Quantum Pirates

A Quantum Game-Theory Approach to The Pirate Game

Daniela Fontes

Instituto Superior Técnico
daniela.fontesl@ist.utl.pt

Abstract. In this document, we develop a model and a simulation of a quantization scheme for the mathematical puzzle created by Omohundro and Stewart - "A puzzle for pirates ", also known as Pirate Game. This game is a multi-player version of the game "Ultimatum ", where the players (Pirates), must distribute fixed number of gold coins acording to some rules.

Keywords: Quantum Game Theory; Pirate Game; Quantum Mechanics; Quantum Computing; Game Theory; Probability Theory

1 Introduction

2 Pirate Game

The original Pirate Game is a multi-player version of the Ultimatum game that was first published as a mathematical problem in the Scientific American as a mathematical problem posed by Omohundro [1]. The problem can be formulated as it follows:

> Suppose there are 5 rational pirates: A; B; C; D; E. The pirates have a loot of 100 indivisible gold coins to divide among themselves.
>
> As the pirates have a strict hierarchy, in which pirate A is the captain and E has the lowest rank, the highest ranking pirate alive will propose a division. Then each pirate will cast a vote on whether or not to accept the proposal.
>
> If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise

the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.

We consider that each pirate privileges her survival, and then will want to maximize the number of coins received. When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.

2.1 Analysis of the Pirate Game for 3 Players

In Figure 1 we have an extensive form representation of the classic Pirate Game for 3 players without taking into account the proposals. Each node in the game tree has the number of the player who will make the decision, either to Cooperate (vote yes to the proposal), or Defect.
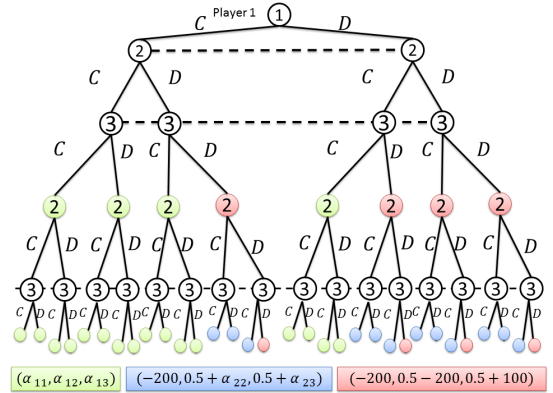

Fig. 1. Extensive form representation of the classic Pirate Game for 3 players.

The green accent, in Figure 1, shown in the nodes represent a state where the first captain (player 1), will see her proposal accepted, the utility associated. The dashed lines are a common representation for imperfect information at the nodes united. The blue accent denotes the outcomes where the second captain makes a proposal and has seen it accepted. The red accent color represents the outcomes where the player 3 will be the remaining pirate.

The number of coins will translate directly the utility associated receiving them. The highest ranking pirate in the hierarchy is responsible to make a proposal to divide the 100 gold coins. A captain i chooses the amount of coins a player j get; this amount will be represented by α_{ij}. In the initial stage of the game the captain will define $\alpha_{11}, \alpha_{12}, \alpha_{13}$, according to Equation 1 (N the number of

pirates in the game). To account for all possible proposals α_{ij} would be overwhelming from the point of the game tree, and from the formulation of the problem the pirates know the share proposal before voting. This means we will concentrate our analysis in the voting aspect. This means that for each specification of the values α_{ij} we will have a diferent game.

$$\forall i \in \{1, 2, ..., N-1\} : \sum_{j=i}^{N} \alpha_{ij} = 100, \forall i, j : \alpha_{ij} \in \mathbb{N}_0 \quad (1)$$

The values for $(\alpha_{11}, \alpha_{12}, \alpha_{13}) = (99, 0, 1)$ will be the allocation that results in an equilibrium for the 3 player game.

If the proposal is rejected the captain will be thrown off board, and the captain he will receive a negative payoff of -200. The pirates have also an incentive to climb the hierarchy; accounted by adding an expected value of half a coin (0.5), to the payoff of the remaining players if the voting fails. This tie breaker and death penalty is shown in Figure 1.

3 Quantum Model

3.1 Game system: Setting up the Initial State

A game Γ can be viewed as a system composed by qubits manipulated by players. We will use the definition of quantum game proposed by [2], to model extensive form games in a normal form representation. Akin to the Quantum Ultimatum game desbribed in [3], our objective is to apply that quantization scheme to the normal form representation of the game tree in Figure 1.

The representation assumes that each qubit, as in the equation bellow, can be manipulated to represent an action. The pure basis $|0\rangle = |C\rangle$ ("C" from "Cooperate"), and $|1\rangle = |D\rangle$ ("D" from "Defect"), representing a yes/no decision or a cooperate/defeat as those found in many classical game theory problems.

$$\varphi = \alpha.|C\rangle + \beta.|D\rangle, \{\alpha, \beta\} \in \mathbb{C} : \alpha^2 + \beta^2 = 1 \quad (2)$$

A game in this form is represented by the following a six-tuple:

$$\Gamma = (\mathcal{H}^{2^a}, N, |\psi_{in}\rangle, \xi, \{\mathcal{U}_j\}, \{E_i\}) \quad (3)$$

Actions a is the number of actions (qubits), in the game. In this 3 player game there will be 5 qubits representing the actions or the players decision; three qubits will represent the first voting round, the other two will portray the actions of the second voting round. The number of qubits needed to represent the game grows exponentially with the number of players. For N players we need $\sum_{i=2}^{N} i$ qubits. With 8 players, this game would already be impractical to simulate in a classical computer. In this regard a quantum computer may enhance our power to simulate this kinds of experiments [4];

Game Space $\mathcal{H}^{2^a}$ is a 2^a-dimensional Hilbert space constructed as $\otimes_{j=1}^{a} \mathbb{C}^2$, with basis $\mathcal{B}$. With 3 players and 5 actions our system with be represented in a $\mathcal{H}^{32}$ using a state ψ. This means that to represent our system we will need $2^5 \times 1$ vectors, our system grows exponentially with the number of players/qubits. Each pure basis of $\mathcal{H}^{32}$ will represent a possible outcome in the game;

Initial State $|\psi_{in}\rangle$ is the initial state of the compound-system composed by a qubits: $|\varphi_1\rangle, |\varphi_2\rangle, ..., |\varphi_j\rangle, ..., |\varphi_a\rangle$;

Mapping the Qubits to the Players ξ is a mapping function that assigns each action to a player. In the Pirate Game (with 3 players), the mapping function ξ that assigns each action/qubit φ_j (with $j = \{1, 2, 3, 4, 5\}$), to a player is represented on :

$$\xi(j) = \begin{cases} 1 & , \ if \ j = 1; \\ 2 & , \ if \ j \in \{2, 4\}; \\ 3 & , \ if \ j \in \{3, 5\}. \end{cases} \quad (4)$$

Manipullating the Quabits For each qubit j $\mathcal{U}_j$ is a subset of unitary operators from $\mathsf{SU}(2)$ (the general for the 2 dimensional Special Unitary Group is presented in Equation 5).

