1

Quantum Pirate Game

In this chapter we describe the Pirate Game and the steps to model a quantum approach to the problem.

1.1 Pirate Game

1.1.1 Problem Description

The original Pirate Game is a multi-player version of the Ultimatum game that was first published as a mathematical problem in the Scientific American as a mathematical problem posed by Omohundro [1]. The main objective of the Pirate Game was to present a fully explainable problem with a non-obvious solution. The problem can be formulated as it follows:

> Suppose there are 5 rational pirates: A; B; C; D; E. The pirates have a loot of 100 indivisible gold coins to divide among themselves.
>
> As the pirates have a strict hierarchy, in which pirate A is the captain and E has the lowest rank, the highest ranking pirate alive will propose a division. Then each pirate will cast a vote on whether or not to accept the proposal.
>
> If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.
>
> We consider that each pirate privileges her survival, and then will want to maximize the number of coins received. When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.

1.1.2 Analysis

We can arrive at the sub-game perfect Nash equilibrium in this problem by using backward induction. At the end of the problem, supposing there are two pirates left, the equilibrium is very straight forward. This sub-game is represented in Table 1.1, and its Nash Equilibrium is (C, D).

	Player 2: C	Player 2: D
Player 1: C	(100, 0)	(100, 0)
Player 1: D	(100, 0)	(-200, 100.5)

Table 1.1: Representation of the 2 player sub-game in normal form.

As the highest ranking pirate can pass the proposal in spite of the other's decision, her self-interest dictates that she will get the 100 gold coins. Knowing this, pirate E knows that any bribe other higher ranking pirate offers her will leave her better than if the game arrives to the last proposal.

When applying this reasoning to the three pirate move, as pirate C knows she needs one more vote to pass her proposal and avoiding death, she will offer the minimum amount of coins that will make pirate E better off than if it comes to the last stage with two pirates. This means that pirate C will offer 1 gold coin to pirate E, and keep the remaining 99 coins.

With 4 pirates, B would rather bribe pirate D with 1 gold coin, because E would rather like climb on the hierarchy and getting the same payoff. Finally, with 5 pirates the captain (A), will keep 98 gold coins and rely on pirate C and E to vote in favour of the proposal, by giving 1 gold coin each.

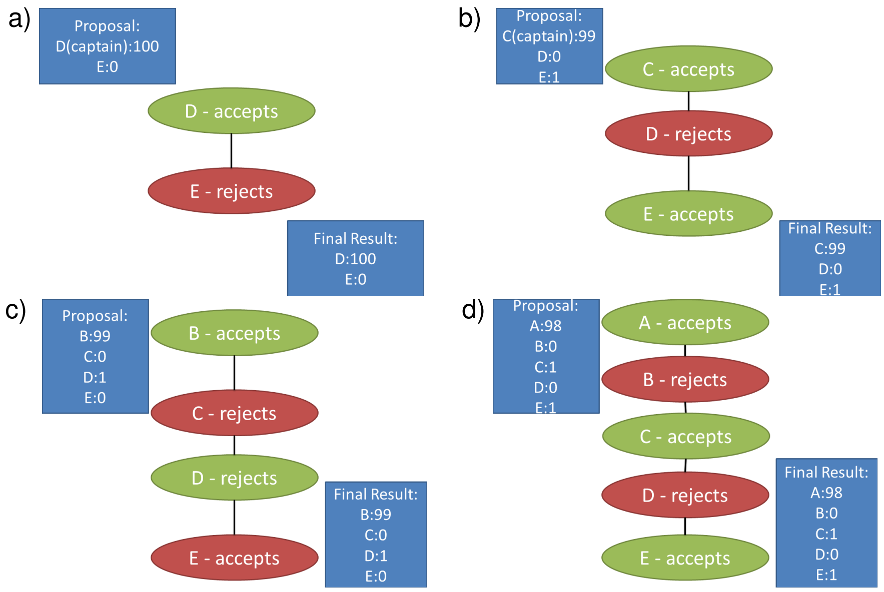

Table 1.2: The equilibrium for the Pirate Game can be found through backward induction. From a), where there's only two pirates left, to d), that corresponds to the initial problem, we define the best response.

