Chapter 1

Hurwitz existence problem

1.1 Branched coverings of surfaces

According to standard terminology, a *surface* is simply a topological 2-manifold. We will, however, only be concerned with compact, connected surfaces without boundary. For the sake of conciseness, unless otherwise stated, we will always implicitly assume that the surfaces we mention have these properties.

Orientable and non-orientable surfaces are completely classified by the following structure theorem (see [7, Theorem 12.1]).

- An orientable surface is (homeomorphic to) a connected sum of $g \geq 0$ tori. We call such a connected sum a *surface of genus* g, and we denote it by Σ_g. By definition, we say that Σ_0 is the 2-sphere $\mathbb{S}$; this is consistent with the formula $\chi(\Sigma_g) = 2 - 2g$ for the Euler characteristic.

- A non-orientable surface is (homeomorphic to) a connected sum of $g \geq 1$ real projective planes. We denote such a connected sum by N_g. The Euler characteristic for a non-orientable surfaces is given by $\chi(N_g) = 2 - g$.

Loosely speaking, given two surfaces Σ, $\widetilde{\Sigma}$, a *covering map* between them is a continuous function $f\colon \widetilde{\Sigma} \to \Sigma$ which is locally modeled on the identity function $\mathbb{R}^2 \to \mathbb{R}^2$. Sometimes, however, the notion of covering map can be too restrictive. Consider, for instance, the sphere $\mathbb{S}$: being simply connected, it does not admit any non-trivial coverings. However, every (non-constant) homolorphic function $\mathbb{S} \to \mathbb{S}$ is *almost* a covering map, in the sense that it is locally modeled on the identity $\mathbb{C} \to \mathbb{C}$, except for a finite number of *branching points*, where it looks like the map

$$\begin{aligned} F_k\colon \mathbb{C} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \xi^k \end{aligned}$$

for some $k \geq 2$. In fact, it turns out that every (non-constant) holomorphic function between two Riemann surfaces has this remarkable property (see [16, Section 3.2]). This motivates the following definition.

Definition 1.1. Let $\Sigma, \widetilde{\Sigma}$ be two surfaces. A continuous function $f\colon \widetilde{\Sigma} \to \Sigma$ is a *branched covering map* (or simply a *branched covering*) if the following property holds: for every $x \in \Sigma$, $\widetilde{x} \in f^{-1}(x)$ there exist a positive integer k, open neighborhoods $U, \widetilde{U}$ of $x, \widetilde{x}$ respectively, and homeomorphisms $\varphi\colon U \to \mathbb{C}$, $\widetilde{\varphi}\colon \widetilde{U} \to \mathbb{C}$ such that $\varphi(x) = 0$, $\widetilde{\varphi}(\widetilde{x}) = 0$, $f(\widetilde{U}) = U$ and the diagram

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "f"] \arrow[d, "\widetilde{\varphi}"] & U \arrow[d, "\varphi"] \\
\mathbb{C} \arrow[r, "F_k"] & \mathbb{C}
\end{tikzcd}$$

commutes (F_k is the map defined above). We say that $\widetilde{U}$ is a *trivializing neighborhood* of $\widetilde{x}$.

More informally, we can say that a branched covering is a continuous function between surfaces which is locally modeled on the complex map $\xi \mapsto \xi^k$, where $k \geq 1$ depends on the point. Note that, for each point $\widetilde{x} \in \widetilde{\Sigma}$, the integer k is well-defined, independently of the charts φ and $\widetilde{\varphi}$: using the notation from the definition, we have that k is equal to the cardinality of $f^{-1}(y) \cap \widetilde{U}$, where y is any point in $U \setminus \{x\}$. We call this integer the *local degree* of $\widetilde{x}$; to emphasize the dependence on $\widetilde{x}$, we will denote it by $k(\widetilde{x})$.

A point $\widetilde{x} \in \widetilde{\Sigma}$ is called a *branching point* if $k(\widetilde{x}) > 1$; in other words, if f is *not* a local homeomorphism in a neighborhood of $\widetilde{x}$. We also say that a point $x \in \Sigma$ is a *branching point* if $f^{-1}(x)$ contains at least one branching point; usually, no ambiguity will arise as to which kind of branching point we are referring to.

It is not hard to see that branching points are quite rare. If $\widetilde{x} \in \widetilde{\Sigma}$ is a branching point, then (using again the notation from the definition) every other point in $\widetilde{U}$ is not a branching point; by compactness, it follows that the set of branching points in $\widetilde{\Sigma}$ is finite. As a consequence, the set of branching points in Σ is finite as well.

Given a branched covering $f\colon \widetilde{\Sigma} \to \Sigma$, we denote by $\Sigma^\bullet$ the subspace of Σ containing all the non-branching points; since the set of branching points is finite, $\Sigma^\bullet$ is a non-compact connected surface with finitely many punctures. We also set $\widetilde{\Sigma}^\bullet = f^{-1}(\Sigma^\bullet)$, and we denote by $f^\bullet$ the restriction of f to $\widetilde{\Sigma}^\bullet$. By construction, $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ is a covering map and, as such, has a well-defined degree d (the number of preimages of an arbitrary point); we call this integer the *degree* of the branched covering f. The following proposition shows that the notion of degree extends nicely to branching points.

Proposition 1.1. *Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d. Then for every point $x \in \Sigma$ we have that the set $f^{-1}(x)$ is finite and*

$$\sum_{\widetilde{x} \in f^{-1}(x)} k(\widetilde{x}) = d.$$

Proof. If x is not a branching point, the conclusion follows immediately, since x has exactly d preimages, all of which have local degree k equal to 1. Assume now that x is a branching point; it is clear from the definition that the set $f^{-1}(x) \subseteq \widetilde{\Sigma}$ is discrete and, hence, finite. Let $f^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_r\}$. Fix disjoint trivializing neighborhoods $\widetilde{U}_1, \ldots, \widetilde{U}_r$ of $\widetilde{x}_1, \ldots, \widetilde{x}_r$ respectively; a routine compactness argument shows that there exists an open neighborhood U of x such that $f^{-1}(U) \subseteq \widetilde{U}_1 \cup \ldots \cup \widetilde{U}_r$. Fix a point $y \in U \setminus \{x\}$: it follows from the discussion above that y is not a branching point and that $|f^{-1}(y) \cup \widetilde{U}_i| = k(\widetilde{x}_i)$ for every $1 \leq i \leq r$. Since $|f^{-1}(y)| = d$, we immediately conclude that

$$\sum_{i=1}^{r} k(\widetilde{x}_i) = \sum_{i=1}^{r} |f^{-1}(y) \cap \widetilde{U}_i| = |f^{-1}(y)| = d. \qquad \square$$

1.2 Branching data

Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering. For each point $x \in \Sigma$, as we have just seen in Proposition 1.1, the sum $k(\widetilde{x}_1)+\ldots k(\widetilde{x}_r)$ of the local degrees of its preimages is equal to the degree d of the branched covering. Since there is no natural ordering on the set $f^{-1}(x)$, the appropriate combinatorial object for representing the collection $k(\widetilde{x}_1), \ldots, k(\widetilde{x}_r)$ is a *partition*.

Definition 1.2. Let d be a positive integer. A *partition* of d is an unordered finite multiset $\pi = [k_1, \ldots, k_r]$, where $k_i > 0$ is an integer for every $1 \leq i \leq r$ and $k_1 + \ldots + k_r = d$.

Given a positive integer d, we denote the set of all partitions of d by $\Pi(d)$. If $\pi = [k_1, \ldots, k_r]$ is a partition of d, we call the integer r the *length* (or *size*, or *cardinality*) of π, and we denote it by $\ell(\pi)$. We also say that the *sum* of π, denoted by $\sum \pi$, is $k_1 + \ldots + k_r$ or, in other words, d. Finally, we introduce a new quantity, the *branching number* $v(\pi) = d - \ell(\pi)$, whose purpose will soon become apparent.

For every point $x \in \Sigma$, if $f^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_r\}$, we can define the *associated partition* $\pi(x) = [k(\widetilde{x}_1), \ldots, k(\widetilde{x}_r)] \in \Pi(d)$. For all non-branching points, the associated partition will simply be $[1, \ldots, 1]$. On the contrary, if x is a branching point, then $\pi(x) \neq [1, \ldots, 1]$ (or, equivalently, $v(\pi(x)) > 0$).

Definition 1.3. Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d. Let $x_1, \ldots, x_n \in \Sigma$ be the branching points of f. The *branching datum* of f is the tuple

$$\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi(x_1), \ldots, \pi(x_n)),$$

well-defined up to a permutation of the branching points $x_1, \ldots, x_n$.

Branching data are a way to extract some combinatorial information from branched coverings. Even though the exact location of the branching points (both in Σ and in $\widetilde{\Sigma}$) is not encoded in the branching datum, this piece of information is completely irrelevant, since surfaces are *homogeneous*[1]. However, the reader should not be induced to believe that the combinatorial information provided by the branching datum is enough to fully reconstruct the topology of f. In fact, it turns out that this is not the case: there can be many inequivalent branched coverings sharing the same branching datum. See [14] for an in-depth discussion of this topic.

This entire thesis is devoted to the problem of determining what values can actually be attained by $\mathcal{D}(f)$ as f ranges over all the possible branched covering maps. We start with a very general definition.

Definition 1.4. A *combinatorial datum* is a tuple

$$\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n),$$

where $\Sigma, \widetilde{\Sigma}$ are surfaces, d and n are positive integers, and $\pi_1, \ldots, \pi_n$ are partitions of d different from $[1, \ldots, 1]$.

Again, a combinatorial datum is defined up to a permutation of the partitions $\pi_1, \ldots, \pi_n$. In other words, we will consider two combinatorial data equal if they have the same partitions, irrespective of the ordering.

Technically, we could also allow combinatorial data to have $n = 0$ partitions. However, data of this kind would correspond to standard covering maps, without any branching points. Since

[1] By *homogeneous*, we mean that, for each $n \geq 1$, the group of homeomorphisms of a surface Σ acts transitively on the set of n-uples of pairwise distinct points $(y_1, \ldots, y_n)$.

covering maps between surfaces are completely understood from a topological point of view (even in the non-empty boundary case, as shown in [5]), we will always assume that $n \geq 1$. For the same reason, when we say "branched covering", we will be implicitly excluding the trivial cases of standard covering maps.

We say that a combinatorial datum $\mathcal{D}$ is *realizable* if $\mathcal{D} = \mathcal{D}(f)$ for some branched covering $f\colon \widetilde{\Sigma} \to \Sigma$, *exceptional* otherwise. A first, naive guess could be that every combinatorial datum can be realized. However, it turns out that there are a few easy and general necessary conditions for a combinatorial datum to be associated to a branched covering. The first one, and arguably the most important, is known as the *Riemann-Hurwitz formula*; we state it in the next proposition.

Proposition 1.2. *Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering, and let $\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be its branching datum. Then we have the equality*

$$d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = v(\pi_1) + \ldots + v(\pi_n).$$

Proof. Using the same notation as above, consider the covering map $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$. Since $f^\bullet$ has degree d, the Euler characteristics of $\Sigma^\bullet$ and $\widetilde{\Sigma}^\bullet$ are related by the formula $d\chi(\Sigma^\bullet) = \chi(\widetilde{\Sigma}^\bullet)$. Note that there are n branching points in Σ, and the number of points in $\widetilde{\Sigma} \setminus \widetilde{\Sigma}^\bullet$ is $\ell(\pi_1) + \ldots + \ell(\pi_n)$; therefore

$$\chi(\Sigma^\bullet) = \chi(\Sigma) - n, \qquad\qquad \chi(\widetilde{\Sigma}^\bullet) = \chi(\widetilde{\Sigma}) - \ell(\pi_1) - \ldots - \ell(\pi_n).$$

As a consequence, we have that

$$d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = d\chi(\Sigma^\bullet) + dn - \chi(\widetilde{\Sigma}^\bullet) - \ell(\pi_1) - \ldots - \ell(\pi_n) = v(\pi_1) + \ldots + v(\pi_n). \qquad \square$$

There are three more conditions that every branching datum must satisfy: two of them concern the orientability of Σ and $\widetilde{\Sigma}$, and the other one provides an additional constraint for the *total branching number* $v(\pi_1) + \ldots + v(\pi_n)$; we group these four requirements in the following definition.

Definition 1.5. A combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ is a *candidate datum* if it satisfies the following conditions:

(i) $d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = v(\pi_1) + \ldots + v(\pi_n)$;

(ii) $v(\pi_1) + \ldots + v(\pi_n)$ is even;

(iii) if Σ is orientable, then $\widetilde{\Sigma}$ is orientable as well;

(iv) if Σ is non-orientable and $\widetilde{\Sigma}$ is orientable, then d is even, and each π_i can be written as $\pi_i' \cup \pi_i''$, where π_i' and π_i'' are partitions of $d/2$.

Remark 1.1. In Proposition 1.10, we will show that conditions (i)–(iv) are necessary for a combinatorial datum to be realizable (actually, the first condition was already proved in Proposition 1.2). For now, let us focus on condition (iii), which is perhaps the most obvious one. If Σ is orientable, then so is $\Sigma^\bullet$, since removing a finite number of points does not affect orientability. The non-compact surface $\widetilde{\Sigma}^\bullet$, being a covering space of an orientable manifold, is itself orientable. Finally, this implies that $\widetilde{\Sigma}$ is orientable as well.

It would be natural to ask whether the necessary conditions we have enumerated are also sufficient for a candidate datum to be realizable. We will see in the following chapters that the answer is negative, and that, in general, determining the full list of exceptional data is remarkably difficult; we call this the *Hurwitz existence problem*[2], and we state it in the following deliberately vague fashion.

Hurwitz existence problem. Determine necessary and sufficient conditions for a candidate datum $(\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ to be realizable.

In the remainder of this thesis, we will try to address some instances of this problem. We will present a variety of techniques, which will provide us with a full solution for some classes of candidate data.

1.3 Symmetric group and partitions

The first approach we will describe is based on the monodromy action. Before describing this action and how it relates to branched coverings, we will recall a few elementary facts about permutations and partitions.

Given a set X, we denote by $\mathfrak{S}(X)$ the set of bijective functions $X \to X$. The set $\mathfrak{S}(X)$ is naturally endowed with a group structure, where the product is given by the composition operator $(\circ)$; we call this group the *symmetric group* of X. Elements of $\mathfrak{S}(X)$ are also called *permutations*.

We will sometimes need to work in a setting where the product in $\mathfrak{S}(X)$ is reversed. Given a group G with product $(*)$, we define the *opposite group* G^{op}, which has the same underlying set as G, but the product $(*^{\mathrm{op}})$ is defined as $g_1 *^{\mathrm{op}} g_2 = g_2 * g_1$.

If d is a positive integer, we will employ the notation $\mathfrak{S}_d$ as a shorthand for $\mathfrak{S}(\{1, \ldots, d\})$. Whenever X is a finite (non-empty) set, we have an isomorphism $\mathfrak{S}(X) \simeq \mathfrak{S}_d$, where d is the cardinality of X. We will therefore restrict out attentions to the groups $\mathfrak{S}_d$, keeping in mind that everything we say also holds for symmetric groups of arbitrary finite sets.