These operators can be used by the player to manipulate her qubit(s);

Expected Utility Finally, for each player i, E_i is a utility functional that specifies her payoff. This is done by attributing a real number (representing a expected utility $u_i(b)$, in 6), to the measurement for the projection of the final state (7), on a basis from the $\mathcal{B}6$.

$$|\psi_{ini}(\gamma)\rangle = exp\left\{i\frac{\gamma}{2}D \otimes D \otimes D \otimes D \otimes D\right\}|00000\rangle$$
$$= cos(\frac{\gamma}{2})|00000\rangle + isin(\frac{\gamma}{2})|11111\rangle, \gamma \in (0, \frac{\pi}{2})$$
(10)

$$\mathcal{U}_j(w, x, y, z) = w.I + ix.\sigma_x + iy.\sigma_y + iz.\sigma_z,$$
$$w, x, y, z \in \mathbb{R} \wedge w^2 + x^2 + y^2 + z^2 = 1$$
(5)

$$E_i = \sum_{b \in \mathcal{B}} u_i(b)|\langle b|\psi_{fin}\rangle|^2, u_i(b) \in \mathbb{R}$$
(6)

The strategy of a player i is a map τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j \in \xi^{-1}(i)$). The simultaneous move is represented in 7.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{N} \otimes_{j \in \xi^{-1}(i)} \mathcal{U}_j|\psi_{in}\rangle$$
(7)

The tensor product of all the operators chosen by the players is referred as a super-operator, which act upon the game system.

The initial system ($|\psi_0(\gamma)\rangle$), will be set up by defining an entanglement coefficient γ, that affect the way the five qubits (belonging to the three pirate players), are related; this is shown in Equation 10. We will entangle our state by applying the gate $\mathcal{J}$ [5]. The parameter γ becomes a way to measure the entanglement in the system [6].

We analysed the role of the entanglement of the system since other examples researched pointed to it being the prominent factor regarding behaviour changes from the classsical perspective [2] [3] [5] [7] [8].

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
(8)

$$\mathcal{J} = exp\left\{i\frac{\gamma}{2}D \otimes D \otimes D \otimes D \otimes D\right\}$$
(9)

Strategic Space The strategic space is a set of unitary operators that the players can use to manipulate their assigned qubits.

Each player will have a strategy τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j \in \xi^{-1}(i)$). $\tau_2 = \{D_2, C_4\}$ represents the strategy where the player 2 votes D in the first stage and C in the second stage.

$$\varphi_j = a.|C\rangle + b.|D\rangle,$$
$$j \in \{1, 2, 3, 4, 5\},$$
$$\{a, b\} \in \mathbb{C} : a^2 + b^2 = 1$$
(11)

Each player will be able to manipulate her assigned qubits j with an unitary operator of the form shown in Equation 5 (shown in Section **??** $\mathcal{U}_j(w, x, y, z) = w.I + ix.\sigma_x + iy.\sigma_y + iz.\sigma_z$, with $w, x, y, z \in \mathbb{R} \wedge w^2 + x^2 + y^2 + z^2 = 1$). However in order to explore the potential of quantum strategies, [6] proposes that it is sufficient to restrict the strategic space span by to the 2-parameter (polar coordinates), set of matrices in Equation 12, with $\theta \in (0, \pi)$, and $\phi \in (0, \frac{\pi}{2})$. We will try to use the strategic space $\mathcal{U}_j(\theta, \phi)$ to represent our player's actions.

$$\mathcal{U}_j(\theta, \phi) = \begin{bmatrix} cos(\frac{\phi}{2}) & e^{i\phi}sin(\frac{\phi}{2}) \\ -e^{-i\phi}sin(\frac{\phi}{2}) & cos(\frac{\phi}{2}) \end{bmatrix},$$
$$j \in \{1, 2, 3, 4, 5\}, \theta \in (0, \pi), \phi \in (0, \frac{\pi}{2})$$
(12)

However the two operators that correspond to the original classical actions of voting "Yes" or to Cooperate, and voting "No" (Defect) are not entirely characterized by the subset $\mathcal{U}_j(\theta, \phi)$. These classical actions belong to a subset S_j described in Equation 13. The classical co-operation operator will be represented by the Identity operator (o_{j0}, where j identifies the qubit that the respective player will act upon). When assigned to a qubit this operator will leave it unchanged. This operator is described by Equation 12 when $\mathcal{U}(0,0)$.

The defection operator (D), is represented by one of Pauli's Operators - the Bit-flip operator. This operator was chosen because it performs the classical operation NOT on a qubit. Within the restricted space $\mathcal{U}_j$, approximate alternative for the defect operator in the set $\mathcal{U}_j$ is $\mathcal{U}_j(\pi, 0)$; they are interchangeable when $\gamma = 0$. For $\gamma > 0$ we will include the pure strategy D represented by the Bit-flip operator into our restricted strategic scape $\mathcal{U}_j$.

$$S_j = \begin{cases} C_j = o_{j0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D_j = o_{j1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}, j \in \{1, 2, 3, 4, 5\} \quad (13)$$

The notation for the pure basis as $|C\rangle$ refers to a quantum state and should not be confused as a Cooperate operator C that is a matrix (the identity matrix). A original player quantum state (the qubits of the form φ_j) is defined by Equation 11.

Final State This final state is calculated by constructing a super-operator, by performing the tensor product of each player chosen strategy, from Equation 12 or 13. The super-operator, containing each player's strategy, will then be applied to the initial state, as shown in Equation 14.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{3} \otimes_{j \in \xi^{-1}(i)} \mathcal{U}_j |\psi_{ini}(\gamma)\rangle \quad (14)$$

In order to calculate the expected payoff functions we need to de-entangle the system, before measuring. The act of measuring, in quantum computing, gives an expected value that can be understood as the probability of the system collapsing into that state. In the Figure 2 we have represented the way we entangle and de-entangle the system.

