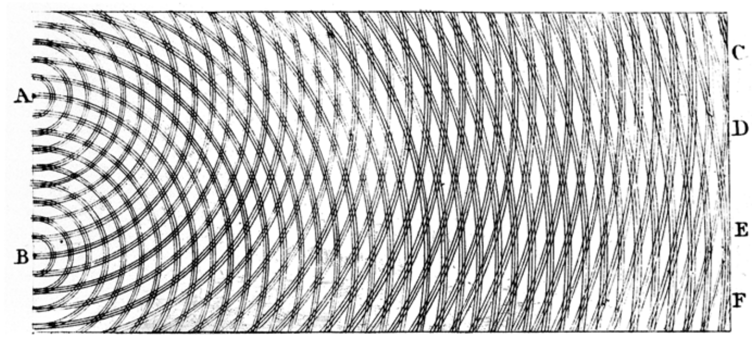

On a Quantum Pirate Game

A Quantum Game-Theory Approach to The Pirate's Game

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I can live with doubt and uncertainty and not knowing. I think it is much more interesting to live not knowing than to have answers that might be wrong. If we will only allow that, as we progress, we remain unsure, we will leave opportunities for alternatives. We will not become enthusiastic for the fact, the knowledge, the absolute truth of the day, but remain always uncertain... In order to make progress, one must leave the door to the unknown ajar.

Richard P. Feynman

Acknowledgments

This work marks the end of an arc in a personal story. The story of a path started naively our of curiosity for the world around me, a path filled hardships, a path filled with good momments.

This journey was marked by self-doubt, but it was also rich in moments where wonderfull travel companions walked alongside. I want to thank all these travel companions. In particular I give thanks to my loved ones who never cessed believing even when I doubted myself. I would like to thank my advisor, Professor Andreas Wichert, for giving me the freedom to fail and learn from my mistakes, while teaching me the meaning of the word Scientist. I would also like to thank my co-workers at Connect Coimbra for teaching me how to balance work and life, and the desire to make a difference in my community.

Abstract

In this document, we develop a model and we simulate a quantization scheme for the mathematical puzzle created by Omohundro and Stewart - “A puzzle for pirates ”. This game is a multi-player version of the game “ultimatum ”.

The Quantum Theory of Games is a field that seeks to introduce the mathematical formalism of Quantum Mechanics in order to explore models of conflict that arise when rational beings make decisions. These models of conflict are pervasive in the structural make-up of our society. The combination of game theory and Quantum Probability, despite not having a practical application, can help in the development of new quantum algorithms. Furthermore the fact that Game Theory is transversal to many areas of knowledge can provide insights to future application of these models.

We focused mainly on the role of quantum entanglement phenomenon in the game system. We found that this phenomenon introduces variations in expected utility by players, for some strategies similar to other models in the area. However we also verified that there are strategies in which there is no interference.

Keywords

Quantum Game Theory; Pirate Game; Quantum Mechanics; Game Theory; Probability Theory

Resumo

Neste trabalho desenvolvemos e simulámos um modelo quâtico para o puzzle matemático criado por Omohundro e Stewart, “Um puzzle para piratas”(original em inglês “A Puzzle for Pirates”). Este jogo consiste numa versão multi-jogador do jogo “Ultimato”.

A Teoria de Jogos Quântica é uma área que procura introduzir o formalismo matemático na base da Mecânica Quântica para explorar modelos de conflito que surgem quando seres racionais tomam decisões. Estes modelos de conflito estão na base da estrutura da nossa sociedade. A combinação de Teoria de Jogos e a Teoria de Probabilidade Quântica apesar de ainda não ter uma aplicação prática pode ajudar no desenvolvimento de novos algoritmos quânticos. O facto da Teoria de Jogos ser ruma disciplina transversal a muitas áreas do conhecimento pode fazer com que estes modelos possam eventualmente vir a ter relevância.

Focámo-nos sobretudo no papel do fenómeno quântico entrelaçamento no sistema do jogo. Verificámos que este fenómeno introduz variações na utilidade esperada pelos jogadores, para algumas estratégias à semelhança de outros modelos na área. Contudo tambÃ©m verificámos a existência de estratégias nas quais não existe interferência.

Palavras Chave

Teoria de Jogos Quântica; Jogo dos Piratas; Mecânica Quântica; Teoria de Jogos; Teoria de Probabilidade

Contents

List of Figures

List of Tables

Abbreviations

CLT - Central Limit Theorem

1

Introduction

Contents

In this section we lay the motivation for this work, its relevance, our objectives, and the general outline of this document.

1.1 Motivation

It's not always easy to understand and introduce new paradigms. Some things we find natural nowadays, were once the source of controversy. Taking for example the number zero 0. As numbers were introduced to help count physical objects, the idea of representing "nothingness" was once considered strange. In the beginning there were numerous ways devised to deal with this mathematical inconvenience, a special case. However the need to use "zero" as a number in its own right lead to the popularization of its concept as a number and opened door to new breakthroughs in mathematics [1].

Quantum mechanics shared this same problem and the mathematical formalization grew out of the need to explain phenomena at the atomic scale [2]. Nowadays we accepted quantum mechanics as a tool, to analyse and predict behaviour at a microscopic level. However at our scale some quantum mechanics phenomena seem almost "ridicule" and paradoxical.

The Quantum Computing started atracting interest in the decade of 1980, with the works of Yuri Manin and Richard Feynman. The objective was to create a computational machine that could use the entanglement and superposition of the wave function to perform calculations that are currently impossible with classic computers. A popular example of that kind of computation is the prime factorization, which is in the base for modern cryptography. However we must bear in mind that our current computers are in fact quantic at the microscopic level. Siliccon, a semiconductor, in in the base of modern chips, and its properties arise from quantum mechanics (it is neither a pure conductor, not a isolant material). Sillicon is used to construct transistors that are based on the concept of acurate measuring; in this process the heat productiong is unavoidable. It is the heat produced by the microchips that make them hard to integrate and scale, and despite the Moore's Law that predicts that the number of transistors double every year (in the industry), there are physical limits that cannot be surpassed. When the transistors become too small they will start behaving in a quantum way, introducing errors in our deterministic computations [3].

The idea of trying to develop in quantum computing arises from the desire to explore this paradigm. The standard curriculum of an undergraduate computer scientist is focused on the current computational paradigm, which has its roots in von Neumman Architecture [4]. While the construction and the inner-workings of computational systems that rely on new paradigms may fall outside of the scope of computer science, understanding it from the point of view of Information Technology and pushing boundaries on model representation are areas where computer scientist might contribute.

A model provides a way to abstract the reality, which is ofen too complex to analyse in its breadth. The Game Theory is an area that tried to find ways to represent and analyse sittuations of conflict generated when intelligent rational beings make decisions. This discipline has major applications in Economics, Biology, Political Science, and Artificial Intelligence. Combining both the mathematical foundations of Quantum Mechanics, and the Game Theory is an ideea that is starting to attract attention. These two areas share the same founding father, von Neumman, and creating a Quantum Model for a game seems to be a relevant way to explore the theory behing Quantum Mechanics while having a controlled and creative way to apply it. It is also interesting to analyse how these quantum games differ from their well defined classical counterparts. Furthermore simulating quantum algorithms on a classical computer is usualy impractical because this systems grow exponencially with the number of quibits. In game theory we have relevant problems that can be modelled using few quibits thus making it possible to simulate in a classical computer.

1.2 Problem Description

The Pirate Game is a mathematical, Game Theory, problem; with this work want to explore it in the light of quantum game theory. This means developing a quantization scheme, which is a way to transform the original game in a quantum simulation. Our simulations will be implemented on Matlab.

Despite not having a clear “real-world” application yet, modelling games with quantum mechanics rules may aid the development of new algorithms that would be ideally deployed using quantum computers.

Furthermore applying quantum probabilities to a well established area as Game Theory, which has applications in fields such as Economy, Political Science, Psychology, Biology, might introduce new insights and even relevant practical applications [5].

1.3 Objectives

The main objectives with this work are:

- To learn how to quantize a classical Game Theory problem;
- To investigate and compile a set of relevant works on Quantum Models;
- To compare the Quantum and the Classical versions of a game. Namely to observe how quantum phenomena, such as superposition and entanglement, may alter the Nash equilibria in a quantum game.

1.4 Contributions

The main contributions expected from this work are:

- To provide a comprehensive state of the art and related work with examples and executable simulations, in order to facilitade the entry in this field;
- A Quantum Model of one unique problem, and do a comparative study with the original version. The Pirate Game is an example of a mathematical game that has not been modelled in a quantum domain yet;
- To study the the possible outcomes, given the initial setup in a quantum game. Specifically to study the role of entanglement and the emergent results it produces.

1.5 Thesis Outline

This document is organized in six main chapters: Introduction, Background, Related Work, Pirate Game, Analysis and Results, and Conclusion.

In the introductory chapter we lay out our problem, its relevance, the mains objectives of this work.

The second chapter, “Background”, will pose an overview of theoretical concepts, and definitions needed to understand this work. We will start by making a comparative study between the Classical Probability Theory (Bayesian Probability), and the Quantum Probability (also known as von Neumann Probability). Then we will lay the fundamental concepts needed to work and understand quantum computing, namely the Superposition and Entanglement phenomena, the role of Operators, how to scale quantum systems. Finally we will present a Game Theory background, and some Quantum Game Theory concepts.

After laying the theoretical construct which is the foundation of this work we will analyse some “Related Work”. In this chapter we select, analyse, and experiment with some Quantum Computing and Quantum Game Theory Models. We tried to privilege a practical approach which might help an unfamiliar reader consolidate some theoretical concepts from the previous chapter.

In the fourth chapter we present the original Pirate Game and our quantum version, emphasizing our chosen definition of quantum game.

The fifth chapter will be reserved for analysing and discussing the results captured from the simulation of the Quantum Pirate Game.

Finally we will wrap up this work by reflecting on our principle results, the relevance of this model. We will also try to set new starting points for future work.

2

Background

Contents

In this section we present the theorectical concepts needed in order to understand this work.

2.1 Bayesian Probability

The probability theory has its roots in the 16^{th} century with attempts to analyse games of chance by Cardano. It is not hard to understand why games of chance are in the foundations of the probability theory. Throughout History the concept of probability has fascinated the Human being. Luck, fate were words that reflect things we feel we have no control upon and are associated with those games, and almost paradoxically we evolved in a way that we do not feel comfortable around them.

2.1.1 Kolmogorov axioms

In spite of the fact that the problem of games of chance kept attracting numerous mathematicians (with some of the most influential ones being Fermat, Pascal and Laplace), it was not until the 20^{th} century that the Russian mathematician Kolmogorov laid the foundations of the modern probability theory (first published in 1933) introducing three axioms [6]:

1. The probability of an event is a non-negative real number:

$$P(A) \in \mathbb{R} \ \wedge \ P(A) \geq 0 \tag{2.1}$$

   This number represents the likelihood of that event happening, the greater the probability the more certain is its associated outcome.

2. The sum of probabilities of all possible outcomes in a space is always 1 ($P(\Omega) = 1$). These first two axioms leave us with the corollary that probabilities are bounded:

$$0 \leq P(A) \leq 1 \tag{2.2}$$

3. The probability of a sequence of pairwise disjoint events is the sum these events. A corollary of this axiom is:

4. Any countable sequence of mutually exclusive events satisfies:

$$P(A_1 \vee A_2 \vee ... \vee A_n) = P(A_1) + P(A_2) + ... + P(A_n) \tag{2.3}$$

Another important corollary derived from the axioms is the addition law of probability:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B) \tag{2.4}$$

2.1.2 Conditional Probability

When some evidence is presented we have what we can conditional probability (or posterior probability). Conditional probability is represented as $P(A|B)$, that could be read as: the probability of A after evidence B is presented.

The product rule is used to calculate posterior probability2.5.

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \tag{2.5}$$

2.1.3 Markov Chains

Markov Chains define a system in terms of states and the probabilistic transitions from one state to another.

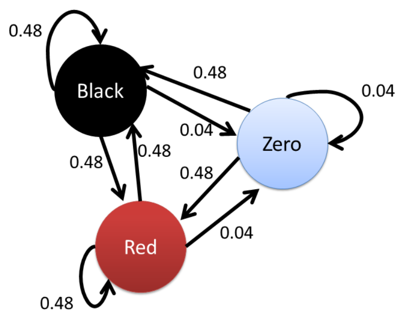

Figure 2.1: Markov Chain of a perspective on a roulette.

In 1913, at Monte Carlo Casino (Monaco), black came up twenty-six times in succession in roulette. Knowing that the probability of the ball landing on a red or on a black house is approximately 0.48 (the zero is a neutral house), many a gambler lost enormous sums while betting red as the they believed that the roulette was ripe with red, given the history. Players didn't want to believe that this and insisted that the roulette was biased; this became known as the Gamblers fallacy.

In the Markov Chain represented on Figure 2.1, we can see that in the state black the probability of transitioning to red is the same as to stay on the state black.

Given a system represented by the states $\{x_0, x_1, ..., x_i, ..., x_n\}$, and considering p_{ij} the probability of being in the state j and transitioning to the state i, the mixed state vector 2.6, which represents the probabilities of the system in the that i to transition to the other states. .

$$\overrightarrow{x_i} = \{p_{0i}x_0, p_{1i}x_1, ..., p_{ii}x_i, ..., p_{ni}x_n\} \tag{2.6}$$

The law of total probability is verified as $\sum_{j=0}^{n} p_{ji} = 1$, by specifying the every transition we get a stochastic matrix P, named the Markov matrix.

To illustrate how to construct a Markov Chain we will pick up on the example of Figure 2.1.

In this simplification of the Roulette we have 3 states:

- Black (B);
- Red (R);
- Zero (0).

We indifferently assign an index to each state, in order to construct the mixed state vector as in 2.6. Having the mixed state vectors defined the next step is to use them to create the stochastic matrix that has specified every transition 2.7.

$$R = \begin{bmatrix} p_{BB} & p_{BR} & p_{B0} \\ p_{RB} & p_{RR} & p_{R0} \\ p_{0B} & p_{0R} & p_{00} \end{bmatrix} = \begin{bmatrix} 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \end{bmatrix} \tag{2.7}$$

2.2 Von Neumann Probability

In the beginning on the 20^{th} century the nature of light was once again in the spotlight. The question whether light would be a particle (corpuscular theory), or a wave (undulatory theory), was posed throughout History. Newton, notoriously, considered light to be a particle and presented arguments such as the fact that light travels in a straight like, not bending when presented with obstacles, unlike waves, and gave an interpretation of the diffraction mechanism by resorting to a special medium (aether), where the light corpuscles could create a localized wave [7].

The idea of light as a particle stood up until the 18^{th}century as many scientists (Robert Hooke, Christian Huygens and Leonhard Euler to name a few) tried to explain contradictions found in corpuscular theory. This brought back the idea that light behaves like a wave.

One of the most famous experiments that corroborates the undulatory theory is the Young's experiments (19^{th} century), or the double-slit interferometer.

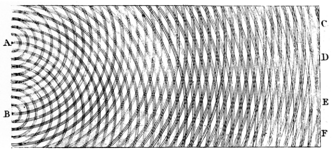

Figure 2.2: Thomas Young's sketch of two-slit diffraction of light.

The apparatus for the double-slit experiment can be seen in Figure 2.2. A light source is placed in such a way that "two portions" of light arrive at same time at the slits. Behind the barrier is a "wall" placed to intercept the light. The light captured at a wall will sport an interference pattern similar to the pattern when two waves interfere. The double-slit experiment was considered for a while the full stop on the discussion on the nature of light. However with experiments on the spectres of the light emitted by diverse substances and its relation with temperature, a new problem was posed.

The black body radiation problem was the theoretical problem where a body that absorbs light in all the electromagnetic spectrum, this makes the body acting as an natural vibrator, where each mode would have the same energy, according to the classic theory.

When a black body is at a determined temperature the frequency of the radiation it emits depends on the temperature. The classic theory predicted that most of the energy of the body would be in the high frequency part of the spectrum (violet part) where most modes would be found, this led to a prediction called the ultraviolet catastrophe. According to the classic theory the black body would emit radiation with an infinite power for temperatures above approximately 5000K. Max Plank(1901), provided an explanation where the light was exchanged in discrete amounts called quanta, so that each frequency

[0]Source: Young, Thomas: Probability. `http://en.wikipedia.org/wiki/File:Young_Diffraction.png(1803)`

would only have specific levels of energy. Plank also determined through experimentation the value of the energy of the quanta that became known as photons later, that value became the physical constant called Plank constant:

$$h = 6.62606957(29) \times 10^{-34} J.s \tag{2.8}$$

In 1905, Einstein used the concept of quanta (photons) to explain the photoelectric effect. De Broglie(1924), suggested that all the matter had a wave-particle duality. This prediction was confirmed by studying the interference patterns caused by electron diffraction.

2.2.1 Mathematical Foundations of Quantum Probability

As previously explained, Quantum Theory is a branch of physics that has arised from the need to explain certain phenomena that could not be explained with the current classical theory. In the beginning of the 20th century Dirac and von Neumann helped to create the mathematical formalisms for this theory [8] [9].

Von Neumann's contributions were focused in the mathematical rigour, as is framework is strongly based in Hilbert's theory of operators. Dirac's concerns were more of a practical nature. Their combined contributions were invaluable to establish this area.

From Dirac it is important to point the Dirac's notation (also known as Bra-ket notation or $\langle Bra|c|ket\rangle$) (introduced in 1939 [8]), that is widely used in literature based on quantum theory. This notation uses angle brackets and vertical bars to represent quantum states (or abstract vectors) as it can be seen in the formulas (2.9) and (2.10).

$$|z\rangle = \begin{bmatrix} z_1 \\ z_2 \\ ... \\ z_n \end{bmatrix} \tag{2.9}$$

$$\langle z| = (|z\rangle)^* = \begin{bmatrix} z_1^* & z_2^* & ... & z_n^* \end{bmatrix} \tag{2.10}$$

This notation provides for an elegant representation of the inner product (2.11) and verifies the has linearity as it follows in equation (2.12). While the Bra-ket notation can be useful in terms of condensing information, using vectors and matrices to represent the states turns out to be a more approachable way to understand and manipulate data.

$$\langle z|z\rangle = \sum_{i=1}^{n} \bar{z}_i z_i \tag{2.11}$$

where $\bar{z}_i$ is the complex conjugate of z_i.

$$\langle z|(\alpha|x\rangle + \beta|y\rangle) = \alpha\langle z|x\rangle + \beta\langle z|y\rangle \tag{2.12}$$

2.2.2 Born rule

The Born rule was formulated by Born in 1926. This law allows to predict the probability that a measurement on a quantum system will yield a certain result. This law provides a link between the mathematical foundation of Quantum Mechanics and the experimental evidence [10] [11].

A quantum system is represented by a n-dimensional Hilbert Space, a complex vector space in which the inner product is defined.

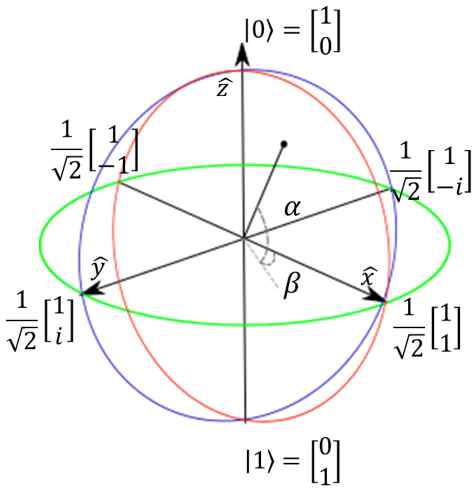

Figure 2.3: Representation of a two-dimensional Hilbert Space($\mathcal{H}^2$)

A 2-dimensional Hilbert Space (corresponding to a qubit for example), can be represented by a Bloch Sphere. In the Figure 2.3 we have a representation of quantum state $|v\rangle$ in a two-dimensional Hilbert Space.