Recall that every permutation $\alpha \in \mathfrak{S}_d$ has a *cycle decomposition* (that is, it can be written as a product of disjoint, possibly trivial cycles):

$$\alpha = (x_{1,1}, \ldots, x_{1,k_1})(x_{2,1}, \ldots, x_{2,k_2}) \cdots (x_{r,1}, \ldots, x_{r,k_r}).$$

The natural action of (the subgroup generated by) α on the set $A = \{1, \ldots, d\}$ induces a decomposition $A = A_1 \sqcup \ldots \sqcup A_r$ into orbits, where $A_i = \{x_{i,1}, \ldots, x_{i,k_i}\}$. This decomposition, in turn, gives rise to a partition $\pi = [|A_1|, \ldots, |A_r|] \in \Pi(d)$. We say that α *matches* the partition π, and we use the notation $[\alpha]$ to refer to the (unique) partition matched by α. We also define the *branching number* $v(\alpha)$ as the branching number of $[\alpha]$; note that $v(\alpha) = 0$ if and only if $\alpha = \mathrm{id} \in \mathfrak{S}_d$ is the trivial permutation.

It is well known that two permutations are conjugate if and only if they match the same partition. In other words, there is a natural bijection between conjugacy classes of $\mathfrak{S}_d$ and the set of partitions $\Pi(d)$.

We conclude this section with a very simple result relating the branching number of two permutations to the branching number of their product. Recall that every permutation has a well-defined *sign*: *even* permutations can only be written as a product of an even number of transpositions (2-cycles), while *odd* permutations can only be written as a product of an odd number of transpositions. The *alternating group* of order d is the index-2 subgroup $\mathfrak{A}_d \leq \mathfrak{S}_d$ containing all the even permutations.

Proposition 1.3. *Let $\alpha, \beta \in \mathfrak{S}_d$ be permutations. Then $v(\alpha\beta) \equiv v(\alpha) + v(\beta) \pmod 2$.*

[2] The question originally posed by Hurwitz in [3] was actually even harder: the task was counting *how many* branched coverings (up to a suitable notion of isomorphism) realize a given candidate datum. We will, however, ignore this point of view, considering that the existence problem is hard enough as it is. The interested reader may find some relevant results on the *Hurwitz enumeration problem* in [6, 4, 11, 12, 13].

Proof. We show that $v(\alpha)$ is even if and only if α is even; the conclusion will then follow trivially. Fix a cycle decomposition

$$\alpha = (x_{1,1}, \dots, x_{1,k_1})(x_{2,1}, \dots, x_{2,k_2}) \cdots (x_{r,1}, \dots, x_{r,k_r}).$$

Since a k-cycle can be written as a product of $k-1$ transpositions, we have that α is the product of $d-r = v(\alpha)$ transpositions. This shows that $v(\alpha)$ is even if and only if $\alpha \in \mathfrak{A}_d$, which concludes the proof. $\square$

1.4 Branched covering action of the fundamental group

We now introduce the monodromy action in the general setting of topological spaces; we will assume that all the spaces we mention are locally path-connected and locally simply-connected. For a much more in-depth discussion of the monodromy action, and especially for a proof of Theorem 1.4, see [16, Section 2.3] (but note that Szamuely, unlike us, uses the right-to-left notation for path concatenation in the fundamental group).

Let $f\colon \widetilde{X} \to X$ be a covering map between topological spaces, with X path-connected. Fix a base-point $x_0 \in X$, and let $\widetilde{x}_0 \in \widetilde{X}$ be a point in $f^{-1}(x_0)$. Given a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$, denote by $\mathrm{lift}(\gamma, \widetilde{x}_0)\colon [0,1] \to \widetilde{X}$ the unique lift of γ such that $\mathrm{lift}(\gamma, \widetilde{x}_0)(0) = \widetilde{x}_0$. There is a very natural right action of the fundamental group $\pi_1(X, x_0)$ on the fiber $f^{-1}(x_0)$, defined by[3]

$$\widetilde{x} \bullet \gamma = \mathrm{lift}(\gamma, \widetilde{x})(1) \qquad \text{for } \widetilde{x} \in f^{-1}(x_0),\ \gamma \in \pi_1(X, x_0).$$

We call this the *monodromy action* of the covering map. It is easy to see that $\widetilde{X}$ is path-connected if and only if the monodromy action on $f^{-1}(x_0)$ is transitive.

There is a relation between monodromy and the fundamental group of $\widetilde{X}$. Fix a base-point $\widetilde{x}_0 \in f^{-1}(x_0) \subseteq \widetilde{X}$. By elementary properties of covering spaces, we have that

$$\begin{aligned} f_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) &= \{\gamma \in \pi_1(X, x_0) : \mathrm{lift}(\gamma, \widetilde{x}_0) = \widetilde{x}_0\} \\ &= \{\gamma \in \pi_1(X, x_0) : \widetilde{x}_0 \bullet \gamma = \widetilde{x}_0\} \\ &= \mathrm{Stab}_{\pi_1(X,x_0)}(\widetilde{x}_0). \end{aligned}$$

The monodromy action induces a group homomorphism $\mathfrak{m}\colon \pi_1(X, x_0) \to \mathfrak{S}(f^{-1}(x_0))^{\mathrm{op}}$. If the covering map has a finite degree d, we will sometimes implicitly fix a bijection between $f^{-1}(x_0)$ and $\{1, \dots, d\}$, and consider the homomorphism $\mathfrak{m}\colon \pi_1(X, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ instead; of course, this map is well-defined up to a conjugation in $\mathfrak{S}_d^{\mathrm{op}}$. The following existence theorem shows that every group homomorphism of this kind is induced by the monodromy action of some covering space.

Theorem 1.4. *Let X be a path-connected topological space, $x_0 \in X$ a point. Let d be a positive integer, and let $\psi\colon \pi_1(X, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ be a group homomorphism. Then there exists a covering map $f\colon \widetilde{X} \to X$ with $\mathfrak{m} = \psi$ (up to conjugation).*

After introducing monodromy for general topological spaces, we turn our attention to the case of branched coverings of surfaces. Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering. As we have seen, by removing the branching points from Σ, we get an associated covering map $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$. Fix a base-point $x_0 \in \Sigma^\bullet$, and let $x \in \Sigma$ be any point. There is a path $\gamma \in \pi_1(\Sigma^\bullet, x_0)$ which "goes

[3] Since the lifting operation is, in some sense, invariant up to homotopy, we will often use the same symbol (γ, in this case) to interchangeably represent a class in the fundamental group $\pi_1(X, x_0)$ and a representative of that class.

around x once"; to be more precise, we can construct it as follows. Pick a small neighborhood $U \subseteq \Sigma$ of x which is homeomorphic to $\mathbb{R}^2$ and does not contain any other branching points; here we can define a loop $\beta\colon [0,1] \to U \smallsetminus \{x\}$ that goes around x once (we do not care about the orientation). Finally, pick a path $\alpha\colon [0,1] \to \Sigma^\bullet$ connecting x_0 to $\beta(0) = \beta(1)$, and define $\gamma = \alpha * \beta * \iota(\alpha)$, where $(*)$ is the concatenation of paths and $\iota(\alpha)$ denotes the inverse path $\iota(\alpha)(t) = \alpha(1-t)$. Of course, the homotopy class of $\gamma \in \pi_1(\Sigma^\bullet, x_0)$ depends on α and even β (we could have chosen $\iota(\beta)$ instead); we will say that any path constructed with the procedure we have described is a *loop around* x (based at x_0).

We are finally ready to prove the connection between branched coverings and the monodromy of the associated covering map.

Proposition 1.5. *Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d, and let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ be the associated covering map. Fix a base-point $x_0 \in \Sigma^\bullet$ and a point $y \in \Sigma$. Let γ be a loop around y. Consider the monodromy homomorphism $\mathfrak{m}\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}(f^{-1}(x_0))^{\mathrm{op}}$. Then the permutation $\mathfrak{m}(\gamma)$ matches $\pi(y)$.*

Proof. Let the preimages $\widetilde{y}_1, \dots, \widetilde{y}_r$ of y have local degrees $k_1, \dots, k_r$. Fix disjoint trivializing neighborhoods $\widetilde{U}_1, \dots, \widetilde{U}_r$ of $\widetilde{y}_1, \dots, \widetilde{y}_r$ respectively. Let $U \subseteq \Sigma$ be an open neighborhood of x which is homeomorphic to $\mathbb{R}^2$ and such that $f^{-1}(U) \subseteq \widetilde{U}_1 \cup \ldots \cup \widetilde{U}_r$. Since γ is a loop around y, we can write $\gamma = \alpha * \beta * \iota(\alpha)$ as described above. Up to homotopy, we can assume that $\beta(t) \in U$ for every $t \in [0,1]$. Let $z = \alpha(1) = \beta(0) = \beta(1) \in U$; note that every preimage of z lies exactly in one $\widetilde{U}_l$. Finally, let $\widetilde{x}_1, \dots, \widetilde{x}_d$ be the preimages of x_0. We will show that $\widetilde{x}_i$ and $\widetilde{x}_j$ belong to the same orbit of $\mathfrak{m}(\gamma)$ if and only if $\mathrm{lift}(\alpha, \widetilde{x}_i)(1)$ and $\mathrm{lift}(\alpha, \widetilde{x}_j)(1)$ lie in the same $\widetilde{U}_l$. Since $f^{-1}(z) \cap \widetilde{U}_l$ has cardinality k_l and $\pi(y) = [k_1, \dots, k_r]$, this would complete the proof.

Let $\widetilde{x}_i \in f^{-1}(x_0)$, $\widetilde{z}_1 = \mathrm{lift}(\alpha, \widetilde{x}_i)(1)$. Let l be the unique index such that $\widetilde{z}_1 \in \widetilde{U}_l$. For each $m \geq 1$, inductively define $\widetilde{z}_{m+1} = \mathrm{lift}(\beta, \widetilde{z}_m)(1)$. Since the support of β lies entirely in U, we have that the support of $\mathrm{lift}(\beta, \widetilde{z}_m)$ lies entirely in $\widetilde{U}_l$ and, therefore, $\widetilde{z}_{m+1} \in \widetilde{U}_l$ as well. Since $\widetilde{U}_l$ is a trivializing neighborhood of $\widetilde{y}_l$, which has local degree k_l, it is easy to see that the sequence $\widetilde{z}_1, \widetilde{z}_2, \dots$ is periodic of period k_l, and that $f^{-1}(z) \cap \widetilde{U}_l = \{\widetilde{z}_1, \dots \widetilde{z}_{k_l}\}$. It is also clear that

$$\mathfrak{m}(\gamma)(\widetilde{x}_i) = \mathrm{lift}(\iota(\alpha), \widetilde{z}_2)(1)$$

and, by induction, that

$$\mathfrak{m}(\gamma)^s(\widetilde{x}_i) = \mathrm{lift}(\iota(\alpha), \widetilde{z}_{s+1})(1) \qquad \text{for every } s \geq 1.$$

This shows that, if $\widetilde{x}_j = \mathfrak{m}(\gamma)^s(\widetilde{x}_i)$, then $\mathrm{lift}(\alpha, \widetilde{x}_j)(1) = \widetilde{z}_{s+1} \in \widetilde{U}_l$.

Conversely, assume that $\mathrm{lift}(\alpha, \widetilde{x}_j)(1) \in \widetilde{U}_l$. This implies that $\mathrm{lift}(\alpha, \widetilde{x}_j)(1) = \widetilde{z}_s$ for some $s \geq 1$. But then

$$\widetilde{x}_j = \mathrm{lift}(\iota(\alpha), \widetilde{z}_s)(1) = \mathfrak{m}(\gamma)^{s-1}(\widetilde{x}_i),$$

so $\widetilde{x}_i$ and $\widetilde{x}_j$ belong to the same orbit of $\mathfrak{m}(\gamma)$. □

1.5 Monodromy and realizability

From Proposition 1.5, we can derive a group-theoretic criterion for the realizability of a given combinatorial datum.

Proposition 1.6. *Let Σ be a surface, $d \geq 1$ an integer, $\pi_1, \dots, \pi_n \in \Pi(d)$ partitions of d. Let $x_1, \dots x_n \in \Sigma$ be distinct points; define $\Sigma^\bullet = \Sigma \smallsetminus \{x_1, \dots, x_n\}$. Fix a base-point $x_0 \in \Sigma^\bullet$ and loops $\gamma_1, \dots, \gamma_n \in \pi_1(\Sigma^\bullet, x_0)$ around $x_1, \dots, x_n$ respectively. Then the following are equivalent.*

(i) There exist a surface $\widetilde{\Sigma}$ and a branched covering $f\colon \widetilde{\Sigma} \to \Sigma$ with branching datum $\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \dots, \pi_n)$.

(ii) There exists a group homomorphism $\psi\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ such that $\psi(\pi_1(\Sigma^\bullet, x_0)) \leq \mathfrak{S}_d^{\mathrm{op}}$ acts transitively on $\{1, \dots, d\}$ and $[\psi(\gamma_i)] = \pi_i$ for every $1 \leq i \leq n$.

Proof. Assume we are given a branched covering $f\colon \widetilde{\Sigma} \to \Sigma$ with $\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \dots, \pi_n)$. Since surfaces are homogeneous, we can assume that the branching points are exactly $x_1, \dots, x_n$, with associated partitions $\pi_1, \dots, \pi_n$ respectively. Let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ be the induced covering map, and let $\mathfrak{m}\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ be the monodromy homomorphism. By Proposition 1.5 we have $[\mathfrak{m}(\gamma_i)] = \pi_i$ for every $1 \leq i \leq n$, and the action of $\mathfrak{m}(\pi_1(\Sigma^\bullet, x_0))$ on $\{1, \dots, d\}$ is transitive since $\widetilde{\Sigma}^\bullet$ is connected. Therefore setting $\psi = \mathfrak{m}$ gives the desired homomorphism.

Assume conversely that we are given a group homomorphism $\psi\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$. Theorem 1.4 gives a covering map $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ of degree d with monodromy $\mathfrak{m} = \psi$. The non-compact surface $\widetilde{\Sigma}^\bullet$ is connected by the transitivity hypothesis. We can "fill the holes" in $\Sigma^\bullet$ and $\widetilde{\Sigma}^\bullet$ to get a branched covering between compact surfaces; here are the details. Consider one of the points x_i, and fix an open neighborhood $U \subseteq \Sigma$ of x_i homeomorphic to $\mathbb{R}^2$. Now $(f^\bullet)^{-1}(U \smallsetminus \{x_i\})$ is a covering space of $U \smallsetminus \{x_i\} \simeq \mathbb{R}^2 \smallsetminus \{0\}$. By the classification of covering spaces, we get that $(f^\bullet)^{-1}(U \smallsetminus \{x_i\})$ is a disjoint union of punctured disks $\widetilde{V}_1 \sqcup \ldots \sqcup \widetilde{V}_r$; moreover, for each $1 \leq j \leq r$ we have charts $\varphi_j\colon U \smallsetminus \{x_i\} \to \mathbb{C} \smallsetminus \{0\}$, $\widetilde{\varphi}_j\colon \widetilde{V}_j \to \mathbb{C} \smallsetminus \{0\}$ and a positive integer k_j such that the diagram

$$\begin{tikzcd} \widetilde{V}_j \arrow[r, "f^\bullet"] \arrow[d, "\widetilde{\varphi}_j"] & U \smallsetminus \{x_i\} \arrow[d, "\varphi_j"] \\ \mathbb{C} \smallsetminus \{0\} \arrow[r, "F_{k_j}"] & \mathbb{C} \smallsetminus \{0\} \end{tikzcd}$$

commutes, where $F_{k_j}(z) = z^{k_j}$. We can "fill the hole" in $\widetilde{V}_j$ by considering the surface $\widetilde{\Sigma}_1^\bullet = \widetilde{\Sigma}^\bullet \sqcup \mathbb{C}/\sim$, where $y \sim \widetilde{\varphi}_j(y)$ for each $y \in \widetilde{V}_j$. It is clear that $f^\bullet$ extends to a continuous function $\widetilde{\Sigma}_1^\bullet \to \Sigma$ by sending $0 \in \mathbb{C}$ to x_i; this map is locally modeled on $\xi \mapsto \xi^{k_j}$ near 0. If we "fill the hole" in $\widetilde{V}_j$ for $1 \leq j \leq r$, and then repeat the process for each x_i, we end up with a surface $\widetilde{\Sigma}$ and a map $f\colon \widetilde{\Sigma} \to \Sigma$. It is easy to see from the construction that $\widetilde{\Sigma}$ is compact and connected, and that f is a branched covering. Since the associated covering map is exactly $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$, Proposition 1.5 implies that $\pi(x_i)$ matches $\mathfrak{m}(\gamma_i) = \psi(\gamma_i)$, so $\pi(x_i) = \pi_i$. □

By combining Proposition 1.6 with the classification of surfaces, we obtain the following criteria for realizability.