Utility To build the expected payoff functionals for the three player situation we must take into account the sub-games created when the proposal is rejected, in Figure 1.

These utility functions will represent the degree of satisfaction for each pirate after game by attributing a real number to a measurement performed to the system.
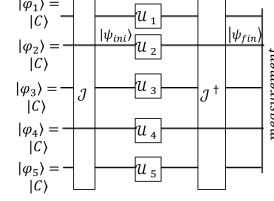

Fig. 2. Scheme that represents the set-up of the 3-player Pirate Game. Before we measure the final result we need to apply the transpose operator $\mathcal{J}^\dagger$.

We can observe in Figure 1 that in that we have three separate groups (denoted by the colour accents), of outcomes that share the same payoff, in the original problem. In our quantum scheme we can aggregate the pure-basis quantum states ($\mathcal{B}$), associates with a payoff in the following manner:

- States where the first proposal is accepted - "Accepted 1' (or A_1), with a green colour accent in Figure 1. If ($P(A_1) = 1$) the players can expect to receive the following payoff: (a_{11}, a_{12}, a_{13}).

- States where the first captain will be eliminated and the second player gets her proposal accepted - "Accepted 2" (or A_2) with a blue colour accent in Figure 1. If $P(A_2) = 1$, then the expected payoff is ($-200, a_{22} + 0.5, a_{23} + 0.5$).

- States where all proposals are rejected - "Rejected 2' (or R_2)' with a red colour accent in Figure 1. If both proposals are rejected ($P(R_2) = 1$), the players might expect to receive ($-200, -200 + 0.5, 100 + 0.5$).

In order to calculate the probability of the final state collapsing onto a basis state $b \in \mathcal{B}$ we perform a projection of the state in the chosen basis and we measure the squared length of the projection, $P(b) = |\langle b|\psi_{fin}\rangle|^2$ [9]. The expected utility function for each player will be a weighted average of all possible outcomes. The expected utility for player 1, shown in Equation 18, give a real number that represents the payoff associated with a final state. The same goes to player 2 that has her expected utility functional specified in Equation 19, and the player 3 in Equation 20.

$$P(A_1) = \sum_{x_3} \sum_{x_4} |\langle 0,0,0,x_4,x5|\psi_{fin}\rangle|^2 +$$
$$+ \sum_{x_3} \sum_{x_4} |\langle 1,0,0,x_4,x5|\psi_{fin}\rangle|^2 +$$
$$+ \sum_{x_3} \sum_{x_4} |\langle 0,1,0,x_4,x5|\psi_{fin}\rangle|^2 + \quad (15)$$
$$+ \sum_{x_3} \sum_{x_4} |\langle 0,0,1,x_4,x5|\psi_{fin}\rangle|^2$$

$$P(A_2) = \sum_{x_5} |\langle 1,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,1,1,0|\psi_{fin}\rangle|^2 +$$
$$+ \sum_{x_5} |\langle 1,1,0,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,0|\psi_{fin}\rangle|^2 +$$
$$+ \sum_{x_5} |\langle 1,0,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,0,1,1,0|\psi_{fin}\rangle|^2 +$$
$$+ \sum_{x_5} |\langle 0,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,0|\psi_{fin}\rangle|^2 \quad (16)$$

$$P(R_2) = |\langle 1,1,1,1,1|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,1|\psi_{fin}\rangle|^2 +$$
$$+ |\langle 1,0,1,1,1|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,1|\psi_{fin}\rangle|^2 \quad (17)$$

$$E_1(|\psi_{fin}\rangle) = \alpha_{11} \times P(A_1) - $$
$$-200 \times (P(A_2) + P(R_2))) \quad (18)$$

$$E_2(|\psi_{fin}\rangle) = \alpha_{12} \times P(A_1) - $$
$$+(0.5 + \alpha_{22}) \times P(A_2) + \quad (19)$$
$$+(0.5 - 200) \times P(R_2)$$

$$E_3(|\psi_{fin}\rangle) = \alpha_{13} \times P(A_1) + $$
$$+(0.5 + \alpha_{23}) \times P(A_2) + \quad (20)$$
$$+(100.5) \times P(R_2)$$

The gate $\mathcal{J}$ is chosen to be commutative with the super-operators created by the tensor product of the classical actions C (cooperate, indicated by the identity matrix and $\mathcal{U}(0,0)$)), and D (defect, indicated by the Bit-flip

operator matrix, and $\mathcal{U}(\pi,0)$ when $\gamma = 0$)). For example $[\mathcal{J}, C \otimes D \otimes C \otimes C \otimes D] = 0$.

This condition implies that choosing any operator from the sub-set $O = \{\mathcal{U}(\theta,0), \theta \in (0,\pi)\}$ with $\gamma = 0$ is the equivalent of a classical mixed action. A strategy τ_i is a classical pure strategy iff all operators in the strategy belong to the subset S (defined in 13). A strategy τ_i is a classical mixed strategy iff all operators in the strategy belong to O. If the parameter ϕ in the operator $U_j(\theta, \phi)$ differs from 0, and the entanglement parameter $\gamma = 0$ we are able to explore quantum strategies that have no counterpart in the classical domain [6].

4 Analysis and Results

In order to test our model we developed a Matlab simulation for the 2-player version of the game and for the 3-player version, the simulations are available in the Appendix **??**. When defining our game space Γ in Section 3.1 we defined a number of variable parameters, such as: the entanglement coefitient γ; the coin distributions α_{ij}; the strategies $\mathcal{U}_j$ that the players might use (restricting the operator space $\mathsf{SU}(2)$).

According to [10] when the entanglement is maximal, $\mathcal{U}(\pi,0)$ is the optimal counter-strategy for C (represented by the identity marix), $\mathcal{U}(0, \frac{\pi}{2})$ is the optimal counter strategy for $\mathcal{U}(\pi,0)$, D becomes the optimal counter-strategy for $\mathcal{U}(0, \frac{\pi}{2})$, and C becomes an optimal counter strategy for D.

4.1 2 Player Game

We simulated the Pirate Game for 2-players according to the rules defined for our quantum model in Section . The simulation can be consulted for reference in Appendix **??**. The classical sub-game with 2-players can be represented in Table 1. It is a strictly determined game, that means that there is at least a Nash Equilibrium when the players choose pure strategies [11]. In this case the outcome of the game is $(100, 0)$, when the players, use the strategies C_1 and D_2. In the quantum game we can find the following 4 pure quantum strategies :

– I

– D (also a Pauli operator σ_x)

- $\mathcal{U}(\pi, 0)$ - a quantum equivalent of a defection operator, (also $i\sigma_y$).
- $\mathcal{U}(0, \frac{\pi}{2})$ - the quantum equivalent of a cooperation operator, (also $i\sigma_z$).

