



## **Quantum Pirates**

A Quantum Game-Theory Approach to The Pirate's Game

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***I can live with doubt and uncertainty and not knowing. I think it is much more interesting to live not knowing than to have answers that might be wrong. If we will only allow that, as we progress, we remain unsure, we will leave opportunities for alternatives. We will not become enthusiastic for the fact, the knowledge, the absolute truth of the day, but remain always uncertain... In order to make progress, one must leave the door to the unknown ajar.***

Richard P. Feynman



# Acknowledgments

This work marks the end of an arc in a personal story. The story of a path started naively out of curiosity for the world around me, a path filled with hardships, a path filled with good moments.

This journey was marked by self-doubt, but it was also rich in moments where wonderful travel companions helped. I want to thank all these travel companions. In particular I give thanks to my loved ones who never ceased believing even when I doubted myself. I would like to thank my advisor, Professor Andreas Wichert, for giving me the freedom to fail and learn from my mistakes, while teaching me the meaning of the word Scientist. I would also like to thank my co-workers at Connect Coimbra for teaching me how to balance work and life, and the desire to make a difference in my community. I would also like to thank Professor Chenyong Ju for his openness.



# Abstract

## Keywords

Quantum Game Theory

Pirate Game

Quantum Mechanics

Game Theory

Probability Theory





# Resumo

## Palavras Chave

Teoria de Jogos Quântica

Jogo dos Piratas

Mecânica Quântica

Teoria de Jogos

Teoria de Probabilidade



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# Abbreviations

**CLT** - Central Limit Theorem



# 1

## Introduction

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## Foreword

### 1.1 Motivation

It's not always easy to understand and introduce new paradigms. Some things we find natural nowadays, were once the source of controversy. Taking as an example the number zero 0, as numbers were introduced to help count physical objects, the idea of representing “nothingness” was once considered strange. In the beginning there were numerous ways devised to deal with this mathematical inconvenience, a special case. However the need to use “zero” as a number in its own right led to the popularization of its concept as a number and opened the door to new breakthroughs in mathematics [1].

Quantum mechanics shared this same problem and the mathematical formalization grew out of the need to explain phenomena [2]. Nowadays we currently accept quantum mechanics as a tool, while still trying to understand why it works at an epistemological level.

The idea of trying to do some contribution in quantum computing arose from the desire to explore this paradigm. The standard curriculum of a computer scientist undergraduate is focused on the current computational paradigm, which has its roots in von Neumann Architecture [3]. While the construction and the inner workings of computational systems that rely on new paradigms may fall outside of the scope of computer science, understanding and taking advantage from the point of view of Information Technology and pushing boundaries on model representation are areas where computer scientists might contribute.

### 1.2 Problem Description

The Pirate Game is a mathematical problem. We want to explore it in the light of quantum game theory.

### 1.3 Objectives

Despite not having a clear “real-world” application modelling games with quantum mechanics rules may aid the development of new algorithms that would be ideally deployed using quantum computing.

Furthermore applying quantum probabilities to a well established area as Game Theory, which has applications in fields such as Economy, Political Science, Psychology, Biology, might introduce new insights

and even relevant practical applications [4].

## **1.4 Contributions**

[TO DO]

- Provide a comprehensive state of the art and related work with examples and executable simulations, in order to facilitate the entry in this field.
- Quantum Model of one unique problem.
- Study of the possible outcomes, given the initial setup in a quantum game.

## **1.5 Thesis Outline**

[TO DO]



# 2

## Background

### Contents

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*In this section we present the theoretical concepts needed in order to understand this work.*

## 2.1 Bayesian Probability

The probability theory has its roots in the 16<sup>th</sup> century with attempts to analyze games of chance by Cardano. It is not hard to understand why games of chance are in the foundations of the probability theory. Throughout History the concept of probability has fascinated the Human being. Luck, fate were words that reflect things we feel we have no control upon and are associated with those games, and almost paradoxically we evolved in a way that we do not feel comfortable around them.

### 2.1.0.A Kolmogorov axioms

In spite of the fact that the problem of games of chance kept attracting numerous mathematicians (with some of the most influential ones being Fermat, Pascal and Laplace), it was not until the 20<sup>th</sup> century that the Russian mathematician Kolmogorov laid the foundations of the modern probability theory (first published in 1933) introducing three axioms [5]:

1. The probability of an event is a non-negative real number:

$$P(A) \in \mathbb{R} \wedge P(A) \geq 0 \quad (2.1)$$

This number represents the likelihood of that event happening, the greater the probability the more certain is its associated outcome.

2. The sum of probabilities of all possible outcomes in a space is always 1 ( $P(\Omega) = 1$ ). These first two axioms leave us with the corollary that probabilities are bounded:

$$0 \leq P(A) \leq 1 \quad (2.2)$$

3. The probability of a sequence of pairwise disjoint events is the sum these events. A corollary of this axiom is:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B) \quad (2.3)$$

### 2.1.0.B Conditional Probability

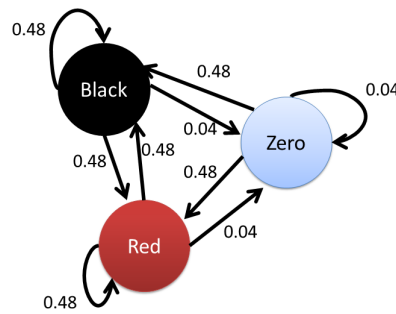
When some evidence is presented we have what we can conditional probability (or posterior probability). Conditional probability is represented as  $P(A|B)$ , that could be read as: the probability of A after evidence B is presented.

The product rule is used to calculate posterior probability 2.4.

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \quad (2.4)$$

### 2.1.1 Markov Chains

Markov Chains define a system in terms of states and the probabilistic transitions from one state to another.



**Figure 2.1:** Markov Chain of a perspective on a roulette.

In 1913, at Monte Carlo Casino (Monaco), black came up twenty-six times in succession in roulette. Knowing that the probability of the ball landing on a red or on a black house is approximately 0.48 (the zero is a neutral house), many a gambler lost enormous sums while betting red as they believed that the roulette was ripe with red, given the history. Players didn't want to believe that this and insisted that the roulette was biased; this became known as the Gamblers fallacy.

In the Markov Chain represented on Figure 2.1, we can see that in the state black the probability of transitioning to red is the same as to stay on the state black.

Given a system represented by the states  $\{x_0, x_1, \dots, x_i, \dots, x_n\}$ , and considering  $p_{ij}$  the probability of being in the state  $j$  and transitioning to the state  $i$ , the mixed state vector 2.5, which represents the probabilities of the system in the that  $i$  to transition to the other states. .

$$\vec{x}_i = \{p_{0i}x_0, p_{1i}x_1, \dots, p_{ii}x_i, \dots, p_{ni}x_n\} \quad (2.5)$$

The law of total probability is verified as  $\sum_{j=0}^n p_{ji} = 1$ , by specifying the every transition we get a stochastic matrix  $P$ , named the Markov matrix.

To illustrate how to construct a Markov Chain we will pick up on the example of Figure 2.1.

In this simplification of the Roulette we have 3 states:

- Black (B);
- Red (R);
- Zero (0).

We indifferently assign an index to each state, in order to construct the mixed state vector as in 2.5. Having the mixed state vectors defined the next step is to use them to create the stochastic matrix that has specified every transition 2.6.

$$R = \begin{bmatrix} p_{BB} & p_{BR} & p_{B0} \\ p_{RB} & p_{RR} & p_{R0} \\ p_{0B} & p_{0R} & p_{00} \end{bmatrix} = \begin{bmatrix} 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \\ 0.48 & 0.48 & 0.04 \end{bmatrix} \quad (2.6)$$

## 2.2 Von Neumann Probability

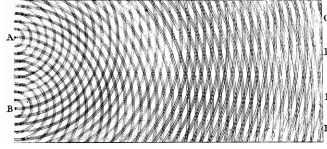
In the beginning on the 20<sup>th</sup> century the nature of light was once again in the spotlight. The question whether light would be a particle (corpuscular theory), or a wave (undulatory theory), was posed throughout History. Newton, notoriously, considered light to be a particle and presented arguments such as the fact that light travels in a straight line, not bending when presented with obstacles, unlike waves, and gave an interpretation of the diffraction mechanism by resorting to a special medium (aether), where the light corpuscles could create a localized wave [6].

The idea of light as a particle stood up until the 18<sup>th</sup> century as many scientists (Robert Hooke, Christian Huygens and Leonhard Euler to name a few) tried to explain contradictions found in corpuscular theory. This brought back the idea that light behaves like a wave.

One of the most famous experiments that corroborates the undulatory theory is the Young's experiments (19<sup>th</sup> century), or the double-slit interferometer.

The apparatus for the double-slit experiment can be seen in Figure 2.2. A light source is placed in such a way that "two portions" of light arrive at same time at the slits. Behind the barrier is a "wall" placed to intercept the light. The light captured at a wall will sport an interference pattern similar to the pattern

<sup>0</sup>Source: Young, Thomas: Probability. [http://en.wikipedia.org/wiki/File:Young\\_Diffraction.png\(1803\)](http://en.wikipedia.org/wiki/File:Young_Diffraction.png(1803))



**Figure 2.2:** Thomas Young's sketch of two-slit diffraction of light.

when two waves interfere. The double-slit experiment was considered for a while the full stop on the discussion on the nature of light. However with experiments on the spectres of the light emitted by diverse substances and its relation with temperature, a new problem was posed.

The black body radiation problem was the theoretical problem where a body that absorbs light in all the electromagnetic spectrum, this makes the body acting as an natural vibrator, where each mode would have the same energy, according to the classic theory.

When a black body is at a determined temperature the frequency of the radiation it emits depends on the temperature. The classic theory predicted that most of the energy of the body would be in the high frequency part of the spectrum (violet part) where most modes would be found, this led to a prediction called the ultraviolet catastrophe. According to the classic theory the black body would emit radiation with an infinite power for temperatures above approximately 5000K. Max Plank(1901), provided an explanation where the light was exchanged in discrete amounts called quanta, so that each frequency would only have specific levels of energy. Plank also determined through experimentation the value of the energy of the quanta that became known as photons later, that value became the physical constant called Plank constant:

$$h = 6.62606957(29) \times 10^{-34} J.s \quad (2.7)$$

In 1905, Einstein used the concept of quanta (photons) to explain the photoelectric effect. De Broglie(1924), suggested that all the matter had a wave-particle duality. This prediction was confirmed by studying the interference patterns caused by electron diffraction.

## 2.2.1 Mathematical Foundations of Quantum Probability

As previously explained, Quantum Theory is a branch of physics that has arised from the need to explain certain phenomena that could not be explained with the current classical theory. In the beginning of the 20<sup>th</sup> century Dirac and von Neumann helped to create the mathematical formalisms for this theory [7] [8].

Von Neumann's contributions were focused in the mathematical rigor, as is framework is strongly based

in Hilbert's theory of operators. Dirac's concerns were more of a practical nature. Their combined contributions were invaluable to establish this area.

From Dirac it is important to point the Dirac's notation (also known as Bra-ket notation or  $\langle Bra|c|ket \rangle$ ) (introduced in 1939 [7]), that is widely used in literature based on quantum theory. This notation uses angle brackets and vertical bars to represent quantum states (or abstract vectors) as it can be seen in the formulas (2.8) and (2.9).

$$\langle z| = [ z_1^* \quad z_2^* \quad \dots \quad z_n^* ] \quad (2.8)$$

$$|z\rangle = (\langle z|)^* = \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{bmatrix} \quad (2.9)$$

This notation provides for an elegant representation of the inner product (2.10) and verifies the has linearity as it follows in equation (2.11). While the Bra-ket notation can be useful in terms of condensing information, using vectors and matrices to represent the states turns out to be a more approachable way to understand and manipulate data.

$$\langle z|z\rangle = \sum_{i=1}^n \bar{z}_i z_i \quad (2.10)$$

where  $\bar{z}_i$  is the complex conjugate of  $z_i$ .

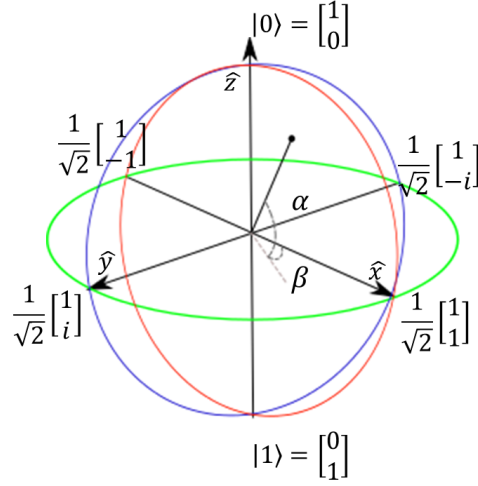
$$\langle z|(\alpha|x\rangle + \beta|y\rangle) = \alpha\langle z|x\rangle + \beta\langle z|y\rangle \quad (2.11)$$

### 2.2.1.A Born rule

The Born rule was formulated by Born in 1926. This law allows to predict the probability that a measurement on a quantum system will yield a certain result. This law provides a link between the mathematical foundation of Quantum Mechanics and the experimental evidence [9] [10].