Corollary 1.7. *Let Σ_g be the connected sum of $g \geq 0$ tori, $d \geq 1$ an integer, $\pi_1, \dots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \dots, \pi_n)$ if and only if there exist permutations $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_g \in \mathfrak{S}_d$ such that:*

(i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;

(ii) $[\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g] \cdot \alpha_1 \cdots \alpha_n = 1$;

(iii) the subgroup $\langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \dots, d\}$.

In this case, $\widetilde{\Sigma}$ is necessarily orientable.

Proof. Let $\Sigma_g^\bullet$ be the non-compact surface obtained by removing n points $x_1, \ldots, x_n$ from Σ_g. Fix a base-point $x_0 \in \Sigma_g^\bullet$. Once we observe that $\pi_1(\Sigma_g^\bullet, x_0)$ has a presentation

$$\pi_1(\Sigma_g^\bullet, x_0) = \langle a_1, \ldots, a_n, b_1, \ldots, b_g, c_1, \ldots, c_g \mid [b_1, c_1] \cdots [b_g, c_g] \cdot a_1 \cdots a_n \rangle ,$$

where a_i is a loop around x_i, the criterion follows immediately from Proposition 1.6. The orientability of $\widetilde{\Sigma}$ was already addressed in Remark 1.1. □

Remark 1.2. Given Σ_g, d and $\pi_1, \ldots, \pi_n$, there is at most one surface $\widetilde{\Sigma}$ such that $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum. In fact, the Riemann-Hurwitz formula gives

$$\chi(\widetilde{\Sigma}) = d\chi(\Sigma_g) - v(\pi_1) - \ldots - v(\pi_n)$$

which, in turn, uniquely determines the orientable surface $\widetilde{\Sigma}$. As an application, assume that we have a candidate datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$, and we find permutations in $\mathfrak{S}_d$ satisfying conditions (i)–(iii) of Corollary 1.7: this immediately implies that $\mathcal{D}$ is realizable.

Before stating the monodromy criterion for non-orientable surfaces, we recall a few basic facts about covering maps and orientability.

Given a non-orientable topological manifold M, let $\omega\colon \widehat{M} \to M$ be its orientable double covering. Fix a base-point $x_0 \in M$, and consider the monodromy homomorphism

$$w\colon \pi_1(M, x_0) \longrightarrow \mathfrak{S}(\omega^{-1}(x_0)) \simeq \mathbb{Z}/2.$$

We denote the kernel of w (which is a subgroup of $\pi_1(M, x_0)$ of index 2) by $W(M, x_0)$. Note that $W = \omega_*(\pi_1(\widehat{M}, \widehat{x}_0))$ for every base-point $\widehat{x}_0 \in \omega^{-1}(x_0)$, where $\omega_*\colon \pi_1(\widehat{M}, \widehat{x}_0 \to \pi_1(M, x_0)$ denotes the map induced by ω on the fundamental groups. Given a connected topological manifold N and a covering map $f\colon N \to M$, elementary properties of covering spaces imply that the following are equivalent:

(i) N is orientable;

(ii) $f_*(\pi_1(N, y_0)) \leq W(M, x_0)$ for every base-point $y_0 \in f^{-1}(x_0)$;

(iii) there exists a covering map $\widehat{f}\colon N \to \widehat{M}$ such that f factors as $f = \omega \circ \widehat{f}$.

Corollary 1.8. *Let N_g be the connected sum of $g \geq 1$ projective planes, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \in \mathfrak{S}_d$ such that:*

(i) $[\alpha_i] = \pi_i$ *for each* $1 \leq i \leq n$*;*

(ii) $\beta_1^2 \cdots \beta_g^2 \cdot \alpha_1 \cdots \alpha_n = 1$*;*

(iii) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \rangle \leq \mathfrak{S}_d$ *acts transitively on* $\{1, \ldots, d\}$*.*

In this case, $\widetilde{\Sigma}$ is non-orientable if and only if there exists a permutation $\gamma \in \mathfrak{S}_d$ which has a fixed point and can be written as the product of an odd number of β_j and any number of α_i.

Proof. The proof of the criterion is identical to that of Corollary 1.7: simply let $N_g^\bullet = N_g \setminus \{x_1, \ldots, x_n\}$ and observe that there is a presentation

$$\pi_1(N_g^\bullet, x_0) = \left\langle a_1, \ldots, a_n, b_1, \ldots, b_g \mid b_1^2 \cdots b_g^2 \cdot a_1 \cdots a_n \right\rangle ,$$

where a_i is a loop around x_i. As far as the orientability of $\widetilde{\Sigma}$ is concerned, note that $w(a_1) = \ldots = w(a_n) = 0$, while $w(b_1) = \ldots = w(b_g) = 1$. Let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to N_g^\bullet$ be the covering map associated to the branched covering, and let $\mathfrak{m}\colon \pi_1(N_g^\bullet, x_0) \to \mathfrak{S}_d$ be the monodromy homomorphism. Fix a base-point $\widetilde{x}_0 \in f^{-1}(x_0)$; we have that $\widetilde{\Sigma}^\bullet$ is orientable if and only if $f_*(\pi_1(\widetilde{\Sigma}^\bullet, \widetilde{x}_0)) \leq W(N_g^\bullet, x_0)$. Since $f_*(\pi_1(\widetilde{\Sigma}^\bullet, \widetilde{x}_0)) = \mathrm{Stab}_{\pi_1(N_g^\bullet, x_0)}(\widetilde{x}_0)$, it follows that $\widetilde{\Sigma}^\bullet$ is non-orientable if and only if there exists a loop $c \in \pi_1(N_g^\bullet, x_0)$ such that $w(c) = 1$ and $\widetilde{x}_0 \bullet c = \widetilde{x}_0$. By applying $\mathfrak{m}$, we see that this is equivalent to the existence of a permutation $\gamma \in \mathfrak{S}_d$ which has a fixed point and can be written as the product of an odd number of β_j and an arbitrary number of α_i; note that the exact fixed point does not matter, since the subgroup generated by the α_i and β_j acts transitively on $\{1, \ldots, d\}$. Finally, observe that $\widetilde{\Sigma}^\bullet$ is orientable if and only if $\widetilde{\Sigma}$ is. □

Remark 1.3. Given N_g, d and $\pi_1, \ldots, \pi_n$, there are at most two surfaces $\widetilde{\Sigma}$, one orientable and one non-orientable, such that $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum: like in Remark 1.2, the Riemann-Hurwitz formula fixes $\chi(\widetilde{\Sigma})$ which, together with orientability, uniquely determines the surface $\widetilde{\Sigma}$.

Moreover, we can show that realizability in the case where Σ is non-orientable and $\widetilde{\Sigma}$ is orientable can be reduced to a situation where both surfaces are orientable.

Proposition 1.9. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots \pi_n)$ be a combinatorial datum, with Σ non-orientable and $\widetilde{\Sigma}$ orientable. Then $\mathcal{D}$ is realizable if and only if d is even and there exist partitions $\pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n \in \Pi(d/2)$ with $\pi_i = \pi'_i \cup \pi''_i$ for each $1 \leq i \leq n$, such that the combinatorial datum*

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_i, \ldots, \pi''_n)$$

is realizable, where $\widehat{\Sigma}$ is the orientable double covering of Σ.

owever, not every litting leads to a re-zable datum.

Proof. Assume that $\mathcal{D}$ is realized by a branched covering $f\colon \widetilde{\Sigma} \to \Sigma$. Let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ be the associated covering map, and let $\omega\colon \widehat{\Sigma} \to \Sigma$ be the orientable double covering. Define $\widehat{\Sigma}^\bullet = \omega^{-1}(\Sigma^\bullet)$, and denote by $\omega^\bullet$ the restriction of ω to $\widehat{\Sigma}^\bullet$; clearly, $\omega^\bullet\colon \widehat{\Sigma}^\bullet \to \Sigma^\bullet$ is the orientable double covering of $\Sigma^\bullet$. Since $\widetilde{\Sigma}^\bullet$ is orientable while $\Sigma^\bullet$ is not, $f^\bullet$ factors as $f^\bullet = \widehat{f}^\bullet \circ \omega^\bullet$ for some covering map $\widehat{f}^\bullet\colon \widetilde{\Sigma}^\bullet \to \widehat{\Sigma}^\bullet$; in particular, d is even and $\widehat{f}^\bullet$ has degree $d/2$. The conclusion follows from a routine topological argument; for the sake of completeness, we will now report the details. Fix a branching point $x \in \Sigma \setminus \Sigma^\bullet$; let $f^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_r\}$ and $\omega^{-1}(x) = \{\widehat{x}_1, \widehat{x}_2\}$. There exists a small open neighborhood $U \subseteq \Sigma$ of x homeomorphic to $\mathbb{R}^2$ such that:

- $f^{-1}(U) = \widetilde{U}_1 \sqcup \ldots \sqcup \widetilde{U}_r$, where $\widetilde{U}_i$ is a trivializing neighborhood of $\widetilde{x}_i$;
- $\omega^{-1}(U) = \widehat{U}_1 \sqcup \widehat{U}_2$, where $\widehat{U}_i$ is a trivializing neighborhood of $\widehat{x}_i$.

It is now clear that $\widehat{f}^\bullet\colon \widetilde{\Sigma}^\bullet \to \widehat{\Sigma}^\bullet$ extends to a branched covering $\widehat{f}\colon \widetilde{\Sigma} \to \widehat{\Sigma}$: for each $1 \leq i \leq r$, we simply set $\widehat{f}(\widetilde{x}_i) = \widehat{x}_1$ if $\widehat{f}^\bullet(\widetilde{U}_i \setminus \{\widetilde{x}_i\}) = \widehat{U}_1 \setminus \{\widehat{x}_1\}$, and $\widehat{f}(\widetilde{x}_i) = \widehat{x}_2$ otherwise; obviously, we must repeat this process for every branching point x. The branched covering $\widehat{f}$ has degree $d/2$; moreover, it is easy to see from the construction that $\pi_f(x) = \pi_{\widehat{f}}(\widehat{x}_1) \cup \pi_{\widehat{f}}(\widehat{x}_2)$ (we use the subscript in order to clarify which branched covering we are referring to). Therefore the realizable combinatorial datum $\mathcal{D}(\widehat{f})$ has the required form.

Conversely, assume that d is even and we are given a branched covering $\widehat{f}\colon \widetilde{\Sigma} \to \widehat{\Sigma}$ with branching datum

$$\mathcal{D}(\widehat{f}) = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n),$$

where $\pi_i = \pi'_i \cup \pi''_i$ for every $1 \leq i \leq n$. Let $x'_1, \ldots, x'_n, x''_1, \ldots, x''_n \in \widehat{\Sigma}$ be the branching points corresponding to the similarly named partitions of $\mathcal{D}(\widehat{f})$. Since surfaces are homogeneous, we can assume that $\omega(x_i') = \omega(x_i'')$ for each $1 \leq i \leq n$, where $\omega\colon \widehat{\Sigma} \to \Sigma$ is the covering map. It is now easy to see that

$$\mathcal{D}(\omega \circ \widehat{f}) = (\widetilde{\Sigma}, \Sigma; d; \pi_1' \cup \pi_1'', \dots, \pi_n' \cup \pi_n''). \qquad \square$$

We conclude this introductory chapter by showing that the four conditions described in Definition 1.5 are actually necessary for a combinatorial datum to be realizable.

Proposition 1.10. *Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering. Then its branching datum $\mathcal{D}(f)$ is a candidate datum.*

Proof. Conditions (i) and (iii) of Definition 1.5 were already addressed, respectively, in Proposition 1.2 and Remark 1.1. Condition (iv) follows immediately from Proposition 1.9, so we only have to show that the total branching number $v(\pi_1) + \ldots + v(\pi_n)$ is even. Using the notations of Corollaries 1.7 and 1.8, by Proposition 1.3 we have that

$$v(\alpha_1 \cdots \alpha_n) \equiv \begin{cases} v([\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g]) & \text{if } \Sigma \text{ is orientable,} \\ v(\beta_1^2 \cdots \beta_g^2) & \text{if } \Sigma \text{ is non-orientable} \end{cases} \pmod 2.$$

But commutators and squares are even permutations, therefore

$$v(\pi_1) + \ldots + v(\pi_n) = v(\alpha_1) + \ldots v(\alpha_n) \equiv v(\alpha_1 \cdots \alpha_n) \equiv 0 \pmod 2. \qquad \square$$

Chapter 2

Realizablility by monodromy

2.1 Non-positive Euler characteristic

The monodromy approach developed in the previous chapter will allow us to completely solve the Hurwitz existence problem for a large class of candidate data. For the majority of this chapter, we will closely follow [1], with a slight rewording of the exposition. Perhaps, the most notable deviation will occur in Sections 2.4 and 2.5, where we will introduce *combinatorial moves* and apply them extensively in our proofs. It should be noted, however, that combinatorial moves can make some arguments easier to follow and less repetitive, but they do not provide any conceptual innovations; the core ideas of the proofs are the same as those presented in [1].

Say “aforementioned article” instead of citi again?

With the following elementary result about symmetric groups, we will already be able to fully address the cases where $\chi(\Sigma) \leq 0$.

Lemma 2.1. *Let $\alpha \in \mathfrak{S}_d$ be a permutation. Set $r = d - v(\alpha)$, and let $t \geq 0$ be an integer such that $2t \leq v(\alpha)$. Then α can be written as the product of a $(r+2t)$-cycle and a d-cycle.*

Proof. Without loss of generality, assume that

$$\alpha = (1, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{r-1} + 1, \dots, d_r),$$

where $d_r = d$. Fix

$$\beta_0 = (1, b_1, b_2, \dots, b_{2t}), \qquad \beta_1 = (1, d_1 + 1, d_2 + 1, \dots, d_{r-1} + 1),$$

where $b_1 < b_2 < \ldots < b_{2t}$ are elements of $\{1, \dots, d\} \smallsetminus \{d_1 + 1, \dots, d_{r-1} + 1\}$. Set

$$\beta = \beta_0\beta_1 = (1, d_1 + 1, d_2 + 1, \dots, d_{r-1} + 1, b_1, b_2, \dots, b_{2t}).$$

An easy computation shows that

$$\begin{aligned}\beta\alpha &= \beta_0\beta_1\alpha \\ &= (1, b_1, b_2, \dots, b_{2t})(1, 2, \dots, d) \\ &= (1, \dots, b_1 - 1, b_2, \dots, b_3 - 1, \dots, b_4, \dots, b_{2t-1} - 1, b_{2t}, \dots, d, b_1, \dots, b_2 - 1, b_3, \dots, b_{2t} - 1).\end{aligned}$$

Writing $\alpha = \beta^{-1}(\beta\alpha)$ gives the desired decomposition. $\square$

Corollary 2.2. *Let $\alpha \in \mathfrak{A}_d$ be an even permutation. Then α can be written as:*

(i) a commutator $[\beta, \gamma]$, where γ is a d-cycle;

(ii) a product of two squares $\delta^2\epsilon^2$, where $\delta\epsilon$ is a d-cycle.