1.1.2.A Analysis of the Pirate Game for 3 Players

In Figure 1.1 we have an extensive form representation of the classic Pirate Game for 3 players. Each node in the game tree has the number of the player who will make the decision, either to Cooperate (vote yes to the proposal), or Defect. The dashed arrows represent states where the player does not have information of the current state (simoultaneous move).

The green accent, in Figure 1.1, shown in the nodes represent a state where the first captain (player 1), will see her proposal accepted, the utility associated . The blue accent denotes the outcomes where the second captain makes a proposal and has seen it accepted. The red accent color represents the outcomes where the player 3 will be the remaining pirate.

The number of coins will translate directly the utility associated with getting those coins. For example if a pirate receives 5 gold coins and the proposal is accepted he will get a utility of 5. The highest ranking pirate in the hierarchy is responsible to make a proposal to divide the 100 gold coins. This proposal means choosing the number of coins each player gets if the proposal is accepted. A captain i chooses the amount of coins a player j get; this amount will be represented by α_{ij}.

In the initial stage of the game the captain will define $\alpha_{11}, \alpha_{12}, \alpha_{13}$, and they will obey to the Equation 1.1, that imposes the rule that the captain i must allocate all the 100 coins, to the players that are still alive. N the number of pirates in the game.

$$\forall i \in \{1, 2, ..., N-1\} : \sum_{j=i}^{N} \alpha_{ij} = 100, \forall i, j : \alpha_{ij} \in \mathbb{N}_0 \tag{1.1}$$

The values for $(\alpha_{11}, \alpha_{12}, \alpha_{13}) = (99, 0, 1)$ will be the allocation that results in an equilibrium for the 3 player game .

The proposed goods allocation will be executed if there is a majority (or a tie), in the voting step. A step