The expected utility when players can choose only pure quantum strategies and the game is maximaly entangled is shown in Figures 3 and 4. When each player has access to the 4 quantum strategies and the entanglement is maximum ($\gamma = \frac{\pi}{2}$), the game is not strictly determined anymore. The mixed strategy will make the players choose with equal probability between the 4 pure quantum strategies described above giving the players an expected payoff of $(25, 25.125)$, this outcome constitutes a Pareto optimal outcome because it is impossible to improve one player's expected utility without lowering the other's.

As there were initialy 100 coins we decided to study what would happen if the captain decided to share the treasure fairly: 50 coins for player 1, 50 for player 2. A Nash equilibrium of the game would also consist in a mixed strategy where the players would choose indiferently which strategy to use, but in this case $\frac{3}{4}$ of the time player 1 would get 50 coins and $\frac{1}{4}$ he would die and receive a payoff of -200. This means that the meek captain would rather have a division "supervised" by a contract (entanglement), than to trust her crew and propose a fair division from the start.

	Player 2: C	Player 2: D
Player 1: C	(100, 0)	(100, 0)
Player 1: D	(100, 0)	(-200, 100.5)

Table 1. Representation of the 2 player sub-game in normal form.

4.2 3 Player Game

The captain proposes: $(99, 0, 1)$ The move $(C_1, D_2, C_3, C_4, D_5)$ with a proposal of $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$ represents Nash Equilibrium of the classic Pirate Game (for 3 players). In the classical version, when the players chose at least 2 operators *Cooperate* on the initial proposal the game ends right away, and the first proposal, made by player 1, is accepted (*Accepted*1 or A_1). Once again the original version of this game game is a strictly determined game because it has a pure-strategy Nash Equilibrium.
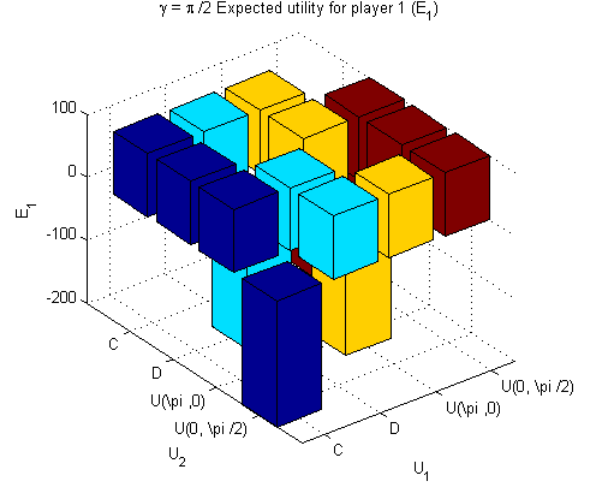

Fig. 3. Expected utility for player 1, when both players have access to pure quantum strategies and the entanglement is maximum.
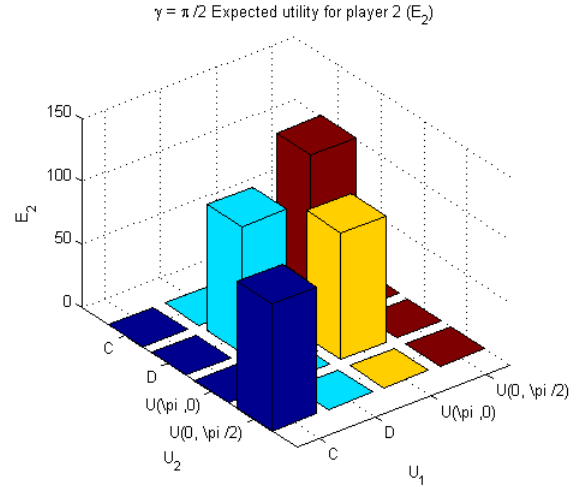
Fig. 4. Expected utility for player 2, when both players have access to pure quantum strategies and the entanglement is maximum.

After the players make their strategic moves and the disentangle operator $\mathcal{J}^\dagger$ is applied, and the payoff functionals are calculated given the final state. The final state will be calculated as shown in Equation 21.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{3} \otimes_{j \in \xi^{-1}(i)} \mathcal{U}_j |\psi_{ini}(\gamma)\rangle \qquad (21)$$

As expected from the problem definition when players use only classical strategies the entanglement coefficient will not affect the final expected utilities ($[\mathcal{J}, C \otimes D \otimes C \otimes C \otimes D] = 0$). An example of this behaviour is shown

Fig. 5. Expected utility for player 1 (the captain), when the system is maximaly entangled and he proposes a fair division of the treasure $(50, 50)$.
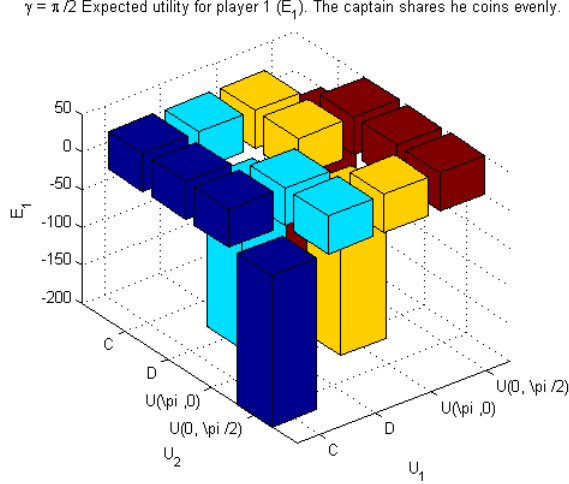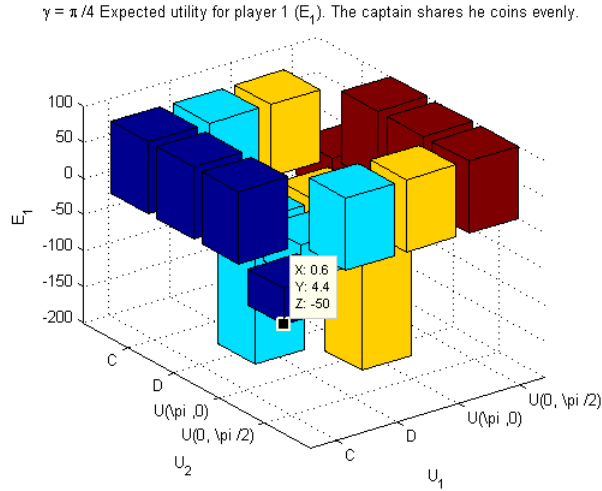

Fig. 6. Expected utility for player 1 (the captain), when the entanglemet coefficient is $\gamma = \frac{\pi}{4}$ and he proposes a division of the treasure of $(100, 0)$.

in Figure 7, where we measure the probability A_1 (first proposal is accepted), A_2, and R_2, with different entanglement levels ($\gamma = \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$).