Proof. Since α is an even permutation, its branching number $v(\alpha)$ is even. By Lemma 2.1, there exist two d-cycles $\tau, \sigma \in \mathfrak{S}_d$ such that $\alpha = \tau\sigma$.

(i) Since τ and σ^{-1} are conjugated, there exists a permutation $\beta \in \mathfrak{S}_d$ such that $\tau = \beta\sigma^{-1}\beta^{-1}$. Setting $\gamma = \sigma^{-1}$, we immediately get that

$$\alpha = \tau\sigma = \beta\sigma^{-1}\beta^{-1}\sigma = \beta\gamma\beta^{-1}\gamma^{-1} = [\beta, \gamma].$$

(ii) Since τ and σ are conjugated, there exists a permutation $\delta \in \mathfrak{S}_d$ such that $\tau = \delta\sigma\delta^{-1}$. Setting $\epsilon = \delta^{-1}\sigma$, we have that

$$\alpha = \tau\sigma = \delta\sigma\delta^{-1}\sigma = \delta^2(\delta^{-1}\sigma)^2 = \delta^2\epsilon^2.$$ □

Theorem 2.3. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\chi(\Sigma) \leq 0$, then $\mathcal{D}$ is realizable.*

Proof. Let us first assume that Σ is orientable; this means that $\Sigma = \Sigma_g$ is the connected sum of $g \geq 1$ tori, and that $\widetilde{\Sigma}$ is orientable as well. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Since $\mathcal{D}$ is a candidate datum, we have that

$$v(\alpha) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

By Corollary 2.2, we can find permutations $\beta_1, \gamma_1 \in \mathfrak{S}_d$ such that $\alpha = [\gamma_1, \beta_1]$ and β_1 is a d-cycle. Set $\beta_2 = \ldots = \beta_g = \gamma_2 = \ldots = \gamma_g = \mathrm{id} \in \mathfrak{S}_d$. All the conditions of Corollary 1.7 are satisfied; since $\widetilde{\Sigma}$ is orientable, this implies that $\mathcal{D}$ is realizable (see Remark 1.2).

Assume now that Σ and $\widetilde{\Sigma}$ are both non-orientable; this means that $\Sigma = N_g$ is the connected sum of $g \geq 2$ projective planes. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Similarly to what we did for the previous case, we can find $\beta_1, \beta_2 \in \mathfrak{S}_d$ such that $\alpha = \beta_2^{-2}\beta_1^{-2}$ and $\beta_1\beta_2$ is a d-cycle. Note that, since $\beta_1\beta_2$ is a d-cycle, there exists an integer $m \geq 0$ such that $\beta_1(\beta_1\beta_2)^m$ has a fixed point. By setting $\beta_3 = \ldots = \beta_g = \mathrm{id} \in \mathfrak{S}_d$, Corollary 1.8 (together with Remark 1.3) implies the realizability of $\mathcal{D}$.

Finally, consider the case where Σ is non-orientable and $\widetilde{\Sigma}$ is orientable. Since $\mathcal{D}$ is a candidate datum, d is even and there exist partitions $\pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n \in \Pi(d/2)$ with $\pi_i = \pi'_1 \cup \pi''_i$ for each $1 \leq i \leq n$. Let $\widehat{\Sigma}$ be the double orientable covering of Σ; by the first case we analyzed, the candidate datum

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n)$$

is realizable. By Proposition 1.9, $\mathcal{D}$ is realizable as well. □

2.2 Products in symmetric groups

The monodromy approach can prove to be quite fruitful also in the cases where $\chi(\Sigma) > 0$ (namely, when Σ is either the sphere $\mathbb{S}$ or the projective plane $\mathbb{RP}^2$). In order to deal with this instances of the Hurwitz existence problem, however, we will need a few more technical results about products of permutations.

Lemma 2.4. *Let X, Y be finite sets; denote by h the cardinality of Y, and by k the cardinality of $X \cap Y$. Let $\alpha \in \mathfrak{S}(X)$, $\beta \in \mathfrak{S}(Y)$, $\gamma \in \mathfrak{S}(X \cap Y)$. Assume that $\beta = (b_1, \ldots, b_h)$ is a h-cycle, and that γ is a k-cycle of the form $\gamma = (b_{i_1}, \ldots, b_{i_k})$ with $1 \le i_1 \le \ldots \le i_k \le h$. Then $\alpha \in \mathfrak{S}(X)$ and $\alpha\gamma^{-1}\beta \in \mathfrak{S}(X \cup Y)$ have the same number of cycles.*

Proof. Write $\gamma = (u_1, \ldots, u_k)$, where $u_j = b_{i_j}$. Without loss of generality, assume that $i_1 = 1$. Then we can write

$$\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{k-1}, u_k, \ldots, w_k),$$

possibly with $u_j = w_j$ for some values of j. We immediately get that

$$\gamma^{-1}\beta = (u_1, \ldots, w_1)(u_2, \ldots, w_2) \cdots (u_k, \ldots, w_k).$$

If we denote by $A_1, \ldots, A_r \subseteq X$ the orbits of α, it is then easy to see that the orbits of $\alpha\gamma^{-1}\beta$ are $A'_1, \ldots, A'_r \subseteq X \cup Y$, where

$$A'_j = A_j \cup \bigcup_{u_l \in A_j} \{u_l, \ldots, w_l\}. \qquad \square$$

Proposition 2.5. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \le d - 1$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = v(\alpha) + v(\beta)$.*

Proof. First of all, note that the conclusion is trivial whenever $v(\pi) = 0$ or $v(\rho) = 0$. This already solves the cases $d = 1$ and $d = 2$. We now proceed by induction on $d \ge 3$, assuming that $v(\pi) > 0$ and $v(\rho) > 0$. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$; without loss of generality, assume that $b_1 > 1$. Fix

$$\beta = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where $b_1 = d_1 \ge 2$, $b_i = d_i - d_{i-1}$ for $2 \le i \le s$ (in particular, $d_s = d$). Note that

$$d - 1 \ge v(\pi) + v(\rho) = (a_1 - 1) + \ldots + (a_r - 1) + d - s \ge a_1 - 1 + d - s,$$

hence $a_1 \le s$. Fix $\alpha_1 = (1, d_1 + 1, \ldots, d_{a_1 - 1} + 1)$, and let $A = \{1, d_1 + 1, \ldots, d_{a_1-1} + 1\}$ be the support of α_1. Define $Q = \{1, \ldots, d_{a_1}\} \smallsetminus A$; note that Q is non-empty, since $d_1 + 1 \ge 3$ implies that $2 \in Q_1$. Consider the partitions

$$\pi' = [a_2, a_3, \ldots, a_r], \qquad \rho' = [|Q|, b_{a_1+1}, \ldots, b_s].$$

We have that

$$\sum \pi' = \sum \rho' = d - a_1, \qquad v(\pi') = v(\pi) - a_1 + 1, \qquad v(\rho') = v(\rho) - 1.$$

Since $d - a_1 < d$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - a_1 < d - a_1$, by induction we find $\alpha', \beta' \in \mathfrak{S}(\{1, \ldots, d\} \smallsetminus A)$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $v(\alpha'\beta') = v(\alpha') + v(\beta')$. Up to conjugation, we may assume that

$$\beta' = \beta_1(d_{a_1} + 1, \ldots, d_{a_1+1})(d_{a_1+1} + 1, \ldots, d_{a_1+2}) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where β_1 is the $|Q|$-cycle whose entries are the elements of Q in increasing order. An easy computation shows that

$$\alpha_1\beta = (1, \ldots, d_{a_1})(d_{a_1} + 1, \ldots, d_{a_1+1}) \cdots (d_{s-1} + 1, \ldots, d_s).$$

Therefore, setting $\alpha = \alpha'\alpha_1$, we have that

$$\begin{aligned}\alpha\beta &= \alpha_1\alpha'\beta \\ &= \alpha'(1, \dots, d_{a_1})(d_{a_1}+1, \dots, d_{a_1+1}) \cdots (d_{s-1}+1, \dots, d_s) \\ &= \alpha'\beta'\beta_1^{-1}(1, \dots, d_{a_1})\end{aligned}$$

By Lemma 2.4, this implies that $\alpha\beta$ has the same number of cycles as $\alpha'\beta'$, so that

$$\begin{aligned}v(\alpha\beta) &= a_1 + v(\alpha'\beta') \\ &= a_1 + v(\alpha') + v(\beta') \\ &= a_1 + (v(\pi) - a_1 + 1) + (v(\rho) - 1) \\ &= v(\pi) + v(\rho).\end{aligned}$$

Since $[\alpha] = \pi$ and $[\beta] = \rho$, the conclusion follows. □

Proposition 2.6. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d - 1$ and $v(\pi) + v(\rho) \equiv d - 1 \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\alpha\beta$ is a d-cycle.*

Proof. If $v(\pi) + v(\rho) = d - 1$, the conclusion follows immediately from Proposition 2.5; therefore, we can assume that $v(\pi) + v(\rho) \geq d$.

Write $\pi = [a_1, \dots, a_r]$. Since $v(\rho) \leq d - 1$ and $v(\pi) + v(\rho) \geq d$, there exists a largest integer $0 \leq i \leq r$ such that $(a_1 - 1) + \dots (a_i - 1) + v(\rho) \leq d - 1$. Define

$$z = d - v(\rho) - (a_1 - 1) - \dots - (a_i - 1).$$

Consider the partition $\pi' = [a_1, \dots, a_i, z, 1, \dots, 1] \in \Pi(d)$. Since by construction $v(\pi') + v(\rho) = d - 1$, thanks to Proposition 2.5 we can find permutations $\alpha', \beta \in \mathfrak{S}_d$ with $[\alpha'] = \pi'$ and $[\beta] = \rho$ such that $v(\alpha'\beta) = d - 1$; in other words, $\alpha'\beta$ is a d-cycle. Consider now the partition $\pi'' = [a_{i+1} - z + 1, a_{i+2}, \dots, a_r]$, whose branching number is $v(\pi'') = v(\pi) + v(\rho) - d + 1$. Let $n = \sum \pi''$; fix an element u_1 of the z-cycle of α', and let $u_2, \dots, u_n$ be the fixed points of α' corresponding to the last ones of π' (it is easy to see that there are exactly $n - 1$ such ones). Since $v(\pi'')$ is even, Lemma 2.1 gives permutations $\alpha'', \gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$ such that $[\alpha''] = \pi''$ and γ and $\alpha''\gamma$ are n-cycles. Up to conjugation, we can assume that $\gamma = (u_1, \dots, u_n)$. Moreover, it is not restrictive to assume that $u_1, \dots, u_n$ appear in this order in the d-cycle $\alpha'\beta$. Therefore, setting $\alpha = \alpha''\alpha'$, we have that

$$\alpha\beta = (\alpha''\gamma)\gamma^{-1}(\alpha'\beta);$$

by Lemma 2.4, this implies that $\alpha\beta$ has the same number of cycles as $\alpha''\gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$, that is 1. Since $[\alpha] = \pi$ and $[\beta] = \rho$, the proof is complete. □

Remark 2.1. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$; assume that $b_1 \geq 2$. By directly examining the proof of Proposition 2.6, we see that the proposed construction yields permutations $\alpha, \beta \in \mathfrak{S}_d$ such that 1 belongs to both the a_1-cycle of α and to the b_1-cycle of β. As a consequence, the statement of that proposition can be enhanced by adding the following line: *α and β can be chosen in such a way that 1 belongs to the a_1-cycle of α and to the b_1-cycle of β, provided that* $b_1 \geq 2$. We will need this improvement for the upcoming proof.

Proposition 2.7. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d$ and $v(\pi) + v(\rho) \equiv d \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \dots, d\}$ and*

$$[\alpha\beta] = \begin{cases} [d/2, d/2] & \text{if } \pi = \rho = [2, \dots, 2], \\ [1, d-1] & \text{otherwise.} \end{cases}$$

Proof. Assume first that $\pi = \rho = [2, \dots, d]$. We can choose

$$\alpha = (2,3)(4,5)\cdots(d,1), \qquad \beta = (1,2)(3,4)\cdots(d-1,d).$$

The action of $\langle \alpha, \beta \rangle$ is obviously transitive, and

$$\alpha\beta = (1,3,\dots,d-1)(2,4,\dots,2).$$

Otherwise, since $v(\pi) + v(\rho) \geq d$, at least one of π and ρ has an entry which is greater than 2; without loss of generality, we can assume it is ρ. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$ with $a_1 \geq 2$ (since $v(\pi) \geq 1$) and $b_1 \geq 3$. Fix

$$\beta = (1,\dots,d_1)(d_1+1,\dots,d_2)\cdots(d_{s-1}+1,\dots,d_s),$$

where $b_1 = d_1 \geq 3$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Consider the partitions

$$\pi' = [a_1 - 1, \dots, a_r], \qquad \rho' = [b_1 - 1, \dots, b_s].$$

Since $\sum \pi' = \sum \rho' = d - 1$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - 2$, by Proposition 2.6 we can find permutations $\alpha', \beta' \in \mathfrak{S}(\{2, \dots, d\})$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $\alpha'\beta'$ is a $(d-1)$-cycle. Up to conjugation, we can assume that

$$\beta' = (2,\dots,d_1)(d_1+1,\dots,d_2)\cdots(d_{s-1}+1,\dots,d_s)$$

or, in other words, $\beta = (1,2)\beta'$; moreover, as explained in Remark 2.1, we can choose α' in such a way that its (a_1-1)-cycle contains 2. By setting $\alpha = \alpha'(1,2)$, we immediately get that $\alpha\beta = \alpha'\beta'$ is a $(d-1)$-cycle fixing 1. Finally, the action of $\langle \alpha, \beta \rangle$ is transitive since α does not fix 1. □

The following result sums up the statements of Propositions 2.6 and 2.7, at the cost of some generality. However, for many of the upcoming applications, it will be more than enough.

Corollary 2.8. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d - 1$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \dots, d\}$ and*

$$[\alpha\beta] \in \{[d], [1, d-1], [d/2, d/2]\}.$$

Proof. The conclusion follows immediately from Proposition 2.6 or Proposition 2.7 depending on the parity of $v(\pi) + v(\rho) + d$. □

2.3 Projective plane

As a first application of the results established in the previous section, we fully solve the Hurwitz existence problem in the cases where $\Sigma = \mathbb{RP}^2$ and $\widetilde{\Sigma}$ is non-orientable.