The Born rule states that if there is a system is in a state $|v\rangle$ (in a given n-dimensional Hilbert Space $\mathcal{H}$), and an Hermitian operator A is applied then the probability of measuring a specific eigenvalue λ_i associated with the i-th eigenvector of A (ψ_i), will be given by [10]:

$$P_v(\lambda_i) = \langle v|Proj_i|v\rangle \tag{2.13}$$

where $Proj_i$ is a projection matrix corresponding to ψ_i :

$$Proj_i = |\psi_i\rangle\langle\psi_i| \tag{2.14}$$

Given the properties of A, the set of eigenvectors $\{\psi_1, \psi_2, ..., \psi_i, ..., \psi_n\}$ forms a orthogonal basis of the n-dimensional Hilbert Space considered. Thus the state $|v\rangle$ can be written as a linear combination of the eigenvectors of A:

$$|v\rangle = \alpha_1\psi_1 + \alpha_2\psi_2 + ... + \alpha_i\psi_i + ... + \alpha_n\psi_n \tag{2.15}$$

The coefficients α_i are complex numbers called probability amplitudes, and their squared sum is equal to 1:

$$\sum_{i=0}^{n} |\alpha_i|^2 = 1 \tag{2.16}$$

this brings us to:

$$P_v(\lambda_i) = \langle v|\psi_i\rangle\langle\psi_i|v\rangle = |\langle v|\psi_i\rangle|^2 = |\alpha_i^*\alpha_i|^2 \tag{2.17}$$

So the determination of the probability of an event ($P(A)$), is made by projecting the quantum state on the eigenvectors corresponding to the operator A of the Hilbert Space and measuring the squared length of the projection. [12]

$$P(A) = (Proj_A|z\rangle)^2 \tag{2.18}$$

Applying an operator can be seen as applying a rotation matrix on the system ($|v\rangle$) and measuring the projection of $|v\rangle$ onto the imaginary axis and the real axis (Figure 2.3), or considering that we have a determined state vector and rotating the orthogonal basis of the Hilbert Space according to an operator and then to do a projection on the new chosen orthogonal basis.

According to Leiffer [13] “quantum theory can be thought of as a non-commutative, operator-valued, generalization of classical probability theory”.

As in the classical probability theory where from a random variable it is possible to establish a probability distribution, also known as density function, in the Hilbert space there is a equivalent density operator. The density operator (ρ) is a Hermitian operator that has the particularity of having its trace equal to 1 [10].

$$\rho = \sum_{i=0}^{n} \alpha_i |\psi_i\rangle\langle\psi_i| \tag{2.19}$$

$$tr(\rho) = 1 \tag{2.20}$$

2.2.2.A Lüders Rule

Lüders Rule defines the state of a quantum system after a partial measurement. We can establish a parallel between this selective measurement and conditional probability2.1.2 [14].

To measure the state of a system we first use a projection operator on the quantum state. This is also the first stage of applying Lüders Rule2.21.

$$A = Proj_A|s\rangle \tag{2.21}$$

After the measurement the resulting state is normalized2.22.

$$|s_A\rangle = \frac{A}{|A|} \tag{2.22}$$

2.2.3 Example of the double-slit experiment with electrons

Like the Young's Experiment with light created an interference pattern similar to a wave, firing electrons one at the time produces a similar pattern. The unobserved fired electron behaved like a wave and after passing the slits the wavelets interfered with one another to create a interference pattern. However if a measuring device was active while the electron was fired the interference pattern wasn't registered.

The fact that the electron was measured while passing through a slit produced a particle behaviour, explained by the classical theory (Figure 2.4).

In this experiment a single electron is shoot at a time. So in the start of the experiment (S), we know the initial position of the electron.

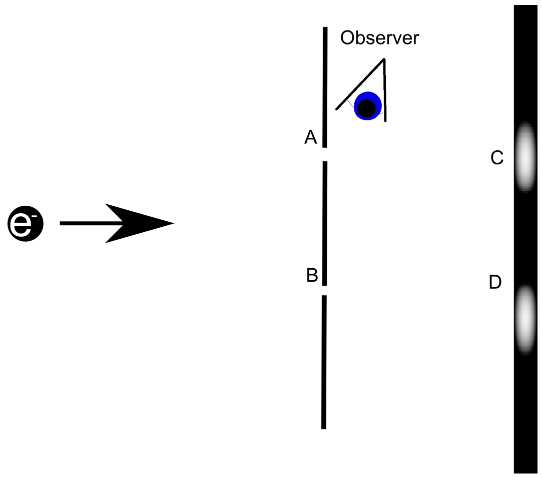

Figure 2.4: Double-slit experiment where there is a measuring device that allows to know through which slit the electron passed.

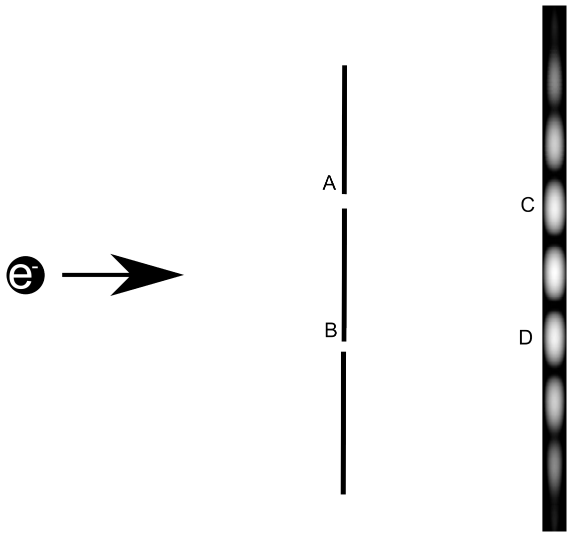

Figure 2.5: Double-slit experiment, where electrons exhibit the interference pattern characteristic in waves

A final measurement (F) is made when the electron hits the wall behind the slits, where we know the final position of the electron. [1]

If this experiment is observed, there is an intermediate measure that tells us whether the electron went through the slit A or B. The corresponding probability amplitudes related to this measurement are ω_A and ω_B, and:

$$\omega_A = \langle F|A\rangle\langle A|S\rangle \tag{2.23}$$

$$\omega_B = \langle F|B\rangle\langle B|S\rangle \tag{2.24}$$

If we consider the intermediate measurement the probability $P(F|S)$ will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle|^2 + |\langle F|B\rangle\langle B|S\rangle|^2 \tag{2.25}$$

[1] Mohrhoff, U.: Two Slits. `http://thisquantumworld.com/wp/the-mystique-of-quantum-mechanics/two-slit-experiment/#fn1back`

But if we only measure the position of the electron at the end of the experiment that probability will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle + \langle F|B\rangle\langle B|S\rangle|^2 \tag{2.26}$$

The latter equation will be dependent on a interference coefficient that will be responsible the interference pattern observed in the unobserved experiment.

2.2.4 Example of the Polarization of Light

The photons in a beam of light don't vibrate all the same direction in most of the natural sources of light. To filter the light polaroids are used. A polaroid only allows the passage of light in a well-defined direction and thus reducing the intensity of the light. In the Figure 2.6 we can observe that the introduction of the oblique polaroid in the third situation led to a passage of light. Although there is a classical explanation to this phenomenon if we consider waves when we are considering a beam of light, if our light source emits one photon at the time a quantum mechanical explanation is needed [15].

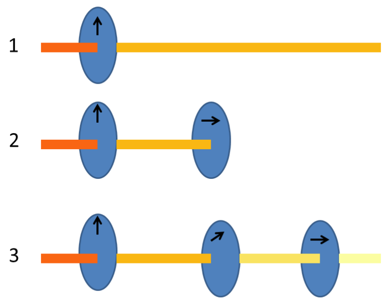

Figure 2.6: 1. With one vertical polaroid the unpolarized light is attenuated by a half. 2. Vertical polarization followed by a horizontal polarization will block all the passing light. 3. Inserting a oblique polaroid between the vertical and horizontal polaroids will allow light to pass.

To model the polarization of the photon in a quantum setting, we will use a vector $|v\rangle$ in a two-dimensional Hilbert Space:

$$|v\rangle = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.27}$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ would represent the vertical direction (could also be represented by the state vector $|\uparrow\rangle$), and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the horizontal one (another possible representation to this basis could be $|\rightarrow\rangle$). We can consider the Figure 2.3 as a graphical representation for this system.

In the first situation if $a = \frac{1}{\sqrt{2}}$, that would mean that the probability of passing the vertical polaroid would be $a = (\frac{1}{\sqrt{2}})^2 = 0.5$, that would light to the expected reduction of a half of the intensity of light. After passing through the vertical polaroid the photon will have a polarization of $|v\rangle = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Considering now the second situation after we have our photon polarized vertically (like in the end on the first situation), the probability of being vertically polarized is 1, thus making the probability of passing through the horizontal polaroid 0. In the third situation, after being vertically polarized the photon will pass through an oblique polaroid that makes its direction

$$|v\rangle = cos(\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i.sen(\theta)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.28}$$

θ being the angle of the polaroid. The photon filtered by the vertical polaroid will pass this second polaroid with a probability of $(cos(\theta))^2$, becoming polarized according to the filter, as we can observe depending on the value of θ we will now have a horizontal component in the vector that describes the state of the photon. This will make the photon pass the horizontal polaroid with a probability of $i.sen(\theta)^2$.

2.3 Quantum Computing

Quantum Computing is an area that tries to take advantage of Quantum Mechanics phenomena to perform calculations. This idea of using the properties of a wave function to compute was first introduced by the works of Feynman [16]. Shor (at Bell Labs), developed an algorithm that, running on a quantum computer, could factor large numbers that can provide theoretical speedups over classical algoritmhs [15].

The quantum equivalent of a bit (the basic unit of information in computers), is a qubit (or quantum bit). A qubit is a two-state quantum system that can be interpreted as normalized vectors in a 2-dimensional Hilbert space. This Hilbert Space($\mathcal{H}^2$), an element in that space can be uniquely specified by resorting to two orthonormal basis (also known as pure states). In quantum computing, the basis used are $|0\rangle, |1\rangle$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

2.3.1 Superposition

The qubit can be described in terms of linear transformations of pure states as in 2.29, ω_0 and ω_1 are complex numbers called probability amplitudes. When a system is in a mixture of this pure states (as the general representation in 2.29 might point), it is in a phenomenon known as superposition. Measuring forces the system to collapse and assume one of the pure states with a certain probability.

$$|\psi\rangle = \omega_0|0\rangle + \omega_1|1\rangle = \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix} \tag{2.29}$$

In the example stated in the representation 2.29 the probability of the system falling into the state $|0\rangle$ would be $|\omega_0|^2$. The probability of the system falling into $|1\rangle$ would be $|\omega_1|^2$. The second axiom of Kolmogorov (law of total probability) is verified (2.30).

$$|\omega_0|^2 + |\omega_1|^2 = 1 \tag{2.30}$$

ω_0 and ω_1 (2.29) are complex numbers, the so-called probability amplitudes. When squared, the probability amplitude represents a probability.

2.3.2 Operators

In order to perform transformations in the qubit systems, we can define linear operators in the Hilbert space. One of the most important classes of operators being the self-adjoint operators, $A = A^*$, that have the property stated in Equation 2.31. The Hermitian operator is one that satisfies the property of being equal to its conjugate transposed, $A = A^{*T} = A^\dagger$. In a finite-dimensional Hilbert space defined by a set of orthonormal basis every self-adjoint operator is Hermitian.

$$\langle A^* z | x \rangle = \langle z | Ax \rangle \tag{2.31}$$

Unitary and Hermitian operators are used in the majority of quantum algorithms because they can be reversible, and because they insure that no rule of quantum mechanics is violated while applying the transformations. For that matter we arrive at an interesting difference between classical computing and quantum computing: it is impossible to copy or clone unknown quantum states [15]. This is known as: The No-Cloning Principle.

We can prove the "The No-Cloning Principle" by *reductio ad impossibilem.* Suppose we have an operator U which is unitary and clones quantum states, and two unknown quantum states $|a\rangle$ and $|b\rangle$. The transformation U means that if we apply it to $|a\rangle$ we have $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$. The result when applying to $|b\rangle$ is $U(|b\rangle|0\rangle) = |b\rangle|b\rangle$.

If we consider a state $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle+|b\rangle)$, by the principle of linearity $U(|c\rangle|0\rangle) = \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle)+U(|b\rangle|0\rangle))$, giving the final result $U(|c\rangle|0\rangle) = \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle)$. However if U is a cloning operator then $U(|c\rangle|0\rangle) = |c\rangle|c\rangle$.

$|c\rangle|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle)$ is different from $\frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle)$, thus we can affirm there is no unitary operator U that can clone unknown quantum states [15].

When presented with q qubits, a q-dimensional square matrix is called a Quantum Gate.

Some important Quantum Gates that operate on a single qubit are:

- The Identity Matrix 2.32.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.32}$$

- The Pauli Operators. This operators are used to perform the NOT operation.

– The Bit-Flip Operator

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.33}$$

– The Phase-Flip Operator

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.34}$$

– The Bit and Phase-Flip Operator

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{2.35}$$

- The Hadamard Gate, belongs to a general class of Fourier Transforms. This 2×2 particular case is also a Discrete Fourier Transform matrix.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.36}$$

2.3.3 Compound Systems

The representation of a system comprising multiple qubits grows exponentially. If to represent a single qubit system there is a 2-dimensional Hilbert space($\mathcal{H}^2$), to represent a system with m qubits a 2^m-dimension space would be required. To represent a higher dimension multiple-qubit system composed by single-qubits, one can perform a tensor product of single-qubit systems.

The tensor product is an operation denoted by the symbol $\otimes$. Given two vector spaces V and W with basis 2.37 and 2.38 respectively, their tensor product would be the mn-dimensional vector space with a basis with elements from the set 2.39 [15].

$$A = \{|\alpha_1\rangle, |\alpha_2\rangle, ..., |\alpha_m\rangle\} \tag{2.37}$$

$$B = \{|\beta_1\rangle, |\beta_2\rangle, ..., |\beta_n\rangle\} \tag{2.38}$$

$$C = \{|\alpha_i\rangle \otimes |\beta_j\rangle\} \tag{2.39}$$

For example, if we consider two Hilbert spaces $\mathcal{H}^2$ with basis $A = \{|0\rangle, |1\rangle\}$ and $B = \{|-\rangle, |+\rangle\}$, their tensor product would be a $\mathcal{H}^4$ with basis 2.40.

$$AB = \{|0-\rangle, |0+\rangle, |1-\rangle, |1+\rangle\} \tag{2.40}$$

Now taking the former Hilbert space, supposing we have the qubits 2.41 and 2.42.

$$|v\rangle = a_0|0\rangle + a_1|1\rangle \tag{2.41}$$

$$|w\rangle = b_0|-\rangle + b_1|+\rangle \tag{2.42}$$

$$|v\rangle \otimes |w\rangle = a_0 b_0|0-\rangle + a_0 b_1|0+\rangle + a_1 b_0|1-\rangle + a_1 b_1|1+\rangle \tag{2.43}$$

The Bra-ket notation provides a way to prevent the escalation of the basis notation. When specified the vector space the basis can be specified in base 10 for simplicity sake. According to this the basis of the last example would be $AB = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ and a system with 3 qubit could be represented in a Hilbert space, $\mathcal{H}^8$, with basis $H = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}$.

2.3.4 Entanglement

In multiple qubit systems, qubits can interfere with each other, thus making impossible to determine the state of part of the system without “disturbing” the whole. In other words, there are states in a multi-qubit system that cannot be described as a probabilistic mixture of the tensor product of single-qubit systems; when this happens this state is not separable (with respect to the tensor product decomposition), this phenomenon is called quantum entanglement [15]. If a mixed quantum state ψ in a quantum system constructed by $V_1, V_2, ..., V_n$ is separable it can be written as:

$$|\psi\rangle = \sum_{j=1}^{m} p_j |\varphi_j^1\rangle\langle\varphi_j^1| \otimes ... \otimes |\varphi_j^n\rangle\langle\varphi_j^n|, \sum_i p_i = 1, |\varphi_j^i\rangle \in V_i \tag{2.44}$$

Quantum entanglement is one of the main differences from the classical theory [15]. In an entangled pair each member is described with relation to the other members. This property is not local as transformations that act separately in different parts of an entangled system cannot break the entanglement. However if we measure a part of an entangled system the system collapses, and if we measure the other part in any point of time from that moment we will find a correlation with the outcome of the first measurement.

For example, supposing we consider the following quantum states, known as Bell states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.45}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.46}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.47}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.48}$$

These states form a particular basis in $\mathcal{H}^4$ as they are all entangled states and they are maximally entangled. If we have a system of two qubits in a Hilbert space in the mixed state $|\Phi^+\rangle$ and we measure the qubit A (by deciding the outcome 0 or 1 with a probability of 0.5 for each),we are automatically uncovering the value of the qubit B. In this state we know the second qubit will always yield the same value of the first measured qubit. The second value is correlated with the first one.

2.4 Game Theory

Game Theory did not exist as a field of own right before von Neumann published the article " On the Theory of Games of Strategy". This field deals mainly with the study of interactions between rational decision-makers [17]. Game Theory knows numerous applications in areas such as Economics, Political Science, Biology, and Artificial Intelligence.

2.4.1 Definition of a Game

A game (Γ), is a model of conflict between players characterized by [18] [19] [20]:

- A set of Players: $P = \{1, 2, .., N\}$.
- For each player there is set of Actions.
- There are preferences , for each player, over the set of Actions.

The preferences in the previous definitions are defined with resort to concept of utility, expected utility, or payoff derives from the theory that we can use real numbers (Equation 2.49), to model and represent the wants and needs of the players.

$$u : X \to \mathbb{R} \tag{2.49}$$

This simplified mathematical model allows to compare different states and rank preferred outcomes. If $u_i(x)$ is a utility function for player i, and $u_i(A) > u_i(B)$, that would mean that the player would strictly prefer A over B. The concept of expected utility is fundamental to analyse games, as a rational player would try to maximize her expected utility [17] [18].

The options a player i has when choosing an action, when the outcome depends not only on their choice (but also on the actions taken by the other players), is referred as strategy s_i. The set of strategies available to a player is represented by the set S_i.

When a strategy gives strictly a higher expected utility in comparrison to other strategies, we have a strategy that dominates, or a dominant strategy. The mathematical definition of dominance is represented in Equation 2.50; where, for a player i, in spite of the strategies chosen by the other players (denoted by s_{-i}), there is a strategy s^* which gives always an higher expected utility in comparrison with other strategies $s^{'}$ available to player i.

$$\forall s_{-i} \in S_{-i}[u_i(s^*, s_{-i}) > u_i(s^{'}, s_{-i})] \tag{2.50}$$

The strategy that leads to the most favourable outcome for a player, taking into account other players strategies, is known as best response.

A pure strategy defines deterministically how the player will play the game. In the game represented in Table2.1 , in a pure strategy, the players either they choose "Cooperate" (C), or "Defect" (D).

If there is a probability distribution associated with probabability of playing with a determined pure strategy, we have a mixed strategy.

There are two standard representations of games:

- Normal Form - lists what payoffs the players get as a function of their actions as if they all make their moves simultaneously. This games are usually represented by a matrix, for example 2.1.
- Extensive Form - extensive form games can be represented by a tree and represent timed actions.

	Player 2: C	Player 2: D
Player 1: C	(2,2)	(0,3)
Player 1: D	(3,0)	(1,1)

Table 2.1: Example of a Normal Form game. This is a possible representation of the Prisoner's Dilemma game.