A quantum system is represented by a  $n$ -dimensional Hilbert Space, a complex vector space in which the inner product is defined.

A two-dimensional Hilbert Space (corresponding to a qubit for example), can be represented by a Bloch



**Figure 2.3:** Representation of a two-dimensional Hilbert Space( $\mathcal{H}^2$ )

Sphere. In the Figure 2.3 we have a representation of quantum state  $|v\rangle$  in a two-dimensional Hilbert Space.

The Born rule states that if there is a system is in a state  $|v\rangle$  (in a given  $n$ -dimensional Hilbert Space  $\mathcal{H}$ ), and an Hermitian operator  $A$  is applied then the probability of measuring a specific eigenvalue  $\lambda_i$  associated with the  $i$ -th eigenvector of  $A$  ( $\psi_i$ ), will be given by [9]:

$$P_v(\lambda_i) = \langle v | Proj_i | v \rangle \quad (2.12)$$

where  $Proj_i$  is a projection matrix corresponding to  $\psi_i$  :

$$Proj_i = |\psi_i\rangle\langle\psi_i| \quad (2.13)$$

Given the properties of  $A$ , the set of eigenvectors  $\{\psi_1, \psi_2, \dots, \psi_i, \dots, \psi_n\}$  forms a orthogonal basis of the  $n$ -dimensional Hilbert Space considered. Thus the state  $|v\rangle$  can be written as a linear combination of the eigenvectors of  $A$ :

$$|v\rangle = \alpha_1\psi_1 + \alpha_2\psi_2 + \dots + \alpha_i\psi_i + \dots + \alpha_n\psi_n \quad (2.14)$$

The coefficients  $\alpha_i$  are complex numbers called probability amplitudes, and their squared sum is equal to 1:

$$\sum_{i=0}^n |\alpha_i|^2 = 1 \quad (2.15)$$



this brings us to:

$$P_v(\lambda_i) = \langle v|\psi_i\rangle\langle\psi_i|v\rangle = |\langle v|\psi_i\rangle|^2 = |\alpha_i^*\alpha_i|^2 \quad (2.16)$$

So the determination of the probability of an event ( $P(A)$ ), is made by projecting the quantum state on the eigenvectors corresponding to the operator  $A$  of the Hilbert Space and measuring the squared length of the projection. [11]

$$P(A) = (\text{Proj}_A|z\rangle)^2 \quad (2.17)$$

Applying an operator can be seen as applying a rotation matrix on the system ( $|v\rangle$ ) and measuring the projection of  $|v\rangle$  onto the imaginary axis and the real axis (Figure 2.3), or considering that we have a determined state vector and rotating the orthogonal basis of the Hilbert Space according to an operator and then to do a projection on the new chosen orthogonal basis.

According to Leiffer [12] “quantum theory can be thought of as a non-commutative, operator-valued, generalization of classical probability theory”.

As in the classical probability theory where from a random variable it is possible to establish a probability distribution, also known as density function, in the Hilbert space there is a equivalent density operator. The density operator ( $\rho$ ) is a Hermitian operator that has the particularity of having its trace equal to 1 [9].

$$\rho = \sum_{i=0}^n \alpha_i |\psi_i\rangle\langle\psi_i| \quad (2.18)$$

$$\text{tr}(\rho) = 1 \quad (2.19)$$

### 2.2.1.B Lüders Rule

Lüders Rule defines the state of a quantum system after a partial measurement. We can establish a parallel between this selective measurement and conditional probability 2.1.0.B [13].

To measure the state of a system we first use a projection operator on the quantum state. This is also the first stage of applying Lüders Rule 2.2.0.

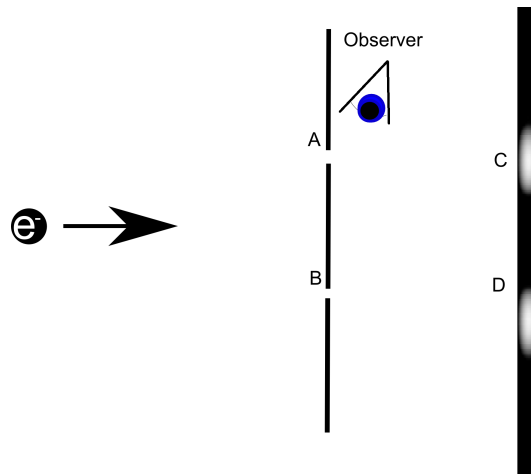
$$A = Proj_A |s\rangle \quad (2.20)$$

After the measurement the resulting state is normalized<sup>2.21</sup>.

$$|s_A\rangle = \frac{A}{|A|} \quad (2.21)$$

### 2.2.1.C Example of the double-slit experiment with electrons

Like the Young's Experiment with light created an interference pattern similar to a wave, firing electrons one at the time produces a similar pattern. The unobserved fired electron behaved like a wave and after passing the slits the wavelets interfered with one another to create a interference pattern. However if a measuring device was active while the electron was fired the interference pattern wasn't registered.



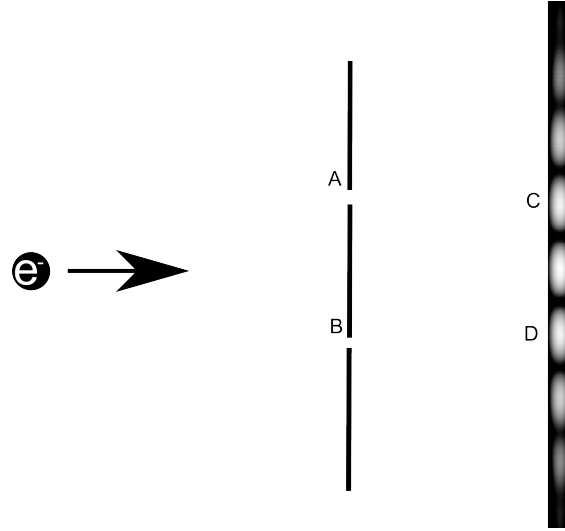
**Figure 2.4:** Double-slit experiment where there is a measuring device that allows to know through which slit the electron passed.

The fact that the electron was measured while passing through a slit produced a particle behaviour, explained by the classical theory (Figure 2.4).

In this experiment a single electron is shoot at a time. So in the start of the experiment (S), we know the initial position of the electron.

A final measurement (F) is made when the electron hits the wall behind the slits, where we know the final position of the electron. <sup>1</sup>

<sup>1</sup>Mohrhoff, U.: Two Slits. <http://thisquantumworld.com/wp/the-mystique-of-quantum-mechanics/two-slit-experiment/#fn1back>



**Figure 2.5:** Double-slit experiment, where electrons exhibit the interference pattern characteristic in waves

If this experiment is observed, there is an intermediate measure that tells us whether the electron went through the slit A or B. The corresponding probability amplitudes related to this measurement are  $\omega_A$  and  $\omega_B$ , and:

$$\omega_A = \langle F|A\rangle\langle A|S\rangle \quad (2.22)$$

$$\omega_B = \langle F|B\rangle\langle B|S\rangle \quad (2.23)$$

If we consider the intermediate measurement the probability  $P(F|S)$  will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle|^2 + |\langle F|B\rangle\langle B|S\rangle|^2 \quad (2.24)$$

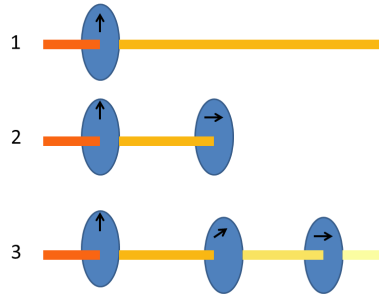
But if we only measure the position of the electron at the end of the experiment that probability will be:

$$P(F|S) = |\langle F|A\rangle\langle A|S\rangle + \langle F|B\rangle\langle B|S\rangle|^2 \quad (2.25)$$

The latter equation will be dependent on an interference coefficient that will be responsible for the interference pattern observed in the unobserved experiment.

#### 2.2.1.D Example of the Polarization of Light

The photons in a beam of light don't vibrate all the same direction in most of the natural sources of light. To filter the light polaroids are used. A polaroid only allows the passage of light in a well-defined direction and thus reducing the intensity of the light. In Figure 2.6 we can observe that the introduction of the oblique polaroid in the third situation led to a passage of light. Although there is a classical explanation to this phenomenon if we consider waves when we are considering a beam of light, if our light source emits one photon at the time a quantum mechanical explanation is needed [14].



**Figure 2.6:** 1. With one vertical polaroid the unpolarized light is attenuated by a half. 2. Vertical polarization followed by a horizontal polarization will block all the passing light. 3. Inserting a oblique polaroid between the vertical and horizontal polaroids will allow light to pass.

To model the polarization of the photon in a quantum setting, we will use a vector  $|v\rangle$  in a two-dimensional Hilbert Space:

$$|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (2.26)$$

where  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  would represent the vertical direction (could also be represented by the state vector  $|\uparrow\rangle$ ), and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  the horizontal one (another possible representation to this basis could be  $|\rightarrow\rangle$ ). We can consider the Figure 2.3 as a graphical representation for this system.

In the first situation if  $a = \frac{1}{\sqrt{2}}$ , that would mean that the probability of passing the vertical polaroid would be  $a = (\frac{1}{\sqrt{2}})^2 = 0.5$ , that would light to the expected reduction of a half of the intensity of light. After passing through the vertical polaroid the photon will have a polarization of  $|v\rangle = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Considering now the second situation after we have our photon polarized vertically (like in the end on the first situation), the probability of being vertically polarized is 1, thus making the probability of passing through the horizontal polaroid 0. In the third situation, after being vertically polarized the photon will pass through an oblique polaroid that makes its direction

$$|v\rangle = \cos(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i.\sin(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (2.27)$$

$\theta$  being the angle of the polaroid. The photon filtered by the vertical polaroid will pass this second polaroid with a probability of  $(\cos(\theta))^2$ , becoming polarized according to the filter, as we can observe depending on the value of  $\theta$  we will now have a horizontal component in the vector that describes the state of the photon. This will make the photon pass the horizontal polaroid with a probability of  $i.\sin(\theta)^2$ .

## 2.3 Quantum Computing

The quantum equivalent of a bit (the basic unit of information in computers), is a qubit (or quantum bit). A qubit is a two-state quantum system that can be interpreted as normalized vectors in a two-dimensional Hilbert space. This Hilbert Space( $\mathcal{H}^2$ ), an element in that space can be uniquely specified by resorting to two orthonormal basis (also known as pure states). In quantum computing, the basis used are  $|0\rangle, |1\rangle$  or  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

### 2.3.1 Superposition

The qubit can be described in terms of linear transformations of pure states as in 2.28,  $\omega_0$  and  $\omega_1$  are complex numbers called probability amplitudes. When a system is in a mixture of this pure states (as the general representation in 2.28 might point), it is in a phenomenon known as superposition. Measuring forces the system to collapse and assume one of the pure states with a certain probability.

$$|\psi\rangle = \omega_0|0\rangle + \omega_1|1\rangle = \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix} \quad (2.28)$$

In the example stated in the representation 2.28 the probability of the system falling into the state  $|0\rangle$  would be  $|\omega_0|^2$ . The probability of the system falling into  $|1\rangle$  would be  $|\omega_1|^2$ . The second axiom of Kolmogorov (law of total probability) is verified (2.29).

$$|\omega_0|^2 + |\omega_1|^2 = 1 \quad (2.29)$$

$\omega_0$  and  $\omega_1$  (2.28) are complex numbers, the so-called probability amplitudes. When squared, the probability amplitude represents a probability.

### 2.3.2 Operators

In order to perform transformations in the qubit systems, we can define linear operators in the Hilbert space. One of the most important classes of operators being the self-adjoint operators,  $A = A^*$ , that have the property stated in 2.30. The Hermitian operator is one that satisfies the property of being equal to its conjugate transposed,  $A = A^{*T} = A^\dagger$ . In a finite-dimensional Hilbert space defined by a set of

orthonormal basis every self-adjoint operator is Hermitian. Unitary and Hermitian operators are used in the majority of quantum algorithms because they can be reversible.