Theorem 2.9. *Let $\mathcal{D} = (\widetilde{\Sigma}, \mathbb{RP}^2; d; \pi_1, \dots, \pi_n)$ be a candidate datum. If $\widetilde{\Sigma}$ is non-orientable, then $\mathcal{D}$ is realizable.*

Proof. First of all, since $\mathcal{D}$ is a candidate datum, the Riemann-Hurwitz formula implies that

$$v(\pi_1) + \ldots + v(\pi_n) = d\chi(\mathbb{RP}^2) - \chi(\widetilde{\Sigma}) \geq d - 1$$

(recall that $\widetilde{\Sigma}$ is non-orientable, so $\chi(\widetilde{\Sigma}) \leq 1$). Moreover, the total branching $v(\pi_1) + \ldots + v(\pi_n)$ is even. In order to apply Corollary 1.8, we will now inductively define permutations $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$, satisfying the following invariant: for every $0 \leq i \leq n$, either

$$v(\alpha_1 \cdots \alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$$

or

$$[\alpha_1 \cdots \alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\} \text{ and } \langle \alpha_1, \ldots, \alpha_i \rangle \text{ acts transitively.}$$

Assume we have already defined $\alpha_1, \ldots, \alpha_{i-1}$; we want to suitably choose α_i. Let $\alpha = \alpha_1 \cdots \alpha_{i-1}$; there are two cases.

- If $v(\alpha) + v(\pi_i) < d$, by Proposition 2.5 we can find $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $v(\alpha\alpha_i) = v(\alpha) + v(\alpha_i)$. The invariant is still satisfied: if $v(\alpha) = v(\pi_1) + \ldots + v(\pi_{i-1})$ then obviously $v(\alpha\alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$. If instead $[\alpha] \in \{[d], [1, d-1], [d/2, d/2]\}$, then either α_i is the identity, or $\alpha\alpha_i$ is a d-cycle; either way, $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$. Note that if the action of $\langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ is transitive, then the action of $\langle \alpha_1, \ldots, \alpha_i \rangle$ is transitive as well.

- If $v(\alpha) + v(\alpha_i) \geq d$, Corollary 2.8 gives a permutation $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$ and the action of $\langle \alpha, \alpha_i \rangle$ is transitive. The invariant is obviously satisfied.

By induction, we can find $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that

$$[\alpha_1 \cdots \alpha_n] \in \{[d], [1, d-1], [d/2, d/2]\}$$

and $\langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$ (note that $v(\alpha_1 \cdots \alpha_n) = v(\alpha_1) + \ldots + v(\alpha_n)$ also implies that $[\alpha_1 \cdots \alpha_n] = [d]$). By Proposition 1.3 we have that

$$v(\alpha_1 \cdots \alpha_n) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

We now prove that $\alpha = \alpha_1 \cdots \alpha_n$ is a square.

- If $[\alpha] = [d]$, then d is odd, so α is the square of $\alpha^{(d+1)/2}$.
- If $[\alpha] = [1, d-1]$, then d is even, so α is the square of $\alpha^{d/2}$.
- If $[\alpha] = [d/2, d/2]$, then d is even, and it is easy to see that α is the square of a d-cycle.

In any case, we obtain a permutations $\beta_1 \in \mathfrak{S}_d$ such that $\alpha = \beta_1^{-2}$ By Corollary 1.8, this implies that there exists a realizable candidate datum $\mathcal{D}' = (\widetilde{\Sigma}', \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$. To see that $\widetilde{\Sigma}'$ is non-orientable (and, therefore, equal to $\widetilde{\Sigma}$, as shown in Remark 1.3), simply note that there exists a permutation $\gamma \in \langle \alpha_1, \ldots, \alpha_n \rangle$ such that $\gamma\beta_1$ has a fixed point, since $\langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$. □

2.4 Combinatorial moves

With Theorems 2.3 and 2.9, we have shown that every candidate datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ is realizable whenever

- $\chi(\Sigma) \leq 0$, or
- $\Sigma = \mathbb{RP}^2$ and $\widetilde{\Sigma}$ is non-orientable.

Moreover, Proposition 1.9 shows that the cases where $\Sigma = \mathbb{RP}^2$ and $\widetilde{\Sigma}$ is orientable can be reduced to the analysis of realizable candidate data on the sphere. As a consequence, it stands to reason that, from now on, we will only be concerned with candidate data where $\Sigma = \mathbb{S}$; in fact, a classification of the realizable data of this kind would lead to a full solution of the Hurwitz existence problem. However, we will soon find out that realizability on the sphere can be a very

be fair, even after lving the existence oblem on the sphere, mpiling the full list of ceptional data with $= \mathbb{RP}^2$ and $\widetilde{\Sigma}$ ori- table could still be n-trivial.

delicate matter: there are several seemingly unrelated families of exceptional data, and even to this day we are still far from a complete understanding of the problem.

Since, from now on, we will only deal with branched coverings on the sphere, we adopt the following convention: whenever we have a combinatorial datum $\mathcal{D} = (\Sigma_g, \mathbb{S}; d; \pi_1, \ldots \pi_n)$, we can omit the $\mathbb{S}$ and simply write $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots \pi_n)$.

For convenience, we will now briefly discuss once and for all the cases where $n \leq 2$.

- If $n = 1$, Corollary 1.7 immediately implies that every candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1)$ is not realizable (recall that we are always assuming $\pi_1 \neq [1, \ldots, 1]$).
- If $n = 2$, the only realizable candidate data are of the form $\mathcal{D} = (\mathbb{S}; d; [d], [d])$, again by Corollary 1.7.

Finally, we turn to the topic promised by the title of this section. *Combinatorial moves* are a conceptually trivial tool, but they offer a convenient framework for presenting proofs based on a reduction technique. The definition is extremely simple.

Definition 2.1. Let $\mathcal{D}, \mathcal{D}'$ be combinatorial data. We say that there is a *combinatorial move* between them, and we write $\mathcal{D} \rightsquigarrow \mathcal{D}'$, if the realizability of $\mathcal{D}'$ implies that $\mathcal{D}$ is realizable as well.

In this thesis we will encounter two kinds of combinatorial moves. The first one, which we introduce in this section, focuses on reducing the number of partitions of a combinatorial datum by means of products in the symmetric group. In Section 3.3, we will describe another kind of combinatorial moves, which relies instead on topological constructions to reduce the genus of the surface $\widetilde{\Sigma}$.

Combinatorial move A.1. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \ldots, \pi_n)$ be a candidate datum. Assume that $v(\pi_1) + v(\pi_2) \leq d - 1$. Then there exists a partition $\pi_1' \in \Pi(d)$ such that $v(\pi_1') = v(\pi_1) + v(\pi_2)$, $\mathcal{D}' = (\Sigma_g; d; \pi_1', \pi_3, \ldots, \pi_n)$ is a candidate datum and $\mathcal{D} \rightsquigarrow \mathcal{D}'$.*

Proof. By Proposition 2.5, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\pi_1) + v(\pi_2)$. Let $\pi_1' = [\alpha_1\alpha_2]$; it is easy to check that $\mathcal{D}' = (\Sigma_g; d; \pi_1', \pi_3, \ldots, \pi_n)$ is a candidate datum. Assume that $\mathcal{D}'$ is realizable; by Corollary 1.7, this implies the existence of permutations $\alpha_1', \alpha_3, \ldots, \alpha_n \in \mathfrak{S}_d$ matching $\pi_1', \pi_3, \ldots, \pi_n$ respectively, such that $\alpha_1'\alpha_3 \cdots \alpha_n = 1$ and $\langle \alpha_1', \alpha_3, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$. Since $[\alpha_1'] = [\alpha_1\alpha_2]$, up to conjugation we may assume that $\alpha_1' = \alpha_1\alpha_2$. It is clear that the permutations $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ imply the realizability of $\mathcal{D}$, again by Corollary 1.7 (see also Remark 1.2). □

Remark 2.2. The proof of Combinatorial move A.1 shows that we have some freedom in selecting the partition π_1': given two permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ matching, respectively, π_1 and π_2, we have the combinatorial move

$$\mathcal{D} \rightsquigarrow (\Sigma_g; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n),$$

provided that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$.

Combinatorial move A.2. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \ldots, \pi_n)$ be a candidate datum. Assume that:*

- $v(\pi_1) + v(\pi_2) \geq d - 1$*;*
- $v(\pi_3) + \ldots + v(\pi_n) \geq d - 1$.

Then there exist an orientable surface $\Sigma_{g'}$ and a partition $\pi_1' \in \Pi(d)$ such that $\mathcal{D}' = (\Sigma_{g'}; d; \pi_1', \pi_3, \ldots, \pi_n)$ is a candidate datum and $\mathcal{D} \rightsquigarrow \mathcal{D}'$. Moreover, if $\pi_1 = \pi_2 = [2, \ldots, 2]$ then π_1' can be chosen equal to $[d/2, d/2]$, otherwise it can be chosen in such a way that $\pi_1' \in \{[d], [1, d-1]\}$.

Proof. By Proposition 2.6 or Proposition 2.7, depending on the parity of $v(\pi_1) + v(\pi_2) + d$, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] \in \{[d], [1, d-1], [d/2, d/2]\}$. More specifically, if $\pi_1 = \pi_2 = [2, \dots, 2]$ we can assume that $[\alpha_1\alpha_2] = [d/2, d/2]$, otherwise we can choose $[\alpha_1\alpha_2] \in \{[d], [1, d-1]\}$. Let $\pi_1' = [\alpha_1\alpha_2]$; if we want $\mathcal{D}' = (\Sigma_{g'}; d; \pi_1', \pi_3, \dots, \pi_n)$ to be a candidate datum, the Riemann-Hurwitz formula requires that

$$g' = \frac{1}{2}(v(\pi_1') + v(\pi_3) + \ldots + v(\pi_n)) - d + 1.$$

But $v(\pi_1') \geq d - 2$ and $v(\pi_3) + \ldots + v(\pi_n) \geq d - 1$, so

$$v(\pi_1') + v(\pi_3) + \ldots + v(\pi_n) \geq 2d - 3.$$

Moreover, $v(\pi_1')$ and $v(\pi_1) + v(\pi_2)$ have the same parity, so $v(\pi_1') + v(\pi_3) + \ldots + v(\pi_n)$ is even and actually

$$v(\pi_1') + v(\pi_3) + \ldots + v(\pi_n) \geq 2d - 2.$$

Therefore g' is a non-negative integer, and $\mathcal{D}'$ is a candidate datum. The very same argument used in the proof of Combinatorial move A.1 shows that $\mathcal{D} \rightsquigarrow \mathcal{D}'$. □

2.5 Some results on the sphere

In this section, we will make extensive use of the combinatorial moves we have just described to solve the Hurwitz existence problem for some specific families of candidate data. As we have already discussed the realizability of candidate data with $n \leq 2$, from now on we will always assume that $n \geq 3$. We start by addressing the cases where one partition (without loss of generality, the last one) has length 1.

Proposition 2.10. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \dots, \pi_{n-1}, [d])$ be a candidate datum. Then $\mathcal{D}$ is realizable.*

Proof. We proceed by induction on n, starting with the base case $n = 3$. If $\mathcal{D} = (\Sigma_g, \mathbb{S}; d; \pi_1, \pi_2, [d])$ is a candidate datum, by the Riemann-Hurwitz formula we have that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d - 1) \geq d - 1.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 1$ is even. Proposition 2.6 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] = [d]$. By Corollary 1.7, $\mathcal{D}$ is realizable (see Remark 1.2).

We now turn to the case $n \geq 4$. Fix a candidate datum $\mathcal{D} = (\Sigma_g, \mathbb{S}; d; \pi_1, \dots, \pi_{n-1}, [d])$; there are two cases.

- Assume that $v(\pi_1) + v(\pi_2) \leq d - 1$. Combinatorial move A.1 gives

  $$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_g; d; \pi_1', \pi_3, \dots, \pi_{n-1}, [d])$$

  for a suitable candidate datum $\mathcal{D}'$. By induction, $\mathcal{D}'$ is realizable, therefore the same holds for $\mathcal{D}$.

- Otherwise, we have that $v(\pi_1) + v(\pi_2) \geq d$. Note that $v([d]) = d - 1$, so Combinatorial move A.2 gives

  $$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_g; d; \pi_1', \pi_3, \dots, \pi_{n-1}, [d])$$

  for a suitable candidate datum $\mathcal{D}'$. Similarly to the previous case, this implies that $\mathcal{D}$ is realizable. □

We now turn to candidate data containing the partition $[1, d-1]$. This is the first step towards the goal of this thesis, which will be further developed in Chapter 4: the full classification of exceptional data containing a partition of length 2.

Proposition 2.11. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \dots, \pi_{n-1}, [1, d-1])$ be a candidate datum. Then $\mathcal{D}$ is exceptional if and only if it satisfies one of the following.*

(1) $\mathcal{D} = (\mathbb{S}; 2k; [2, \dots, 2], [2, \dots, 2], [1, 2k-1])$ *with* $k \geq 2$.

(2) $\mathcal{D} = (\Sigma_{n-3}; 4; [2, 2], \dots, [2, 2], [1, 3])$.

Proof. The exceptionality of the listed candidate data is addressed in Propositions 4.6 and 4.9 respectively. In order to prove that every other candidate datum is realizable, we proceed by induction on n, starting from the base case $n = 3$. Let

$$\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [1, d-1]) \neq (\mathbb{S}; d; [2, \dots, 2], [2, \dots, 2], [1, d-1])$$

be a candidate datum. The Riemann-Hurwitz formula implies that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d-2) \geq d.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 2$ is even. Proposition 2.7 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1 \alpha_2] = [1, d-1]$ and the action of $\langle \alpha_1, \alpha_2 \rangle$ is transitive. As usual, Corollary 1.7 implies that $\mathcal{D}$ is realizable.

We now turn to the case $n \geq 4$; we will employ a reduction technique, similar to the proof of Proposition 2.10. Fix a candidate datum

$$\mathcal{D} = (\Sigma_{n-3}; d; \pi_1, \dots, \pi_{n-1}, [1, d-1]) \neq (\Sigma_{n-3}; 4; [2, 2], \dots, [2, 2], [1, 3]).$$

There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) \leq d-1$ holds for a pair of indices $1 \leq i < j \leq n-1$; up to reindexing, we can assume that $i = 1$ and $j = 2$. Combinatorial move A.1 gives

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_g; d; \pi'_1, \pi_3, \dots, \pi_{n-1}, [1, d-1])$$

  for a suitable candidate datum $\mathcal{D}'$, where $v(\pi'_1) = v(\pi_1) + v(\pi_2)$. By induction, $\mathcal{D}'$ is realizable unless one of the following happens.

  - $\mathcal{D}' = (\mathbb{S}; 2k; [2, \dots, 2], [2, \dots, 2], [1, 2k-1])$ with $2k = d$. If this is the case, then $n = 4$, $g = 0$ and $v(\pi'_1) = v(\pi_3) = k$. This implies that

$$k < 1 + k \leq v(\pi_1) + v(\pi_3) < 2k = d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

  - $\mathcal{D}' = (\Sigma_{n-3}; 4; [2, 2], \dots, [2, 2], [1, 3])$. This implies that $\pi_1 = \pi_2 = [1, 1, 2]$ and $\pi_3 = \dots = \pi_{n-1} = [2, 2]$. Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

  Therefore we can assume that $\mathcal{D}'$ is realizable. Since $\mathcal{D} \rightsquigarrow \mathcal{D}'$, we have that $\mathcal{D}$ is realizable as well.