When each player chooses an operator from the set $O = \{\mathcal{U}(\theta, 0), \theta \in (0, \pi)\}$ and $\gamma = 0$ we have a separable game. This is an aquivalent situation to the classical game, but contrarily to the classical game where we have only a Nash Equilibrium (C, D, C, C, D), we have more Equilibria, all with an outcome of $(99, 0, 1)$. This is verifyable because in the separable game the players act only in their qubits φ_j, and $D|0\langle = |1\langle$ and $\mathcal{U}(\pi, 0)|0\rangle = -|1\rangle$, when measuring the probability of the operators D and
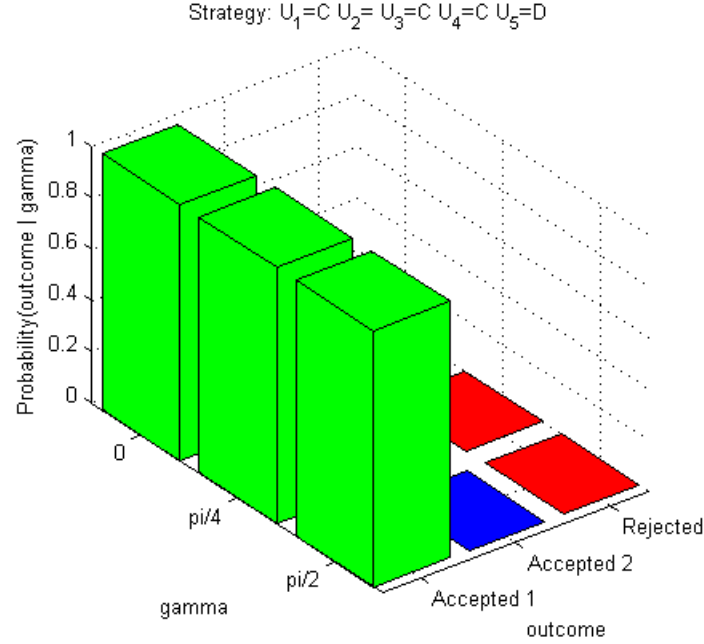
Fig. 7. When player play classical strategies the entanglement will not affect the final result.

$\mathcal{U}(\pi, 0)$ tranforming a pure state $|0\rangle$ into state $|1\rangle$ we square the probability amplitude associated with the final state. When $\gamma = 0$ C is interchangeble with $\mathcal{U}(0, \frac{\pi}{2})$, and D is interchangeble wih $\mathcal{U}(\pi, 0)$.

However for $\gamma > 0$ the $\mathcal{U}(\theta, 0)$ may present an interference behaviour that is inherently quantum. This effect is shown in Figures 14 and 14, allowing us to understand the difference between the Bit-flip operator D and $\mathcal{U}(\pi, 0)$.

The Tables 2 and 3 represent a 3-player pirate game where given the classical equilibrium ($CDCCD$) we test what are the outcomes if the players can use quantum strategies. By considering the quantum operators that constitute an alternative to the classical cooperation ($\mathcal{U}(0, \frac{\pi}{2})$), or defection ($\mathcal{U}(\pi, 0)$), like in the 2-player analysis when the entanglement is maximum we found that there is only a mixed strategy Nash Equilibrium because we detect a restricted cycled as suggested by [10].

Strategy: $U_1=D$ $U_2=U(\pi,0)$ $U_3=C$ $U_4=C$ $U_5=U(\pi,0)$

Fig. 8. Players use the operators .