When presented with  $q$  qubits, a  $q$ -dimensional square matrix is called a Quantum Gate.

$$\langle A^* z | x \rangle = \langle z | Ax \rangle \quad (2.30)$$

Some important Quantum Gates that operate on a single qubit are:

- The Identity Matrix 2.31.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.31)$$

- The Pauli Operators. These operators are used to perform the NOT operation.

- The Bit-Flip Operator

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2.32)$$

- The Phase-Flip Operator

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (2.33)$$

- The Bit and Phase-Flip Operator

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (2.34)$$

- The Hadamard Gate, belongs to a general class of Fourier Transforms. This  $2 \times 2$  particular case is a Discrete Fourier Transform matrix.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (2.35)$$

### 2.3.3 Compound Systems

The representation of a system comprising multiple qubits grows exponentially. If to represent a single qubit system there is a two-dimensional Hilbert space, to represent a system with  $m$  qubits a  $2^m$ -dimension space would be required. To represent a higher dimension multiple-qubit system one can perform a tensor product of single-qubit and/or lower dimension systems.

The tensor product is an operation denoted by the symbol  $\otimes$ . Given two vector spaces  $V$  and  $W$  with basis 2.36 and 2.37 respectively, their tensor product would be the  $mn$ -dimensional vector space with a basis with elements of the form 2.38 [14].

$$A = \{|\alpha_1\rangle, |\alpha_2\rangle, \dots, |\alpha_m\rangle\} \quad (2.36)$$

$$B = \{|\beta_1\rangle, |\beta_2\rangle, \dots, |\beta_n\rangle\} \quad (2.37)$$

$$|\alpha_i\rangle \otimes |\beta_j\rangle \quad (2.38)$$

For example, if we consider two Hilbert spaces  $\mathcal{H}^2$  with basis  $A = \{|0\rangle, |1\rangle\}$  and  $B = \{|-\rangle, |+\rangle\}$ , their tensor product would be a  $\mathcal{H}^4$  with basis 2.39.

$$AB = \{|0-\rangle, |0+\rangle, |1-\rangle, |1+\rangle\} \quad (2.39)$$

Now taking the former Hilbert spaces, supposing we have the qubits 2.40 and 2.41.

$$|v\rangle = a_0|0\rangle + a_1|1\rangle \quad (2.40)$$

$$|w\rangle = b_0|-\rangle + b_1|+\rangle \quad (2.41)$$

$$|v\rangle \otimes |w\rangle = a_0b_0|0-\rangle + a_0b_1|0+\rangle + a_1b_0|1-\rangle + a_1b_1|1+\rangle \quad (2.42)$$

The Bra-ket notation provides a way to prevent the escalation of the basis notation. When specified the vector space the basis can be specified in base 10 for simplicity sake. According to this the basis of the last example would be  $AB = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$  and a system with 3-qubit could be represented in a Hilbert space with basis  $H = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}$ .

### 2.3.4 Entanglement

In multiple qubit systems, qubits can interfere with each other, thus making impossible to determine the state of part of the system without “disturbing” the whole. In other words, there are states in a multi-qubit system that cannot be described as a probabilistic mixture of the tensor product of single-qubit systems; when this happens this state is not separable (with respect to the tensor product decomposition), this

phenomenon is called quantum entanglement [14]. If a mixed quantum state  $\psi$  in a quantum system constructed by the tensor product of  $n$  qubits is separable it can be written as:

$$|\psi\rangle = \sum_{j=1}^m p_j |\varphi_j^1\rangle \langle \varphi_j^1| \otimes \dots \otimes |\varphi_j^n\rangle \langle \varphi_j^n|, \sum_i p_i = 1 \quad (2.43)$$

This property is not local as transformations that act separately in different parts of an entangled system cannot break the entanglement. Quantum entanglement is one of the main differences from the classical theory [14].

For example, supposing we consider the following quantum states, known as Bell states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad (2.44)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad (2.45)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad (2.46)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \quad (2.47)$$

These states form a particular basis in  $\mathcal{H}^4$  as they are all entangled states and they are maximally entangled. If we have a system of two qubits in a Hilbert space in the mixed state  $|\Phi^+\rangle$  and we measure the qubit A (by deciding the outcome 0 or 1 with a probability of 0.5 for each), we are automatically uncovering the value of the qubit B. In this state we know the second qubit will always yield the same value of the first measured qubit. Se second value is correlated with the first one.

## 2.4 Game Theory

Game Theory did not exist in its own right before von Neumann published the article "On the Theory of Games of Strategy". This field deals mainly with the study of interactions between racional decision-makers [15]. Game Theory knows numerous applications in areas such as Economics, Political Science, Biology, and Artificial Intelligence.



### 2.4.1 Definition of a Game

A game  $(\Gamma)$ , is a model of conflict between players characterized by [16] [17] [18]:

- A set of Players:  $P = \{1, 2, \dots, N\}$ .
- For each player there is set of Actions.
- There are preferences, for each player, over the set of Actions.

The preferences in the previous definitions are defined with resort to concept of utility, expected utility, or payoff derives from the theory that we can use real numbers (Equation 2.48), to model and represent the wants and needs of the players. This simplified mathematical model allows to compare different states and rank preferred outcomes. For example if  $u(x)$  is a utility function for a player, and  $u(A) > u(B)$ , that would mean that the player would strictly prefer  $A$  over  $B$ . The concept of expected utility is fundamental to analyse games, as a rational player would try to maximize her expected utility [15] [16].

There are two standard representations of games:

- Normal Form - lists what payoffs the players get as a function of their actions as if they all make their moves simultaneously. These games are usually represented by a matrix.
- Extensive Form - extensive form games can be represented by a tree and represent timed actions.

A finite game is a game that has a finite set of actions, a finite number of players, and it does not go on indefinitely.

A zero-sum game is a mathematical representation of a system where the gains are completely evened out by the losses; this means that the sum of the utilities of all players will always be zero.

$$u : X \rightarrow \mathbb{R} \tag{2.48}$$

### 2.4.2 Nash Equilibrium

When all players cannot improve their utility by changing their strategy unilaterally, we have an equilibrium point. This equilibrium point is named after John Nash, who proved that it exists at least one mixed

strategy Nash equilibrium in a finite game [19] [20]. This concept is used to analyse game where several decision makers interact simultaneously and the final outcome depends on the players strategy [16].

In a sequential game

### 2.4.3 Pareto Optimal

When every player cannot improve her payoff without lowering another player's expected utility we have a Pareto Optimal solution.

## 2.5 Quantum Game Theory

[4] [Intro SAUCES]

The definition of to normal representation of extensive games proposes a representation for finite extensive form games [18] [21]. The representation assumes that there are only two available actions, which can be represented by a qubit. Those actions could be a yes/no decision or a cooperate/defeat as those found in many classical game theory problems. A game in this form is represented by a six-tuple 2.49, where:

$$\Gamma = (\mathcal{H}^a, N, |\psi_{in}\rangle, \xi, \{\mathcal{U}_j\}, \{E_i\}) \quad (2.49)$$

- $a$  is the number of actions (qubits), in the game;
- $\mathcal{H}^a$  is a  $a$ -dimensional Hilbert space constructed as  $\otimes_{j=1}^a \mathbb{C}^2$ , with basis  $\mathcal{B}$ ;
- $|\psi_{in}\rangle$  is the initial state of the compound-system composed by  $a$  qubits:  $|\varphi_1\rangle, |\varphi_2\rangle, \dots, |\varphi_j\rangle, \dots, |\varphi_a\rangle$ ;
- $\xi$  is a mapping function that assigns each action to a player;
- For each qubit  $j$   $\mathcal{U}_j$  is a subset of unitary operators that can be used by the player to manipulate her qubit;
- Finally, for each player  $i$   $E_i$  is a utility functional that specifies her payoff. This is done by attributing a real numbered utility to the measurement of the projection of a basis in the final state 2.50.

$$E_i = \sum_{b \in \mathcal{B}} u_i(b) |\langle b | \psi_{fin} \rangle|^2, u_i(b) \in \mathbb{R} \quad (2.50)$$

The strategy of a player  $i$  is a map  $\tau_i$  which assigns a unitary operator  $U_j$  to every qubit  $j$  that is manipulated by the player ( $j \in \xi^{-1}(i)$ ). The simultaneous move is represented in 2.51.

$$|\psi_{fin}\rangle = \otimes_{i=1}^N \otimes_{j \in \xi^{-1}(i)} \mathcal{U}_j |\psi_{in}\rangle \quad (2.51)$$

## 2.6 Overview



# 3

## Related Work

### Contents

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*In this section we present relevant work that it is related with the problem. This constitutes an in-depth analysis of some important problems but also tries to present a broad representative view on the path towards building our solution.*

### 3.1 Quantum Walk on a Line

Quantum Walk is the quantum version of random walks, which are mathematical formalisms that describe a path composed of random steps. A Markov Chain might be used to describe these processes.

We can define the Discrete Quantum Walk on a Line as a series of Left/Right decisions. Understanding this algorithm is important towards being able to define and design more complex algorithms that make use of quantum properties.

We followed an approach suggested by [22] towards simulating n-steps of a Quantum Walk on a Line. The algorithm can be consulted on A. In a discrete quantum walk in a line we want to preserve the fact that the probability of turning left is equal to the probability of turning right. To represent a state in this algorithm we will need the number of the node and a direction (identified as L,R) 3.1.

$$|\psi\rangle = |n, L\rangle \quad (3.1)$$

With two equally possible direction choices in each step, we can use a coin metaphor [22] [23] to approach the decision. We toss a coin and go either Left or Right depending on the result.

In a quantum version we need to define a Coin Operator (Coin Matrix), which is responsible to imprint a direction to the current state. This operator is a unitary matrix in a 2-dimension Hilbert space. Some examples of Coin Operators are the Hadamard matrix 3.2 and a symmetric unitary matrix 3.3.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (3.2)$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad (3.3)$$

Taking the Hadamard matrix as an example 3.2, the coin matrix will operate on the state in the following way 3.4 3.5 [22].

$$C|n, L\rangle = \frac{1}{\sqrt{2}}|n, L\rangle + \frac{1}{\sqrt{2}}|n, R\rangle \quad (3.4)$$

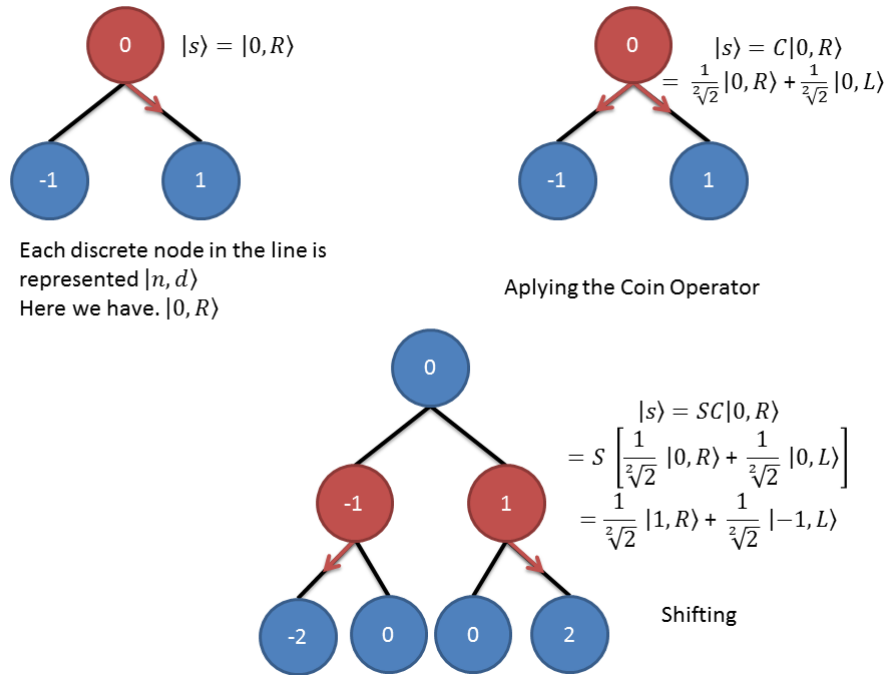
$$C|n, R\rangle = \frac{1}{\sqrt{2}}|n, L\rangle - \frac{1}{\sqrt{2}}|n, R\rangle \quad (3.5)$$

The Coin Matrix obtains its name by being the quantum equivalent of flipping a classic coin. After tossing a coin comes an operator that will move the node in the direction assigned. The operator responsible for this modification is commonly referred as Shift Operator 3.63.7.

$$S|n, L\rangle = \frac{1}{\sqrt{2}}|n-1, L\rangle \quad (3.6)$$

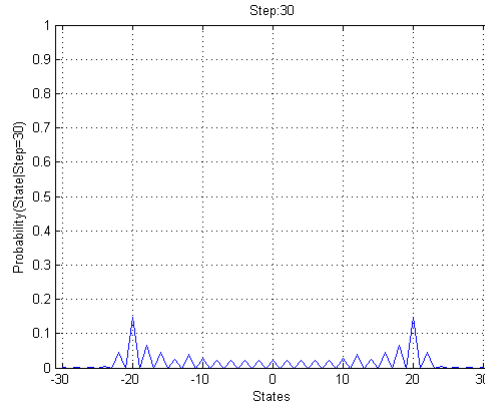
$$S|n, R\rangle = \frac{1}{\sqrt{2}}|n+1, R\rangle \quad (3.7)$$

These matrices (Coin Matrix and Shift Operator) are referred conceptually ubiquitously throughout the literature [?], therefore it is important to be familiar with them. A single step of the algorithm A is illustrated in Figure 3.1.