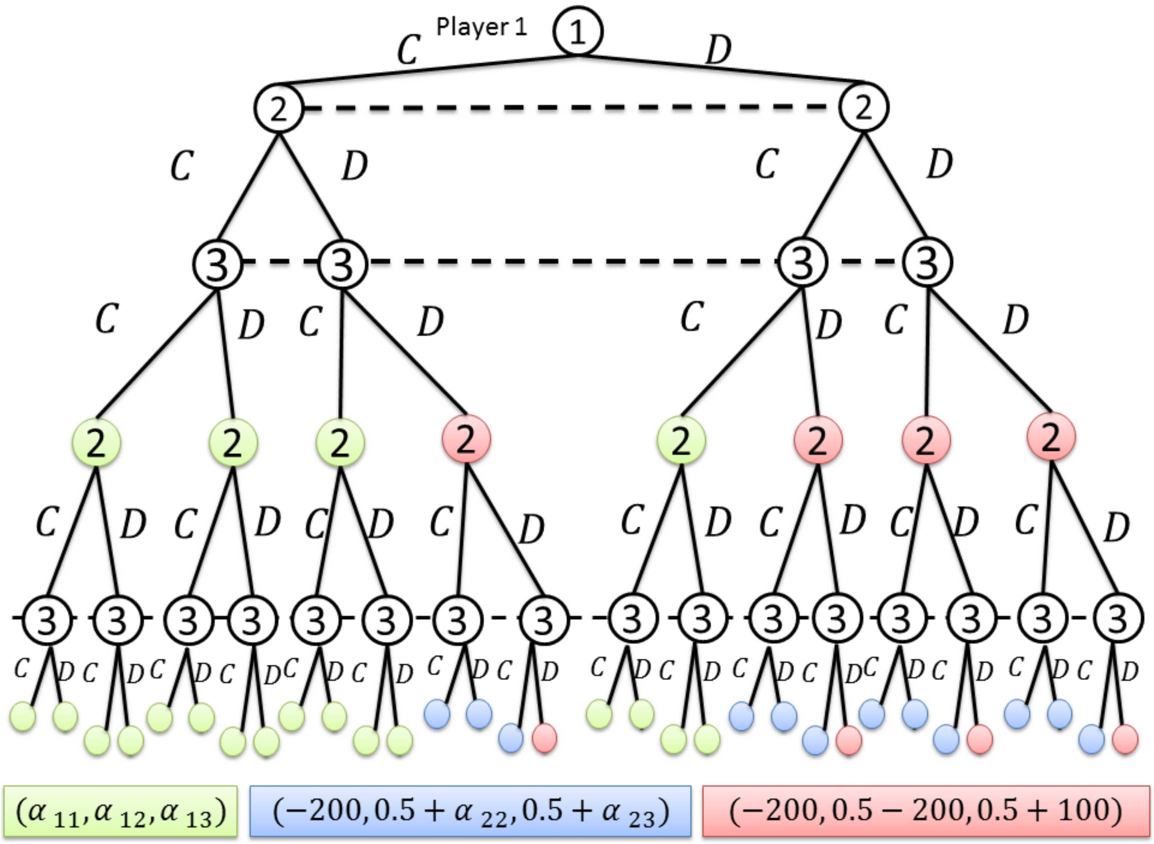

Figure 1.1: Extensive form representation of the classic Pirate Game for 3 players.

in the game consists on the highest ranking pirate defining a proposal and the subsequent vote, where all players choose simultaneously an operator.

If the proposal is rejected the captain will be thrown off board, to account for the fact that this situation is very undesirable for the captain he will receive a negative payoff of -200 (that can be seen in the blue and red outcome in Figure 1.1). This value was derived from the fact that a pirate values her integrity more than any number of coins she might receive.

> "When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy."

This means that the pirates have a small incentive to climb the hierarchy. For example in the three player classical game, the third player, who has the lowest rank, will prefer to defect the initial proposal if the player 1 doesn't give her a coin, even knowing that in the second round the player 2 will be able to keep the 100 coins. We will account for this preference by assigning an expected value of half a coin (0.5), to the payoff of the players that will climb on the hierarchy if the voting fails. This tie breaker is shown in the blue and red outcomes in Figure 1.1.

1.1.2.B Consideration on the generalization for N players

We can generalize this problem for N pirates. If we assign a number to each pirate, where the captain is number 1 and the lower the number the higher the rank. If the number of coins is superior to the number of pirates, the equilibrium will have the captain (highest ranking pirate), giving a gold coin to each odd pirate, in case the number of players alive is odd, while keeping the rest to herself. When we have a even number of players the captain will assign a gold piece to each pirate with a even number, and the the remaining coins to herself.

If the number of pirates is greater than two times the amount of coins $N > 2C$, a new situation arises. If we have 100 coins and 201 pirates, the captain will not get any coin. By the same reasoning with 202 pirates the captain will still be able to survive by bribing the majority of the pirates and keeping no coins for herself. With 203 pirates the first captain will die. However with 204 pirates, the first captain will be able to survive even though he won't be able to bribe the majority, because her second in command knows that when she makes a proposal, she'll be thrown off board. In the game with 205 pirates, however the captain is not able to secure the vote from the second in command on the 204 pirate game, because the second captain is safe and she is able to make a have her proposal accepted and have the third pirate safe.

We can generalize this problem for $N > 2C$ as [1], as the games with a number of pirates equal to $2C$ plus a power of two will have an equilibrium in the first round, in the others every captain until a sub-game with a number of pirates equal to $2C$ plus a power of two will be thrown off board.

1.2 Quantum Pirate Game

The original Pirate Game is posed from the point of view of the captain. How should she allocate the treasure to the crew in order to maximize her payoff. We can find the a equilibrium to the original Pirates Game and, while the solution may seem unexpected at first sight, it is fully described using backwards induction.

When modelling this problem from a quantum theory perspective we are faced with some questions, such as:

- Will the initial conditions provide different equilibria?
- What are the similarities with the classical problem?
- Is it possible for a captain, in a situation where we have more than two pirates left, to acquire all the coins?

The main difference from the original problem will rely on how the system is set up and the fact that we will allow quantum strategies. We propose to study this problem for a 3 player game and trying to extrapolate for N players.