A finite game is a game that has a finite set of actions, a finite number or players, and it does not go on indefinitely.

A zero-sum game is a mathematical representation of a system where the gains are completely evened out by the losses; this means that the sum of the utilities of all players will always be zero.

2.4.2 Nash Equilibrium

If we observe the representation of the classic game "Prisoner's Dilemma", we can observe that each player has two strategies; either they choose "Cooperate" (C), or "Defect" (D). The "Defect" strategy, for both players (the game is symmetrical), always yield a higher payoff in spite of the other player's strategy, this means that it is a dominant strategy and also constitutes a best response to the game. If both players chose their best response the final outcome (D, D) becomes an equilibrium solution, more specifficaly a Nash Equilibrium.

When all players cannot improve their utility by changing their strategy unilaterally, we have an equilibrium point. This equilibrium point is named after John Nash, who proved that it exists at least one mixed strategy Nash equilibrium in a finite game [21] [22]. This concept is used to analyse game where several decision makers interact simultaneously and the final outcome depends on the players strategy [18].

2.4.3 Pareto Optimal

From the point of view of an observer outside the game system some outcomes may seem better than others. For example on the game represented in Table 2.1 (Prisioners' Dilemma), the outcome (C, C) seems better than the outcome (D, D) because it provides a strictly higher utility to the players. However we know that the outcome (D, D) is the Nash Equilibrium of the game. If both players use their best response their outcome might not be the best outcome for both players.

When every player cannot improve her payoff without lowering another player's expected utility we have a Pareto Optimal solution.

For example if we have two players and 10 units of a finite resource to distribute among the players, the pareto optimal sollution is $(5, 5)$. Any other attemp to redistribute -$(6, 4)$, $(1, 9)$, etc...- would always leave a player worse than the pareto optimal distribution of the resource.

2.5 Quantum Game Theory

In the article “Quantum information approach to normal representation of extensive games” [23], the authors propose a representation for both normal and finite extensive form games [20]. This definition is based on the premise that any strategic game can be represented as an extensive form game where all the players have no knowledge about the actions taken by other players.

The representation assumes that there are only two available actions, which can be represented by a qubit. Those actions could be a yes/no decision or a cooperate/defeat as those found in many classical game theory problems.

A game in this form is represented by a six-tuple3.2.3.A, where:

$$\Gamma = (\mathcal{H}^{2^a},\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\}) \tag{2.51}$$

- a is the number of actions (qubits), in the game;
- $\mathcal{H}^{2^a}$ is a 2^a-dimensional Hilbert space constructed as $\otimes_{j=1}^{a}\mathbb{C}^2$, with basis $\mathcal{B}$;
- $|\psi_{in}\rangle$ is the initial state of the compound-system composed by a qubits: $|\varphi_1\rangle,\ |\varphi_2\rangle, ..., |\varphi_j\rangle, ..., |\varphi_a\rangle$;
- ξ is a mapping function that assigns each action to a player;
- For each qubit j $\mathcal{U}_j$ is a subset of unitary operators that can be used by the player to manipulate her qubit;
- Finally, for each player i, E_i is a utility functional that specifies her payoff. This is done by attributing a real number (representing a expected utility $u_i(b)$, in 2.52), to the measurement for the projection of the final state (2.53), on a basis from the $\mathcal{B}$2.52.

$$E_i = \sum_{b\in\mathcal{B}} u_i(b)|\langle b|\psi_{fin}\rangle|^2, u_i(b) \in \mathbb{R} \tag{2.52}$$

The strategy of a player i is a map τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j\in \xi^{-1}(i)$). The simultaneous move is represented in 2.53.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{N} \otimes_{j\in\xi^{-1}(i)} \mathcal{U}_j|\psi_{in}\rangle \tag{2.53}$$

The tensor product of all the operators chosen by the players is referred as a super-operator, which act upon the game system.

In the article ”Quantum Games and Quantum Strategies” [5], the authors describe a quantization scheme for the Prisioner’s Dilemma game. The way the initial state, $|\psi_{in}\rangle$, is set-up in the game system provide a way to entangle the game system, thus allowing the study of this phenomenon. For a 2-player game, this is accomplished using Equation 2.55. $\mathcal{J}$ is a matrix exponential that is chosen because it can commute with a super-operator, made from the subset of unitary operators $\mathcal{U}_j$ [24]. According to [5], the parameter γ becomes a way to measure the entanglement in the system.

$$\mathcal{J} = exp\left\{ i\frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{2.54}$$

$$\begin{aligned} |\psi_{in}(\gamma)\rangle &= exp\left\{ i\frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} |00\rangle \\ &= cos(\frac{\gamma}{2})|00\rangle + isin(\frac{\gamma}{2})|11\rangle, \gamma \in (0, \pi) \end{aligned} \tag{2.55}$$

2.6 Overview

In this chapter we provided a theoretical background that can help the reader to grasp the contents presented in the subsequent chapters.

We started by laying down a comparison between the classical probability theory and the quantum probability theory. The von Neumann probability is the mathematical foundation behind Quantum Mechanics.

The von Neumann probability differs from the classical mainly because mutually exclusive events can interfere, this happens in the Double-slit Experiment, this means that the third axiom of Kolmogorov does not hold true in this probability theory. The concept of probability is deeply intertwined with Quantum Mechanics.

In the Quantum Computing we presented the fundamental concepts. The book "'Quantum Computing - A Gentle Introduction" [15] provides a more in-depth resource on this subject; including the Shor's algorithm that is used to find prime factorizations, and the Deutsch-Jozsa algorithm, that with a single query decides if an unknown function is constant or balanced. These algorithms fall outside the scope of this document.

The Game Theory section contains a brief description of some fundamental concepts from Game Theory needed in order to understand this work. A comprehensive reference such as [18] can be consulted for a deeper insight on the subject.

3

Related Work

Contents

In this section we present relevant work that it is related with the problem. This constitutes an in-depth analisys of some important problems but also tries to present a broad representative view on the path towards building our solution.

3.1 Quantum Walk on a Line

Quantum Walk is the quantum version of random walks, which are mathematical formalisms that describe a path composed of random steps. A Markov Chain might be used to describe these processes.

We can define the Discrete Quantum Walk on a Line as a series of Left/Right decisions. Understanding this algorithm is important towards being able to define and design more complex algorithms that make use of quantum properties.

We followed an approach suggested by [25] towards simulating n-steps of a Quantum Walk on a Line. The Matlab algorithm can be consulted on Appendix A. In a discrete quantum walk in a line we want to preserve the fact that the probability of turning left is equal to the probability of turning right. To represent a state in this algorithm we will need the number of the node and a direction (identified as L,R) 3.1.

$$|\psi\rangle = |n, L\rangle \tag{3.1}$$

With two equally possible direction choices in each step, we can use a coin metaphor [25] [26] to approach the decision. We toss a coin and go either Left or Right depending on the result.

In a quantum version we need to define a Coin Operator (Coin Matrix), which is responsible to imprint a direction to the current state. This operator is a unitary matrix in a 2-dimension Hilbert space. Some examples of Coin Operators are the Hadamard matrix3.2 and a symmetric unitary matrix 3.3.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3.2}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{3.3}$$

Taking the Hadamard matrix as an example3.2, the coin matrix will operate on the state in the following way 3.43.5 [25].

$$C|n, L\rangle = \frac{1}{\sqrt{2}}|n, L\rangle + \frac{1}{\sqrt{2}}|n, R\rangle \tag{3.4}$$

$$C|n, R\rangle = \frac{1}{\sqrt{2}}|n, L\rangle - \frac{1}{\sqrt{2}}|n, R\rangle \tag{3.5}$$

The Coin Matrix obtains its name by being the quantum equivalent of flipping a classic coin. After tossing a coin comes an operator that will move the node in the direction assigned. The operator responsible for this modification is commonly referred as Shift Operator 3.63.7.

$$S|n, L\rangle = \frac{1}{\sqrt{2}}|n - 1, L\rangle \tag{3.6}$$

$$S|n, R\rangle = \frac{1}{\sqrt{2}}|n + 1, R\rangle \tag{3.7}$$

These matrices (Coin Matrix and Shift Operator) are used conceptually in various algorithms [15], therefore it is important to be familiar with them. A single step of the algorithm A is illustrated in Figure 3.1.

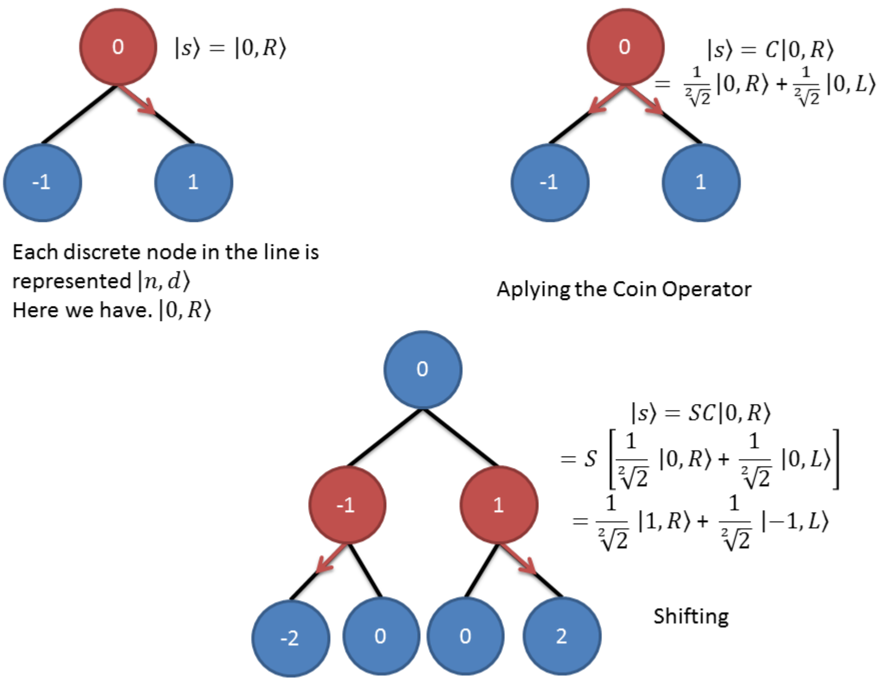

Figure 3.1: Simulating a step of a discrete quantum walk on a line. In the beginning we have a state characterized by the position (0) and a direction (either Left or Right).

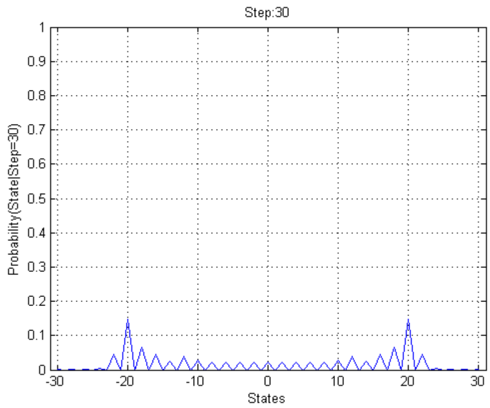

Figure 3.2: 30 Step of the Simulation A using Matrix 3.2 as a Coin Operator.

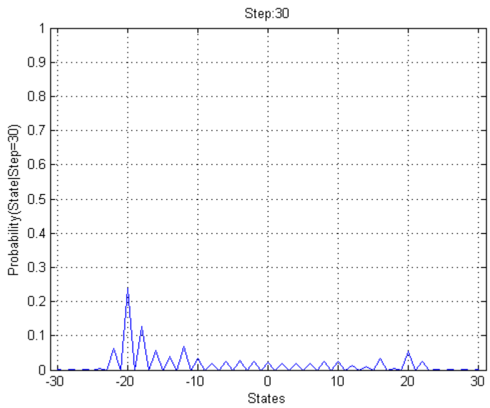

Figure 3.3: 30 Step of the Simulation A using a Hadamard Matrix 3.2 as a Coin Operator.

If we took the classical approach in which we tossed a fair coin to decide to go either left or right, and after n-steps we measured the final node repeatedly, by the Central Limit Theorem (CLT) the final distribution would converge to a normal distribution. However, in the quantum approach, depending on the Coin Matrix we can get different distributions. In this simple problem we have the basis for some quantum algorithms.

In Figures 3.3 and 3.2, depending on the coin operator we get two different results. Despite that, on one moment the probability of shifting left, is always equal to the probability of shifting right. The main difference from the classical approach lies on the fact there is a quantum interference during the walk, when we measure the results after N steps, somewhat like the electrons interfere in the “double-slit experiment”, presented in Section 2.2.3.

3.2 Quantum Models

The rationale behind building a quantum version of a Game Theory and/or Statistics problem lays in bringing phenomena like quantum superposition, and entanglement into known frameworks. This creates [][][][][][][]AQUI

The effort put in converting known classical problems also enables the familiarization with the potential differences these models bring.

3.2.1 Quantum Roulette

In the arbitrary N-State quantum roulette, [27] presented a N-State roulette model using permutation matrices.

This model is interesting because in captures the usage of permutation matrices to manipulate and change the state of the system.

To verify this model with two players we developed a Matlab simulation C, that followed the steps taken in [27].

The game in represented in a N-Dimensional Hilbert Space. There is a basis in the space that represents each of the equally probable entries as shown in 3.8. In a sense this is a generalization of a quantum coin flip that is also used in Section 3.1.

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ |N\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{3.8}$$

Each state transition is obtained using a permutation matrix denoted by P^i. There are $N!$ permutation matrices, so in the particular case of having a 3-State roulette, there are 6 possible transition choices. The classical strategy considered will rely on choosing an arbitrary probability distribution, that verifies3.10, and that maps the usage of the permutation matrices. This step will not affect the density matrix (ρ) of the roulette3.9.

$$\rho = \frac{1}{N!} \sum_{i=0}^{N!-1} P^i \tag{3.9}$$

$$\sum_{i=0}^{N!-1} p_i = 1 \tag{3.10}$$

The density matrix is diagonalizable by a Discrete Fourier Transform because it is a kind of circulant matrix [28], as we can see in 3.11. In 3.11 λ_k are eigenvalues of ρ. $\lambda_1 = 1$ while $\lambda_2 = \lambda_3 = \lambda_k = \lambda_{N-1} = 0$. Each column i of the Fourier matrix will represent a eigenvector $|\lambda_i\rangle$. If we construct the diagonalizing matrix by rotating the columns of the Fourier Matrix we can obtain the projection states as in 3.12.

$$F^\dagger \rho F = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N-1} \end{bmatrix} \tag{3.11}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = F^\dagger \rho F \tag{3.12}$$

The quantum strategy advantage in this case is that the first player will not alter the density matrix3.13.

$$\rho = \sum_{i=0}^{N!-1} p_i P^i \rho P^{i\dagger}, \ \sum_{i=0}^{N!-1} p_i = 1 \tag{3.13}$$

This means that if the second player knows the initial state and the first player plays with a classical strategy, thus never modifying the system density matrix, the second player will be able to manipulate the game under optimal conditions. This result is confirms the demonstration done in [29]; on Quantum Strategies, where in a classical 2 player zero-sum game, if one player adopts a quantum strategy, she increases her chances of winning the game.

	Player 2: C	Player 2: D
Player 1: C	(R,R)	(S,T)
Player 1: D	(T,S)	(P,P)

Table 3.1: The canonical normal form representation for the Prisoner's Dilemma must respect $T > R > P > S$.

3.2.2 Prisoner's Dilemma

The Prisoner's Dilemma is a classic example of a game that can be represented in normal form [18]. This problem has received a great deal of attention because, in its simple form, rational individuals will seem to deviate from solutions that would represent the best social interest, the Pareto Optinal solution. In this game the Pareto Optimal solution is not a Nash Equilibrium. The Prisoner's Dilemma can be formulated as it follows:

> Two suspects of being partners in a crime are arrested. The police needs more evidence in order to prosecute the prisoners. So each prisoner is locked in solitary confinement and has no means of communicating with the other suspect. The police will then try to extort a confession from the prisoners. A bargain will be proposed to the suspects:
>
> - If the suspect testifies against the other suspect (Defects) and the other denies, he will go free and the second will get three years sentence.
> - If they both testify against one another (both Defect), both will be convicted and they will get two years.
> - In case both suspects deny the involvement (both Cooperate) of the other, they will get a one year sentence.

A matrix representation of the problem is in Table 3.1, here the payoff represented by the letter R would be the standard reward for the game, T would be the temptation to deviate from a cooperation profile, P represents the punishment when both entities do not cooperate, and finally S would be a sucker's payoff. A particular case for this problem is represented in Table 3.2. In each cell of the matrix we have a pair of the expected utility for the players for every outcome. A higher utility represents a more desirable state.

In this game the Pareto Optimal solution happens when both players chose to Cooperate (the pair (2,2) in Table 3.2). However both players have the incentive to Defect, because regardless what the opponent choses they will have always a strictly higher payoff. Defecting becomes a dominant strategy in the Prisioner's Dillema and the outcome $(Defect, Defect)$ is a Nash Equilibrium to the game.

	Player 2: C	Player 2: D
Player 1: C	(2,2)	(0,3)
Player 1: D	(3,0)	(1,1)

Table 3.2: One possible normal form representation of Prisoner's Dilemma.

$\bigotimes$	C-$\|0\rangle$	D-$\|1\rangle$
C-$\|0\rangle$	$\|0,0\rangle$	$\|0,1\rangle$
D-$\|1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$

Table 3.3: Construction of the basis for the game space; $\mathcal{H}^4$.

3.2.2.A Quantum Prisoner's Dilemma

The importance of the Prisoner's Dilemma for the study of Game Theory made it a prime target for investigation in Quantum Game Theory. The problem has been modelled several times [5] [23]. Therefore we will use it in order to exemplify and consolidate the definition described in Section 2.5 [23].

Each player i in the quantum version of Prisoner's Dilemma will be able to manipulate one qubit (φ_1 and φ_2) in Equations 3.14 and 3.15, with two possible operators (shown in Equation 3.16) corresponding to the classical actions: Cooperate (C), and Defect (D).

$$\varphi_1 = a_0|0\rangle + a_1|1\rangle, \sum_{i=0}^{1} |a_i|^2 = 1 \tag{3.14}$$

$$\varphi_2 = b_0|0\rangle + b_1|1\rangle, \sum_{j=0}^{1} |b_j|^2 = 1 \tag{3.15}$$

$$\mathcal{U}_i = \begin{cases} C = O_{i0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D = O_{i1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}, i \in \{1, 2\} \tag{3.16}$$

The system that holds the game is represented in a $\mathcal{H}^4$. Each basis ($|1\rangle, |2\rangle, |3\rangle, |4\rangle$), represents a final outcome as Table 3.3 suggests.

The fundamental difference from the classical version lies in the way the initial state is formulated in Equation 3.18. We will entangle our state by applying the gate $\mathcal{J}$ [30] [5]. The gate $\mathcal{J}$ is chosen to be commutative with the super-operators

The parameter γ becomes a way to measure the entanglement in the system [5].

$$\mathcal{J} = exp\left\{i\frac{\gamma}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\} \tag{3.17}$$

$$\begin{aligned} |\psi_{in}(\gamma)\rangle &= exp\left\{i\frac{\gamma}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}|00\rangle \\ &= cos(\frac{\gamma}{2})|00\rangle + isin(\frac{\gamma}{2})|11\rangle, \gamma \in (0, \pi) \end{aligned} \tag{3.18}$$

$$|\psi_{fin}\rangle = \mathcal{J}^{\dagger} \otimes_{i=1}^{2} \mathcal{U}_i|\psi_{in}\rangle \tag{3.19}$$

The entanglement in quantum game theory can be viewed as an intrinsic unbreakable contract. Furthermore measuring an entangled state will cause the wave function that describes the state to collapse. Before measuring the final result we will de-entangle the system by applying the operator $\mathcal{J}^{\dagger}$, as shown in Equation 3.19.