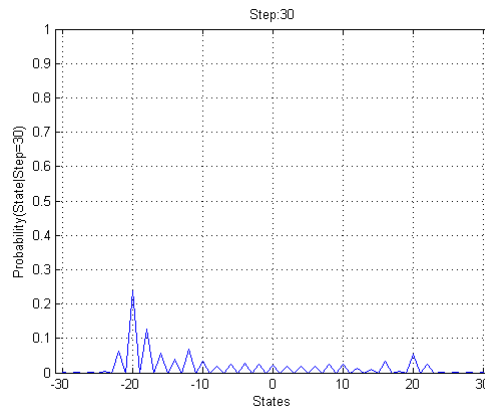


**Figure 3.1:** Simulating a step of a discrete quantum walk on a line. In the beginning we have a state characterized by the position (0) and a direction (either Left or Right).





**Figure 3.2:** 30 Step of the Simulation A using Matrix 3.2 as a Coin Operator.



**Figure 3.3:** 30 Step of the Simulation A using a Hadamard Matrix 3.2 as a Coin Operator.

If we took the classical approach in which we tossed a fair coin to decide to go either left or right, and after  $n$ -steps we measured the final node repeatedly, by the Central Limit Theorem (CLT) the final distribution would converge to a normal distribution. However, in the quantum approach, depending on the Coin Matrix we can get different distributions. In this simple problem we have the basis for some quantum algorithms.

In Figures 3.3 and 3.2, depending on the coin operator we get two different results. Despite that, on one moment the probability of shifting left, is always equal to the probability of shifting right. The main difference from the classical approach lies on the fact there is a quantum interference during the walk, when we measure the results after  $N$  steps, somewhat like the electrons interfere in the “double-slit experiment”, presented in Section 2.2.1.C.

## 3.2 Quantum Models

The rationale behind building a quantum version of a Game Theory and/or Statistics problem lays in bringing phenomena like quantum superposition, and entanglement into known frameworks allowing

different results, that seemingly perform at least as good as their classical versions.

[To Introduce more SOURCES]

The effort put in converting known classical problems also enables the familiarization with the potential differences these models bring.

### 3.2.1 Quantum Roulette

In the arbitrary  $N$ -State quantum roulette, [24] presented a  $N$ -State roulette model using permutation matrices.

This model is interesting because it captures the usage of permutation matrices to manipulate and change the state of the system.

To verify this model with two players we developed a Matlab simulation C, that followed the steps taken in [24].

The game is represented in a  $N$ -Dimensional Hilbert Space. There is a basis in the space that represents each of the equally probable entries as shown in 3.8. In a sense this is a generalization of a quantum coin flip that is also used in Section 3.1.

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, |N\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (3.8)$$

Each state transition is obtained using a permutation matrix denoted by  $P^i$ . There are  $N!$  permutation matrices, so in the particular case of having a 3-State roulette, there are 6 possible transition choices. The classical strategy considered will rely on choosing an arbitrary probability distribution, that verifies 3.10, and that maps the usage of the permutation matrices. This step will not affect the density matrix ( $\rho$ ) of the roulette 3.9.

$$\rho = \frac{1}{N!} \sum_{i=0}^{N!-1} P^i \quad (3.9)$$

$$\sum_{i=0}^{N!-1} p_i = 1 \quad (3.10)$$

The density matrix is diagonalizable by a Discrete Fourier Transform because it is a kind of circulant matrix [25], as we can see in 3.11. In 3.11  $\lambda_k$  are eigenvalues of  $\rho$ .  $\lambda_1 = 1$  while  $\lambda_2 = \lambda_3 = \lambda_k = \lambda_{N-1} = 0$ . Each column  $i$  of the Fourier matrix will represent an eigenvector  $|\lambda_i\rangle$ . If we construct the diagonalizing matrix by rotating the columns of the Fourier Matrix we can obtain the projection states as in 3.12.

$$F^\dagger \rho F = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N-1} \end{bmatrix} \quad (3.11)$$

$$|1\rangle\langle 1| = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = F^\dagger \rho F \quad (3.12)$$

The quantum strategy advantage in this case is that the first player will not alter the density matrix 3.13.

$$\rho = \sum_{i=0}^{N!-1} p_i P^i \rho P^{i\dagger}, \quad \sum_{i=0}^{N!-1} p_i = 1 \quad (3.13)$$

This means that if the second player knows the initial state and the first player plays with a classical strategy, thus never modifying the system density matrix, the second player will be able to manipulate the game under optimal conditions. This result confirms the demonstration done in [26], on Quantum Strategies, where in a classical 2 player zero-sum game, if one player adopts a quantum strategy, she increases her chances of winning the game.

### 3.2.2 Prisoner's Dillema

The Prisoner's Dillema is one example of a game that can be represented in normal form [16]. This problem has received a great deal of attention because, in its simple form, rational individuals will seem to deviate from solutions that would represent the best interest overall solution, the Pareto Optimal solution. In this game the pareto optimal solution is not a Nash Equilibrium. The Prisoners Dillema can be formulated as it follows:

Two suspects of being partners in a crime are arrested. The police needs more evidence in order to prosecute the prisoners. So each prisoner is locked in solitary confinement and

	Player 2: C	Player 2: D
Player 1: C	(R,R)	(S,T)
Player 1: D	(T,S)	(P,P)

**Table 3.1:** The canonical normal form representation for the Prisoner's Dillema must respect  $T > R > P > S$ .

	Player 2: C	Player 2: D
Player 1: C	(2,2)	(0,3)
Player 1: D	(3,0)	(1,1)

**Table 3.2:** One possible normal form representation of Prisoner's Dillema.

has no means of communicating with the other suspect. The police will then try to extort a confession from the prisoners. A bargain will be proposed to the suspects:

- If the suspect testifies against the other suspect (Defects) and the other denies, he will go free and the second will get three years sentence.
- If they both testify against one another (both Defect), both will be convicted and they will get two years.
- In case both suspects deny the involvement (both Cooperate) of the other, they will get a one year sentence.

A matrix representation of the problem is in Table 3.1, a particular case for this problem is represented in Table 3.2. In each cell of the matrix we have a pair of the expected utility for the players for every outcome. A higher utility represents a more desirable state.

In this game the Pareto Optimal solution happens when both players chose to Cooperate (the pair (2,2) in Table 3.2). However both players have the incentive to Defect, because regardless what the opponent choses they will have always a strictly higher payoff. Defecting becomes a dominant strategy in the Prisoner's Dillema and the outcome (Defect, Defect) is a Nash Equilibrium to the game.

### 3.2.2.A Quantum Prisoner's Dillema

The importance of the Prisoner's Dillema for the study of Game Theory made it a prime target for investigation in Quantum Game Theory. The problem has been modelled several times [4] [21]. Therefore we will use it iin order to exemplify and consolidate the definition described in Section 2.5 [21].

$\otimes$	C- $ 0\rangle$	D- $ 1\rangle$
C- $ 0\rangle$	$ 0, 0\rangle$	$ 0, 1\rangle$
D- $ 1\rangle$	$ 1, 0\rangle$	$ 1, 1\rangle$

**Table 3.3:** Construction of the basis for the game space;  $\mathcal{H}^4$ .

Each player  $i$  in the quantum version of Prisoner's Dillema will be able to manipulate one qubit ( $\varphi_1$  and  $\varphi_2$ )3.143.15, with two possible operators (shown in 3.16) corresponding to the classical actions: Cooperate (C), and Defect (D).

$$\varphi_1 = a_0|0\rangle + a_1|1\rangle, \sum_{i=0}^1 |a_i|^2 = 1 \quad (3.14)$$

$$\varphi_2 = b_0|0\rangle + b_1|1\rangle, \sum_{j=0}^1 |b_j|^2 = 1 \quad (3.15)$$

$$\mathcal{U}_i = \begin{cases} O_{i0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ O_{i1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}, i \in \{1, 2\} \quad (3.16)$$

The system that holds the game is represented in a  $\mathcal{H}^4$ . Each basis ( $|1\rangle, |2\rangle, |3\rangle, |4\rangle$ ), represents a final outcome as Table 3.3 suggests.

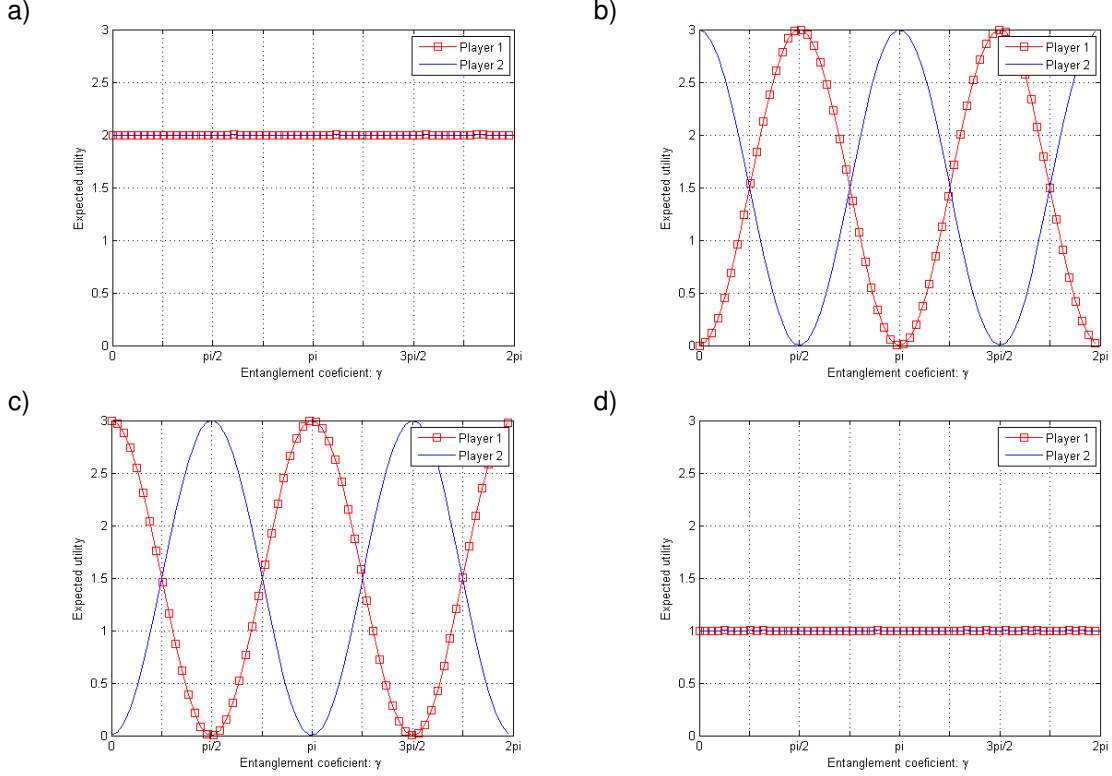
The fundamental difference from the classical version lies in the way the initial state is formulated3.18. We will entangle our state by applying the gate  $\mathcal{J}$  [27]. The parameter  $\gamma$  becomes a way to measure the entanglement in the system [4].

$$\mathcal{J} = \exp \left\{ i \frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \quad (3.17)$$

$$\begin{aligned} |\psi_{in}(\gamma)\rangle &= \exp \left\{ i \frac{\gamma}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} |00\rangle \\ &= \cos(\frac{\gamma}{2})|00\rangle + i \sin(\frac{\gamma}{2})|11\rangle, \gamma \in (0, \pi) \end{aligned} \quad (3.18)$$

$$|\psi_{fin}\rangle = \mathcal{J}^\dagger \otimes_{i=1}^2 \mathcal{U}_i |\psi_{in}\rangle \quad (3.19)$$

The entanglement in quantum game theory can be viewed as an intrinsic unbreakable contract. Furthermore measuring an entangled state will cause the wave function that describes the state to collapse.