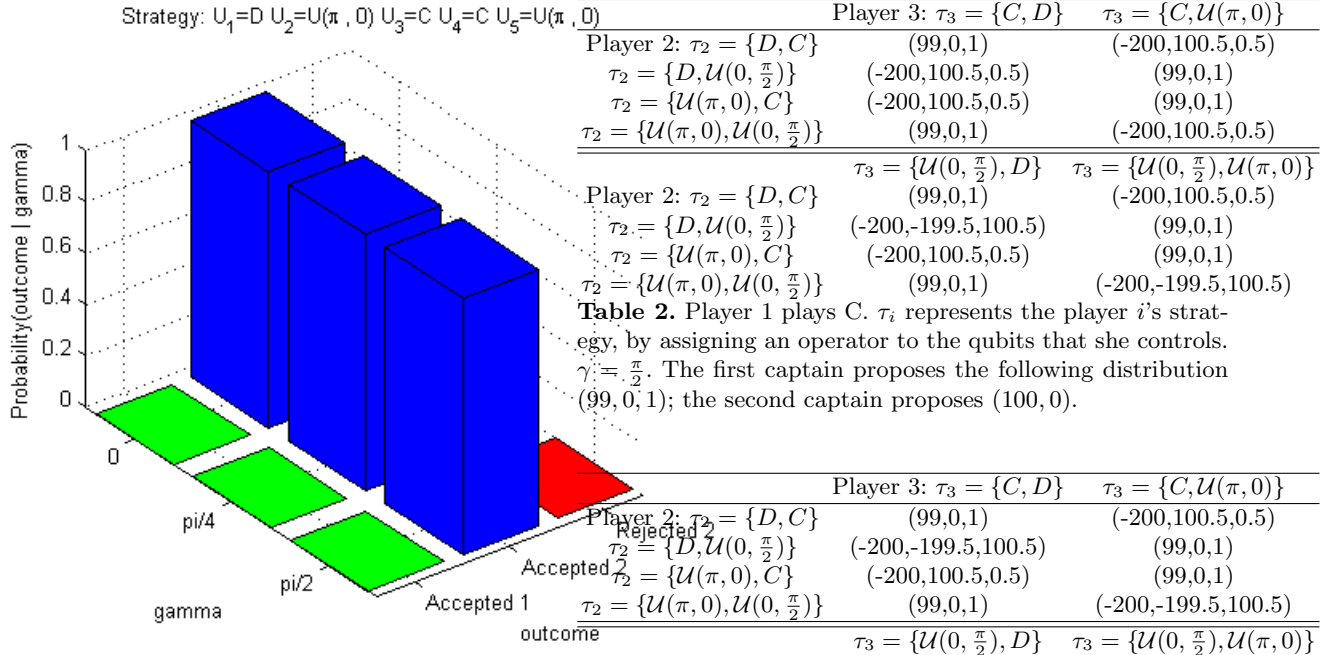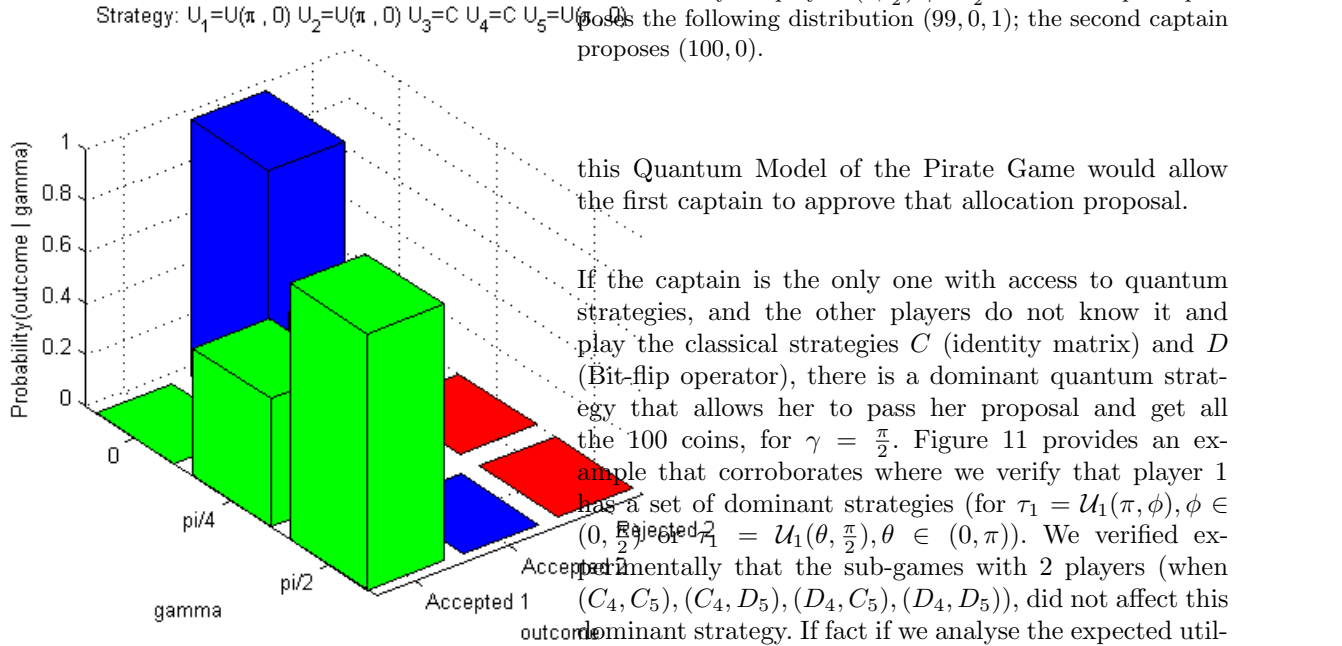|	Player 3: $\tau_3 = \{C,D\}$	$\tau_3 = \{C,\mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D,C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{D,\mathcal{U}(0,\frac{\pi}{2})\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0),C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0),\mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,100.5,0.5)
	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}),D\}$	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}),\mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D,C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{D,\mathcal{U}(0,\frac{\pi}{2})\}$	(-200,-199.5,100.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0),C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0),\mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,-199.5,100.5)

Table 2. Player 1 plays C. τ_i represents the player i's strategy, by assigning an operator to the qubits that she controls. $\gamma = \frac{\pi}{2}$. The first captain proposes the following distribution $(99,0,1)$; the second captain proposes $(100,0)$.

	Player 3: $\tau_3 = \{C,D\}$	$\tau_3 = \{C,\mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D,C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{D,\mathcal{U}(0,\frac{\pi}{2})\}$	(-200,-199.5,100.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0),C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{\mathcal{U}(\pi,0),\mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,-199.5,100.5)
	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}),D\}$	$\tau_3 = \{\mathcal{U}(0,\frac{\pi}{2}),\mathcal{U}(\pi,0)\}$
Player 2: $\tau_2 = \{D,C\}$	(-200,100.5,0.5)	(99,0,1)
$\tau_2 = \{D,\mathcal{U}(0,\frac{\pi}{2})\}$	(99,0,1)	(-200,-199.5,100.5)
$\tau_2 = \{\mathcal{U}(\pi,0),C\}$	(99,0,1)	(-200,100.5,0.5)
$\tau_2 = \{\mathcal{U}(\pi,0),\mathcal{U}(0,\frac{\pi}{2})\}$	(-200,-199.5,100.5)	(99,0,1)

Table 3. Player 1 plays $\mathcal{U}(0,\frac{\pi}{2})$. $\gamma = \frac{\pi}{2}$. The first captain proposes the following distribution $(99,0,1)$; the second captain proposes $(100,0)$.

Strategy: $U_1=U(\pi,0)$ $U_2=U(\pi,0)$ $U_3=C$ $U_4=C$ $U_5=U(\pi,0)$

Fig. 9. Unlike with the Bit-flip operator D, $\mathcal{U}(\pi,0)$ is affected by the entanglement coefficient γ in this case.

this Quantum Model of the Pirate Game would allow the first captain to approve that allocation proposal.

If the captain is the only one with access to quantum strategies, and the other players do not know it and play the classical strategies C (identity matrix) and D (Bit-flip operator), there is a dominant quantum strategy that allows her to pass her proposal and get all the 100 coins, for $\gamma = \frac{\pi}{2}$. Figure 11 provides an example that corroborates where we verify that player 1 has a set of dominant strategies (for $\tau_1 = \mathcal{U}_1(\pi,\phi), \phi \in (0,\frac{\pi}{2})$ or $\tau_1 = \mathcal{U}_1(\theta,\frac{\pi}{2}), \theta \in (0,\pi)$). We verified experimentally that the sub-games with 2 players (when $(C_4,C_5),(C_4,D_5),(D_4,C_5),(D_4,D_5)$), did not affect this dominant strategy. If fact if we analyse the expected utility function for player 1 (E_1.), and the Figure 1 representing the extensive form game, we verify that the the first captain is indifferent to the results of the sub-game with 2-players, when both player play a classical strategy, because she is already dead.