We will analyse the role of entanglement and superposition in the game system.

Another aspect worth studying is the variation in the coin distribution on the payoff functions for the players. We are particularly interested in studying the classical equilibrium where the captain retains 99 coins and gives a single coin to the player with the lowest rank. Moreover we want to study what happens when the captain tries to get all the coins.

1.2.1 Quantum Model

In order to model the problem we will start by defining it using the definition of quantum game (Γ), referred in **??**, Section **??** [2].

We want to keep the problem as close to the original as possible in order to better compare the results. Thus we will analyse the game from the point of view of the captain. Will her best response change?

For the purpose of demonstration this problem could be described using 3 players; the lowest number of players that has an equilibrium in which the captain has to bribe another pirate.

We begin by assigning an offset to each pirate (in order to identify her), as in the Section 1.1.1. The captain is number 1 and the lower the number the higher the rank.

1.2.1.A Game system: Setting up the Initial State

A game Γ can be viewed as a system composed by qubits manipulated by players. We will use the definition of quantum game discussd in Section **??** ($\Gamma = (\mathcal{H}^{2^a},\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\})$), to model our game system.

In this 3 player game there will be 5 qubits representing the actions or the players decision. The number of qubits needed to represent the game grows exponentially with the number of players. For N players we need $\sum_{i=2}^{N} i$ qubits. With 8 players, this game would already be impractical to simulate in a classical computer. In this regard a quantum computer may enhance our power to simulate this kinds of experiments [3].

The mapping function ξ that assigns each action/qubit φ_j (with $j = \{1, 2, 3, 4, 5\}$), to a player is represented on Equation 1.2.

$$\xi(j) = \begin{cases} 1 & , \; if \; j = 1; \\ 2 & , \; if \; j \in \{2, 4\}; \\ 3 & , \; if \; j \in \{3, 5\}. \end{cases} \tag{1.2}$$

With 3 players and 5 actions our system with be represented in a $\mathcal{H}^{32}$ using a state ψ. This means that to represent our system we will need $2^5 \times 1$ vectors, our system grows exponentially with the number of players/qubits. Each pure basis of $\mathcal{H}^{32}$, shown in Equation 1.3, will represent a possible outcome in the game. We assign a pure basis as $|0\rangle = |C\rangle$ ("C" from "Cooperate"), and $|1\rangle = |D\rangle$ ("D" from "Defect").

$$\begin{aligned} \mathcal{B} = \{ & |00000\rangle, |00001\rangle, |00010\rangle, |00011\rangle, |00100\rangle, |00101\rangle, |00110\rangle, |00111\rangle, \\ & |01000\rangle, |01001\rangle, |01010\rangle, |01011\rangle, |01100\rangle, |01101\rangle, |01110\rangle, |01111\rangle, \\ & |10000\rangle, |10001\rangle, |10010\rangle, |10011\rangle, |10100\rangle, |10101\rangle, |10110\rangle, |10111\rangle, \\ & |11000\rangle, |11001\rangle, |11010\rangle, |11011\rangle, |11100\rangle, |11101\rangle, |11110\rangle, |11111\rangle \} \end{aligned} \tag{1.3}$$

The initial system ($|\psi_0(\gamma)\rangle$), will be set up by defining an entanglement coefficient γ, that affect the way the five qubits (belonging to the three pirate players), are related; this is shown in Equation 1.5. We will entangle our state by applying the gate $\mathcal{J}$ [4]. The parameter γ becomes a way to measure the entanglement in the system [5].

The concept of entanglement is crucial to explain some phenomena in Quantum Mechanics (Section **??**).We analysed the role of the entanglement of the system since other examples researched pointed to it being the proeminent factor regarding behaviour changes from the classsical perspective [2] [6] [4] [7] [8].

We can interpret the existence (or non-existence), of entanglement or superposition in the initial system as an unbreakable contract between the players [9]. The initial state starts by revealing a group of pirates that cooperate by default. We chose this initial set-up because it is prevalent in the literature [5] [2] [6] [4], and we want to test if there is any equilibrium situation where the first captain can pass her proposal while taking all the 100 coins.