**Table 3.4:** Expected utility for players 1 and 2 giving the entanglement coefficient  $\gamma$  used in preparing the initial state. a) Player 1: Cooperates, Player 2: Cooperates; b) Player 1: Cooperates, Player 2: Defects; c) Player 1: Defects, Player 2: Cooperates; d) Player 1: Defects, Player 2: Defects.

Before measuring the final result we will deentangle the system by applying the operator  $\mathcal{J}^\dagger$ , as shown in Equation 3.19.

The utility functions for each player is calculated by projecting the final state in each base and atributing a real number to each measurement, as in equation. In order to compare a classical version with this quantum model, the real numbers assigned are those in the classical example in Table 3.1, and the basis are presented in Table 3.3.

$$E_0(|\psi_{fin}\rangle) = 2 \times |\langle 00|\psi_{fin}\rangle|^2 + 3 \times |\langle 10|\psi_{fin}\rangle|^2 + 1 \times |\langle 11|\psi_{fin}\rangle|^2 \quad (3.20)$$

$$E_1(|\psi_{fin}\rangle) = 2 \times |\langle 00|\psi_{fin}\rangle|^2 + 3 \times |\langle 01|\psi_{fin}\rangle|^2 + 1 \times |\langle 11|\psi_{fin}\rangle|^2 \quad (3.21)$$

By varying the parameter  $\gamma$ , we can observe that the expected utility changes as we set up the initial state. Furthermore for  $\gamma = \frac{\pi}{4} + k\frac{\pi}{2}, k \in 0, 1, 2, 3$  the expected utility for each player is 2 if both players Cooperate, thus removing the dilemma. When the entanglement coefficient is  $\gamma = 0 + k\frac{\pi}{2}, k \in 0, 1, 2$  we are left with the classical problem, where there isn't entanglement.

### 3.2.3 Monty Hall Problem

The Monty Hall problem became popularized in 1990 in a column, in the magazine *Parade* [28]. The reason for its notoriety rests mainly in its counter-intuitive nature. Although the Monty Hall problem can be modeled as a Bayesian probability problem, human beings have difficulty in grasping the probabilities involved.

#### 3.2.3.A Problem Description

The Monty Hall problem is loosely based on a television game show hosted by its namesake - Monty Hall. Furthermore it was first attributed to a statistician, named Selvin. The problem is posed as it follows:

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

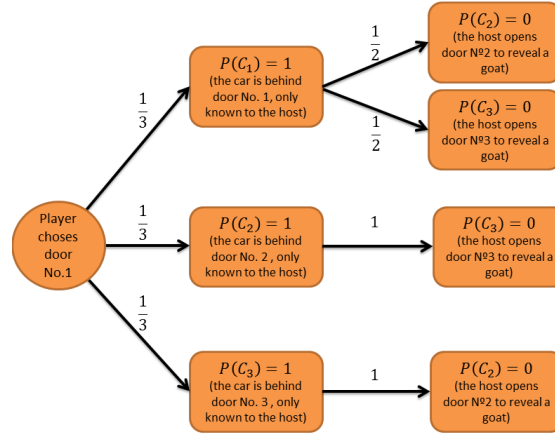
Most people, when faced with this problem, will be indifferent about whether to switch or to stay with the initially picked door.

It is also verified that they will tend to stick with their first choice. According to Granberg and Brow [29], only 13% of 228 subjects decided to switch their initial choice. However, by using probability theory, one can arrive at the conclusion that, in fact, it is advantageous to switch given the previous formulation.

The action of the Host implies a belief update on the probabilities of the variables in the system. This poses a violation of rational decision making; subjects do not seem to follow the best strategy which would maximize their chances of winning the prize.

To understand this exercise, one can look at the decision tree in Figure 3.4. We assumed indifferently that the player chose the door No. 1. The situation is symmetric whichever door she chooses.

Assuming we call  $C_1$  to the variable that describes whether or not the car is behind door No. 1. The variables  $C_2$  and  $C_3$  will respectively describe the probability associated with the car being (or not), behind doors 2 and 3 ( $P(C_2) = 1$  or  $P(C_2) = 0$ , for example).



**Figure 3.4:** Decision tree modelling the Monty Hall problem.

After the player has had her choice, the host will perform an operation on the remaining two doors. The host of the show has complete information of the game, unlike the player.

$$P(C_1) = P(C_1|\neg C_2) + P(C_1|\neg C_3) \quad (3.22)$$

$$P(C_1) = \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) = 1 \quad (3.23)$$

$$P(\neg C_1) = P(C_2|\neg C_3) + P(C_3|\neg C_2) = \left(\frac{1}{3} \times 1\right) + \left(\frac{1}{3} \times 1\right) = 2/3 \quad (3.24)$$

The probability of switching and getting the car is twice 3.24 as likely of staying with the first choice and getting the prize 3.23.

### 3.2.3.B Quantum Model

Various Models have been proposed to describe a quantum version of the Monty Hall problem [30] [31].

As the host reveals information, the initial setup is modified. This is an interesting property. Despite being a counter-intuitive problem, a quantum approach to this problem allows an in-depth comparison between the classical measurement and the quantum measurement. The classic Monty Hall problem is modeled using conditional probability and Bayes Rule. In the quantum version, measuring the outcome of the final state yields the result, instead of taking into account the intermediate actions [18].



Moreover it is important to realize that there is not a unique way to model a classical problem [32]. Therefore, when modelling a classical problem, we need to select properties that could potentially benefit from a quantum approach. In [32] we can observe the attempt to stick as closely to the classical formulation as possible, the host has a system that is correlated to the game system.

### **3.3 Overview**



# 4

## Quantum Pirate Game

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*In this chapter we describe the Pirate Game and the steps to model a quantum approach to the problem.*

## **4.1 Pirate Game**

On selecting the problem...

### **4.1.1 Problem Description**

The original Pirate Game is a multi-player version of the Ultimatum game that was first published as a mathematical problem in the Scientific American as a mathematical problem posed by Omohundro [33]. The main objective of the Pirate Game was to present a fully explainable problem with a non-obvious solution. The problem can be formulated as it follows:

Suppose there are 5 rational pirates: A; B; C; D; E. The pirates have a loot of 100 indivisible gold coins to divide among themselves.

As the pirates have a strict hierarchy, in which pirate A is the captain and E has the lowest rank, the highest ranking pirate alive will propose a division. Then each pirate will cast a vote on whether or not to accept the proposal.

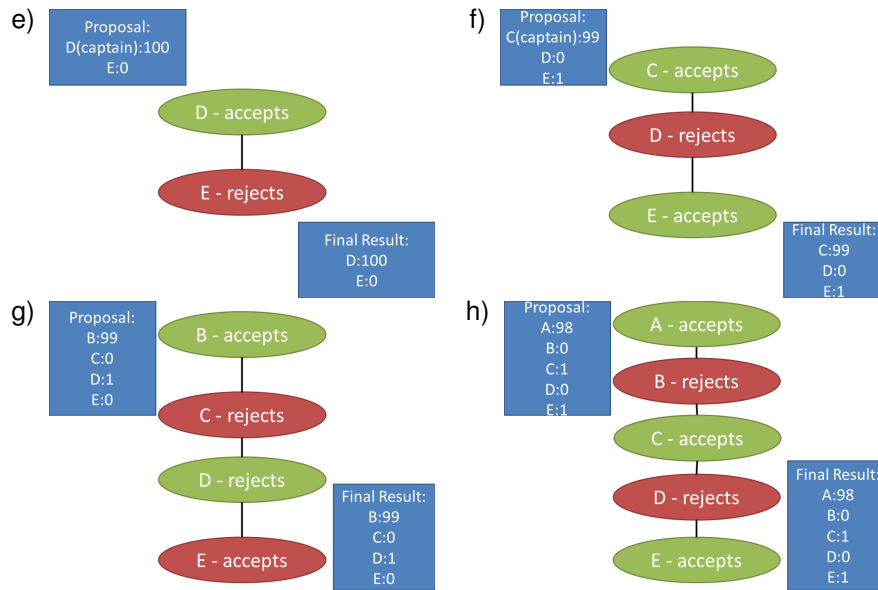
If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.

We consider that each pirate privileges her survival, and then will want to maximize the number of coins received. When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.

### **4.1.2 Analysis**

We can arrive at an equilibrium in this problem by using backward induction. At the end of the problem, supposing there are two pirates left, the equilibrium is very straight forward. As the highest ranking pirate can pass the proposal in spite of the other decision, her self-interest dictates that she will get the 100 gold coins. Knowing this, pirate E knows that any bribe other higher ranking pirate offers her will leave her better than if the game arrives to the last proposal.

When applying this reasoning to the three pirate move, as pirate C knows she needs one more vote to pass her proposal and avoiding death, she will offer the minimum amount of coins that will make pirate



**Table 4.1:** The equilibrium for the Pirate Game can be found through backward induction. From a), where there's only two pirates left, to d), that corresponds to the initial problem we define the best response.

E better off than if it comes to the last stage with two pirates. This means that pirate C will offer 1 gold coin to pirate E, and keep the remaining 99 coins.

With 4 pirates, B would rather bribe pirate D with 1 gold coin, because E would rather like climb on the hierarchy and getting the same payoff. Finally, with 5 pirates the captain (A), will keep 98 gold coins and rely on pirate C and E to vote in favour of the proposal, by giving 1 gold coin each.

We can generalize this problem for  $N$  pirates. If we assign a number to each pirate, where the captain is number 1 and the lower the number the higher the rank. If the number of coins is superior to the number of pirates, the equilibrium will have the captain (highest ranking pirate), giving a gold coin to each odd pirate, in case the number of players alive is odd, while keeping the rest to herself. When we have a even number of players the captain will assign a gold piece to each pirate with a even number, and the the remaining coins to herself.

If the number of pirates is greater than two times the amount of coins  $N > 2C$ , a new situation arises. If we have 100 coins and 201 pirates, the captain will not get any coin. By the same reasoning with 202 pirates the captain will still be able to survive by bribing the majority of the pirates and keeping no coins for herself. With 203 pirates the first captain will die. However with 204 pirates, the first captain will be able to survive even though he won't be able to bribe the majority, because her second in command knows that when she makes a proposal, she'll be thrown off board. In the game with 205 pirates, however the captain is not able to secure the vote from the second in command on the 204 pirate game, because the second captain is safe and she is able to make a have her proposal accepted and have the third pirate safe.

We can generalize this problem for  $N > 2C$  as [33], as the games with a number of pirates equal to  $2C$  plus a power of two will have an equilibrium in the first round, in the others every captain until a subgame with a number of pirates equal to  $2C$  plus a power of two will be thrown off board.

## 4.2 Quantum Pirate Game

### 4.2.1 Hypothesis

The original Pirate Game is posed from the point of view of the captain. How should she allocate the treasure to the crew in order to maximize her payoff. We can find the a equilibrium to the original Pirates Game and, while the solution may seem unexpected at first sight, it is fully described using backwards induction.

When modelling this problem from a quantum theory perspective we are faced with some questions. The main difference from the original problem will rely on how the system is set up. Will the initial conditions provide different equilibria? Is there a condition where we are left with the classical problem? Is it possible for a captain, in a situation where we have more than two pirates left, to acquire all the coins?

We propose to study this problem for the 2 and 3 player games and trying to extrapolate for  $N$  players. We will analyse the role of entanglement and superposition in the game system. Another aspect worth studying is the variation in the coin distribution on the payoff functions for the players.

### 4.2.2 Quantum Model

In order to model the problem we will start by defining it using the definition of quantum game ( $\Gamma$ ), referred in 2.49, Section 2.5 [21].

We want to keep the problem as close to the original as possible in order to better compare the results. Thus we will analyse the game from the point of view of the captain. Will her best response change?

In terms of mechanics and steps, this problem could be described using 3 players and later extended to any number  $N$  of players.

We begin by assigning an offset to each pirate (in order to identify her), as in the Section . The captain is number 1 and the lower the number the higher the rank.

#### 4.2.2.A Strategic Space

In Equation 2.49 there is the notion of a subset of unitary operators that the players can use to manipulate their assigned qubits.

Each player will be able to manipulate a qubit in the system, in this case  $|\varphi_1\rangle$ ,  $|\varphi_2\rangle$ , and  $|\varphi_3\rangle$ , with one of two operators shown in Equation 4.1.

An operator is an unitary  $2 \times 2$  matrix that is used to manipulate a qubit in the system. This restriction of the strategic space is relevant to keep the problem as close to the classical version as possible. The two operators will correspond to the action of voting “Yes” or to Cooperate, and voting “No”, meaning that they will not accept the proposal.