The utility functions for each player is calculated by projecting the final state in each base and attributing a real number to each measurement, as in equation. In order to compare a classical version with this quantum model, the real numbers assigned are those in the classical example in Table 3.1, and the basis are presented in Table 3.3.

$$E_0(|\psi_{fin}\rangle) = 2 \times |\langle 00|\psi_{fin}\rangle|^2 + 3 \times |\langle 10|\psi_{fin}\rangle|^2 + 1 \times |\langle 11|\psi_{fin}\rangle|^2 \tag{3.20}$$

$$E_1(|\psi_{fin}\rangle) = 2 \times |\langle 00|\psi_{fin}\rangle|^2 + 3 \times |\langle 01|\psi_{fin}\rangle|^2 + 1 \times |\langle 11|\psi_{fin}\rangle|^2 \tag{3.21}$$

By varying the parameter γ, we can observe that the expected utility changes as we set up the initial state. Furthermore for $\gamma = \frac{\pi}{4} + k\frac{\pi}{2}, k \in 0, 1, 2, 3$ the expected utility for each player is 2 if both players Cooperate, thus removing the dilemma. When the entanglement coefficient is $\gamma = 0 + k\frac{\pi}{,}k \in 0, 1, 2$ we are left with the classical problem, where there isn't entanglement.

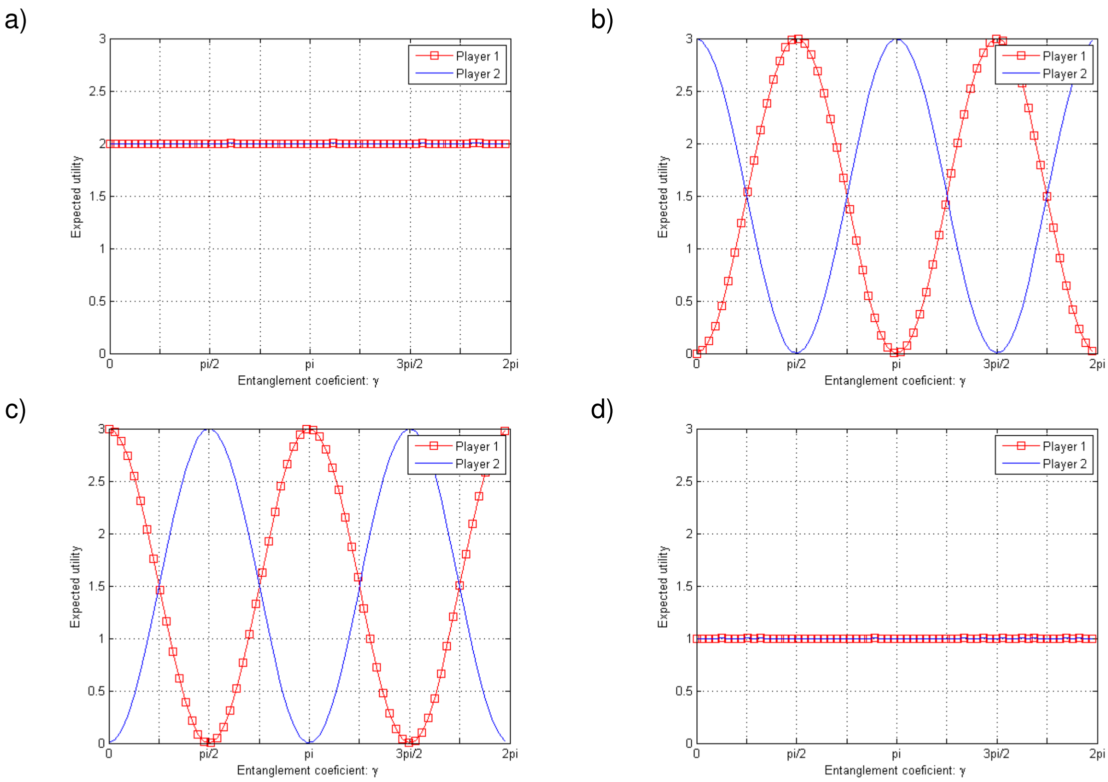

Table 3.4: Expected utility for players 1 and 2 giving the entanglement coefficient γ used in preparing the initial state. a) Player 1: Cooperates, Player 2: Cooperates; b) Player 1: Cooperates, Player 2: Defects; c) Player 1: Defects, Player 2: Cooperates; d) Player 1: Defects, Player 2: Defects.

3.2.3 Ultimatum Game

The ultimatum game is an example of an extensive form game where two players interact in order to divide a sum of money.

A finite amount of money (or other finite resource), is given to the players, and player 1 must propose how the money will be divided between the two players. If the second player agrees with the proposal, the resource will be split accordingly. When the player 2 rejects the proposal, neither player will receive the money.

If we consider that we have 100 coins, the number of coins received can be considered the expected utility associated with the proposal. The first player can either present a fair division (F), where the coins are split evenly, or an unfair division (U) game tree that represents the Ultimatum Game is shown in Figure 3.4.

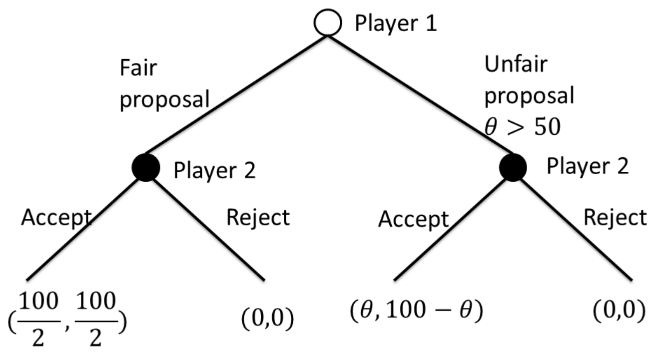

Figure 3.4: Ultimatum Game representation in the extensive form.

3.2.3.A Quantum Model

In a “Quantum information approach to the ultimatum game” [20] we are presented with a quantization scheme for the ultimatum game that uses the definition of quantum game in Section 2.5.

If we present the game in Figure 3.4 in the normal form we get the matrix represented in Table 3.5. The player 2 has 4 possible strategies. The strategy $A_F R_U$ means that the player 2 will accept a fair division proposed by player 1 but will reject a unfair division.

The quantum game representation for this game is $\Gamma_{Ultimatum} = (\mathcal{H}^{2^3},\ 2,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_i\},\ \{E_i\})$. The game system will consist in 3 qubit, which correspond to the number of actions in the game. Player 1

will be able to manipulate the qubit 1, the player 2 can manipulate the remaining qubits.

	Player 2: $A_F A_U$	Player 2: $A_F R_U$	Player 2: $R_F A_U$	Player 2: $R_F R_U$
Player 1: F	$(50, 50)$	$(50, 50)$	$(0, 0)$	$(0, 0)$
Player 1: U	$(\theta, 100 - \theta)$	$(0, 0)$	$(\theta, 100 - \theta)$	$(0, 0)$

Table 3.5: Normal form representation of the ultimatum game.

3.2.4 Monty Hall Problem

The Monty Hall problem became popularized in 1990 in a column, in the magazine Parade [31]. The reason for its notoriety rests mainly in its counter-intuitive nature. Although the Monty Hall problem can be modelled as a Bayesian probability problem, human beings have difficulty in grasping the probabilities involved.

3.2.4.A Problem Description

The Monty Hall problem is loosely based on a television game show hosted by its namesake - Monty Hall. Furthermore it was first attributed to a statistician, named Selvin. The problem is posed as it follows:

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Most people, when faced with this problem, will be indifferent about whether to switch or to stay with the initially picked door.

It is also verified that they will tend to stick with their first choice. According to Granberg and Brow [32], only 13% of 228 subjects decided to switch their initial choice. However, by using probability theory, one can arrive at the conclusion that, in fact, it is advantageous to switch given the previous formulation.

The action of the Host implies a belief update on the probabilities of the variables in the system. This poses a violation of rational decision making; subjects do not seem to follow the best strategy which would maximize their chances of winning the prize.

To understand this exercise, one can look at the decision tree in Figure 3.5. We assumed indifferently that the player chose the door No. 1. The situation is symmetric whichever door she chooses.

Assuming we call C_1 to the variable that describes whether or not the car in behind door No. 1. The variables C_2 and C_3 will respectively describe the probability associated with the car being (or not), behind doors 2 and 3 ($P(C_2) = 1$ or $P(C_2) = 0$, for example).

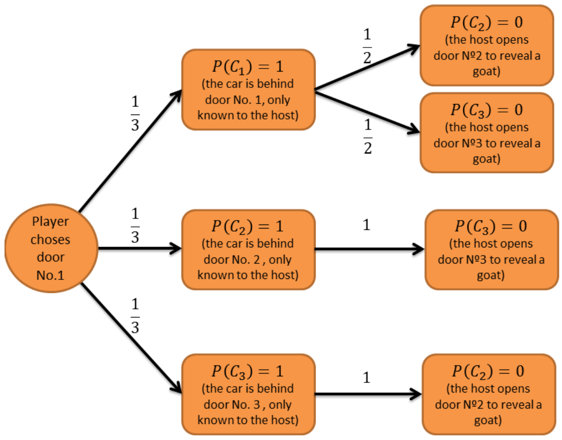

Figure 3.5: Decision tree modelling the Monty Hall problem.

After the player has had her choice, the host will perform an operation on the remaining two doors. The host of the show has complete information of the game, unlike the player.

$$P(C_1) = P(C_1|\neg C_2) + P(C_1|\neg C_3) \tag{3.22}$$

$$P(C_1) = (\frac{1}{3}\times\frac{1}{2}) + (\frac{1}{3}\times\frac{1}{2}) = 1 \tag{3.23}$$

$$P(\neg C_1) = P(C_2|\neg C_3) + P(C_3|\neg C_2) = (\frac{1}{3}\times 1) + (\frac{1}{3}\times 1) = 2/3 \tag{3.24}$$

The probability of switching and getting the car is twice 3.24 as likely of staying with the first choice and getting the prize3.23.

3.2.4.B Quantum Model

Various Models have been proposed to describe a quantum version of the Monty Hall problem [33] [34].

As the host reveals information, the initial set-up is modified. This is an interesting property. Despite being a counter-intuitive problem, a quantum approach to this problem allows an in-depth comparison between the classical measurement and the quantum measurement. The classic Monty Hall problem is modelled using conditional probability and Bayes Rule. In the quantum version, measuring the outcome of the final state yields the result, instead of taking into account the intermediate actions [20].

Moreover it is important to realize that there is not a unique way to model a classical problem [35]. Therefore, when modelling a classical problem, we need to select properties that could potentially benefit from a quantum approach. In [35] we can observe the attempt to stick as closely to the classical formulation as possible, the host has a system that is correlated to the game system.

4

Quantum Pirate Game

Contents

In this chapter we describe the Pirate Game and the steps to model a quantum approach to the problem.

4.1 Pirate Game

4.1.1 Problem Description

The original Pirate Game is a multi-player version of the Ultimatum game that was first published as a mathematical problem in the Scientific American as a mathematical problem posed by Omohundro [36]. The main objective of the Pirate Game was to present a fully explainable problem with a non-obvious solution. The problem can be formulated as it follows:

> Suppose there are 5 rational pirates: A; B; C; D; E. The pirates have a loot of 100 indivisible gold coins to divide among themselves.
>
> As the pirates have a strict hierarchy, in which pirate A is the captain and E has the lowest rank, the highest ranking pirate alive will propose a division. Then each pirate will cast a vote on whether or not to accept the proposal.
>
> If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.
>
> We consider that each pirate privileges her survival, and then will want to maximize the number of coins received. When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.

4.1.2 Analysis

We can arrive at the sub-game perfect Nash equilibrium in this problem by using backward induction. At the end of the problem, supposing there are two pirates left, the equilibrium is very straight forward. This sub-game is represented in Table 4.1, and its Nash Equilibrium is (C, D).

	Player 2: C	Player 2: D
Player 1: C	(100, 0)	(100, 0)
Player 1: D	(100, 0)	(-200, 100.5)

Table 4.1: Representation of the 2 player sub-game in normal form.

As the highest ranking pirate can pass the proposal in spite of the other's decision, her self-interest dictates that she will get the 100 gold coins. Knowing this, pirate E knows that any bribe other higher ranking pirate offers her will leave her better than if the game arrives to the last proposal.

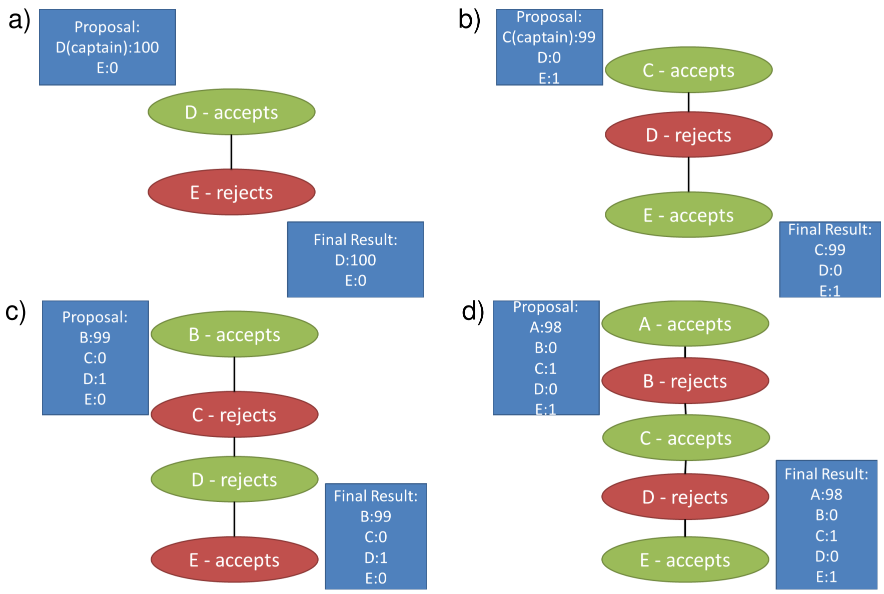

Table 4.2: The equilibrium for the Pirate Game can be found through backward induction. From a), where there's only two pirates left, to d), that corresponds to the initial problem, we define the best response.

When applying this reasoning to the three pirate move, as pirate C knows she needs one more vote to pass her proposal and avoiding death, she will offer the minimum amount of coins that will make pirate E better off than if it comes to the last stage with two pirates. This means that pirate C will offer 1 gold coin to pirate E, and keep the remaining 99 coins.

With 4 pirates, B would rather bribe pirate D with 1 gold coin, because E would rather like climb on the hierarchy and getting the same payoff. Finally, with 5 pirates the captain (A), will keep 98 gold coins and rely on pirate C and E to vote in favour of the proposal, by giving 1 gold coin each.

4.1.2.A Analysis of the Pirate Game for 3 Players

In Figure 4.1 we have an extensive form representation of the classic Pirate Game for 3 players. Each node in the game tree has the number of the player who will make the decision, either to Cooperate (vote yes to the proposal), or Defect. The dashed arrows represent states where the player does not have information of the current state (simoultaneous move).

The green accent, in Figure 4.1, shown in the nodes represent a state where the first captain (player 1), will see her proposal accepted, the utility associated . The blue accent denotes the outcomes where the second captain makes a proposal and has seen it accepted. The red accent color represents the outcomes where the player 3 will be the remaining pirate.

The number of coins will translate directly the utility associated with getting those coins. For example if

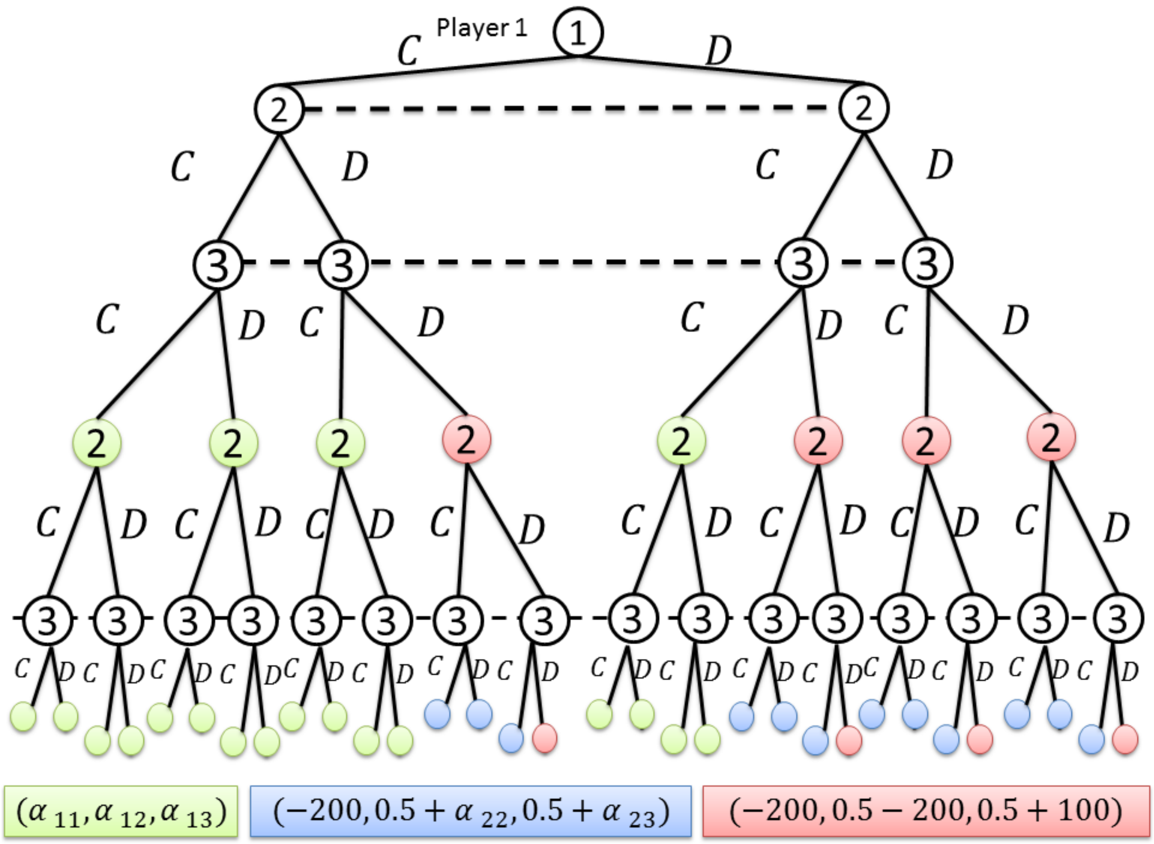

Figure 4.1: Extensive form representation of the classic Pirate Game for 3 players.

a pirate receives 5 gold coins and the proposal is accepted he will get a utility of 5. The highest ranking pirate in the hierarchy is responsible to make a proposal to divide the 100 gold coins. This proposal means choosing the number of coins each player gets if the proposal is accepted. A captain i chooses the amount of coins a player j get; this amount will be represented by α_{ij}.

In the initial stage of the game the captain will define $\alpha_{11}, \alpha_{12}, \alpha_{13}$, and they will obey to the Equation 4.1, that imposes the rule that the captain i must allocate all the 100 coins, to the players that are still alive. N the number of pirates in the game.

$$\forall i \in \{1, 2, ..., N-1\} : \sum_{j=i}^{N} \alpha_{ij} = 100, \forall i, j : \alpha_{ij} \in \mathbb{N}_0 \tag{4.1}$$

The values for $(\alpha_{11}, \alpha_{12}, \alpha_{13}) = (99, 0, 1)$ will be the allocation that results in an equilibrium for the 3 player game .

The proposed goods allocation will be executed if there is a majority (or a tie), in the voting step. A step in the game consists on the highest ranking pirate defining a proposal and the subsequent vote, where all players choose simultaneously an operator.