Due to the nature of quantum mechanics we have to pay attention of how we set-up our architecture; we cannot copy or clone unknown quantum states (No-cloning Theorem) [3].

$$\mathcal{J} = exp\left\{ i\frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{1.4}$$

$$|\psi_{ini}(\gamma)\rangle = exp\left\{i\frac{\gamma}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}|00000\rangle$$
$$= cos(\frac{\gamma}{2})|00000\rangle + isin(\frac{\gamma}{2})|11111\rangle, \gamma \in (0, \pi) \quad (1.5)$$

1.2.1.B Strategic Space

In Equation **??** ($\Gamma = (\mathcal{H}^{2^a},\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\})$), there is the notion of a subset of unitary operators that the players can use to manipulate their assigned qubits.

Each player will be able to manipulate at least one qubit in the system. Those qubits are $|\varphi_1\rangle$, $|\varphi_2\rangle$, and $|\varphi_3\rangle$. The Equation 1.3 assigns the qubit $|\varphi_1\rangle$ to player 1, qubits $|\varphi_2\rangle$ and $|\varphi_4\rangle$ to player 2, the remaing qubits are assigned to player 3. Each player will be able to manipulate her assigned qubits j with an unitary operator of the form shown in Equation 1.6. [5] notices that it is sufficient to restrict the strategic space to the 2-parameter set of matrices in Equation 1.6, with $\theta \in (0, \pi)$, and $\phi \in (0, \frac{\pi}{2})$.

$$\mathcal{U}_j(\theta, \phi) = \begin{bmatrix} cos(\frac{\phi}{2}) & e^{i\phi}sin(\frac{\phi}{2}) \\ -e^{-i\phi}sin(\frac{\phi}{2}) & cos(\frac{\phi}{2}) \end{bmatrix}, j \in \{1, 2, 3, 4, 5\}, \theta \in (0, \pi), \phi \in (0, \frac{\pi}{2}) \quad (1.6)$$

The two operators will correspond to the classical actions of voting “Yes” or to Cooperate, and voting “No” (Defect) are particular cases of the subset U_j, S_j described in 1.7. The classical cooperation operator will be represented by the Identity operator (o_{j0}, where j identifies the qubit that the respective player will act upon). When assigned to a qubit this operator will leave it unchanged.

The defection operator (D), will be represented by one of Pauli’s Operators - the Bit-flip operator. This operator was chosen because it performs the classical operation NOT on a qubit and it is a particular case $\mathcal{U}_j(\pi, 0)$.

$$S_j = \begin{cases} C_j = o_{j0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D_j = o_{j1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{cases}, j \in \{1, 2, 3, 4, 5\} \quad (1.7)$$

Each player will have a strategy τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j\in \xi^{-1}(i)$). $\tau_2 = \{D_2, C_4\}$ represents the strategy where the player 2 votes D in the first stage and C in the second stage.

In Quantization schemas of the Prisoner’s Dilemma [4] [5], and Quantum Ultimatum Game [6], the strategic space, described in Equation 1.6, was analysed allowed a infinity of mixed quantum strategies. In the Quantum Roulette Game [10] and [11] we have a demonstration that in a classical two-person zero-sum strategic game, if one person adopts a quantum strategy, she has a better chance of winning the game.

1.2.1.C Final State

We can play the Pirate Game by considering a succession of steps or voting rounds. In each step we have a simultaneous move(the players sellect their strategies at the same time), however, considering the potential rounds the game has, we have a sequential game.

With three players, the first move will correspond to the player 1 (or the captain), if the proposal fails we will proceed to the second step in the game, where the remaining two players will vote on a new proposal made by player 2 (who will be the new captain).

This final state is calculated by constructing a super-operator, by performing the tensor product of each player chosen strategy, as shown in Equation 1.7. The super-operator, containing each player's strategy, will then be applied to the initial state,as shown in Equation1.8.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{3} \otimes_{j\in\xi^{-1}(i)} \mathcal{U}_j |\psi_{ini}(\gamma)\rangle \tag{1.8}$$

In the Figure 1.2 we have a representation of the game.We start by building our initial state $k-1$, then the players select their strategies, a super operator is constructed by performing a tensor product of the selected operators.