The cooperation operator will be represented by the Identity operator ( $o_{i0}$ , where  $i$  identifies the qubit upon which player  $i$  will act). When assigned to a qubit this operator will leave it unchanged.

The defection operator ( $D$ ), will be represented by one of Pauli's Operators - the Bit-flip operator. This operator was chosen because it performs the classical operation NOT on a qubit.

These operators are also permutation matrices, so our players are in fact permutating the state of their qubit as in the roulette quantum model (Section 3.2.1).

$$\mathcal{U}_i = \begin{cases} C = o_{i0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ D = o_{i1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}, i \in \{1, 2, 3\} \quad (4.1)$$

#### 4.2.2.B Game system: Setting up the Initial State

A game  $\Gamma$  can be viewed as a system composed by qubits manipulated by players. In a 3 player game there will be 3 qubits, each manipulated by a different player.

With 3 players our system will be represented in a  $\mathcal{H}^3$  using a state  $\psi$ . This means that to represent our system we will need vectors  $2^3 \times 1$  vectors, our system grows exponentially with the number of players/qubits. This makes que a game for 200 players (for example), impractical to simulate in a classical computer. In this regard a quantum computer may enhance our power to simulate this kinds of experiments ??.



The initial system ( $|\psi_0(\gamma)\rangle$ ), will be set up by defining an entanglement coefficient  $\gamma$ , that affect the way the three qubits (belonging to the three pirate players), are related 4.2. The index 0 represents the depth of the game tree which can be examined in Figure 4.5.

The concept of entanglement is crucial to explain some phenomena in Quantum Mechanics (Section 2.3.4). Analysing the role of entanglement in the system is also considered to be a major source of different results from the traditional Game Theory approach [21] [18] [27] [34] [35].

$$|\psi_0(\gamma)\rangle = \cos(\gamma)|000\rangle + i\sin(\gamma)|111\rangle, \gamma \in (0, \pi) \quad (4.2)$$

#### 4.2.2.C Final State

We can play the Pirate Game by considering a succession of steps or voting rounds. In each step we have a simultaneous move, however considering the potential rounds the game has we have a sequential game.

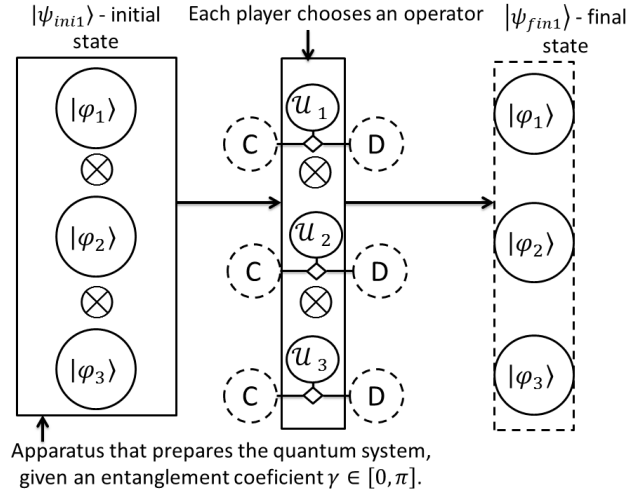
With three players, the first move will correspond to the player 1 (or the captain), if the proposal fails we will proceed to the second step in the game, where the remaining two players will vote on a new proposal made by player 2 (who will be the new captain).

After a move we have a “final state” that can be identified by an index which points to the depth  $k$  of the game ( Figure 4.5). This state is calculated by constructing a super-operator, by performing the tensor product of each player chosen strategy4.1. The super-operator, containing each player strategy, will then be applied to the initial state, this will correspond to the players making a simultaneous move4.3.

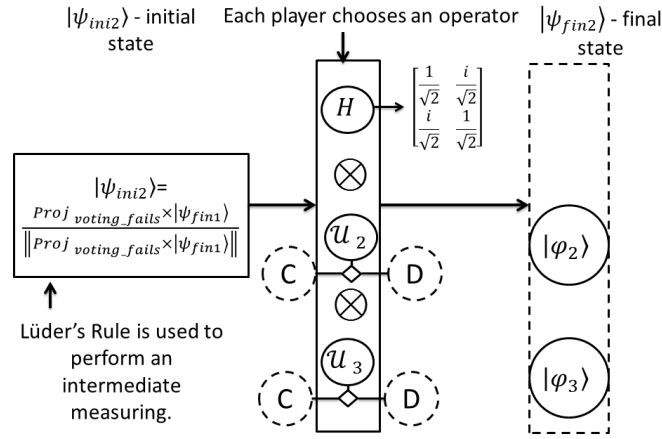
In the Figure 4.1 we have a step in the game. We start by building our initial state, then the players will select their strategy, a super operator is constructed by performing a tensor product of the selected operators.

$$|\psi_k\rangle = \otimes_{i=1}^N \mathcal{U}_i |\psi_{k-1}\rangle \quad (4.3)$$

The second stage in the game can be played in one of two ways. Either we perform an intermediate measuring step on the final state from the first round of voting as in Figure 4.2, or we “ask” what are the two player’s strategy for this round without informing them of the results in the last round, like in the Figure 4.3. In the original game the voting results are displayed in between rounds:



**Figure 4.1:** Playing the first round of the Pirate Game with 3 players.



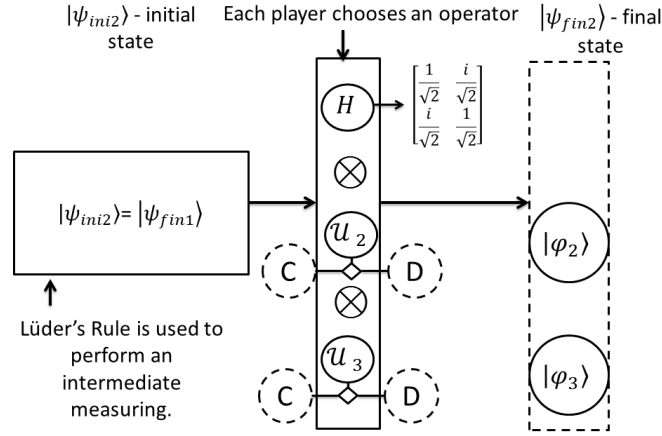
**Figure 4.2:** Playing the second round of the Pirate Game with 3 players by performing an intermediate measuring.

If a majority or a tie is reached the goods will be allocated according to the proposal. Otherwise the proposer will be thrown overboard and the next pirate in the hierarchy assumes the place of the captain.

When modelling this problem we want to study if withholding the results in the intermediate step will give way to new strategies.

#### 4.2.2.D Utility

To build the expected payoff functionals for the three player situation we must take into account the sub-games created when the proposal fail. In Figure 4.5 we can see an extensive form representation of the game.



**Figure 4.3:** Playing the second round of the Pirate Game with 3 players without performing an intermediate measurement.

As defined on Equation 2.50, for each player we must specify a utility functional that attributed a real number to the measurement of the projection of a basis in the quantum state that we get after the game. This measurement can be understood as a probability of the system collapsing into that state (that derives from the Born Rule, Section 2.2.1.A).

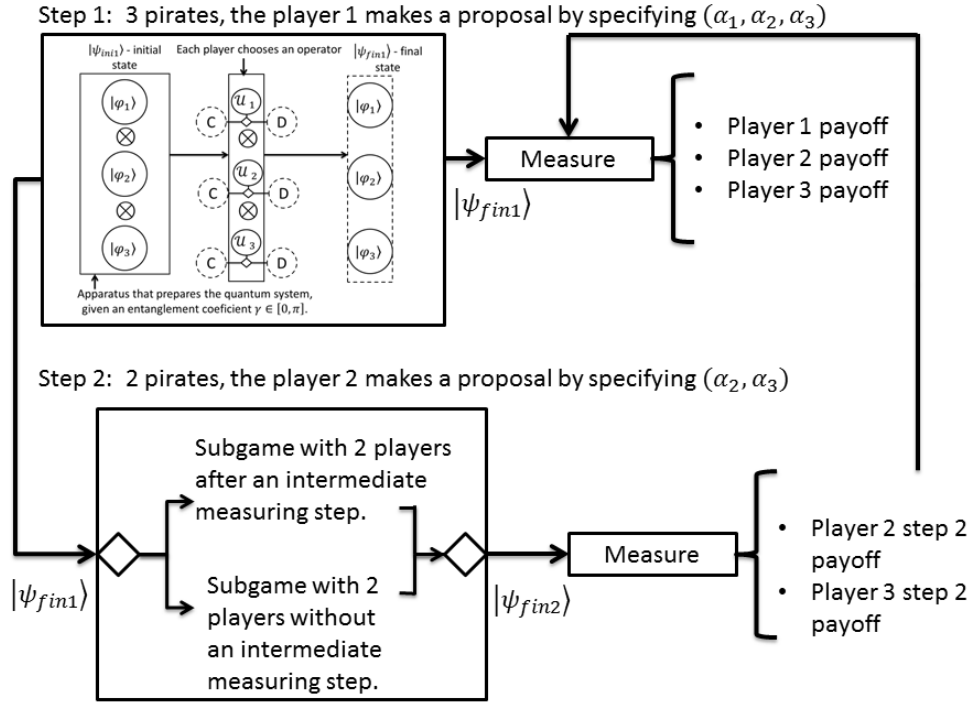
These utility functions will represent the degree of satisfaction for each pirate after game by attributiong a real number to a measurement performed to the system, as in Equation 2.50 (Section 2.5).

As each pirate wants to maximize her utility, the Nash equilibrium will be thoroughly used to find the strategies that the pirates will adopt [19] [20].

The real numbers used convey the logical relations of utility posed by the original problem description. Those numbers will represent the utility associated with the number of coins that a pirate gets, a death penalty, and a small incentive to climb the hierarchy.

The number of coins will translate directly the utility associated with getting those coins. For example if a pirate receives 5 gold coins and the proposal is accepted he will get a utility of 5.

The highest ranking pirate in the hierarchy will be responsible to make a proposal to divide the 100 gold coins. This proposal is modelled as choosing some parameters for the payoff functionals for every player, according to some rules. For the initial step in the game with three pirates these parameters will be  $\alpha_1, \alpha_2, \alpha_3$ , and they will obey to the Equation 4.4, where  $k$  is the offest of the current captain, and  $N$  the number of pirates in the game.



**Figure 4.4:** Overall view on the Quantum Pirate Game architecture.

$$\sum_{i=k}^N \alpha_i = 100, \forall i : \alpha_i \in \mathbb{N}_0 \quad (4.4)$$

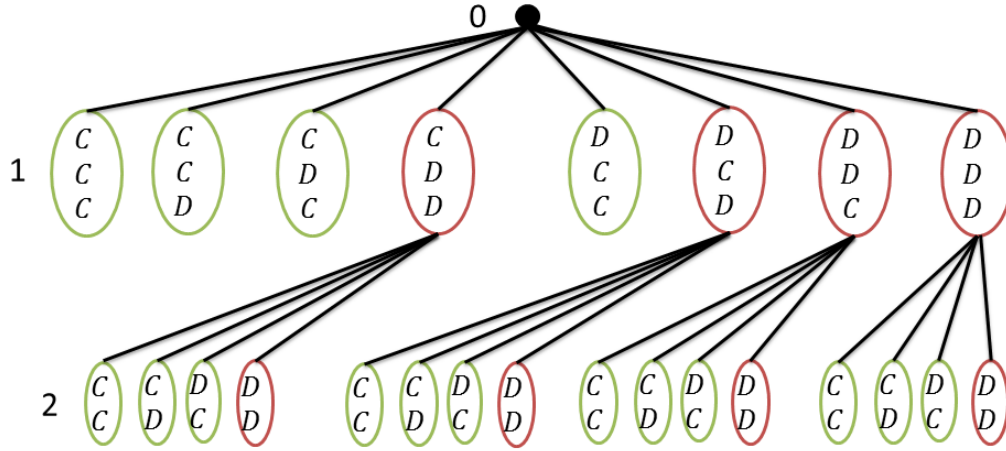
The most interesting values for  $(\alpha_1, \alpha_2, \alpha_3)$  will be the allocation that results in a Nash equilibrium in the original Pirate Game  $(99, 0, 1)$ , and the case where the captain maximizes the number of coins he can get  $(100, 0, 0)$ . Will the game modelled as a quantum system allow the captain to acquire all the coins?

The proposed goods allocation will be executed if there is a majority (or a tie), in the voting step. A step in the game consists on the highest ranking pirate defining a proposal and the subsequent vote, where all players choose simultaneously an operator.

