49            CM(j, 2)= CM(j, 2)+C(2,2)*SM(j, 2);
50        end
51    end
52    CM;
53
54    %Display
55
56    figure
57
58    probability=zeros(steps*2+1);
59
60    for j=1:steps*2+1
61       probability(j)=abs(SM(j,1)).^2+ abs(SM(j,2)).^2;
62    end
63    axisP = -steps : steps;
64    plot(axisP,probability)
65    title(strcat('Step: ' ,num2str(i)))
66    axis([-steps-1 , steps+1, 0, 1]);
67    grid on
68    ylabel(strcat('Probability(State|Step=',num2str(i), ')'));
69    xlabel('States');
70
71    %Display
72 end
```