In order to calculate the expected payoff functions we need to de-entangle the system, before measuring. The act of measuring, in quantum computing, gives an expected value that can be understood as the probability of the system collapsing into that state.

We can de-entangle the our $\mathcal{H}^{32}$ system by applying $\mathcal{J}^\dagger$ (Equation **??**), this will produce a final state that we will be able to measure. If we do not apply the inverse transformation $\mathcal{J}^\dagger$ we are introducing errors in the system (when the entanglement parameter γ is different than 0), because we are introducing a correlation between the qubits in the system. In the Figure 1.2 we have represented the way we entangle and de-entangle the system.

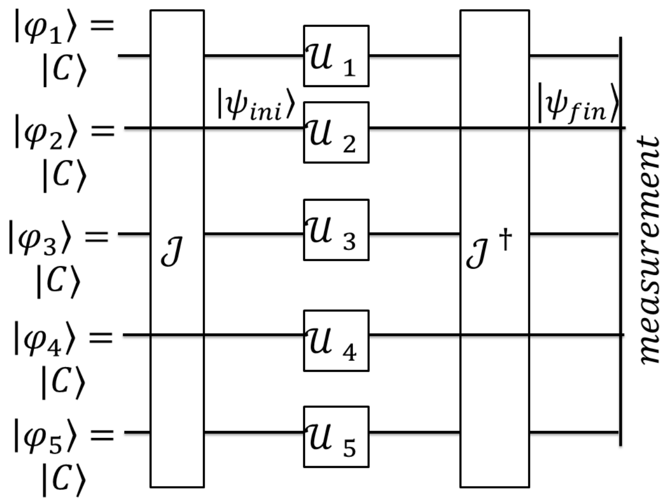

Figure 1.2: Scheme that represents the set-up of the 3-player Pirate Game. Before we measure the final result we need to apply the transpose operator $\mathcal{J}^\dagger$.

1.2.1.D Utility

To build the expected payoff functionals for the three player situation we must take into account the sub-games created when the proposal is rejected. In Figure 1.1 we can see an extensive form representation of the game.

As defined on Equation **??** ($E_i = \sum_{b \in \mathcal{B}} u_i(b)|\langle b|\psi_{fin}\rangle|^2, u_i(b) \in \mathbb{R}$),for each player we must specify a utility functional that attributes a real number to the measurement of the projection of a basis in the quantum state that we get after the game.

This measurement can be understood as a probability of the system collapsing into that state (that derives from the Born Rule, Section **??**).

These utility functions will represent the degree of satisfaction for each pirate after game by attributing a real number to a measurement performed to the system (as in Equation **??**, Section **??**). The real numbers used convey the logical relations of utility posed by the original problem description. Those numbers will represent the utility associated with the number of coins that a pirate gets, a death penalty, and a small incentive to climb the hierarchy. As each pirate wants to maximize her utility, the Nash equilibrium will be thoroughly used to find the strategies that the pirates will adopt [12] [13].

We can observe in Figure 1.1 that in that we have three separate groups (denoted by the colour accents), of outcomes that share the same payoff. In our quantum scheme we can agregate the quantum states associates with a payoff in the following manner:

- States where the first proposal is accepted:
  - $|C, C, C, x_4, x5\rangle$ or $|0, 0, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|D, C, C, x_4, x5\rangle$ or $|1, 0, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|C, D, C, x_4, x5\rangle$ or $|0, 1, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|C, C, D, x_4, x5\rangle$ or $|0, 0, 1, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$.
- States where the first captain will be eliminated and the second player gets her proposal accepted:
  - $|D, D, D, C, x5\rangle$ or $|1, 1, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, D, D, D, C\rangle$ or $|1, 1, 1, 1, 0\rangle$;
  - $|D, D, C, C, x5\rangle$ or $|1, 1, 0, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, D, C, D, C\rangle$ or $|1, 1, 0, 1, 0\rangle$;
  - $|C, D, D, C, x5\rangle$ or $|0, 1, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|C, D, D, D, C\rangle$ or $|0, 1, 1, 1, 0\rangle$;
  - $|D, C, D, C, x5\rangle$ or $|1, 0, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, C, D, D, C\rangle$ or $|1, 0, 1, 1, 0\rangle$.
- States where all proposals are rejected:
  - $|D, D, D, D, D\rangle$ or $|1, 1, 1, 1, 1\rangle$;
  - $|D, D, C, D, D\rangle$ or $|1, 1, 0, 1, 1\rangle$;
  - $|C, D, D, D, D\rangle$ or $|0, 1, 1, 1, 1\rangle$;
  - $|D, C, D, D, D\rangle$ or $|1, 0, 1, 1, 1\rangle$.

In order to calculate the probability of the final state collapsing onto a basis state $b \in \mathcal{B}$ we perform a projection of the state in the chosen basis and we measure the squared length of the projection, $|\langle b|\psi_{fin}\rangle|^2$ [14].

Each player has an expected utility functionals. The expected utility for player 1, shown in Equation 1.9, give a real number that represents the payoff associated with a final state. The same goes to player 2 that has her expected utiity functional specified in Equation 1.10, and the player 3 in Equation 1.11.

$$\begin{aligned} E_1(|\psi_{fin}\rangle) = \alpha_{11} \times (\sum_{x_3}\sum_{x_4} |\langle 0,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 1,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \\ + \sum_{x_3}\sum_{x_4} |\langle 0,1,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 0,0,1,x_4,x5|\psi_{fin}\rangle|^2) - \\ -200 \times (\sum_{x_3}\sum_{x_4} |\langle 1,1,1,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 0,1,1,x_4,x5|\psi_{fin}\rangle|^2 + \\ + \sum_{x_3}\sum_{x_4} |\langle 1,0,1,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 1,1,0,x_4,x5|\psi_{fin}\rangle|^2) \end{aligned} \tag{1.9}$$

$$\begin{aligned} E_2(|\psi_{fin}\rangle) = \alpha_{12} \times (\sum_{x_4}\sum_{x_5} |\langle 0,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_4}\sum_{x_5} |\langle 1,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \\ + \sum_{x_4}\sum_{x_5} |\langle 0,1,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_4}\sum_{x_5} |\langle 0,0,1,x_4,x5|\psi_{fin}\rangle|^2) - \\ +(0.5+\alpha_{22}) \times (\sum_{x_5} |\langle 1,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,1,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 1,1,0,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 1,0,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,0,1,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 0,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,0|\psi_{fin}\rangle|^2) + \\ +(0.5-200) \times (|\langle 1,1,1,1,1|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,1|\psi_{fin}\rangle|^2 + \\ + |\langle 1,0,1,1,1|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,1|\psi_{fin}\rangle|^2) \end{aligned} \tag{1.10}$$

$$\begin{aligned} E_3(|\psi_{fin}\rangle) = \alpha_{13} \times (\sum_{x_3}\sum_{x_4} |\langle 0,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 1,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \\ + \sum_{x_3}\sum_{x_4} |\langle 0,1,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4} |\langle 0,0,1,x_4,x5|\psi_{fin}\rangle|^2) + \\ +(0.5+\alpha_{23}) \times (\sum_{x_5} |\langle 1,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,1,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 1,1,0,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 1,0,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 1,0,1,1,0|\psi_{fin}\rangle|^2 + \\ + \sum_{x_5} |\langle 0,1,1,0,x5|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,0|\psi_{fin}\rangle|^2) + \\ +(100.5) \times (|\langle 1,1,1,1,1|\psi_{fin}\rangle|^2 + |\langle 1,1,0,1,1|\psi_{fin}\rangle|^2 + \\ + |\langle 1,0,1,1,1|\psi_{fin}\rangle|^2 + |\langle 0,1,1,1,1|\psi_{fin}\rangle|^2) \end{aligned} \tag{1.11}$$

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