If the proposal is rejected the captain will be thrown off board, to account for the fact that this situation is very undesirable for the captain he will receive a negative payoff of  $-200$ . This value was chosen to be much less than the highest number of coins a pirate could get.

“When the result is indifferent the pirates prefer to throw another pirate overboard and thus climbing in the hierarchy.”

This means that the pirates have a small incentive to climb the hierarchy. For example in the three player



**Figure 4.5:** Extensive form representation of the game (3 player). Red circles represent failed proposals, green represent accepted proposals.

classical game, the third player, who has the lowest rank, will prefer to defect the initial proposal if the player 1 doesn't give her a coin, even knowing that in the second round the player 2 will be able to keep the 100 coins. We will account for this preference by assigning an expected value of half a coin (0.5), to the payoff of the players that will climb on the hierarchy if the voting fails.

As stated in Section 4.2.2.C, each step will be identified by the offset  $k$ , which will be equal to the offset  $i$  for the highest ranking pirate.

We can observe that in Equations 4.5 and 4.9 that we have two separate groups of outcomes: the one that represents outcomes where the proposal is passed and the other that aggregates the outcomes where the captain will be eliminated and the remaining players will keep playing.

So with 3 pirates we start with step 1 (Figure 4.4), if the voting is rejected we will get to step 2, where 1 vote is enough to pass a proposal. The final payoff function (for example Equation 4.5 for a 3 player game), will be calculated recursively, the base case being the 2 player sub-game in a 3 player system will be Equation 4.9.

$$\left\{ \begin{array}{l} E_{11}(|\psi_{fin1}\rangle, \alpha_1) = \alpha_1 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \quad - 200 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \\ E_{12}(|\psi_{fin1}\rangle, \alpha_2) = \alpha_2 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \quad + (0.5 + E_{22}) \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \\ E_{13}(|\psi_{fin1}\rangle, \alpha_3) = \alpha_3 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \quad + (0.5 + E_{23}) \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \end{array} \right. \quad (4.5)$$

$$\left\{ \begin{array}{l} E_{22}(|\psi_{fin2}\rangle, \alpha_2) = \alpha_2 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \quad + 0.5 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \\ E_{23}(|\psi_{fin2}\rangle, \alpha_3) = \alpha_3 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - \\ \quad + 100.5 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \end{array} \right. \quad (4.6)$$

### 4.2.3 Analysing the 2-Player sub-game

By hypothesis, the way we set up our system may originate different results, so when we have a 3 player system, we will try to analyse the sub-game with 2 players taking into account the original system. In the classical version of the Pirate Game, the problem with 2 players is equivalent to a sub-problem with 2 players of a problem with more players where previous captains are killed.

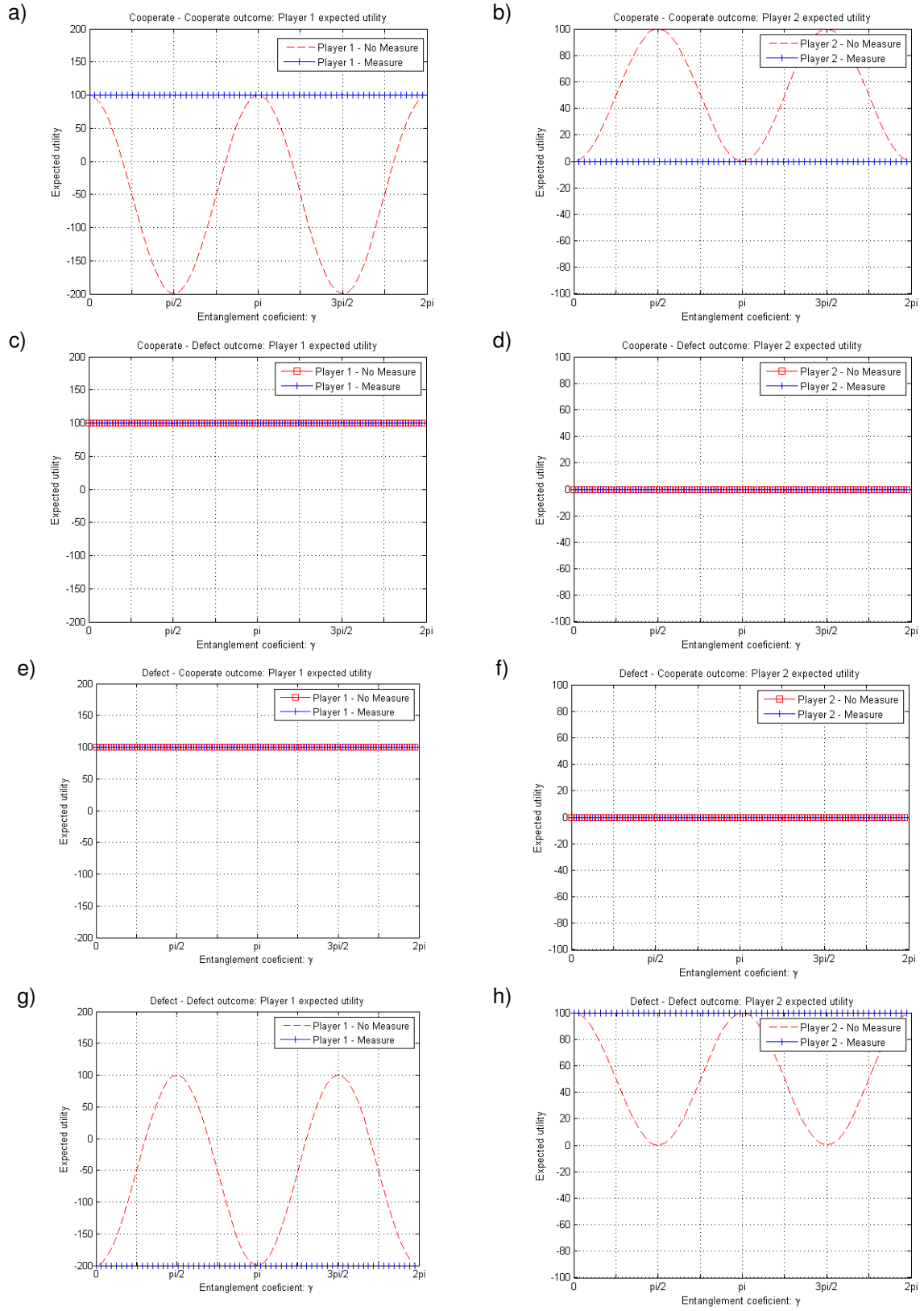
We set up a system with 3 players and in the step 1 the three players chose the operators  $o_{10}, o_{21}, o_{31}$  producing the outcome  $CDD$ , the initial proposal for the gold coin division will be  $(100, 0, 0)$ .

To analyse the step 2, or the sub-game with 2 pirates, the system will be kept in the initial dimension  $(\mathcal{H}^3)$ , because of the following reasons: if we have entangled states, they won't be separable; There may be various mappings in a lower dimension that fit the higher dimension composition. This means that the previous highest ranking player will no longer have access to the voting operators, instead she will use invariably a symmetric coin operator  $H$ .

In one situation we will use the Lüder's Rule, after the voting. This is the equivalent of performing a measurement on the system after the first round 4.2. In the other situation the two players will act on step 2, as in Figure 4.3. The results for this experiment (Table 4.2), allow us to conclude that not measuring the system affects the expected utilities for the players in two of the four possible analysed outcomes.

In this case the act of measuring leaves us with the same utility results one would get in a classic situation with 2 players game, the equilibrium here is  $CD$  outcome.

Without the intermediate measuring step the outcome will depend on the  $\gamma$  - the initial coefficient used



**Table 4.2:** Comparison of the expected utilities when measuring and not measuring between steps 1 and 2 of a 3 player game (the player 1 will be the player 2 in the 3 player game, and the player 2 will be the player 3 in the 3 game situation).

$\gamma = \frac{\pi}{4}$	Player 2: C	Player 2: D
Player 1: C	(-50, 50.25)	(100, 0)
Player 1: D	(100, 0)	(-50, 50.25)

**Table 4.3:** Normal form representation of the second step in a 3 player game with an entanglement coefficient of  $\frac{\pi}{4}$ , where there isn't an intermediate measuring.

to set up the system. For example:

- $\gamma = 0$ , we have the same results as in the original problem. The Nash equilibrium is unique and it is  $CD$ .
- $\gamma = \frac{\pi}{2}$ , the equilibrium is  $DC$ , as the payoffs for the  $CC$  outcome are  $(-200, 0.5)$ , and the outcome  $DD$  will present  $(100, 0)$  for the expected utilities of the players.
- $\gamma = \frac{\pi}{4}$ , all possible outcomes will be Nash equilibria. As we can see in Table 4.3, no player can improve her payoff by unilaterally changing her strategy.

#### 4.2.4 Analysis and Results

If we measure in between steps where we measure the system in between states (as in Figure 4.2), we can replace the expected value for the subgame with 2 players with the subgame perfect equilibrium in that case. The constants  $(e_{22}, e_{23})$  will be the pair of utilities for player 2 and 3 in the subgame equilibrium will be  $(e_{22}, e_{23}) = (100, 0)$ , in Equations 4.8 and 4.9.

$$E_{11}(|\psi_{fin1}\rangle, \alpha_1) = \alpha_1 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - 200 \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \quad (4.7)$$

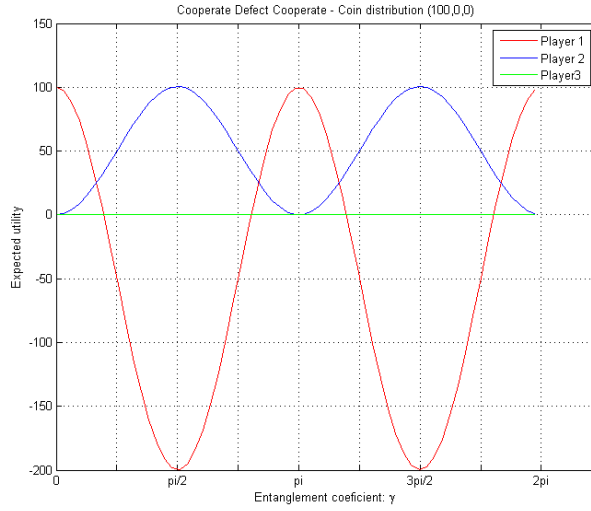
$$E_{12}(|\psi_{fin1}\rangle, \alpha_2) = \alpha_2 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - (0.5 + e_{22}) \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \quad (4.8)$$

$$E_{13}(|\psi_{fin1}\rangle, \alpha_3) = \alpha_3 \times (|\langle 000|\psi_{fin1}\rangle|^2 + |\langle 100|\psi_{fin1}\rangle|^2 + |\langle 010|\psi_{fin1}\rangle|^2 + |\langle 001|\psi_{fin1}\rangle|^2) - (0.5 + e_{23}) \times (|\langle 111|\psi_{fin1}\rangle|^2 + |\langle 110|\psi_{fin1}\rangle|^2 + |\langle 101|\psi_{fin1}\rangle|^2 + |\langle 011|\psi_{fin1}\rangle|^2) \quad (4.9)$$

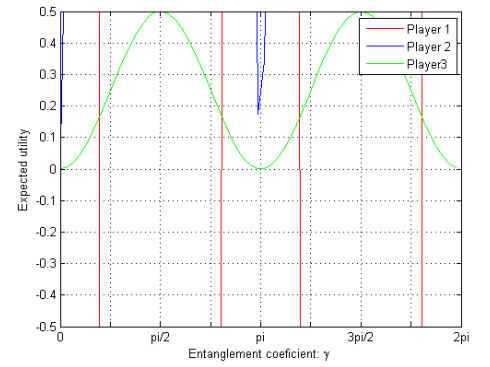
In the Figure 4.5, the outcome  $CDC$  with a proposal of  $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$  would represent the Nash Equilibrium of the classic Pirate Game (for 3 players).



a)

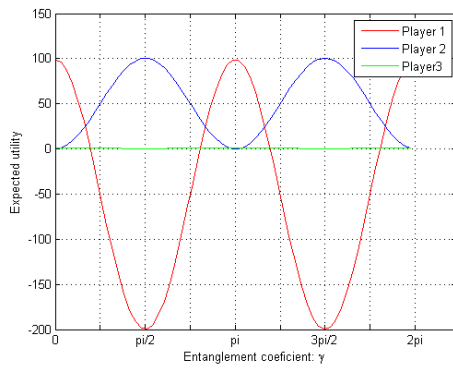


a1)

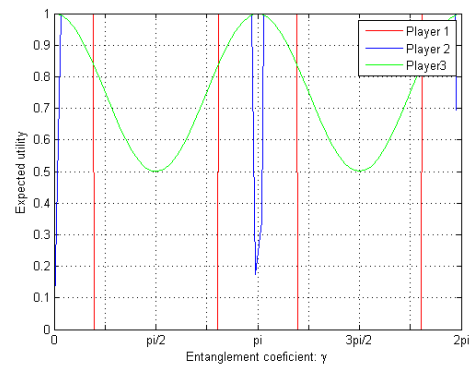


**Table 4.4:** Expected utility in a three player situation with intermediate measure step, where the players will use the *(Cooperate, Defect, Cooperate)* operators. The initial proposal is  $(\alpha_1, \alpha_2, \alpha_3) = (100, 0, 0)$ .

a)



a1)



**Table 4.5:** Expected utility for 3 players with intermediate measure step, where the players will use the *(Cooperate, Defect, Cooperate)* operators. The initial proposal is  $(\alpha_1, \alpha_2, \alpha_3) = (99, 0, 1)$ .