If the proposal is rejected the captain will be thrown off board, to account for the fact that this situation is very undesirable for the captain he will receive a negative payoff of -200 (that can be seen in the blue and red outcome in Figure 4.1). This value was derived from the fact that a pirate values her integrity more than any number of coins she might receive.

> “When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.”

This means that the pirates have a small incentive to climb the hierarchy. For example in the three player classical game, the third player, who has the lowest rank, will prefer to defect the initial proposal if the player 1 doesn't give her a coin, even knowing that in the second round the player 2 will be able to keep the 100 coins. We will account for this preference by assigning an expected value of half a coin (0.5), to the payoff of the players that will climb on the hierarchy if the voting fails. This tie breaker is shown in the blue and red outcomes in Figure 4.1.

4.1.2.B Consideration on the generalization for N players

We can generalize this problem for N pirates. If we assign a number to each pirate, where the captain is number 1 and the lower the number the higher the rank. If the number of coins is superior to the number of pirates, the equilibrium will have the captain (highest ranking pirate), giving a gold coin to each odd pirate, in case the number of players alive is odd, while keeping the rest to herself. When we have a even number of players the captain will assign a gold piece to each pirate with a even number, and the the remaining coins to herself.

If the number of pirates is greater than two times the amount of coins $N > 2C$, a new situation arises. If we have 100 coins and 201 pirates, the captain will not get any coin. By the same reasoning with 202 pirates the captain will still be able to survive by bribing the majority of the pirates and keeping no coins for herself. With 203 pirates the first captain will die. However with 204 pirates, the first captain will be able to survive even though he won't be able to bribe the majority, because her second in command knows that when she makes a proposal, she'll be thrown off board. In the game with 205 pirates, however the captain is not able to secure the vote from the second in command on the 204 pirate game, because the second captain is safe and she is able to make a have her proposal accepted and have the third pirate safe.

We can generalize this problem for $N > 2C$ as [36], as the games with a number of pirates equal to $2C$ plus a power of two will have an equilibrium in the first round, in the others every captain until a sub-game with a number of pirates equal to $2C$ plus a power of two will be thrown off board.

4.2 Quantum Pirate Game

The original Pirate Game is posed from the point of view of the captain. How should she allocate the treasure to the crew in order to maximize her payoff. We can find the a equilibrium to the original Pirates Game and, while the solution may seem unexpected at first sight, it is fully described using backwards induction.

When modelling this problem from a quantum theory perspective we are faced with some questions, such as:

- Will the initial conditions provide different equilibria?
- What are the similarities with the classical problem?
- Is it possible for a captain, in a situation where we have more than two pirates left, to acquire all the coins?

The main difference from the original problem will rely on how the system is set up. We propose to study this problem for a 3 player game and trying to extrapolate for N players.

We will analyse the role of entanglement and superposition in the game system.

Another aspect worth studying is the variation in the coin distribution on the payoff functions for the players. We are particularly interested in studying the classical equilibrium where the captain retains 99 coins and gives a single coin to the player with the lowest rank. Moreover we want to study what happens when the captain tries to get all the coins.

4.2.1 Quantum Model

In order to model the problem we will start by defining it using the definition of quantum game (Γ), referred in 3.2.3.A, Section 2.5 [23].

We want to keep the problem as close to the original as possible in order to better compare the results. Thus we will analyse the game from the point of view of the captain. Will her best response change?

For the purpose of demonstration this problem could be described using 3 players; the lowest number of players that has an equilibrium in which the captain has to bribe another pirate.

We begin by assigning an offset to each pirate (in order to identify her), as in the Section 4.1.1. The captain is number 1 and the lower the number the higher the rank.

4.2.1.A Game system: Setting up the Initial State

A game Γ can be viewed as a system composed by qubits manipulated by players. We will use the definition of quantum game discussd in Section 2.5 ($\Gamma = (\mathcal{H}^{2^a},\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\})$), to model our game system.

In this 3 player game there will be 5 qubits representing the actions or the players decision. The number of qubits needed to represent the game grows exponentially with the number of players. For N players we need $\sum_{i=2}^{N} i$ qubits. With 8 players, this game would already be impractical to simulate in a classical computer. In this regard a quantum computer may enhance our power to simulate this kinds of experiments [15].

The mapping function ξ that assigns each action/qubit φ_j (with $j = \{1, 2, 3, 4, 5\}$), to a player is represented on Equation 4.2.

$$\xi(j) = \begin{cases} 1 & , \ if\ j = 1; \\ 2 & , \ if\ j \in \{2, 4\}; \\ 3 & , \ if\ j \in \{3, 5\}. \end{cases} \tag{4.2}$$

With 3 players and 5 actions our system with be represented in a $\mathcal{H}^{32}$ using a state ψ. This means that to represent our system we will need $2^5 \times 1$ vectors, our system grows exponentially with the number of players/qubits. Each pure basis of $\mathcal{H}^{32}$, shown in Equation 4.3, will represent a possible outcome in the game. We assign a pure basis as $|0\rangle = |C\rangle$ ("C" from "Cooperate"), and $|1\rangle = |D\rangle$ ("D" from "Defect").

$$\begin{aligned} \mathcal{B} = \{&|00000\rangle, |00001\rangle, |00010\rangle, |00011\rangle, |00100\rangle, |00101\rangle, |00110\rangle, |00111\rangle, \\ &|01000\rangle, |01001\rangle, |01010\rangle, |01011\rangle, |01100\rangle, |01101\rangle, |01110\rangle, |01111\rangle, \\ &|10000\rangle, |10001\rangle, |10010\rangle, |10011\rangle, |10100\rangle, |10101\rangle, |10110\rangle, |10111\rangle, \\ &|11000\rangle, |11001\rangle, |11010\rangle, |11011\rangle, |11100\rangle, |11101\rangle, |11110\rangle, |11111\rangle\} \end{aligned} \tag{4.3}$$

The initial system ($|\psi_0(\gamma)\rangle$), will be set up by defining an entanglement coefficient γ, that affect the way the five qubits (belonging to the three pirate players), are related; this is shown in Equation 4.5. We will entangle our state by applying the gate $\mathcal{J}$ [30]. The parameter γ becomes a way to measure the entanglement in the system [5].

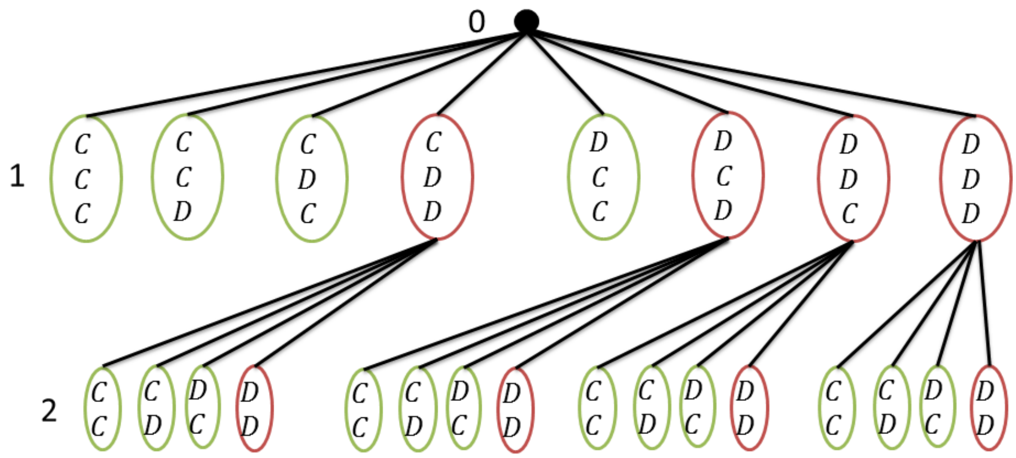

Figure 4.2: Game tree representation for a 3-player game. Red circles represent failed proposals, green represent accepted proposals.

The concept of entanglement is crucial to explain some phenomena in Quantum Mechanics (Section 2.3.4).We analysed the role of the entanglement of the system since other examples researched pointed to it being the proeminent factor regarding behaviour changes from the classsical perspective [23] [20] [30] [37] [38].

We can interpret the existence (or non-existence), of entanglement or superposition in the initial system as an unbreakable contract between the players [39]. The initial state starts by revealing a group of pirates that cooperate by default. We chose this initial set-up because it is prevalent in the literature [5] [23] [20] [30], and we want to test if there is any equilibrium situation where the first captain can pass her proposal while taking all the 100 coins.

The index 0 represents the depth of the game tree which can be examined in Figure 4.2.

Due to the nature of quantum mechanics we have to pay attention of how we set-up our architecture; we cannot copy or clone unknown quantum states (No-cloning Theorem) [15].

$$\mathcal{J} = exp\left\{ i\frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{4.4}$$

$$|\psi_{ini}(\gamma)\rangle = exp\left\{i\frac{\gamma}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}|00000\rangle$$
$$= cos(\frac{\gamma}{2})|00000\rangle + isin(\frac{\gamma}{2})|11111\rangle, \gamma \in (0, \pi) \quad (4.5)$$

4.2.1.B Strategic Space

In Equation 3.2.3.A ($\Gamma = (\mathcal{H}^{2^a},\ N,\ |\psi_{in}\rangle,\ \xi,\ \{\mathcal{U}_j\},\ \{E_i\})$), there is the notion of a subset of unitary operators that the players can use to manipulate their assigned qubits.

Each player will be able to manipulate at least one qubit in the system. Those qubits are $|\varphi_1\rangle,\ |\varphi_2\rangle$, and $|\varphi_3\rangle$. The Equation 4.3 assigns the qubit $|\varphi_1\rangle$ to player 1, qubits $|\varphi_2\rangle$ and $|\varphi_4\rangle$ to player 2, the remaing qubits are assigned to player 3. Each player will be able to manipulate her assigned qubits j with one of two operators shown in Equation 4.6.

An operator is an unitary 2×2 matrix that is used to manipulate a qubit in the system. This restriction of the strategic space is relevant to keep the problem as close to the classical version as possible. The two operators will correspond to the action of voting "Yes" or to Cooperate, and voting "No", meaning that they will not accept the proposal.

The cooperation operator will be represented by the Identity operator (o_{j0}, where j identifies the qubit that the respective player will act upon). When assigned to a qubit this operator will leave it unchanged.

The defection operator (D), will be represented by one of Pauli's Operators - the Bit-flip operator. This operator was chosen because it performs the classical operation NOT on a qubit.

These operators are also permutation matrices, so our players are in fact permuting the state of their qubit as in the roulette quantum model (Section 3.2.1). It is also noteworthy that this operators correspond to pure-strategies.

$$\mathcal{U}_j = \begin{cases} C_j = o_{j0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D_j = o_{j1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases} ,j \in \{1,2,3,4,5\} \quad (4.6)$$

Each player will have a strategy τ_i which assigns a unitary operator U_j to every qubit j that is manipulated by the player ($j \in \xi^{-1}(i)$). $\tau_2 = \{D_2, C_4\}$ represents the strategy where the player 2 votes D in the first stage and C in the second stage.

In Quantization schemas of the Prisoner's Dilemma [30] [5], and Quantum Ultimatum Game [20], the strategic space, described in Equation 4.7, was analysed allowed a infinity of mixed quantum strategies. In the Quantum Roulette Game [27] and [29] we have a demonstration that in a classical two-person zero-sum strategic game, if one person adopts a quantum strategy, she has a better chance of winning the game.

However we claim that allowing mixed quantum strategies would degenerate the voting problem. In the original problem the stages of the game are dependent of the actions taken by the players. Also studying the strategic space in the game system (represented in a $\mathcal{H}^{32}$), would be a daunting process in terms of simulation.

$$\mathcal{U}(\theta, \phi) = \begin{bmatrix} cos(\frac{\phi}{2}) & e^{i\phi}sin(\frac{\phi}{2}) \\ -e^{-i\phi}sin(\frac{\phi}{2}) & cos(\frac{\phi}{2}) \end{bmatrix} \tag{4.7}$$

4.2.1.C Final State

We can play the Pirate Game by considering a succession of steps or voting rounds. In each step we have a simultaneous move(the players sellect their strategies at the same time), however, considering the potential rounds the game has, we have a sequential game.

With three players, the first move will correspond to the player 1 (or the captain), if the proposal fails we will proceed to the second step in the game, where the remaining two players will vote on a new proposal made by player 2 (who will be the new captain).

This final state is calculated by constructing a super-operator, by performing the tensor product of each player chosen strategy, as shown in Equation 4.6. The super-operator, containing each player's strategy, will then be applied to the initial state,as shown in Equation4.8.

$$|\psi_{fin}\rangle = \otimes_{i=1}^{3} \otimes_{j \in \xi^{-1}(i)} \mathcal{U}_j |\psi_{ini}(\gamma)\rangle \tag{4.8}$$

In the Figure 4.3 we have a representation of the game.We start by building our initial state $k-1$, then the players select their strategies, a super operator is constructed by performing a tensor product of the selected operators.

In order to calculate the expected payoff functions we need to de-entangle the system, before measuring. The act of measuring, in quantum computing, gives an expected value that can be understood as the probability of the system collapsing into that state.

We can de-entangle the our $\mathcal{H}^{32}$ system by applying $\mathcal{J}^\dagger$ (Equation **??**), this will produce a final state that we will be able to measure. If we do not apply the inverse transformation $\mathcal{J}^\dagger$ we are introducing errors in the system (when the entanglement parameter γ is different than 0), because we are introducing a correlation between the qubits in the system. In the Figure 4.3 we have represented the way we entangle and de-entangle the system.

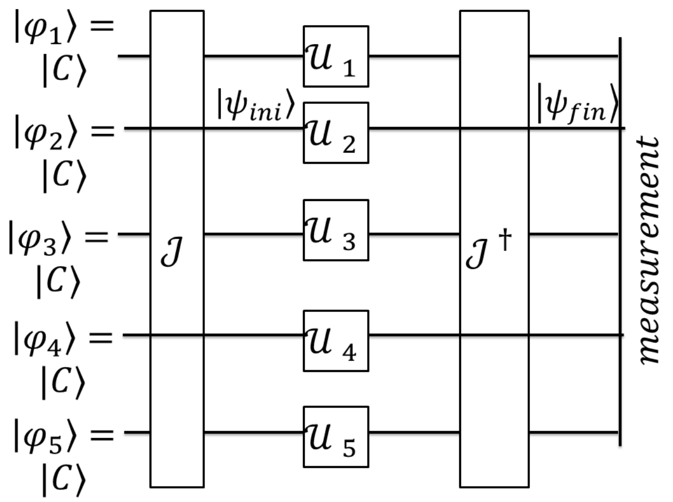

Figure 4.3: Scheme that represents the set-up of the 3-player Pirate Game. Before we measure the final result we need to apply the transpose operator $\mathcal{J}^\dagger$.

4.2.1.D Utility

To build the expected payoff functionals for the three player situation we must take into account the sub-games created when the proposal is rejected. In Figure 4.2 we can see an extensive form representation of the game.

As defined on Equation 2.52,for each player we must specify a utility functional that attributes a real number to the measurement of the projection of a basis in the quantum state that we get after the game.

This measurement can be understood as a probability of the system collapsing into that state (that derives from the Born Rule, Section 2.2.2).

These utility functions will represent the degree of satisfaction for each pirate after game by attributing a real number to a measurement performed to the system (as in Equation 2.52, Section 2.5). The real numbers used convey the logical relations of utility posed by the original problem description. Those numbers will represent the utility associated with the number of coins that a pirate gets, a death penalty, and a small incentive to climb the hierarchy. As each pirate wants to maximize her utility, the Nash equilibrium will be thoroughly used to find the strategies that the pirates will adopt [21] [22].

We can observe in Figure 4.1 that in that we have three separate groups (denoted by the colour accents), of outcomes that share the same payoff. In our quantum scheme we can agregate the quantum states associates with a payoff in the following manner:

- States where the first proposal is accepted:
  - $|C, C, C, x_4, x5\rangle$ or $|0, 0, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|D, C, C, x_4, x5\rangle$ or $|1, 0, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|C, D, C, x_4, x5\rangle$ or $|0, 1, 0, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$;
  - $|C, C, D, x_4, x5\rangle$ or $|0, 0, 1, x_4, x5\rangle$, with $x_4 \in \{0, 1\}$ and $x5 \in \{0, 1\}$.
- States where the first captain will be eliminated and the second player gets her proposal accepted:
  - $|D, D, D, C, x5\rangle$ or $|1, 1, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, D, D, D, C\rangle$ or $|1, 1, 1, 1, 0\rangle$;
  - $|D, D, C, C, x5\rangle$ or $|1, 1, 0, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, D, C, D, C\rangle$ or $|1, 1, 0, 1, 0\rangle$;
  - $|C, D, D, C, x5\rangle$ or $|0, 1, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|C, D, D, D, C\rangle$ or $|0, 1, 1, 1, 0\rangle$;
  - $|D, C, D, C, x5\rangle$ or $|1, 0, 1, 0, x5\rangle$, with $x5 \in \{0, 1\}$;
  - $|D, C, D, D, C\rangle$ or $|1, 0, 1, 1, 0\rangle$.
- States where all proposals are rejected:
  - $|D, D, D, D, D\rangle$ or $|1, 1, 1, 1, 1\rangle$;
  - $|D, D, C, D, D\rangle$ or $|1, 1, 0, 1, 1\rangle$;
  - $|C, D, D, D, D\rangle$ or $|0, 1, 1, 1, 1\rangle$;
  - $|D, C, D, D, D\rangle$ or $|1, 0, 1, 1, 1\rangle$.

In order to calculate the probability of the final state collapsing onto a basis state $b \in \mathcal{B}$ we perform a projection of the state in the chosen basis and we measure the squared length of the projection, $|\langle b|\psi_{fin}\rangle|^2$ [12].

The final payoff function (for example Equation 4.9 for a 3 player game), will be calculated recursively, the base case being the 2 player sub-game in a 3 player system will be Equation 4.10.

$$\begin{cases} E_1(|\psi_{fin}\rangle,\alpha_1) = \alpha_1 \times (\sum_{x_3}\sum_{x_4}|\langle 0,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4}|\langle 1,0,0,x_4,x5|\psi_{fin}\rangle|^2 + \\ \qquad + \sum_{x_3}\sum_{x_4}|\langle 0,1,0,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4}|\langle 0,0,1,x_4,x5|\psi_{fin}\rangle|^2) - \\ \qquad -200 \times (\sum_{x_3}\sum_{x_4}|\langle 1,1,1,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4}|\langle 0,1,1,x_4,x5|\psi_{fin}\rangle|^2 + \\ \qquad + \sum_{x_3}\sum_{x_4}|\langle 1,0,1,x_4,x5|\psi_{fin}\rangle|^2 + \sum_{x_3}\sum_{x_4}|\langle 1,1,0,x_4,x5|\psi_{fin}\rangle|^2) \\ E_{12}(|\psi_{fin}\rangle,\alpha_2) = \alpha_2 \times (|\langle 000|\psi_{fin}\rangle|^2 + |\langle 100|\psi_{fin}\rangle|^2 + |\langle 010|\psi_{fin}\rangle|^2 + |\langle 001|\psi_{fin}\rangle|^2) - \\ \qquad +(0.5+E_{22}) \times (|\langle 111|\psi_{fin}\rangle|^2 + |\langle 110|\psi_{fin}\rangle|^2 + |\langle 101|\psi_{fin}\rangle|^2 + |\langle 011|\psi_{fin}\rangle|^2) \\ E_{13}(|\psi_{fin}\rangle,\alpha_3) = \alpha_3 \times (|\langle 000|\psi_{fin}\rangle|^2 + |\langle 100|\psi_{fin}\rangle|^2 + |\langle 010|\psi_{fin}\rangle|^2 + |\langle 001|\psi_{fin}\rangle|^2) - \\ \qquad +(0.5+E_{23}) \times (|\langle 111|\psi_{fin}\rangle|^2 + |\langle 110|\psi_{fin}\rangle|^2 + |\langle 101|\psi_{fin}\rangle|^2 + |\langle 011|\psi_{fin}\rangle|^2) \end{cases} \tag{4.9}$$

$$\begin{cases} E_{22}(|\psi_{fin2}\rangle,\alpha_2) = \alpha_2 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \qquad +0.5 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \\ E_{23}(|\psi_{fin2}\rangle,\alpha_3) = \alpha_3 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \qquad +100.5 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \end{cases} \tag{4.10}$$

5

Analysis and Results

Contents

In this chapter we will analyse and discuss the results obtained by simulating our quantization scheme for the Pirate Game.