- Otherwise, the inequality $v(\pi_1) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n-1$. In particular, $v(\pi_1) + v(\pi_2) \geq d$ and $v(\pi_3) + v([1, d-1]) \geq d-1$. Combinatorial move A.2 gives

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_{g'}; d; \pi_1', \pi_3, \ldots, \pi_{n-1}, [1, d-1])$$

for a suitable candidate datum $\mathcal{D}'$ with $v(\pi_1') \geq d-2$. Note that, if $d = 4$, we can assume that $\pi_1 \neq [2,2]$; as remarked in the statement of Combinatorial move A.2, this allows us to choose $\pi_1' \neq [2,2]$. It is then easy to see that $\mathcal{D}'$ is realizable by induction: the case $d = 4$ was addressed just now, while $\mathcal{D}' = (\mathbb{S}; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$ is impossible for $d \geq 6$ since

$$v(\pi_1') \geq d - 2 > \frac{d}{2} = v([2, \ldots, 2]).$$

Since $\mathcal{D} \rightsquigarrow \mathcal{D}'$, this implies that $\mathcal{D}$ is realizable as well. □

The next result we are going to prove does not have immediate applications to the study of candidate data with a short partition, but the theoretical consequences it provides are definitely interesting.

Proposition 2.12. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. Assume that $d \neq 4$ and that $2g \geq d-1$. Then $\mathcal{D}$ is realizable.*

Proof. Note that the the condition $2g \geq d-1$ is equivalent to

$$v(\pi_1) + \ldots + v(\pi_n) \geq 3d - 3$$

by the Riemann-Hurwitz formula. Moreover, the cases where $d = 2$ are trivial because of Proposition 2.10; therefore, assume that $d \geq 3$.

We proceed by induction, starting from the base case $n = 3$. If $n = 3$, the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 3d - 3$ implies that $\pi_1 = \pi_2 = \pi_3 = [d]$; by Proposition 2.10, $\mathcal{D}$ is then realizable.

We now turn to the case $n \geq 4$; we will once again employ a reduction technique. Fix a candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_n)$. We can assume that $\pi_i \neq [d]$ for every $1 \leq i \leq n$, otherwise $\mathcal{D}$ is immediately realizable by Proposition 2.10. There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) \leq d-1$ holds for a pair of indices $1 \leq i < j \leq n$. It is now routine to employ Combinatorial move A.1 to show that $\mathcal{D}$ is realizable by induction.

- Otherwise, we have $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. We consider two sub-cases.

  - Assume first that there is a partition, say π_1, which is different from $[2, \ldots, 2]$. Combinatorial move A.1 gives

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_{g'}; d; \pi_1', \pi_3, \ldots, \pi_n)$$

  for a suitable candidate datum $\mathcal{D}'$ with $\pi_1' \in \{[d], [1, d-1]\}$. By Propositions 2.10 and 2.11, $\mathcal{D}'$ is realizable unless $\mathcal{D}' = (\mathbb{S}; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$. If this were the case, we would have $n = 4$ and

$$v(\pi_1) + v(\pi_2) + v(\pi_3) + v(\pi_4) \leq d - 2 + d - 2 + \frac{d}{2} + \frac{d}{2} = 3d - 4,$$

  which contradicts the hypothesis. Therefore $\mathcal{D}'$ is realizable, and the same holds for $\mathcal{D}$.

- Finally, consider the case where $\pi_1 = \ldots = \pi_n = [2, \ldots, 2]$. In this situation $d = 2k \geq 6$ and
$$v(\pi_1) + \ldots + v(\pi_n) = nk > 6k - 3$$
(the inequality is strict since the total branching number is even, while $6k - 3$ is odd). Since $k \geq 3$, this immediately implies that $n \geq 6$. We have that $v(\pi_1) + v(\pi_2) = 2k$ and $v(\pi_3) + \ldots + v(\pi_n) = (n-2)k > 2k$, so we can apply Combinatorial move A.2 to get
$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_{g'}; 2k; [k, k], \pi_3, \ldots, \pi_n)$$
for a suitable candidate datum $\mathcal{D}'$. Now, $v([k, k]) + v(\pi_3) = 2k - 2 + k \geq 2k$ and $v(\pi_4) + \ldots + v(\pi_n) = (n-3)k > 2k$, so we can apply Combinatorial move A.2 once again and find that
$$\mathcal{D}' \rightsquigarrow \mathcal{D}'' = (\Sigma_{g''}; 2k; \pi_1'', \pi_4, \ldots, \pi_n)$$
for a suitable candidate datum $\mathcal{D}''$, where $\pi_1'' \in \{[2k], [1, 2k-1]\}$. Since $n - 2 \geq 4$ and $2k \geq 6$, Propositions 2.10 and 2.11 imply that $\mathcal{D}''$ is realizable. As a consequence, $\mathcal{D}$ is realizable as well. □

Corollary 2.13. *Let d be a positive integer with $d \neq 4$. Then there exist at most finitely many exceptional candidate data of the form $(\Sigma_g; d; \pi_1, \ldots, \pi_n)$.*

Proof. By Proposition 2.12, every candidate datum with $n \geq 3d - 3$ is realizable, and there are only a finite number of candidate data with $n < 3d - 3$. □

Instead, there exist infinitely many exceptional candidate data of degree 4.

Proposition 2.14. *Let $\mathcal{D} = (\Sigma_g; 4; \pi_1, \ldots, \pi_n)$ be a candidate datum. Then $\mathcal{D}$ is realizable if and only if*
$$\mathcal{D} \neq (\Sigma_{n-3}; 4; [2, 2], \ldots, [2, 2], [1, 3]).$$

Proof. The exceptionality of the mentioned candidate datum is addressed in Proposition 4.6. In order to show that every other candidate datum with $d = 4$ is realizable, we proceed by induction, starting from the base case $n = 3$. Note that if any of the partitions is either $[4]$ or $[1, 3]$, we immediately conclude by Propositions 2.10 and 2.11. By the Riemann-Hurwitz formula, we have the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 6$; therefore, the only candidate datum left to consider is $(\mathbb{S}; 4; [2, 2], [2, 2], [2, 2])$. This datum is realizable, for instance, by choosing
$$\alpha_1 = (1, 2)(3, 4), \qquad \alpha_2 = (1, 3)(2, 4), \qquad \alpha_3 = (1, 4)(2, 3),$$
and applying Corollary 1.7.

We now turn to the case $n \geq 4$; once again, we can assume that all the partitions are either $[1, 1, 2]$ or $[2, 2]$. There are three cases.

- If all the partitions are equal to $[2, 2]$, then we can apply Combinatorial move A.2 to combine two partitions into a single $[2, 2]$, and conclude by a reduction argument.
- If all the partitions are equal to $[1, 1, 2]$, then by the Riemann-Hurwitz formula we have $n \geq 6$. We can apply Combinatorial move A.1 twice to combine three partitions into a single $[4]$, and conclude by reduction.
- Otherwise, there is at least one $[2, 2]$ and one $[1, 1, 2]$; using Combinatorial move A.1, we can combine them into a single $[4]$ and conclude by reduction. □

2.6 Prime-degree conjecture

This section is devoted to a fascinating and surprising conjecture related to the Hurwitz existence problem. After studying many examples of exceptional data, a very unexpected pattern begins to emerge: it looks like no such datum can be found when the degree is a prime number. In other words, one comes to suspect the following.

Prime-degree conjecture. *Let p be a prime number. Then every candidate datum $(\Sigma_g; p; \pi_1, \ldots, \pi_n)$ with $n \geq 3$ is realizable.*

It is not immediately clear why prime numbers should be even remotely related to the realizability of candidate data. In fact, to the best of the author's knowledge, there is no number-theoretic argument in favor (or against) this conjecture; all the evidence gathered so far supporting the prime-degree conjecture is "experimental".

- As shown in Proposition 4.7, for every composite number $d \geq 4$ there is an exceptional candidate datum of degree d.

- In general, all the exceptional data found so far have a composite degree; for all the families of candidate data whose realizability is completely understood (including those presented in the previous section), the prime-degree conjecture holds.

- The first efficient algorithmic approach to the Hurwitz existence problem was presented in [17]. Every candidate datum with $n = 3$ and $d \leq 20$ was analyzed by a computer, and no exceptional data with prime degree were found; therefore, the prime-degree conjecture was verified for numbers up to 19 (see Proposition 2.15 below). In Appendix A, with a much heavier computational effort, we extend this result to prime numbers up to 29.

$d \leq 22$, according to
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The following proposition is an essential tool for the computational approach to the existence problem, since it reduces the verification of the prime-degree conjecture to the case of $n = 3$ partitions.

Proposition 2.15. *Let d be a positive integer. Assume that every candidate datum $(\Sigma_g; d; \pi_1, \pi_2, \pi_3)$ is realizable. Then every candidate datum $(\Sigma_g; d; \pi_1, \ldots, \pi_n)$ with $n \geq 3$ is realizable.*

Proof. We proceed by induction on $n \geq 4$, using the standard reduction technique. Fix a candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_n)$; there are two cases.

- If there are two indices $1 \leq i < j \leq n$ such that $v(\pi_i) + v(\pi_j) \leq d - 1$, then a routine application of Combinatorial move A.1 shows that $\mathcal{D}$ is realizable.

- Otherwise, $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. In particular, $v(\pi_1) + v(\pi_2) \geq d$ and $v(\pi_3) + \ldots + v(\pi_n) \geq d$. By applying Combinatorial move A.2, it is easy to see that $\mathcal{D}$ is realizable. $\square$

Chapter 3

Dessins d'enfant

3.1 Child's drawings on surfaces

In Section 2.4, we discussed how the Hurwitz existence problem can be reduced to the analysis of candidate data on the sphere. Moreover, thanks to Combinatorial moves A.1 and A.2, we have devised a relatively reliable technique to decrease the number n of partitions; this technique was successfully employed in Section 2.5 to show the realizability of a wide variety of candidate data by induction on n, starting from the base case $n = 3$. Ignoring the cases where $n \leq 2$, which were fully analyzed in Section 2.4, it should come as no surprise that candidate data with $n = 3$ play a very important role in the study of the existence problem.

Up to this point, we have only approached the Hurwitz existence problem from a group-theoretic point of view, showing realizability by looking for elements of $\mathfrak{S}_d$ with certain properties. In this section, we will present a totally different tool, of a more topological and combinatorial nature, for attacking the same problem. The concept of *dessins d'enfant*[1] was popularized by Grothendieck in [2], in a setting related to, but different from, the Hurwitz existence problem. Dessins d'enfant provide a strikingly elementary tool for showing the realizability of candidate data with $n = 3$ partitions, although they generalize quite nicely to the case $n \geq 4$. However, we will not deal with said generalization, since the reduction technique will prove to be sufficient for our purposes; we refer the interested reader to [10, Section 3].

We start by introducing some basic terminology about graphs. Given a surface Σ, a *graph* embedded in Σ (or, simply, a graph on Σ) is a closed subspace $\Gamma \subseteq \Sigma$ consisting of:

- a finite number of points $x_1, \ldots, x_r \in \Sigma$, called *vertices*;
- a finite number of segments (subspaces homeomorphic to $[0, 1]$) $e_1, \ldots, e_d \subseteq \Sigma$, called *edges*; we require that each edge connects two (not necessarily distinct) vertices, and that the interiors of two edges are disjoint; in other words, two edges may intersect at most at their endpoints; moreover, a vertex cannot lie on the interior of an edge.

The *degree* of a vertex x is the number of edges having x as an endpoint; edges connecting x to itself are counted twice; we denote the degree of x by $k(x)$. In order to avoid unpleasant corner cases, we will always require that there are no *isolated vertices* or, in other words, that $k(x) \geq 1$ for every vertex x.

A *bipartite graph* is a graph whose vertices are colored either black or white, and each edge connects a black vertex and a white one. If we denote the black vertices by $x_1, \ldots, x_r$ and the

[1] "*Dessin d'enfant*" is French for "child's drawing", hence the title of this section.

white vertices by $y_1, \dots, y_s$, an easy counting argument shows that

$$k(x_1) + \ldots + k(x_r) = k(y_1) + \ldots + k(y_s) = d,$$

where d is the number of edges.

Given a graph Γ on a surface Σ, the space $\Sigma \smallsetminus \Gamma$ is a disjoint union of a finite number of non-compact surfaces $S_1 \sqcup \ldots \sqcup S_h$, called *complementary regions* of Γ. We are finally ready to give the definition of the much anticipated dessins d'enfant.

Definition 3.1. Let Σ be a surface. A *dessin d'enfant* on Σ is a bipartite graph $\Gamma \subseteq \Sigma$ whose complementary regions are topological disks.

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enfant.

Let Γ be a dessin d'enfant, and fix one a complementary region D. By traveling along its boundary, always keeping D to the left, we get a cyclic sequence of edges of Γ, which we call *combinatorial boundary* of D, and denote by ∂D. Note that the same edge e can be traveled along twice, once for each direction; in this case, it will appear twice in ∂D, and we will say that e is *enveloped* by D. The number of edges (with multiplicity) of ∂D is the *perimeter* of D, denoted by $|\partial D|$. If $D_1, \dots, D_h$ are the complementary region of Γ, a counting argument shows that

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$$|\partial D_1| + \ldots + |\partial D_h| = 2d.$$

It is also easy to see that the perimeter of each complementary region is even: in fact, when traveling along the boundary of a region, we alternately encounter black and white vertices, so an even number of edges is required to get back to the starting color.

Finally, note that every dessin d'enfant is necessarily connected, otherwise there would be some complementary region with two or more boundary components.

Definition 3.2. Let Γ be a dessin d'enfant on a surface Σ; let d be the number of edges. Let $x_1, \dots, x_r$ be the black vertices, $y_1, \dots, y_s$ the white ones. Denote by $D_1, \dots, D_h$ the complementary regions of Γ. The *branching datum* of Γ is the tuple

$$\mathcal{D}(\Gamma) = (\Sigma, \mathbb{S}; d; [k(x_1), \dots, k(x_r)], [k(y_1), \dots, k(y_s)], [|\partial D_1|/2, \dots, |\partial D_h|/2]).$$

From the discussion above, we immediately see that $\mathcal{D}(\Gamma)$ is a combinatorial datum, since

$$k(x_1) + \ldots + k(x_r) = k(y_1) + \ldots + k(y_s) = |\partial D_1|/2 + \ldots + |\partial D_h|/2 = d.$$

Actually, if Σ is orientable, $\mathcal{D}(\Gamma)$ is a candidate datum: by the Euler formula,

$$\chi(\Sigma) = r + s - d + h = 2d - v(\pi_1) - v(\pi_2) - v(\pi_3),$$

where $\pi_1 = [k(x_1), \dots, k(x_r)]$, $\pi_2 = [k(y_1), \dots, k(y_s)]$ and $\pi_3 = [|\partial D_1|/2, \dots, |\partial D_h|/2]$. This is no coincidence, just like the name "branching datum of Γ" was not picked at random: the following result establishes a strong connection between dessins d'enfant and realizable combinatorial data.