The captain proposes: $(100,0,0)$ Suppose captain is greedy and proposes to get the 100 coins. In the classical Pirate Game this would pose a conflict with his self-preserving needs. A pertinent question would be if

If players 2 and 3 know that the captain has access to quantum strategies, with a maximaly entangled game, and they have the restricted sub-set of pure classical strategies C_j and D_j, if they both play C, the dominant
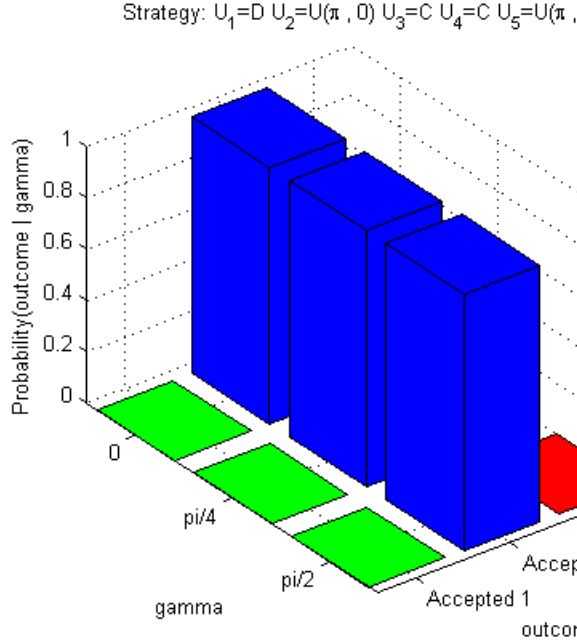
Strategy: $U_1=D$ $U_2=U(\pi,0)$ $U_3=C$ $U_4=C$ $U_5=U(\pi,$

Fig. 10. Player 1 uses the classical operator D. The probabilities don't change with the entanglement.

Strategy: $U_1=U(\theta,\phi)$ $U_2=C$ $U_3=C$ $U_4=C$ $U_5=D$
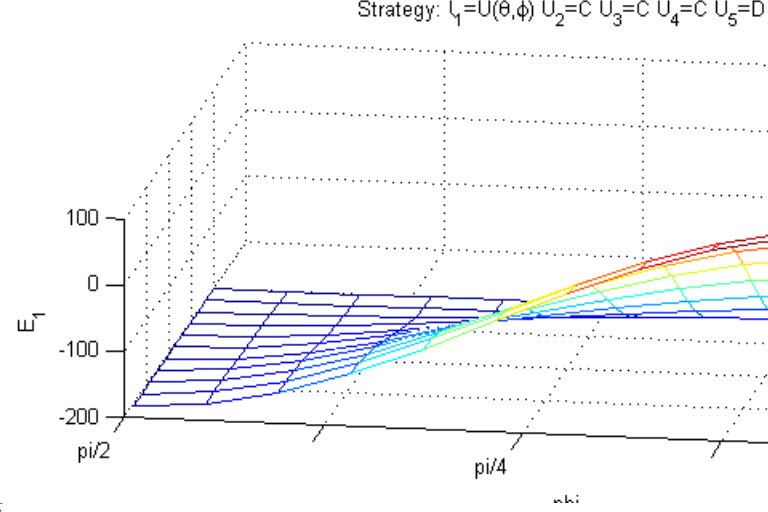
Fig. 12. Players 2 and 3 use a mixed classical strategy to choose the operators O_2 and O_3, where half the time they will play C and the other half D.
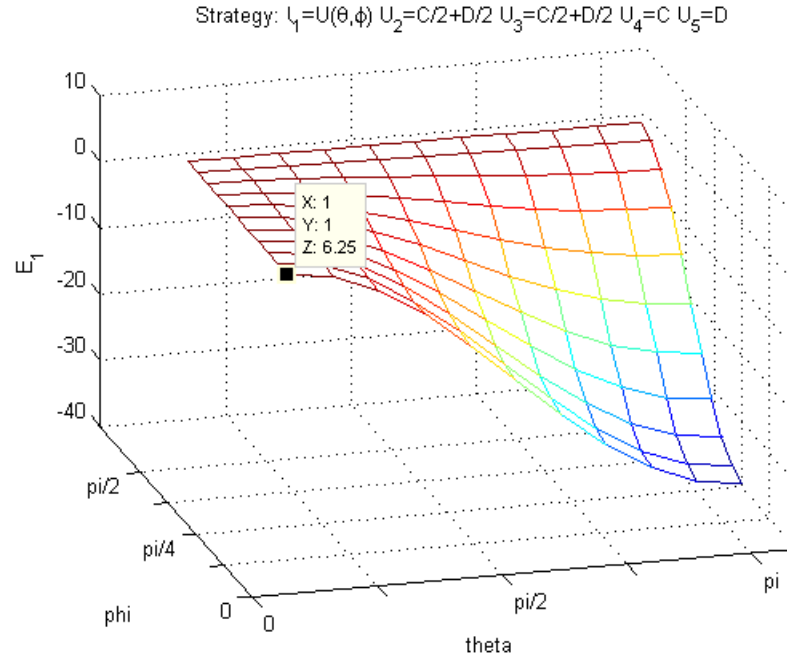

Strategy: $U_1=U(\theta,\phi)$ $U_2=D$ $U_3=D$ $U_4=C$ $U_5=D$

Fig. 11. Expected utility for player 1 when she adopts a strategic move of the form $U_1(\theta,\phi)$ and the other players play D_2, D_3, C_4, D_5.

Strategy: $U_1=U(\theta,\phi)$ $U_2=C/2+D/2$ $U_3=C/2+D/2$ $U_4=C$ $U_5=D$

Fig. 13. Players 2 and 3 use the operators C_2 and C_3.

5   Conclusion

strategy for the captain becomes $\tau_1 = \mathcal{U}_1(0,0)$. Their best response becomes playing a mixed strategy where half the time they will play C and the other half D. The best response for player 1 in this case is to play a mixed quantum strategy where half the time she plays $\mathcal{U}(0,0)$, half $\mathcal{U}(0,pi/2)$.