5.1 Analysis and Results

5.1.1 3 Player Game

5.1.1.A The captain proposes: $(99, 0, 1)$

The outcome CDC with a proposal of $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$ would represent the Nash Equilibrium of the classic Pirate Game (for 3 players).

When the players chose at least 2 operators $Cooperate$ on the initial proposal the game ends right away, the disentangle operator $\mathcal{J}^\dagger$ is applied, and the payoff functionals are calculated given the final state. The final state will be calculated as shown in Equation 5.1.

The Tables D.1, D.2, D.3, and D.4, in the Apendix D ,present the results for the sittuation described above.

In the Table 5.1 we find a reproduction of Table D.1, where the 3 players choose to use the cooperate strategy. We can see that the parameter γ has no effect on the final result. This happens because no player chooses to change her qubit.

In Tables D.2, D.3, and D.4 we can see a pattern on the probability distributions associated with probability of measuring a determined outcome.

The Table 5.2 reproduces the results of Table D.3. The players select the operators (CDC) in Table 5.2, as there two players chose the "Cooperate" operator (represented by a 2×2 identity matrix), the proposal is accepted and the game ends (refer to Figure **??**). The states CDC and DCD vary with γ, such that $P(CDC) = 1 - P(DCD)$. The probability distribution associated with DCD(the state where the qubits are flipped), is approximated by normal distribution as vary γ. The peak for this distribution happens when the initial system is maximally entangled, for $\gamma = frac\pi 2$.

For $\gamma = 0 + k\pi, k \in \{0, 1\}$ we are rendered with the classical situation, this means the proposal suggested by the captain will be enforced exactly as she stated. When $\gamma = \frac{\pi}{2}$ we are before a symmetric outcome where player 1 would die and player 2 would approve her proposal and get the 100 coins.

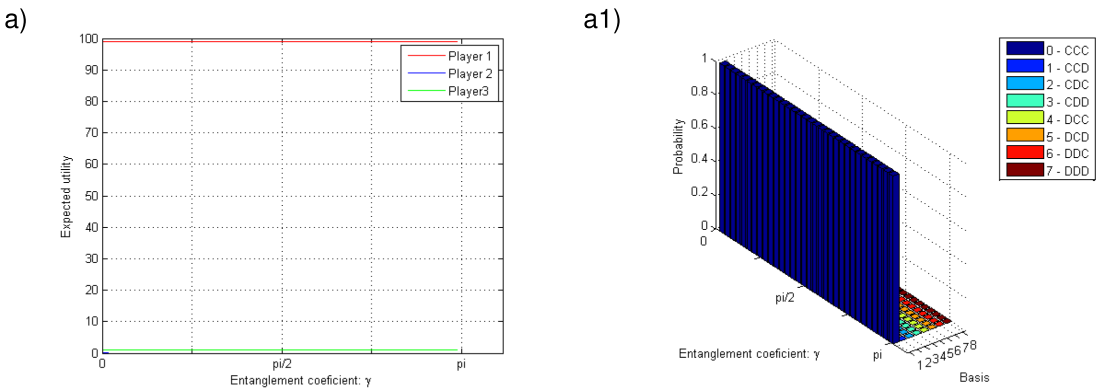

Table 5.1: a) Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Cooperate)$ operators. The initial proposal is $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$. a1) Probability distribution of the final state depending on the entanglement coefficient γ. Reproduction from Table D.1

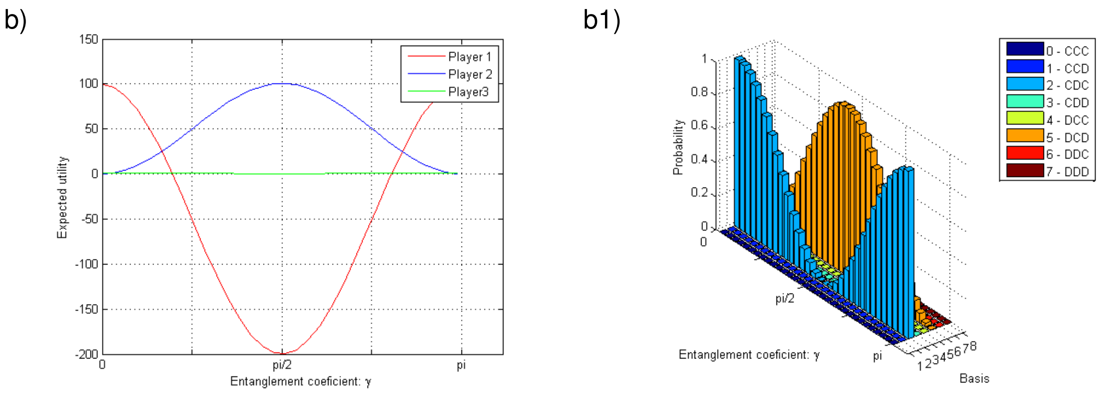

Table 5.2: b) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Cooperate)$ operators. b1) Probability distribution of the final state depending on the entanglement coefficient γ. Reproduction from Table D.3.

$$|\psi_{fin}\rangle = \mathcal{J}^{\dagger}|\psi_1\rangle \tag{5.1}$$

When the first proposal is rejected (more than 1 player chooses to $Defect$), the second round ensues. We calculate the final state for these states with Equation 5.2.

$$|\psi_{fin}\rangle = \mathcal{J}^{\dagger}|\psi_2\rangle \tag{5.2}$$

In a classical game the best response for player 1 is to cooperate. In this quantum version depending of the parameter γ, the strategy that produces the most favourable outcome changes. However if the first proposal is rejected and player 1 voted in favour (using the C operator), we get to a final sub-game where the parameter γ won't influence the expected utility; these results for this can be consulted on Appendix D, Section D.1.2.B. Table 5.3 displays a particular outcome of the sub-game generated when the players chose the operators CDD in the first stage (CD); it shows the expected utility when player 2

accepts her proposal, and player 3 rejects.

This happens due to the nature of the game, more specifically the way we calculate our expected utility for each player, which is described in Section 4.2.1.D, Equation 4.9. The sum of the probability for the system being on states $|CCD\rangle$ or $|CDC\rangle$ is constant ($\frac{1}{2}$), this also happens with the sum of the probability of the system being on state $vertDCD\rangle$ or $|DDC\rangle$.

However the parameter γ affects the probability of the system being measured in a determined state, as in Table 5.3.

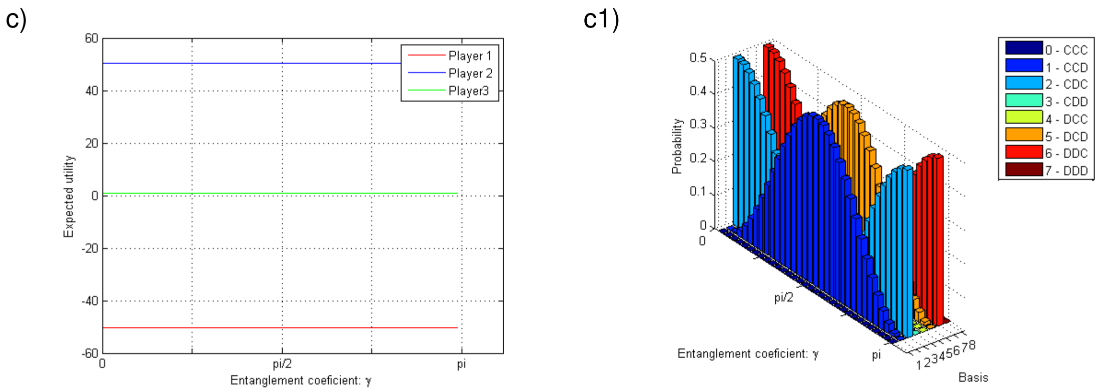

Table 5.3: Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CD).

If in the quantum game the player 1 decides to defect D and the proposal is rejected we get 4 different probability distributions that are permuted given the players chosen operators. In Tables D.11, D.14, and D.17 (in Apendix D)

In Table D.11, the player 2 has assumes the strategy $\tau_2 = (C, D)$, this means that she will play C on the first stage and D on the second stage; In Table D.14 her strategy is $\tau_2 = DC$. As the multiplication by a identity matrix is commutative, in the previous cases, we applied the same operation to player 2 qubit. This result happens because the players are manipulating their qubits with permutation operators C and D; described on Section 4.2.1.B, where we define the strategic space the players have access.

When the parameter γ influences the expected utility for the players, we verify the functions that describe the expected utility for player 1 and 3 tend have maxima and minima for the same values of γ. However if player 1 decides to $Defect$ the two functions will not be synchronized. The function that describes the expected utility for player 2 tends to have opposite behaviour. This happens because the captain bribed player 3 with one gold coin.

5.1.1.B The captain proposes: $(100, 0, 0)$

Suppose captain is greedy and proposes to get the 100 coins. In the classical Pirate Game this would pose a conflict with his self-preserving needs. A pertinent question would be if this Quantum Model of the Pirate Game would allow the first captain to approve that allocation proposal. The initial proposal will be accepted if there is at least 2 players play $\mathcal{U}_i = C$ in a round.

The player 1, the captain, needs two votes to pass her proposal. In order to answer our question we must analyse the expected utilities for all players where the initial proposal is accepted. Moreover we need to analyse an instance where the proposal is refused, when the players select the actions (C, D, D). In the previous analysis we observed that player 2 has a strong motivation to Defect in the first round, because in the second round she can pass the proposal with her vote alone, this means if the player 1 wants to pass is proposal she should choose the C operator.

Our final state will be calculated as shown in Equation 5.1.

In Table 5.4 we notice that player 3 will get a expected payoff of 0.5 if she chooses to Defect.

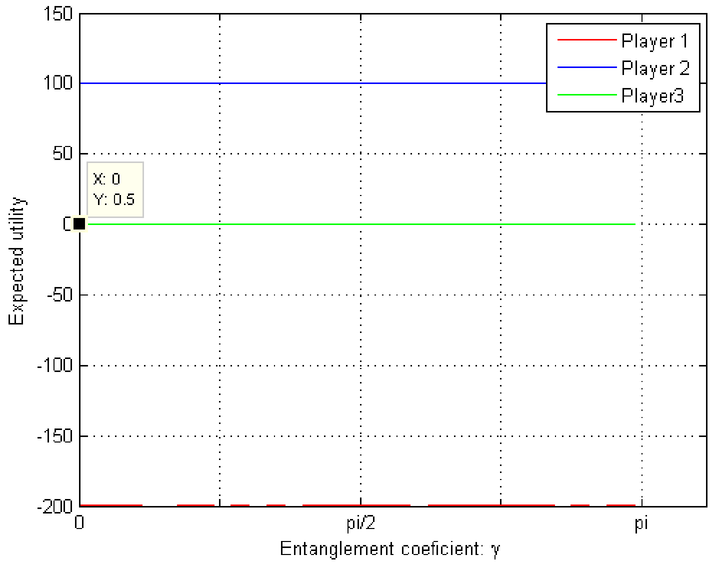

Table 5.4: Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Defect)$ operators in the first stage. In the second stage we represented the sub-game equilibrium of the game.

In Tables 5.6 and 5.5, the expected utility function for player 3 starts of as 0 for $\gamma = 0$ (which corresponds to the classical problem). The function presents a maximum when the system is maximally entangled; this maximum is 0.5. If the proposal is accepted when the entanglement coefficient is $\gamma = \frac{\pi}{2}$ player 1 will get a negative expected payoff (which means he will die under that condition).

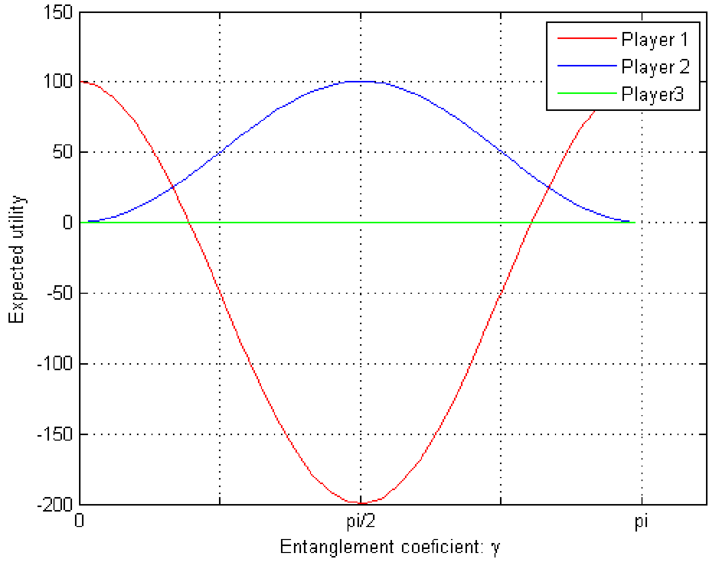

Table 5.5: Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Defect)$ operators.

The Table 5.7 presents an outcome where all the players vote in favour of the proposal. The players 2 and 3 have and expected utility of 0, regardless the role of entanglement in the system.

From this simulation we can conclude that if the captain wants to pass her proposal for $\gamma = \frac{\pi}{2}$ there are 3 equilibria when the player select the operators (CDC), (C, C, D), and $((C, D, D), (C, C))$. (CDC), (C, C, D) correspond to an accepted proposal. However player 1 will get a negative payoff in any of this equilibria.

We can interpret this in the context of the problem with the captain getting the coins as she proposed, but being immediately betrayed and killed by her fellow pirates who conspired in order for player 2 to get all the coins, and player 3 to become second in command.

Otherwise there is one Nash Equilibrium when the players select $((C, D, D), (C, C))$. Even in a quantum version, the captain needs to bribe player 3 in order to approve her proposal and get a maximum of 99 gold coins.

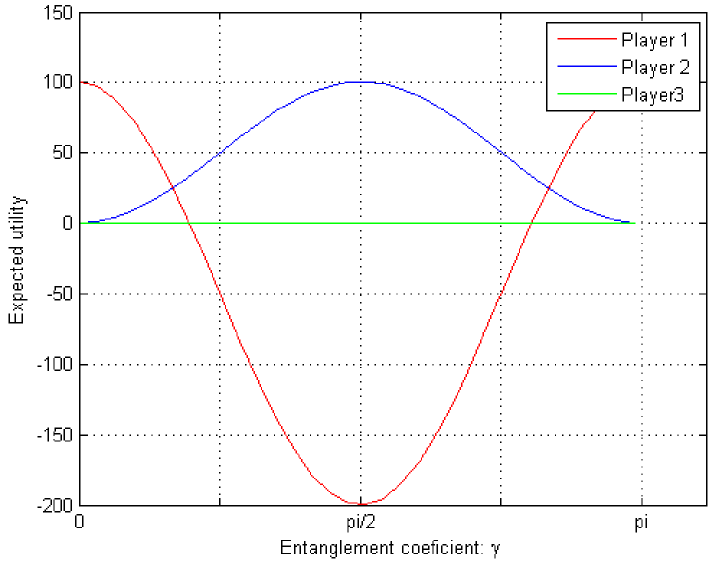

Table 5.6: Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Cooperate)$ operators.

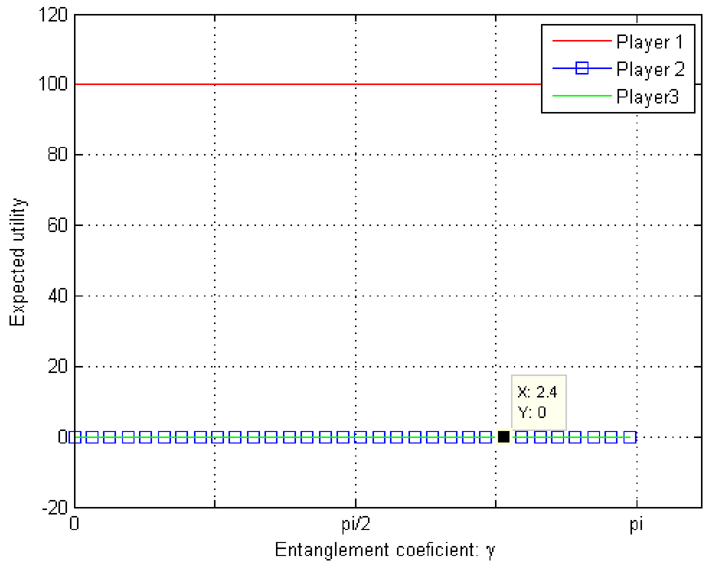

Table 5.7: Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Cooperate)$ operators. The initial proposal is $(\alpha_1, \alpha_2, \alpha_3) = (100, 0, 0)$.

5.1.2 Discussion

We tried to get our Quantum Model of the Pirate Game as close as possible in order to compare the the original game.

In works such as the Prisoner's Dilemma [30] [5], and Quantum Ultimatum Game [20], the strategic space was analysed allowing a infinity of mixed quantum strategies.

When designing the Quantum Model, we claimed that allowing mixed quantum strategies would degenerate the voting problem, as the original problem has stages that are dependent on the actions taken by the players.

The set-up of the initial system was crucial to introduce the phenomenon of entanglement which was the principal study variable in our model.

In the simulation of the Quantum Pirate Game with 3 players we found possible final states where the probability distributions of the final outcomes depended on the entanglement parameter γ, but the expected utility functions did not vary with γ. This particular result is an example that arises from the problem description (the way the expected utility functionals are build for each player), however it is not explored in the related work.

It is impossible to not trace a parallel with our world that seems to have some deterministic rules from interactions governed by quantum mechanics at a fundamental level. The concept of temperature is an example of that. At a fundamental level the temperature is the measurement of radiation emitted by oscillating particles.

This version of the Quantum Pirate Game ended up being a case study of how to modify as little as possible a classical Game Theory problem while introducing mechanics inspired in the Quantum Theory.

6

Conclusions and Future Work

Contents

This section closes this document. It provides an overview on the work, presents a summary of the main results and relevant contributions. Moreover we compiled a list of interesting points worth pursuing in the future.

From an historic point of view there is a character who is active on the fields of Quantum Mechanics, Game Theory, and Computer Science (among others), John von Neumman. von Neumman died at the age of 53 years, so we can only specculate if he would ever try to apply the principles of Quantum Mechanics to Game Theory.

Robel Laughlin uses the concept of emergence to explain the way classical phenomena arises from quantum mechanics [3]. In the Pirate Game we found that derived from the game design and the strategic space (the players could only use pure strategies like Cooperate/Defect), there were possible outcomes to the game where the entanglement parameter did not affect the final result.