Proposition 3.1. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3)$ be a combinatorial datum. Then $\mathcal{D}$ is realizable if and only if there exists a dessin d'enfant $\Gamma \subseteq \Sigma_g$ with $\mathcal{D}(\Gamma) = \mathcal{D}$.*

Proof. Assume that $\mathcal{D}$ is realized by a branched covering $f\colon \Sigma_g \to \mathbb{S}$. Let $\{\widetilde{x}_1, \dots, \widetilde{x}_r\} = f^{-1}(x)$, $\{\widetilde{y}_1, \dots, \widetilde{y}_s\} = f^{-1}(y)$, $\{\widetilde{z}_1, \dots, \widetilde{z}_h\} = f^{-1}(z)$. Fix a segment $e \subseteq \mathbb{S}$ connecting x and y (and avoiding z); we claim that $\Gamma = f^{-1}(e)$ is the desired dessin d'enfant. Let $\mathring{e}$ be the interior of e (that is, $\mathring{e} = e \smallsetminus \{x, y\}$). First of all, note that $f^{-1}(\mathring{e})$ is the disjoint union of d open segments $\mathring{e}_1, \dots, \mathring{e}_d$, since the restriction of f to $\mathbb{S} \smallsetminus \{x, y, z\}$ is a covering map of degree d. Moreover, it is easy to see that the closure of each $\mathring{e}_i$ is a closed segment e_i connecting one point in $f^{-1}(x)$ and one point of $f^{-1}(y)$; it follows that Γ is a bipartite graph on Σ_g, with black vertices $\widetilde{x}_1, \dots, \widetilde{x}_r$ and white vertices $\widetilde{y}_1, \dots, \widetilde{y}_s$. Consider a vertex $\widetilde{x}_i$; recall that f is locally modeled on the complex map $\xi \mapsto \xi^k$, where $k = k(\widetilde{x}_i)$ is the local degree of $\widetilde{x}_i$. As a consequence, we immediately see that there are exactly $k(\widetilde{x}_i)$ edges of Γ with $\widetilde{x}_i$ as an endpoint; of course, the same holds for every $\widetilde{y}_j$. Finally, we turn to the complementary regions of Γ. Let $D = \mathbb{S} \smallsetminus e \simeq \mathbb{R}^2$, $D^\bullet = D \smallsetminus \{z\} \simeq \mathbb{R}^2 \smallsetminus \{0\}$. The restriction of f to $f^{-1}(D^\bullet) = \Sigma_g \smallsetminus (\Gamma \cup \{\widetilde{z}_1, \dots, \widetilde{z}_h\})$ is a covering map of the punctured disk $D^\bullet$. It is then easy to see that the complementary regions of Γ are discs $\widetilde{D}_1, \dots, \widetilde{D}_h$, with $\widetilde{z}_i \in \widetilde{D}_i$ for each $1 \le i \le h$, and that the restriction $f\colon \widetilde{D}_i \to D$ is modeled the complex map $\xi \mapsto \xi^{k(\widetilde{z}_i)}$. Since the perimeter of D is 2, we have that $|\partial \widetilde{D}_i| = 2k(\widetilde{z}_i)$; this concludes the proof of the equality $\mathcal{D}(f) = \mathcal{D}(\Gamma)$.

Conversely, assume that we are given a dessin d'enfant $\Gamma \subseteq \Sigma_g$ with $\mathcal{D}(\Gamma) = \mathcal{D}$. Fix three arbitrary points $x, y, z \in \mathbb{S}$, and let $e \subseteq \mathbb{S}$ be a segment connecting x and y (and avoiding z). First of all, we define f on Γ, sending black vertices to x and white vertices to y, and mapping edges homeomorphically to e. Extending f to all of Σ_g is a relatively easy task: here are the details. Consider the standard closed disk $K = \{a \in \mathbb{R}^2 : \|a\| \le 1\}$, and take a complementary region $\widetilde{D} \subseteq \Sigma_g$; let $\varphi\colon K \to \Sigma_g$ be a continuous map which restricts to a homeomorphism $\varphi\colon \mathring{K} \to \widetilde{D}$, where $\mathring{K}$ denotes the interior of K. There exists a map $\psi\colon K \to \mathbb{S}$ such that $\psi(0) = z$, the diagram Picture much needed.

$$\begin{tikzcd} \partial K \arrow[d, "\varphi"'] \arrow[dr, "\psi"] & \\ \Gamma \arrow[r, "f"] & e \end{tikzcd}$$

commutes, ψ is a local homeomorphism in $\mathring{K} \smallsetminus \{0\}$ and it is modeled on $\xi \mapsto \xi^k$ in a neighborhood of $0 \in K$; in particular k will necessarily be equal to half the perimeter of $\widetilde{D}$. We can now extend f to $\widetilde{D}$ by setting $f(\widetilde{x}) = \psi(\varphi^{-1}(\widetilde{x}))$ for every $\widetilde{x} \in \widetilde{D}$. After repeating the process for all the complementary regions, it is not hard to verify that the map $f\colon \widetilde{\Sigma} \to \mathbb{S}$ we have obtained is a branched covering with branching points $x, y, z \in \mathbb{S}$. Since $\Gamma = f^{-1}(e)$, the first part of the proof Namely, $\Rightarrow$.
implies that $\mathcal{D}(\Gamma) = \mathcal{D}(f)$. □

3.2 Unwinding, joining and fattening

In the next section we will introduce a new kind of combinatorial moves, which operate on dessins d'enfant rather than permutations. In this context, the importance of visual intuition cannot be overstated. Therefore, we will now spend some time describing in detail two operations that will play a major role in the topological explanation of the upcoming combinatorial moves.

Unwinding the boundary. Let Γ be a graph on a surface Σ. Take a complementary region D, and assume D is a topological disk. Intuitively, when we *unwind the boundary* of D, we represent D as the standard closed disk K embedded in $\mathbb{R}^2$; the edges of the combinatorial boundary of D are placed sequentially on the topological boundary of K, possibly with repetitions. For a more formal description, we can follow the strategy presented in the second part of the proof of Proposition 3.1: we consider a continuous map $\varphi\colon K \to \Sigma_g$ which restricts to a homeomorphism $\varphi\colon \mathring{K} \to D$; edges on the topological boundary of D can be pulled back by φ, thus unwinding the combinatorial boundary of D on ∂K.

Joining vertices along edges. Let Γ be a graph on a surface Σ. Consider an edge e, and let x, y be its (distinct) endpoints. *Joining* x and y along e means shrinking e to a single point, so that x and y are merged into a single vertex, say z; it is immediate to check that $k(z) = k(x)+k(y)-2$, while the degrees of the other vertices are left unchanged. The topology of the complementary regions does not change either. To be more precise, there is a natural one-to-one correspondence between regions of $\Sigma \smallsetminus \Gamma$ and regions of $\Sigma \smallsetminus \Gamma'$, where Γ' is the graph obtained after joining x and y along e, and corresponding regions are homeomorphic. The edge e disappears from the combinatorial boundaries, so the perimeters of the two regions touching e decrease by 1 (if the two regions were actually the same, then the perimeter decreases by 2); the other perimeters do not change. Of course, the joining operation can be performed along more edges simultaneously, by joining vertices along one edge at a time.

Fattening graphs. Representing dessins d'enfant on the sphere is easy, since graphs on $\mathbb{S}$ naturally embed in $\mathbb{R}^2$; unfortunately, this is not the case for surfaces of genus $g \geq 1$. However, for a specific class of graphs (including dessins d'enfant), there is a trick we can exploit in order to represent them as diagrams on the plane. Let Γ be a graph on a surface Σ_g; assume that the complementary regions of Γ are disks. Note that the topology of the embedding $\Gamma \subseteq \Sigma_g$ can be completely recovered if we are given Γ as an abstract graph, plus a tubular neighborhood of Γ in Σ_g; we call such a datum a *fat graph*. A fat graph can be represented as a diagram with a finite number of transverse crossings on the plane; each crossing is equipped with the additional information of which edge goes over and which goes under.

In order to reconstruct the embedding of Γ in Σ_g, we simply have to thicken the edges of the diagram, keeping in mind that the two edges involved in a crossing actually go one under the other. This operation yields a fat graph, from which Σ_g can be recovered by gluing a disk along each boundary component.

We will employ this technique in order to represent dessins d'enfant (and more generally, graphs whose complementary regions are disks) embedded in higher genus surfaces.

3.3 Genus-reducing combinatorial moves

As we have already anticipated, the goal of this thesis is a complete classification of the exceptional data with a partition of length 2. Combinatorial moves A.1 and A.2 are often able to reduce the existence problem to instances with $n = 3$ partitions. We will now introduce a few more combinatorial moves, which heavily exploit the machinery of dessins d'enfant. Unlike the aforementioned ones, these moves only work under very restrictive assumptions, namely that $n = 3$ and $\ell(\pi_3) = 2$; on the other hand, they allow a much finer control on the partitions involved, and are often versatile enough to reduce an instance of the existence problem to the case where $\widetilde{\Sigma} = \mathbb{S}$.

In this section, we will only be dealing with candidate data of the form $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ with $1 \leq s \leq d-1$. In this setting, the Riemann-Hurwitz formula can simply be written as

$$\ell(\pi_1) + \ell(\pi_2) = d - 2g.$$

We will adopt the following conventions:

- vertices corresponding to the entries of π_1 (or π_1') will be colored black;
- vertices corresponding to the entries of π_2 (or π_2') will be colored white;
- unnamed vertices will be labeled with their degrees;

- the complementary disk associated to the first entry of π_3 (or π'_3) will be denoted by D_1 and will be colored orange;

- the complementary disk associated to the second entry of π_3 (or π'_3) will be denoted by D_2 and will be colored blue.

Combinatorial move B.1. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 1$. Assume that $[1,1,3] \subseteq \pi_1$. Consider the candidate datum*

$$\mathcal{D}' = (\Sigma_{g-1}; d; \pi'_1, \pi_2, [s, d-s]),$$

where $\pi'_1 = \pi_1 \smallsetminus [3] \cup [1,1,1]$. Then $\mathcal{D} \rightsquigarrow \mathcal{D}'$.

Proof. Assume that $\mathcal{D}'$ is realizable; by Proposition 3.1, there exists a dessin d'enfant $\Gamma' \subseteq \Sigma_{g-1}$ with $\mathcal{D}(\Gamma') = \mathcal{D}'$. Our aim will be to construct a new dessin d'enfant $\Gamma \subseteq g$ with $\mathcal{D}(\Gamma) = \mathcal{D}$; by Proposition 3.1, this will imply that $\mathcal{D}$ is realizable as well.

Note that $[1,1,1,1,1] \subseteq \pi'_1$; therefore, without loss of generality, we can assume that Γ has three black vertices of degree 1 lying on the boundary of the complementary region D_1. Let us represent D_1 with its boundary unwound, and focus on the three black vertices of degree 1. We perform the following operations on Γ'.

(1) Attach a tube to Σ_{g-1} with both endpoints in D_1; to be more precise, remove two disjoint open disks contained in the interior of D_1, and glue a tube $S^1 \times [0,1]$ along the two new boundary components. This effectively increases the genus by 1.

(2) Connect the three black vertices with two new edges, as shown in red in the picture; note that the orange complementary region of the new graph is still a disk.

(3) Join the three black vertices along the red edges.

After these operations, we get a new dessin d'enfant Γ embedded in Σ_g. It is easy to check that $\mathcal{D}(\Gamma) = \mathcal{D}$, therefore $\mathcal{D}$ is realizable. □

In the upcoming proofs, we will often represent complementary disks with their boundaries unwound, without explicitly saying so. The pictures should be clear enough to avoid any ambiguity.

Combinatorial move B.2. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 1$. Assume that:*

- $2 \leq s \leq d-s$;

- $x \in \pi_1$ *for some* $x \geq 4$;

- $2 \in \pi_2$.

Let x_1, x_2 be positive integers whose sum equals $x-2$, and consider the candidate datum

$$\mathcal{D}' = (\Sigma_{g-1}; d-2; \pi'_1, \pi'_2, [s-1, d-s-1]),$$

where $\pi'_1 = \pi_1 \smallsetminus [x] \cup [x_1, x_2]$ and $\pi'_2 = \pi_2 \smallsetminus [2]$. Then $\mathcal{D} \rightsquigarrow \mathcal{D}'$.

Proof. Consider a dessin d'enfant $\Gamma' \subseteq \Sigma_{g-1}$ realizing $\mathcal{D}'$. There are two cases.

Case 1: the black vertex of degree x_1 lies on ∂D_1 and the one with degree x_2 lies on ∂D_2 (or vice versa). Then we perform the following operations on Γ'.

(1) Attach a tube to Σ_{g-1} with one endpoint in D_1 and the other one in D_2.

(2) Add one black vertex, one white vertex and two edges as shown in the picture.

(3) Draw the two red edges shown in the picture.

(4) Perform the joining operation along the red edges.

Case 2: the two black vertices of degrees x_1 and x_2 lie (say) on ∂D_1. Fix an edge $e \subseteq \Gamma'$ which lies on the boundaries of both disks. We perform the following operations on Γ'.

(1) Add one black vertex and one white vertex on e.

(2) Attach a tube to Σ_{g-1} with both endpoints in D_1.

(3) Draw the two red edges shown in the picture.

(4) Perform the joining operation along the red edges.

In both cases, we get a new dessin d'enfant Γ embedded in Σ_g. It is easy to check that Γ realizes the candidate datum $\mathcal{D}$. $\square$

Combinatorial move B.3. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 1$. Assume that:*

- $3 \leq s \leq d-3$;
- $[x, y] \subseteq \pi_1$ *for some* $x \geq 3$, $y \geq 3$;
- $[2, 2] \subseteq \pi_2$.

Consider the candidate datum

$$\mathcal{D}' = (\Sigma_{g-1}; d-4; \pi_1', \pi_2', [s-2, d-s-2]),$$

where $\pi_1' = \pi_1 \setminus [x, y] \cup [x-2, y-2]$ *and* $\pi_2' = \pi_2 \setminus [2, 2]$. *Then* $\mathcal{D} \rightsquigarrow \mathcal{D}'$.

Proof. Consider a dessin d'enfant $\Gamma' \subseteq \Sigma_{g-1}$ realizing $\mathcal{D}'$. There are two cases.

Case 1: the black vertex of degree $x-2$ lies on ∂D_1 and the one with degree $y-2$ lies on ∂D_2 (or vice versa). Then we perform the following operations on Γ'.

(1) Attach a tube to Σ_{g-1} with one endpoint in D_1 and the other one in D_2.

(2) Add two black vertices, two white vertices and four edges as shown in the picture.

(3) Draw the red edge as shown in the picture.

(4) Perform the join operation along the red edges.

Case 2: the two black vertices of degrees $x-2$ and $y-2$ lie (say) on ∂D_1. We perform the following operations on Γ'.

(1) Add two black vertices and two white vertices on e.

(2) Attach a tube to Σ_{g-1} with both endpoints in D_1.

(3) Draw the two red edges shown in the picture.

(4) Perform the joining operation along the red edges.

In both cases, we get a new dessin d'enfant Γ embedded in Σ_g. It is easy to check that Γ realizes the candidate datum $\mathcal{D}$. □

Combinatorial move B.4. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 1$. Assume that:*

- $2 \leq s \leq d-2$*;*
- $x \in \pi_1$ *for some* $x \geq 4$*;*
- $y \in \pi_2$ *for some* $y \geq 3$.

Consider the candidate datum

$$\mathcal{D}' = (\Sigma_{g-1}; d-2; \pi_1', \pi_2', [s-1, d-s-1]),$$

where $\pi_1' = \pi_1 \smallsetminus [x] \cup [x-2]$ *and* $\pi_2' = \pi_2 \smallsetminus [y] \cup [y-2]$*. Then* $\mathcal{D} \rightsquigarrow \mathcal{D}'$.