# 5

## Conclusions and Future Work

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5.1 Future Work . . . . .	lxxv
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*This section closes this document. It provides an overview on the work, presents a summary of the main results and relevant contributions. Moreover we compiled a list of interesting points worth pursuing in the future.*

From an historic point of view there is a character who is active on the fields of Quantum Mechanics, Game Theory, and Computer Science (among others), John von Neumann. von Neumann died at the age of 53 years.

## 5.1 Future Work

The principles of quantum computing provide an extremely rich source of ideas to extend other fields of knowledge. Some suggestions for possible extensions for this work would be:

1. **To implement and test this quantum model with human subjects.** The field of Quantum Game theory tries to explain the Human reasoning process by using principles from quantum game theory, namely quantum probabilities. To expose subjects to a quantum model and having them try to take advantage of the rules of the system might provide valuable insights to the way we reason.
2. **Increase the number of players.** Studying the quantum system with 8 players and 1 gold coin, would be an interesting follow up.
3. **A graphical user interface (GUI) for the Pirate Game.** This would provide a more approachable way to tweak parameters.
4. **To build a generic Quantum Game Theory framework.** Although there are already some theoretical frameworks that facilitate the creation of a quantum models of a game theory problem, there is still not a comprehensive .
5. **To create and structure a platform to promote scientific divulgation of quantum computing.** While investigating and developing this solution, there was often the thought that it would be important for undergraduate Computer Science students to come with contact with this paradigm. This field is still shrouded with mystery to many people, and this makes it a priority topic for scientific divulgation. In this work we tried to present examples that someone with a basic understanding of Algebra may be able to follow, but recollecting my experiences this is by no means enough.



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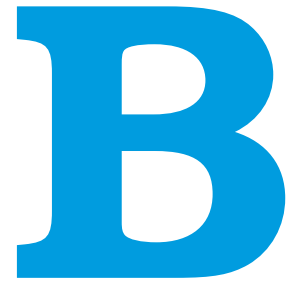
# **Matlab Simulation: Discrete Quantum Walk on a Line**



```

1
2
3 %— number of steps in the simulation
4 steps= 30;
5
6 %— hadamard matrix
7 H = [1/sqrt(2) 1/sqrt(2); 1/sqrt(2) -1/sqrt(2) ];
8 %— symetric matrix
9 M = [1/sqrt(2) 1i/sqrt(2); 1i/sqrt(2) 1/sqrt(2) ];
10 %— coin flip unitary operator
11 C = H;
12 C=M;
13
14 %— coin flip matrix
15 CM= zeros((steps*2+1),2);
16 %— shift matrix
17 SM= zeros(steps*2+1,2);
18
19 %— middle index
20 i0= steps+1;
21
22 %— initialize flip probability amplitudes (coin has 1/2 chance of going +1 or -1)
23 CM(i0,1)=1/sqrt(2);
24 CM(i0,2)=1/sqrt(2);
25
26 for i=1:steps
27     %— clean SM
28     SM= zeros(steps*2+1,2);
29
30     for j=1:(steps*2+1)
31         if CM(j, 1)≠0
32             SM(j-1,1)=CM(j, 1);
33         end
34         if CM(j, 2)≠0
35             SM(j+1,2)=CM(j, 2);
36         end
37     end
38     SM;
39     %disp('————');
40     %— clean CM
41     CM= zeros(steps*2+1,2);
42     for j=1:(steps*2+1)
43         if SM(j, 1)≠0
44             CM(j, 1)= CM(j, 1)+C(1,1)*SM(j, 1);
45             CM(j, 2)= CM(j, 2)+C(1,2)*SM(j, 1);
46         end
47         if SM(j, 2)≠0
48             CM(j, 1)= CM(j, 1)+C(2,1)*SM(j, 2);
49             CM(j, 2)= CM(j, 2)+C(2,2)*SM(j, 2);
50         end
51     end
52     CM;
53
54     %Display
55
56     figure
57
58     probability=zeros(steps*2+1);
59
60     for j=1:steps*2+1
61         probability(j)=abs(SM(j,1)).^2+ abs(SM(j,2)).^2;
62     end
63     axisP = -steps : steps;
64     plot(axisP,probability)
65     title(strcat('Step: ',num2str(i)))
66     axis([-steps-1 , steps+1, 0, 1]);
67     grid on
68     ylabel(strcat('Probability (State|Step=',num2str(i), ')'));
69     xlabel('States');
70
71     %Display
72 end

```



## **Quantum Prisoner's Dillema**



```

1 % Quantum Prisoner's Dillema
2 function out = quantumprisonersdillema(g,T,R,P,S)
3 %quantumprisonersdillema(g)
4 %
5 %           Simulates the payoff of 2-player prisoner's dillema game
6 %
7 %           IN:
8 %           g   : entanglement coefitient, by default g=0 (no
9 %                 entanglement)
10 %          T   : temptation, by default T=3
11 %          R   : reward, by default R=2
12 %          P   : punishment, by default P=1
13 %          S   : suckers, by default S=0
14 %
15 % To have a prisoner's dilemma game according to the cannonical form,
16 % it must respect:
17 % T > R > P > S
18 %           P           Player 2
19 %           l           C           D
20 %           a C: (R, R)   (S, T)
21 %           y D: (T, S)   (P, P)
22 %           e
23 %           r
24 %           l
25 %
26 %           OUT:
27 %           out: 4x2 matrix, column 1 has the expected utility for
28 %                 player 1, column 2 has the expected utility for
29 %                 player 2. A possible outcome corresponds to each
30 %                 line like [CC;CD;DC;DD]
31 %
32 % All parameters are optional
33 %-----%
34
35 %— entanglement coeficient (checks if exists)
36 %— Check variables and set to defaults
37 if exist('g','var')≠1, g=0; end
38
39 %— Utility
40 %— Check variables and set to defaults
41 if exist('T','var')≠1, T=3; end
42 if exist('R','var')≠1, R=2; end
43 if exist('P','var')≠1, P=1; end
44 if exist('S','var')≠1, S=0; end
45
46 %— Actions
47 %   cooperate= [1 0;0 1]
48 C= eye(2);
49 %   defect= [0 1;1 0]
50 D= ones(2)–eye(2);
51
52 %— Building the initial state
53 ini= cos(g)*kron([1 0]',[1 0]') + 1i*sin(g)*kron([0 1]',[0 1]');
54 %— Deentangles to produce a final state
55 Jt = ctranspose( expm(1i*(g)*kron(D,D)));
56 out= zeros(4,2);
57 %— Simulation a outcome where player1 (C)oooperates
58 %   and player 2 (C)oooperates
59 finCC= Jt*kron(C,C)*ini;
60 out(1,1)=payofffunc.player1(finCC, T, R, P, S);
61 out(1,2)=payofffunc.player2(finCC, T, R, P, S);
62 %— Simulation a outcome where player1 (C)oooperates
63 %   and player 2 (D)effects
64 finCD= Jt*kron(C,D)*ini;
65 out(2,1)=payofffunc.player1(finCD, T, R, P, S);
66 out(2,2)=payofffunc.player2(finCD, T, R, P, S);
67 %— Simulation a outcome where player1 (D)effects
68 %   and player 2 (C)oooperates
69 finDC= Jt*kron(D,C)*ini;
70 out(3,1)=payofffunc.player1(finDC, T, R, P, S);
71 out(3,2)=payofffunc.player2(finDC, T, R, P, S);
72 %— Simulation a outcome where player1 (D)effects
73 %   and player 2 (D)effects

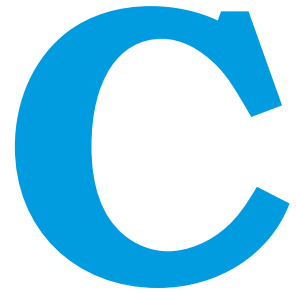
```



```

74     finDD= Jt*kron(D,D)*ini;
75     out(4,1)=payofffunc.player1(finDD, T, R, P, S);
76     out(4,2)=payofffunc.player2(finDD, T, R, P, S);
77 end
78
79 function u = payofffunc.player1(fin, T, R, P, S)
80 %payofffunc.player1(fin)
81 %
82 %           Calculates the payoff for player 1
83 %           IN
84 %           fin: final state
85 %           T, R, P, S: payoff numbers
86 %           OUT
87 %           u: expected utility for player 1
88 % (norm(conj([0 1 1 1 0 1 1 1]).*fin2CC))^2
89 u= R*(norm(conj([1 0 0 0]).*fin))^2 + T*(norm(conj([0 0 1 0]).*fin))^2+ ...
    P*(norm(conj([0 0 0 1]).*fin))^2;
90 end
91
92 function u = payofffunc.player2(fin, T, R, P, S)
93 %payofffunc.player1(fin)
94 %
95 %           Calculates the payoff for player 2
96 %           IN
97 %           fin: final state
98 %           T, R, P, S: payoff numbers
99 %           OUT
100 %           u: expected utility for player 2
101 u= R*(norm(conj([1 0 0 0]).*fin))^2 + T*(norm(conj([0 1 0 0]).*fin))^2+ ...
    P*(norm(conj([0 0 0 1]).*fin))^2;
102 end

```



## **Quantum Roulette**



```

1 function []= quantum_roulette3()
2 %%
3 % Simulation based on
4 % S. Salimi and M. M. Soltanzadeh, "Investigation of quantum roulette arXiv : 0807 . ...
   3142v3 [ quant-ph ] 30 Apr 2009," 2009.
5
6 %%
7
8 N=3
9 %% make states
10 I = eye(N);
11 D = (1/N)*ones(N);
12
13
14 %% permutation matrices change between states
15 X0 = circshift(I, 0);
16 %X0 = I;
17 X1 = circshift(I, 1);
18 %X1 = [0 1 0; 1 0 0; 0 0 1];
19 X2 = circshift(I, 2);
20 %X2 = [0 0 1; 0 1 0; 1 0 0];
21 X3 = circshift(I, 3);
22 %X3 = [1 0 0; 0 0 1; 0 1 0];
23 X4 = circshift(I, 4);
24 %X4 = [0 0 1; 1 0 0; 0 1 0];
25 X5 = circshift(I, 5);
26 %X5 = [0 1 0; 0 0 1; 1 0 0];
27
28
29 %%Fourier Matrix
30 F= (1/sqrt(N))*fft(I)
31
32 T0 = circshift(F,[0 0]);
33 T1 = circshift(F,[0 1]);
34 T2 = circshift(F,[0 2]);
35 %%
36
37 %%Step1
38
39 %Assuming Alice places the roulette in state 2 were both player can see
40 Ro0= [0; 1; 0]*[0; 1; 0]'
41
42 %%Step2
43
44 %As Alice chose state 2, Bob will select T1 to rotate the state
45 Ro1 = T1 * Ro0 * T1'
46
47
48 %step 3 -
49 Ro1 = T2 * Ro1 * T2'
50
51
52
53 %%Step3
54 %Alice will play again
55 %
56
57 Ro2 = (2/6)*X0*Ro1*X0' + (1/12)*X1*Ro1*X1' + (1/12)*X2*Ro1*X2' + (1/6)*X3*Ro1*X3' ...
   + (1/6)*X4*Ro1*X4' + (1/6)*X5*Ro1*X5'
58
59 %%Step4
60 %Bob can choose which state he wants
61
62 Ro3 = T0'*Ro2*T0
63
64 Ro3 = T1'*Ro2*T1
65
66 Ro3 = T2'*Ro2*T2
67
68 Ro4 = (2/6)*X0*Ro3*X0' + (1/12)*X1*Ro3*X1' + (1/12)*X2*Ro3*X2' + (1/6)*X3*Ro3*X3' ...
   + (1/6)*X4*Ro3*X4' + (1/6)*X5*Ro3*X5'
69
70 Ro3 = T0'*Ro3*T0

```

71  
72  
73 end