6.1 Future Work

The principles of quantum computing provide an extremely rich source of ideas to extend other fields of knowledge. Some suggestions for possible extensions for this work would be:

1. **Study different ways to set up the initial state.** Setting the initial state in a Quantum Game Theory game is not trivial. What if the pirates had a prior pre-disposition to cooperate or defect? Possibly exploring various initial superpositions derived from the players pre-disposition to cooperate or defect would provide insights on decision making with non-rational players.
2. **To implement and test this quantum model with human subjects.** The field of Quantum Game theory tries to explain the Human reasoning process by using principles from quantum game theory, namely quantum probabilities. To expose subjects to a quantum model and having them try to take advantage of the rules of the system might provide valuable insights to the way we reason.
3. **Increase the number of players.** Studying the quantum system with 8 players and 1 gold coin, would be an interesting extentions for this work. This would allow to experiment the bizarre survival sittuation that happens when there are not enough coins for the captain to bribe the other pirates.
4. **A graphical user interface (GUI) for the Pirate Game.** This would provide a more approachable way to tweak parameters.
5. **To create and structure a platform to promote scientific divulgation of quantum computing.** While investigating and developing this solution, there was often the thought that it would be important for undergraduate Computer Science students to come with contact with this paradigm. This

field is still shrouded with mystery to many people, and this makes it a priority topic for scientific divulgation. In this work we tried to present examples that someone with a basic understanding of Algebra may be able to follow, but recollecting my experiences it is not enough.

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A

Matlab Simulation: Discrete Quantum Walk on a Line

```
%-- number of steps in the simulation
steps= 30;

%-- hadamard matrix
H = [1/sqrt(2) 1/sqrt(2); 1/sqrt(2) -1/sqrt(2) ];
%-- symetric matrix
M = [1/sqrt(2) 1i/sqrt(2); 1i/sqrt(2) 1/sqrt(2) ];
%-- coin flip unitary operator
C = H;
C=M;

%-- coin flip matrix
CM= zeros((steps*2+1),2);
%-- shift matrix
SM= zeros(steps*2+1,2);

%-- middle index
i0= steps+1;

%-- initialize flip probability amplitudes (coin has 1/2 chance of going +1 or -1)
CM(i0,1)=1/sqrt(2);
CM(i0,2)=1/sqrt(2);

for i=1:steps
   %-- clean SM
   SM= zeros(steps*2+1,2);

   for j=1:(steps*2+1)
       if CM(j, 1)≠0
           SM(j-1,1)=CM(j, 1);
       end
       if CM(j, 2)≠0
           SM(j+1,2)=CM(j, 2);
       end
   end
   SM;
   %disp('-------');
   %-- clean CM
   CM= zeros(steps*2+1,2);
   for j=1:(steps*2+1)
       if SM(j, 1)≠0
           CM(j, 1)= CM(j, 1)+C(1,1)*SM(j, 1);
           CM(j, 2)= CM(j, 2)+C(1,2)*SM(j, 1);
       end
       if SM(j, 2)≠0
           CM(j, 1)= CM(j, 1)+C(2,1)*SM(j, 2);
           CM(j, 2)= CM(j, 2)+C(2,2)*SM(j, 2);
       end
   end
   CM;

   %Display

   figure

   probability=zeros(steps*2+1);

   for j=1:steps*2+1
      probability(j)=abs(SM(j,1)).^2+ abs(SM(j,2)).^2;
   end
   axisP = -steps : steps;
   plot(axisP,probability)
   title(strcat('Step: ' ,num2str(i)))
   axis([-steps-1 , steps+1, 0, 1]);
   grid on
   ylabel(strcat('Probability(State|Step=',num2str(i), ')'));
   xlabel('States');

   %Display
end
```

B

Quantum Prisioner's Dillema

```matlab
% Quantum Prisioner's Dillema
function out = quantumprisionersdillema(g,T,R,P,S)
%quantumprisionersdillema(g)
%
%                 Simulates the payoff of 2-player prisioner's dillema game
%
%         IN:
%                 g   : entanglement coefitient, by defaut g=0 (no
%                 entanglement)
%                 T   : temptation, by defaut T=3
%                 R   : reward, by defaut R=2
%                 P   : punishment, by defaut P=1
%                 S   : suckers, by defaut S=0
%
% To have a prisoner's dilemma game according to the cannonical form,
% it must respect:
% T > R > P > S
%             P      Player 2
%             l        C       D
%             a  C: (R, R)   (S, T)
%             y  D: (T, S)   (P, P)
%             e
%             r
%             1
%
%         OUT:
%                 out: 4x2 matrix, column 1 has the expected utility for
%                      player 1, column 2 has the expected utility for
%                      player 2. A possible outcome corresponds to each
%                      line like [CC;CD;DC;DD]
%
% All parameters are optional
%-------------------------%

%-- entanglement coeficient (checks if exists)
%-- Check variables and set to defaults
if exist('g','var')≠1, g=0; end

%-- Utility
%-- Check variables and set to defaults
if exist('T','var')≠1, T=3; end
if exist('R','var')≠1, R=2; end
if exist('P','var')≠1, P=1; end
if exist('S','var')≠1, S=0; end

%-- Actions
%   cooperate= [1 0;0 1]
C= eye(2);
%   defect= [0 1;1 0]
D= ones(2)-eye(2);

%-- Building the initial state
ini= cos(g)*kron([1 0]',[1 0]') + 1i*sin(g)*kron([0 1]',[0 1]');
%-- Deentangles to produce a final state
Jt = ctranspose( expm(1i*(g)*kron(D,D)));
out= zeros(4,2);
%-- Simulation a outcome where player1 (C)ooperates
%   and player 2 (C)ooperates
   finCC= Jt*kron(C,C)*ini;
   out(1,1)=payofffunc_player1(finCC, T, R, P, S);
   out(1,2)=payofffunc_player2(finCC, T, R, P, S);
%-- Simulation a outcome where player1 (C)ooperates
%   and player 2 (D)efects
   finCD= Jt*kron(C,D)*ini;
   out(2,1)=payofffunc_player1(finCD, T, R, P, S);
   out(2,2)=payofffunc_player2(finCD, T, R, P, S);
%-- Simulation a outcome where player1 (D)efects
%   and player 2 (C)ooperates
   finDC= Jt*kron(D,C)*ini;
   out(3,1)=payofffunc_player1(finDC, T, R, P, S);
   out(3,2)=payofffunc_player2(finDC, T, R, P, S);
%-- Simulation a outcome where player1 (D)efects
%   and player 2 (D)efects
```

```
    finDD= Jt*kron(D,D)*ini;
    out(4,1)=payofffunc_player1(finDD, T, R, P, S);
    out(4,2)=payofffunc_player2(finDD, T, R, P, S);
end

function u = payofffunc_player1(fin, T, R, P, S)
%payofffunc_player1(fin)
%
%                 Calculates the payoff for player 1
%               IN
%                    fin: final state
%                    T, R, P, S: payoff numbers
%               OUT
%                    u: expected utility for player 1
%(norm(conj([0 1 1 1 0 1 1 1]').*fin2CC))^2
    u= R*(norm(conj([1 0 0 0]').*fin))^2 + T(norm(conj([0 0 1 0]').*fin))^2+ ...
        P*(norm(conj([0 0 0 1]').*fin))^2;
end

function u = payofffunc_player2(fin, T, R, P, S)
%payofffunc_player1(fin)
%
%                 Calculates the payoff for player 2
%               IN
%                    fin: final state
%                    T, R, P, S: payoff numbers
%               OUT
%                    u: expected utility for player 2
    u= R*(norm(conj([1 0 0 0]').*fin))^2 + T*(norm(conj([0 1 0 0]').*fin))^2+ ...
        P(norm(conj([0 0 0 1]').*fin))^2;
end
```

C

Quantum Roullete

```
function []= quantum_roullete3()
%%
% Simulation based on
% S. Salimi and M. M. Soltanzadeh, "Investigation of quantum roulette arXiv : 0807 . ...
    3142v3 [ quant−ph ] 30 Apr 2009," 2009.

%%

N=3
%% make states
I = eye(N);
D = (1/N)*ones(N);


%% permutation matrices change between states
 X0 = circshift(I, 0);
%X0 = I;
 X1 = circshift(I, 1);
%X1 = [0 1 0; 1 0 0; 0 0 1];
 X2 = circshift(I, 2);
%X2 = [0 0 1; 0 1 0; 1 0 0];
 X3 = circshift(I, 3);
%X3 = [1 0 0; 0 0 1; 0 1 0];
X4 = circshift(I, 4);
%X4 = [0 0 1 ; 1 0 0; 0 1 0];
 X5 = circshift(I, 5);
%X5 = [0 1 0; 0 0 1; 1 0 0];


%%Fourier Matrix
F= (1/sqrt(N))*fft(I)

T0 = circshift(F,[0 0]);
T1 = circshift(F,[0 1]);
T2 = circshift(F,[0 2]);
%%

%%Step1

%Assuming Alice places the roullete in state 2 were both player can see
Ro0= [0; 1; 0]*[0; 1; 0]'

%%Step2

%As Alice chose state 2, Bob will select T1 to rotate the state
Ro1 = T1 * Ro0 * T1'


%step 3 −
Ro1 = T2 * Ro1 * T2'



%%Step3
%Alice will play again
%

Ro2 = (2/6)*X0*Ro1*X0' + (1/12)*X1*Ro1*X1' +(1/12)*X2*Ro1*X2' +(1/6)*X3*Ro1*X3' ...
    +(1/6)*X4*Ro1*X4' +(1/6)*X5*Ro1*X5'

%%Step4
%Bob can choose which state he wants

Ro3 = T0'*Ro2*T0

Ro3 = T1'*Ro2*T1

Ro3 = T2'*Ro2*T2

Ro4 = (2/6)*X0*Ro3*X0' + (1/12)*X1*Ro3*X1' +(1/12)*X2*Ro3*X2' +(1/6)*X3*Ro3*X3' ...
    +(1/6)*X4*Ro3*X4' +(1/6)*X5*Ro3*X5'

Ro3 = T0'*Ro3*T0
```

```
end
```

D

Results: Pirate Game

D.1 3 Player Game

D.1.1 Simulation

```
function out = simulation3players(a_11,a_12,a_13,a_22,a_23)
%
%        IN:
%-- alocation proposal a_ij -i number of the player that proposes a_ij coins
%   to player j
%-- Check variables and set to defaults
if exist('a_11','var')≠1, a_11=99; end
if exist('a_12','var')≠1, a_12=0; end
if exist('a_13','var')≠1, a_13=1; end
if exist('a_22','var')≠1, a_22=100; end
if exist('a_23','var')≠1, a_23=0; end



%   cooperate= [1 0;0 1]
C= eye(2);
%   defect= [0 1;1 0]
D= ones(2)-eye(2);
% coin matrix
H= [1/sqrt(2) 1i/sqrt(2);1i/sqrt(2) 1/sqrt(2)];



step=0.1;
t=0:step:pi;

prob= zeros(size(t),8);

% variation of the paremeter gamma (the entanglement in the system)
for i=0:step:pi


ini= cos(i/2)*kron([1 0]', kron([1 0]',[1 0]'))+1i*sin(i/2)*kron([0 1]', kron([0 1]',[0 ...
    1]'));
% alternative: ini = J*kron([1 0]', kron([1 0]',[1 0]'));

% entanglement matrix
J= expm(1i*(i/2)*kron([0 1;1 0],kron([0 1;1 0],[0 1;1 0])));
% conjugate transpose of the entanglement matrix
Jd= ctranspose( expm(1i*(i/2)*kron([0 1;1 0],kron([0 1;1 0],[0 1;1 0]))));

% % --- Strategies

%% CCC
   fin = Jd*kron(C,kron(C,C))*ini;
   payofffunc(fin, a_11,a_12,a_13,a_22,a_23)
%% CCD
  fin = Jd*kron(C,kron(C,D))*ini;
  payofffunc(fin, a_11,a_12,a_13,a_22,a_23)
%% CDC
 fin = Jd*kron(C,kron(D,C))*ini;
 payofffunc(fin, a_11,a_12,a_13,a_22,a_23)
%
%% DCC
 fin = Jd*kron(C,kron(D,D))*ini;
 payofffunc(fin, a_11,a_12,a_13,a_22,a_23)

% %% CDD
  fin = kron(C,kron(D,D))*ini

      %% CC
 fn= Jd*kron(H,kron(C,C))*fin;
 payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%     %% CD
```

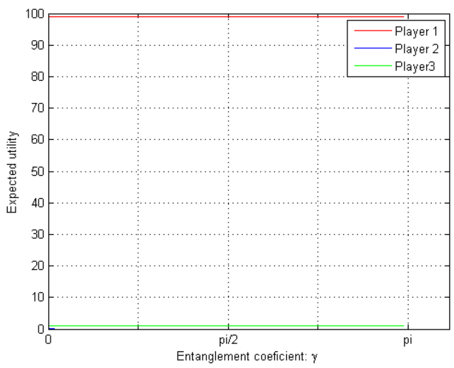
```
    fn= Jd*kron(H,kron(C,D))*fin
     payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DC
   fn= Jd*kron(H,kron(D,C))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DD
    fn= Jd*kron(H,kron(D,D))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)

% %% DCD
  fin = kron(D,kron(C,D))*ini
        %% CC
 fn= Jd*kron(H,kron(C,C))*fin;
 payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% CD
    fn= Jd*kron(H,kron(C,D))*fin
     payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DC
   fn= Jd*kron(H,kron(D,C))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DD
    fn= Jd*kron(H,kron(D,D))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
% %% DDC
  fin = kron(D,kron(D,C))*ini
        %% CC
 fn= Jd*kron(H,kron(C,C))*fin;
 payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% CD
    fn= Jd*kron(H,kron(C,D))*fin
     payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DC
   fn= Jd*kron(H,kron(D,C))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DD
    fn= Jd*kron(H,kron(D,D))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
% %% DDD
 fin = kron(D,kron(D,D))*ini

       %% CC
 fn= Jd*kron(H,kron(C,C))*fin;
 payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% CD
    fn= Jd*kron(H,kron(C,D))*fin
     payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DC
   fn= Jd*kron(H,kron(D,C))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)
%      %% DD
    fn= Jd*kron(H,kron(D,D))*fin;
    payofffunc(fn, a_11,a_12,a_13,a_22,a_23)


end
end

function [] = payofffunc(fin, a_11,a_12,a_13,a_22,a_23)
%payofffunc_player1(fin)
%
%                 Calculates the payoff for player 1
%               IN
%                    fin: final state
%

%-- Check variables and set to defaults
if exist('a_11','var')≠1, a_11=99; end
if exist('a_12','var')≠1, a_12=0; end
if exist('a_13','var')≠1, a_13=1; end
if exist('a_22','var')≠1, a_22=100; end
if exist('a_23','var')≠1, a_23=0; end

```

```
u1= a_11*(norm(conj([1 1 1 0 1 0 0 0]').*fin))^2 -200*(norm(conj([0 0 0 1 0 1 1 ...
    1]').*fin))^2
%
u2=  a_12*(norm(conj([1 1 1 0 1 0 0 0]').*fin))^2 +(0.5+a_22*(norm(conj([0 1 1 1 0 1 1 ...
    1]').*fin))^2 -200*(norm(conj([1 0 0 0 1 0 0 0]').*fin))^2)*(norm(conj([0 0 0 1 0 1 ...
    1 1]').*fin))^2
%
u3=  a_13*(norm(conj([1 1 1 0 1 0 0 0]').*fin))^2 +(0.5+a_23*(norm(conj([0 1 1 1 0 1 1 ...
    1]').*fin))^2 +100.5*(norm(conj([1 0 0 0 1 0 0 0]').*fin))^2)*(norm(conj([0 0 0 1 0 ...
    1 1 1]').*fin))^2



end
```

D.1.2 Captain proposes $(99, 0, 1)$.

D.1.2.A Accepted proposal after 1 round of the game.

a)



a1)

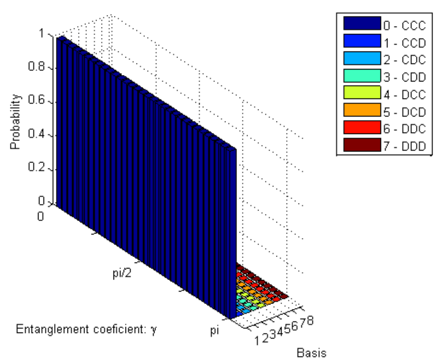
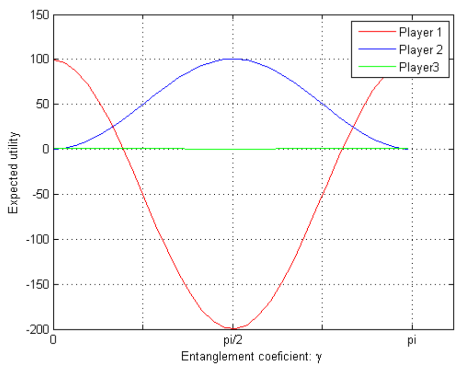


Table D.1: a) Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Cooperate)$ operators. The initial proposal is $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$. a1) Probability distribution of the final state depending on the entanglement coefficient γ.

b)



b1)

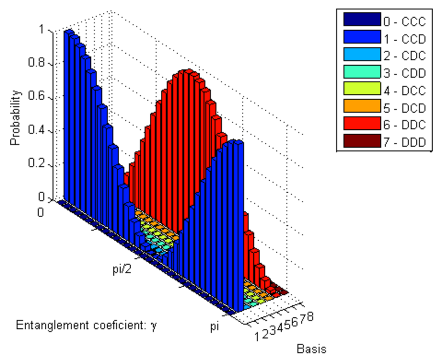
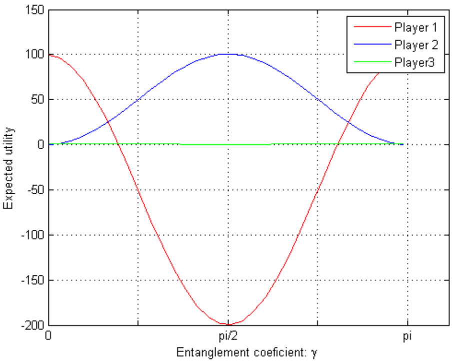


Table D.2: b) Expected utility for 3 players, where the players will use the $(Cooperate, Cooperate, Defect)$ operators. b1) Probability distribution of the final state depending on the entanglement coefficient γ.

c)



c1)

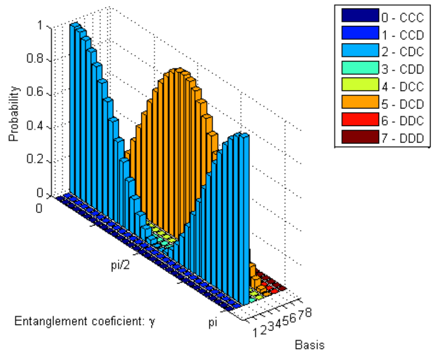
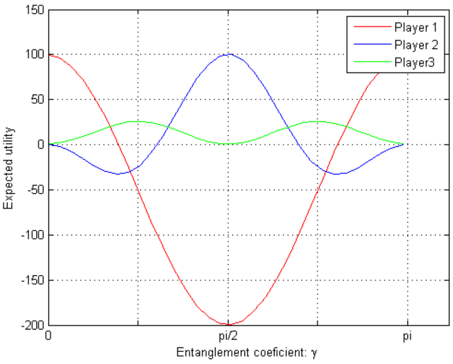


Table D.3: c) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Cooperate)$ operators. c1) Probability distribution of the final state depending on the entanglement coefficient γ.

d)



d1)

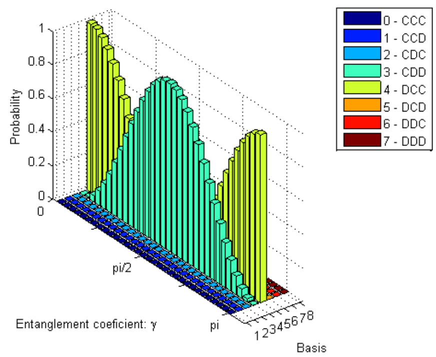
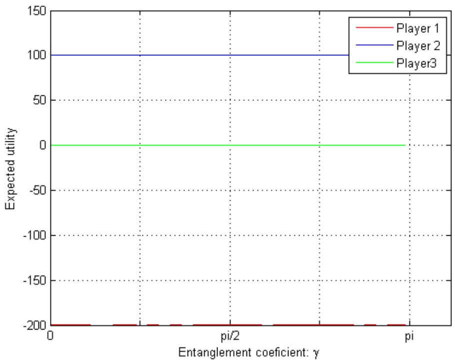


Table D.4: b) Expected utility for 3 players, where the players will use the $(Defect, Cooperate, Cooperate)$ operators. b1) Probability distribution of the final state depending on the entanglement coefficient γ.