We tried to get our Quantum Model of the Pirate Game as close as possible in order to compare the the original game. However we did not incorporate the concept of measuring the results between rounds of voting. In the original problem after each stage the results are accounted. Our approach was deemed more adequate to analyse in [3].
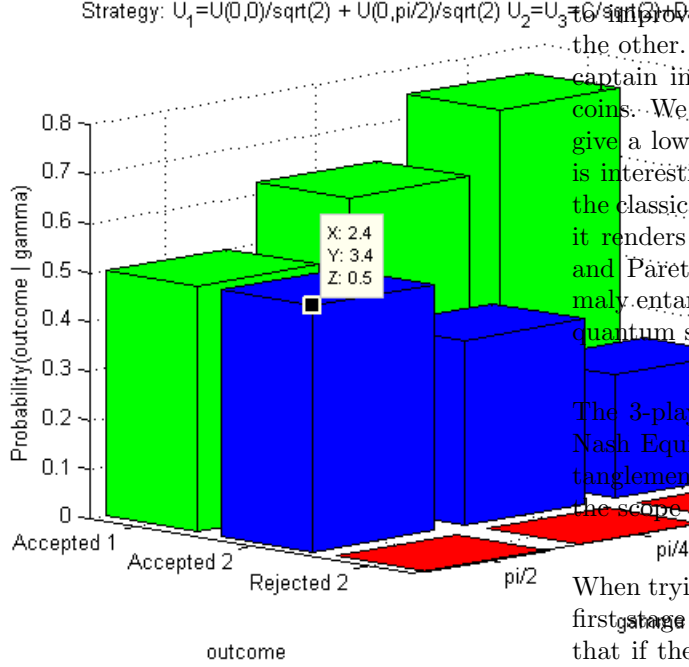
9

Strategy: $U_1=U(0,0)/\text{sqrt}(2) + U(0,\text{pi}/2)/\text{sqrt}(2)$ $U_2=U_3=C/\text{sqrt}(2)+D/\text{sqrt}(2)$ $U_1=C\ U_2=D$

Fig. 14. Players 2 and 3 use the operators C_2 and C_3.

In works such as the Prisoner's Dilemma [5] [6], and Quantum Ultimatum Game [3], the strategic space was analysed allowing a infinity of mixed quantum strategies. Our game space was much more vast than the works previously mencioned, we used 5 qubits to model our system, their models had no more than 3 qubits. This imposed that in order to analyse our system while conceptualy, our strategic space is also infinite, we restricted in order to analyse. In particular we payed special interest in the strategies: C, D, $\mathcal{U}(\pi,0)$, and $\mathcal{U}(0,\frac{\pi}{2})$. The first two mensioned strategies are the equivalent of the classical strategies "Cooperate" and "Defect", the other two are quantum strategies. Toghether these 4 strategies when paired with the coefitients (w,x,y,z) are known to span a class that contains all unitary operators, the $\mathsf{SU}(2)$. According to [10] these 4 operators when the entanglement is maximum provided a cycle where each operator was an optimal counter strategy to other.

The set-up of the initial system was crucial to introduce the phenomenon of entanglement. We concluded that without entanglement the game has a strictly determined solution and behaves as the original problem, however there are more pure strategy Nash Equilibria.

For the 2-player game we concluded that the expected utility when the players use a mixed strategy Nash equilibrium renders an expected utility of $(25, 25.125)$ wich is also a Pareto optimal solution because it is not possible to improve one players expected utility without harm the other. The classical setting is more beneficial to the captain in this game, because she will receive the 100 coins. We also found that distributing 50 coins would give a lower expected utility to the captain. This result is interesting because it is a Pareto optimal solution in the classical version, though not a Nash Equilibrium, and it renders an higher payoff than the Nash Equilibrium and Pareto Optimal solution when the sytem is maximaly entangled and the players have access to the 4 pure quantum strategies discussed in [10] and [5].

The 3-player game also has a mixed quantum strategy Nash Equilibrium, like the 2-player game, when the entanglement is maximum, however its calculus fell beyond the scope of this dissertation.

When trying to find if it is possible for the captain in the first stage of the game to acquire all gold coins we found that if the other players have a restricted set of strategies, they can only use the classical Cooperate or Defect operators, and if they don't know that the captain has access to quantum strategies, the captain will be able to get the 100 gold coins. However if players 2 and 3 have a restricted set where they can only use the classical Cooperation operator, or the classical Defection operator, the player 1 will no longer be able to acquire the 100 coins with certainty. Instead she may be able to acquire the 100 coins with probability $\frac{1}{2}$ or end up thrown off board with equal probability.

These results corroborate the literature in the Chapter 3, while contributing with another case study.

References

1. I. Stewart, "Mathematical recreations - A Puzzle for Pirates," *Scientific American*, no. May, pp. 98–99, 1999. [Online]. Available: http://scholar.google.com/scholar?hl=en\&btnG= Search\&q=intitle:Mathematical+recreations\#1

2. P. Fra, "Quantum information approach to normal representation of extensive games," no. 1, pp. 1–14, 2011.

3. ——, "Quantum information approach to the ultimatum game," pp. 1–14, 2011.

4. E. Rieffel and W. Polak, *Quantum Computing, A Gentle Approach*. Cambridge: Massachusetts Institute of Technology Press, 2011.

5. J. Du, X. Xu, H. Li, X. Zhou, and R. Han, "Playing prisoner's dilemma with quantum rules," *Fluctuation and Noise Letters*, vol. 2, no. 4, 2002. [Online]. Available: http://www.worldscientific.com/doi/pdf/10. 1142/S0219477502000993

6. J. Eisert, M. Wilkens, and M. Lewenstein, "Quantum games and quantum strategies," *arXiv preprint quant-ph/9806088*, vol. 2, no. 1, pp. 2–5, 1998. [Online]. Available: http://arxiv.org/abs/quant-ph/9806088

7. S. Khan and M. K. Khan, "Quantum Stackelberg Duopoly in a Noninertial Frame," *Chinese Physics Letters*, vol. 28, no. 7, July 2011. [Online]. Available: http://stacks.iop.org/0256-307X/28/i=7/a=070202? key=crossref.99b40803901ef6b332c1b5b78d9b9002

8. S. Ricketts, "An Investigation of Quantum Computational Game Theory," pp. 1–13, 2006.

9. J. S. Trueblood and J. R. Busemeyer, "Quantum Information Processing Theory," pp. 1–11.

10. J. Du, H. Li, X. Xu, X. Zhou, and R. Han, "Phase-transition-like behaviour of quantum games," *Journal of Physics A: . . .*, 2003. [Online]. Available: http://iopscience.iop.org/0305-4470/36/23/318

11. K. Leyton-Brown and Y. Shoham, *Essentials of Game Theory: A Concise Multidisciplinary Introduction*. Morgan and Claypool Publishers, 2008.