D.1.2.B Initial proposal rejected; $(Cooperate, Defect, Defect)$

a)



a1)

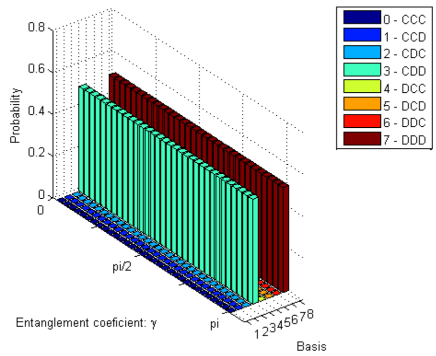
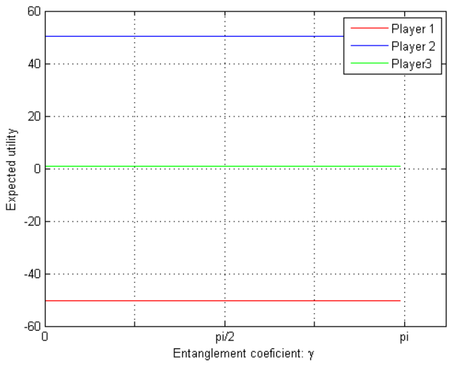


Table D.5: a) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CC).

b)



b1)

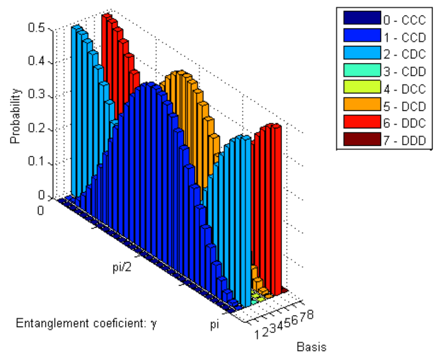
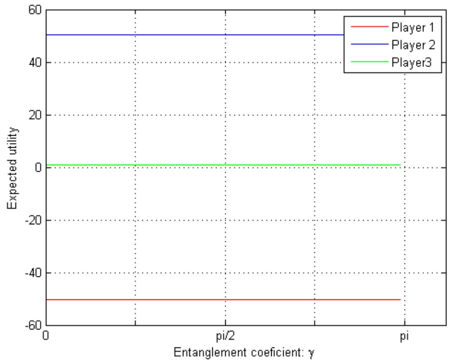


Table D.6: b) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CD).

c)



c1)

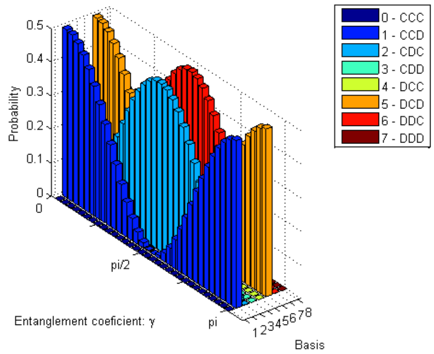
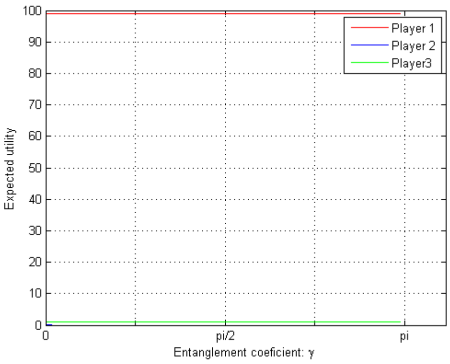


Table D.7: c) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DC).

d)



d1)

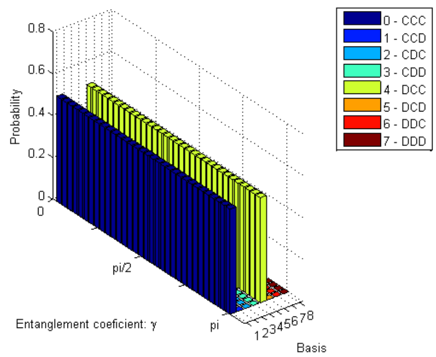
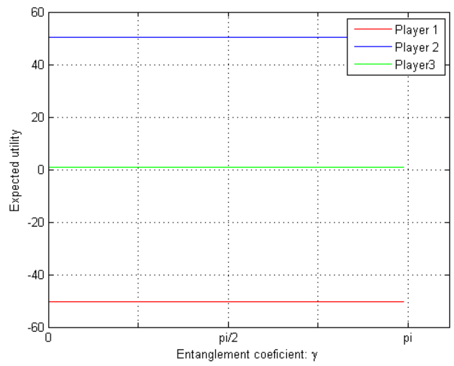


Table D.8: b) Expected utility for 3 players, where the players will use the $(Cooperate, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DD).

D.1.2.C Initial proposal rejected; $(Defect, Cooperate, Defect)$

a)



a1)

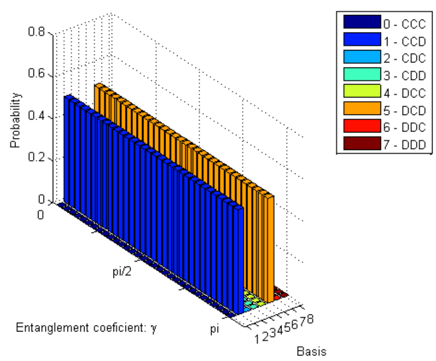
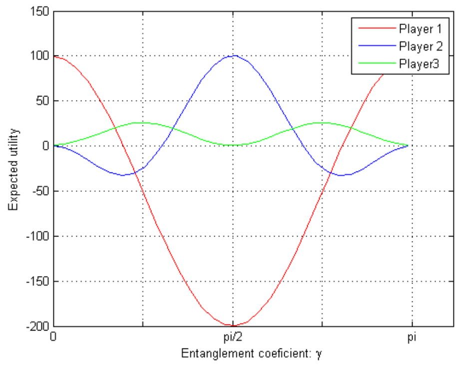


Table D.9: a) Expected utility for 3 players, where the players will use the $(Defect, Cooperate, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CC).

b)



b1)

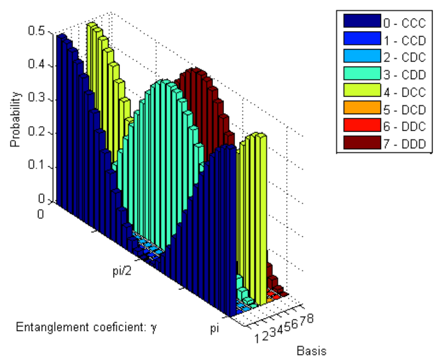


Table D.10: b) Expected utility for 3 players, where the players will use the $(Defect, Cooperate, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CD).

c) c1)

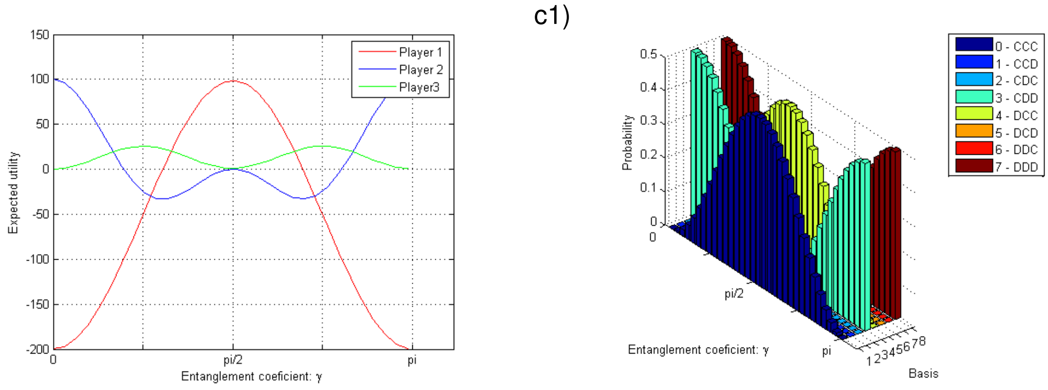


Table D.11: c) Expected utility for 3 players, where the players will use the $(Defect, Cooperate, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DC).

d) d1)

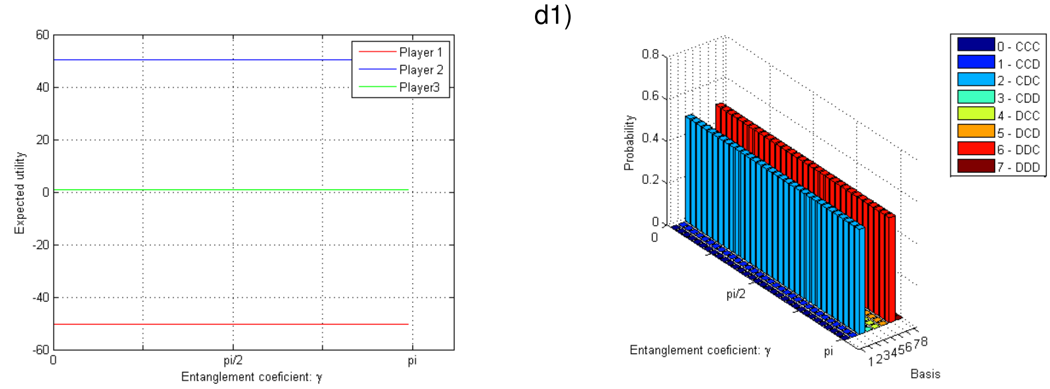
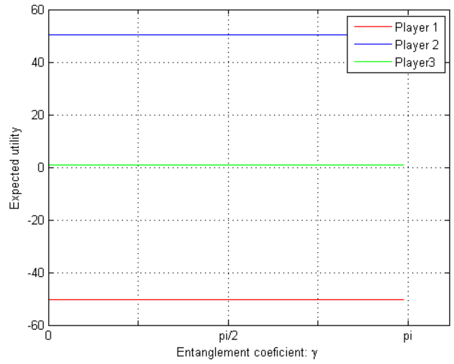


Table D.12: b) Expected utility for 3 players, where the players will use the $(Defect, Cooperate, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DD).

D.1.2.D Initial proposal rejected; $(Defect, Defect, Cooperate)$

a)



a1)

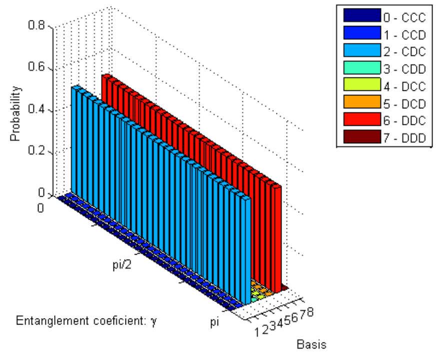
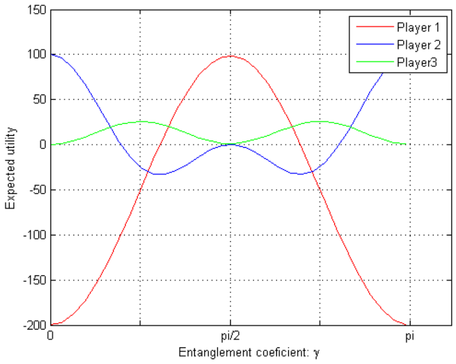


Table D.13: a) Expected utility for 3 players, where the players will use the $(Defect, Defect, Cooperate)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CC).

b)



b1)

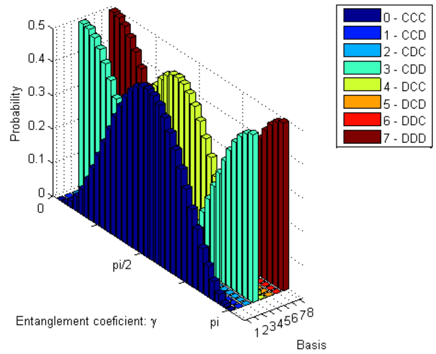
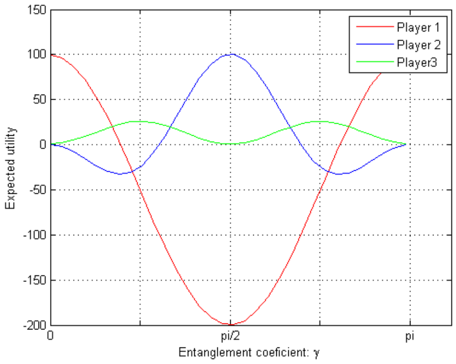


Table D.14: b) Expected utility for 3 players, where the players will use the $(Defect, Defect, Cooperate)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CD).

c)



c1)

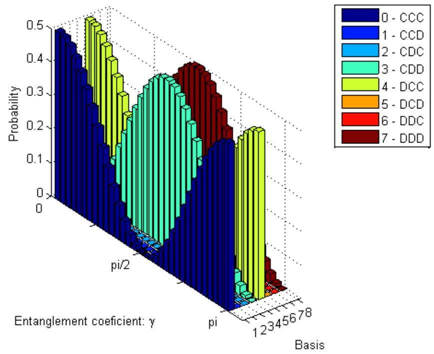
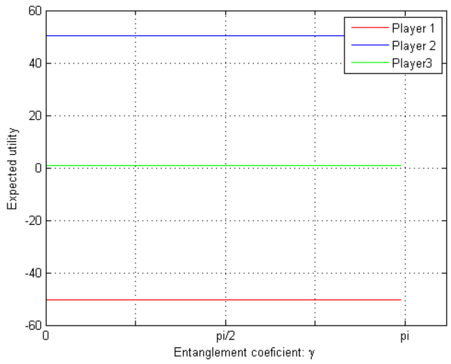


Table D.15: c) Expected utility for 3 players, where the players will use the $(Defect, Defect, Cooperate)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DC).

d)



d1)

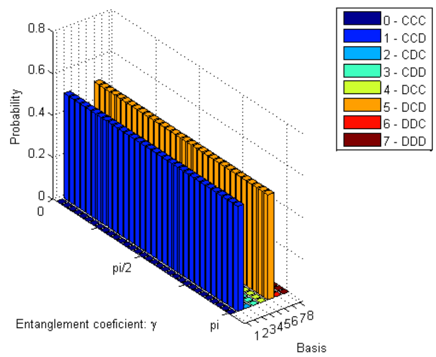
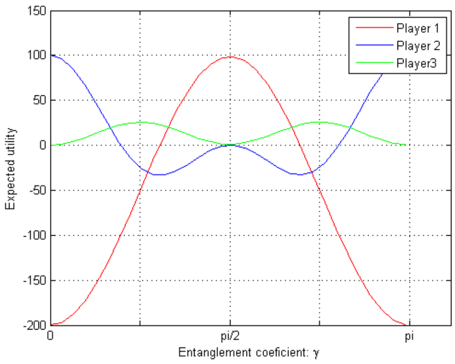


Table D.16: b) Expected utility for 3 players, where the players will use the $(Defect, Defect, Cooperate)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DD).

D.1.2.E Initial proposal rejected; $(Defect, Defect, Defect)$

a)



a1)

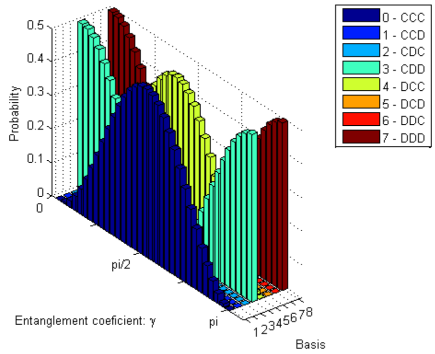
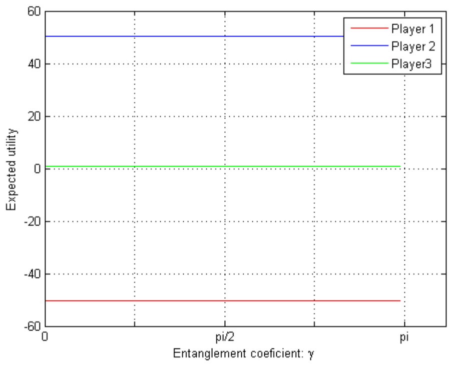


Table D.17: a) Expected utility for 3 players, where the players will use the $(Defect, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CC).

b)



b1)

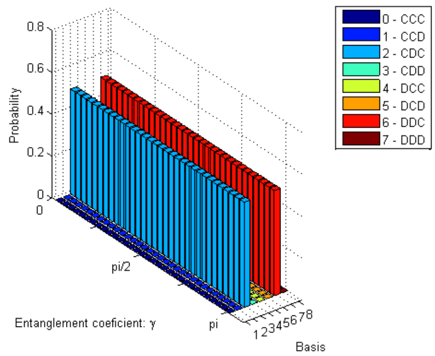
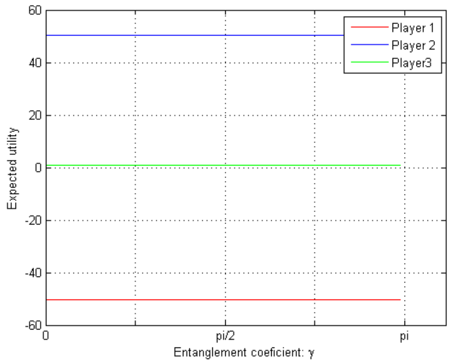


Table D.18: b) Expected utility for 3 players, where the players will use the $(Defect, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (CD).

c)



c1)

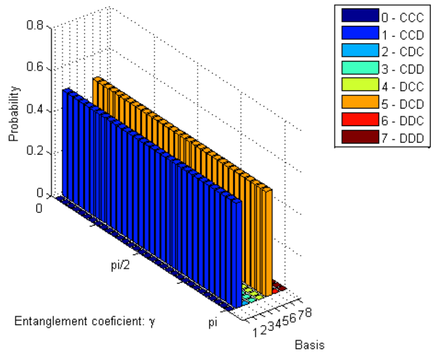
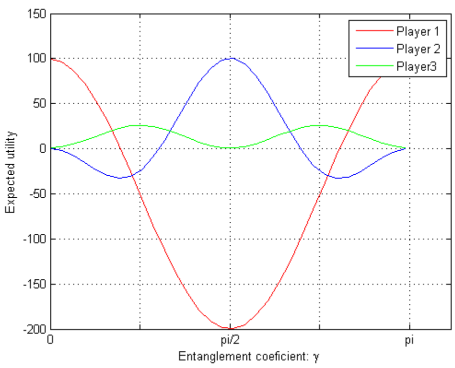


Table D.19: c) Expected utility for 3 players, where the players will use the $(Defect, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DC).

d)



d1)

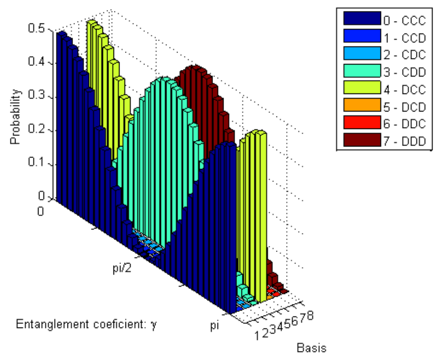


Table D.20: b) Expected utility for 3 players, where the players will use the $(Defect, Defect, Defect)$ operators in the first round of the game; in the second round player 2 and player 3 will play (DD).