Proof. Consider a dessin d'enfant $\Gamma' \subseteq \Sigma_{g-1}$ realizing $\mathcal{D}'$. Let u be the black vertex of degree $x-2$, and let v be the white vertex of degree $y-2$; there are two cases.

Case 1: u lies on ∂D_1 and v lies on ∂D_2 (or vice versa). Then we perform the following operations on Γ'.

(1) Attach a tube to Σ_{g-1} with one endpoint in D_1 and the other in D_2.

(2) Add one black vertex, one white vertex and two edges as shown in the picture.

(3) Draw the two red edges shown in the picture.

(4) Perform the joining operation along the red edges.

Case 2: neither u nor v lie (say) on ∂D_2; analyzing this case will be more involved than usual. We will say that an edge is *shared* if it is not enveloped by D_1 or by D_2; in other words, an edge is shared if it appears exactly once in ∂D_1; in the following pictures, shared edges will be colored green. Our goal will be to prove that we can add two vertices of degree 2 – one black and one white – on a shared edge, in such a way that the vertices $\{u, 2, 2, v\}$ appear in this order on ∂D_1. Let us unwind the boundary of D_1; since u does not lie on the boundary of D_2, it appears exactly $x-2 \geq 2$ times on ∂D_1; similarly, v appears exactly $y-2$ times.

- Assume that $\{u, v, u, \text{shared edge}\}$ appear in this order on ∂D_1. Then, by adding a black and a white vertex on this shared edge, we get the desired result.
- Otherwise, consider the pictures below.

(1) We are in the following situation: there is a contiguous segment A of ∂D_1 that contains all the occurrences of u, and does not contain any occurrence of v or any shared edge. Similarly, there is a segment B of ∂D_1 that contains all the occurrences of v, no occurrence of u and no shared edge. Note that u and v are never adjacent to a shared edge, since by assumption they do not lie on ∂D_2.

(2) Choose an orientation of ∂D_1 (counterclockwise in the picture) and consider the first occurrence of u in A; let α be the edge immediately afterwards in ∂D_1. Since α is not shared, it must appear once more on the combinatorial boundary of D_1, with the opposite orientation. Note that α it cannot occur immediately before the first appearance of u, otherwise u would have degree 1, therefore it will occur somewhere else on A

(3) Consider the first occurrence of v in B; let β be the edge immediately before in ∂D_1. Since β is not shared, it will also occur somewhere else on the combinatorial perimeter of D_1

(4) Let a be the other endpoint of α, and let b be the other endpoint of β. Erase the edges α and β and draw two new ones, connecting u to v and a to b as shown in the picture.

(5) It is now easy to see that the orange region is still a complementary disk, and that its perimeter has not changed. Moreover, by traveling along its boundary, we encounter $\{u, v, u\}$ in this order, without any shared edges in between; since there must be at least one shared edge on the boundary of the new orange region, the argument of the first bullet point applies.

Once we have added the two vertices of degree 2 as explained above, we can perform the following operations.

(1) Attach a tube to Σ_{g-1} with both endpoints in D_1.

(2) Draw the two red edges shown in the picture.

(3) Perform the joining operation along the red edges.

In both cases, we get a new dessin d'enfant Γ embedded in Σ_g. It is easy to check that Γ realizes the candidate datum $\mathcal{D}$. $\square$

We will make extensive use of these combinatorial moves in the next chapter, where a full classification of the exceptional data with $n = 3$ and $\ell(\pi_3) = 2$ will be provided (see Theorems 4.1, 4.3 and 4.4).

3.4 Realizability by dessins d'enfant

We conclude this chapter by proving the realizability of a few families of candidate data by means of dessins d'enfant; these results, while interesting by themselves, will be useful in the next chapter for addressing some cases which are not covered by the combinatorial moves we have introduced.

Proposition 3.2. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [2, d-2])$ be a candidate datum. Assume that:*

- $[x, y] \subseteq \pi_1$ *for some* $x \geq 2$, $y \geq 3$*;*
- $[2, 2] \subseteq \pi_2$.

Then $\mathcal{D}$ is realizable.

Proof. We will show that, under the stated assumptions, there is a combinatorial move[2]

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_g; d-4; \pi_1 \setminus [x,y] \cup [x+y-4], \pi_2 \setminus [2,2], [d-4]).$$

Note that $\mathcal{D}'$ is realizable by Proposition 2.10; let $\Gamma' \subseteq \Sigma_g$ be a dessin d'enfant realizing it. We perform the following operations on Γ'.

(1) Consider the black vertex of degree $x+y-4$ and split it into two vertices of degrees $x-2$ and $y-2$.

(2) Add two white vertices and four edges as shown in the picture. This creates a new complementary disk with perimeter 4.

After these operations, we get a new dessin d'enfant Γ embedded in Σ_g. It is easy to check that Γ realizes the candidate datum $\mathcal{D}$. □

Proposition 3.3. *The following families of candidate data are realizable for every $g \geq 2$.*

(1) $(\Sigma_g; 6g; [3,\dots,3], [3,\dots,3], [s, 6g-s])$.

(2) $(\Sigma_g; 6g+2; [2,3,\dots,3], [2,3\dots,3], [s, 6g+2-s])$.

(3) $(\Sigma_g; 6g+3; [3,\dots,3], [1,2,3,\dots,3], [s, 6g+3-s])$.

(4) $(\Sigma_g; 6g+4; [1,3,\dots,3], [1,3,\dots,3], [s, 6g+4-s])$.

(5) $(\Sigma_g; 6g+6; [1,2,3,\dots,3], [1,2,3,\dots,3], [s, 6g+6-s])$.

Proof. In the scope of this proof we define an *augmented combinatorial datum* as a combinatorial datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3)$ where some distinguished elements of π_3 are called *enveloping*; we will underline the enveloping elements in order to recognize them. An augmented combinatorial datum $\mathcal{D}$ is *realizable* if there exists a dessin d'enfant Γ with $\mathcal{D}(\Gamma) = \mathcal{D}$ such that, for every complementary disk D corresponding to an enveloping element of π_3, there is an edge of Γ which is enveloped by D.

Step 1. The augmented datum

$$\mathcal{D} = (\Sigma_2; 12; [3,3,3,3], [3,3,3,3], \pi_3)$$

is realizable for $\pi_3 \in \{[1, \underline{11}], [2, \underline{10}], [\underline{3}, \underline{9}], [\underline{4}, \underline{8}], [\underline{5}, \underline{7}], [\underline{6}, \underline{6}]\}$. The following pictures display dessins d'enfant realizing each of these augmented data; as usual, the disk associated to the first element of π_3 is colored orange and the other one is colored blue; enveloped edges are drawn in the same color as the corresponding disk.

Step 2. Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3)$ be a realizable augmented datum, and let $\underline{x} \in \pi_3$ be an enveloping element. Then the augmented datum

$$\mathcal{D}' = (\Sigma_{g+1}; d+6; \pi_1 \cup [3,3], \pi_2 \cup [3,3], \pi_3 \setminus [\underline{x}] \cup [\underline{x+6}])$$

is realizable as well. In fact, consider a dessin d'enfant $\Gamma \subseteq \Sigma_g$ realizing $\mathcal{D}$, and fix an edge e enveloped by the disk D associated to $\underline{x}$. We perform the following operations on Γ.

[2]To be precise, when $d = 5$ the tuple $\mathcal{D}'$ is not a combinatorial datum according to our definition. However, the only candidate datum $\mathcal{D}$ of degree 5 satisfying the assumptions is $\mathcal{D} = (\mathbb{S}; 5; [2,3], [1,2,2], [2,3])$, whose realizability can be easily checked by hand with a suitable dessin d'enfant.

(1) Add one black vertex and one white vertex on e.

(2) Attach a tube to Σ_g with both endpoints on D.

(3) Add one black vertex, one white vertex and four edges as shown in the picture.

After these operations, we get a new dessin d'enfant Γ' embedded in Σ_{g+1}. It is easy to check that Γ' realizes the augmented datum $\mathcal{D}'$.

Step 3. For every $g \geq 2$, the augmented datum

$$(\Sigma_g; 6g; [3, \ldots, 3], [3, \ldots, 3], \pi_3)$$

is realizable for $\pi_3 \in \{[1, \underline{6g-1}], [2, \underline{6g-2}]\} \cup \{[\underline{s}, \underline{6g-s}] : 3 \leq s \leq 6g-3\}$. This can be easily shown by induction on g, using step 1 as the base case and step 2 for the induction.

Step 4. Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3)$ be a realizable augmented datum, and let $\underline{x} \in \pi_3$ be an enveloping element. Then the augmented datum

$$\mathcal{D}' = (\Sigma_g; d+2; \pi_1 \cup [2], \pi_2 \cup [2], \pi_3 \smallsetminus [\underline{x}] \cup [\underline{x+2}])$$

is realizable as well. In fact, consider a dessin d'enfant $\Gamma \subseteq \Sigma_g$ realizing $\mathcal{D}$, and fix an edge e enveloped by the disk D associated to $\underline{x}$. Then, add one black vertex and one white vertex on e. The new dessin d'enfant realizes the augmented datum $\mathcal{D}'$.

Step 5. Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3)$ be a realizable augmented datum, and let $\underline{x} \in \pi_3$ be an enveloping element. Then the augmented datum

$$\mathcal{D}' = (\Sigma_g; d+3; \pi_1 \cup [3], \pi_2 \cup [1,2], \pi_3 \smallsetminus [\underline{x}] \cup [\underline{x+3}])$$

is realizable as well. In fact, consider a dessin d'enfant $\Gamma \subseteq \Sigma_g$ realizing $\mathcal{D}$, and fix an edge e enveloped by the disk D associated to $\underline{x}$. Then, add one black vertex and two white vertices as shown in the picture. The new dessin d'enfant realizes the augmented datum $\mathcal{D}'$.

Step 6. Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3)$ be a realizable augmented datum, and let $\underline{x} \in \pi_3$ be an enveloping element. Then the augmented datum

$$\mathcal{D}' = (\Sigma_g; d+4; \pi_1 \cup [1,3], \pi_2 \cup [1,3], \pi_3 \smallsetminus [\underline{x}] \cup [\underline{x+4}])$$

is realizable as well. In fact, consider a dessin d'enfant $\Gamma \subseteq \Sigma_g$ realizing $\mathcal{D}$, and fix an edge e enveloped by the disk D associated to $\underline{x}$. Then, add two black vertices and two white vertices as shown in the picture. The new dessin d'enfant realizes the augmented datum $\mathcal{D}'$.

Finally, it is easy to see that the five families listed in the statement can be obtained by applying steps 4, 5 and 6 zero or more times to an augmented datum which is realizable by step 3, and then forgetting about the augmentation; here are the details.

(1) $(\Sigma_g; 6g; [3, \ldots, 3], [3, \ldots, 3], [s, 6g-s])$ is already realizable by step 3.

(2) If we assume that $s \leq 3g+1$, step 4 gives

$$(\Sigma_g; 6g+2; [2, 3, \ldots, 3], [2, 3 \ldots, 3], [s, \underline{6g+2-s}])$$
$$\rightsquigarrow (\Sigma_g; 6g; [3, \ldots, 3], [3, \ldots, 3], [s, \underline{6g-s}]),$$

which is realizable by step 3.

(3) If we assume that $s \leq 3g+1$, step 5 gives

$$\begin{aligned}&(\Sigma_g; 6g+3; [3,\dots,3], [1,2,3,\dots,3], [s, \underline{6g+3-s}])\\ &\qquad \rightsquigarrow (\Sigma_g; 6g; [3,\dots,3], [3,\dots,3], [s, \underline{6g-s}]),\end{aligned}$$

which is realizable by step 3.

(4) If we assume that $s \leq 3g+2$, step 6 gives

$$\begin{aligned}&(\Sigma_g; 6g+4; [1,3,\dots,3], [1,3,\dots,3], [s, \underline{6g+4-s}])\\ &\qquad \rightsquigarrow (\Sigma_g; 6g; [3,\dots,3], [3,\dots,3], [s, \underline{6g-s}]),\end{aligned}$$

which is realizable by step 3.

(5) If we assume that $s \leq 3g+3$, step 4 and 6 give

$$\begin{aligned}&(\Sigma_g; 6g+6; [1,2,3,\dots,3], [1,2,3,\dots,3], [s, \underline{6g+6-s}])\\ &\qquad \rightsquigarrow (\Sigma_g; 6g+4; [1,3,\dots,3], [1,3,\dots,3], [s, \underline{6g+4-s}])\\ &\qquad \rightsquigarrow (\Sigma_g; 6g; [3,\dots,3], [3,\dots,3], [s, \underline{6g-s}]),\end{aligned}$$

which is realizable by step 3. $\square$

Chapter 4

Exceptional data with a short partition

4.1 Realizability on the sphere

In this final chapter, we will give a full solution of the Hurwitz existence problem for candidate data containing a partition of length 2; as usual, we will assume that $\Sigma = \mathbb{S}$ and $n \geq 3$. This specific instance of the existence problem had already received some interest in the literature, leading to a few partial results.

- Proposition 2.11 addresses the cases where $\pi_n = [1, d-1]$; the proof was actually borrowed from [1, Proposition 5.3].
- Pervova and Petronio [9] dealt with the cases where $n = 3$ and $\pi_3 = [2, d-2]$.
- Pakovich [8] solved the existence problem for $\ell(\pi_n) = 2$ and $\widetilde{\Sigma} = \mathbb{S}$.

In particular, we will consistently exploit the results by Pakovich as a base case for genus-reducing combinatorial moves. We will now state the relevant theorems, but we decide to omit the proofs: while the core ideas are very ingenious, filling in the details is quite tedious and time-consuming[1]. We refer the interested reader to [8].

Theorem 4.1. *Let $\mathcal{D} = (\mathbb{S}; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

(2) $\mathcal{D} = (\mathbb{S}; 2k; [2,\dots,2], [2,\dots,2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

(3) $\mathcal{D} = (\mathbb{S}; 2k; [2,\dots,2], [1,2,\dots,2,3], [k,k])$ *with* $k \geq 2$.

(4) $\mathcal{D} = (\mathbb{S}; 4k+2; [2,\dots,2], [1,\dots,1,k+1,k+2], [2k+1,2k+1])$ *with* $k \geq 1$.

(5) $\mathcal{D} = (\mathbb{S}; 4k; [2,\dots,2], [1,\dots,1,k+1,k+1], [2k-1,2k+1])$ *with* $k \geq 2$.

(6) $\mathcal{D} = (\mathbb{S}; kh; [h,\dots,h], [1,\dots,1,k+1], [lh,(k-l)h])$ *with* $h \geq 2$, $k \geq 2$, $1 \leq l \leq k-1$.

[1] The same could probably be said about the other proofs in this chapter, which cannot be omitted for obvious reasons.

Theorem 4.2. *Let $\mathcal{D} = (\mathbb{S}; d; \pi_1, \dots, \pi_{n-1}, [s, d-s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable.*

In the upcoming proofs, we will also make extensive use of the computational results from Appendix A; although delegating work to the computer is never necessary (and in theory we could prove the same results by hand), doing so will save us a lot of effort in dealing with tricky corner-cases, and allow us to focus more on the reduction-oriented part of the proofs.

4.2 Realizability on the torus for $n = 3$

As anticipated in the title, this section deals with the cases where $n = 3$ and $\widetilde{\Sigma} = \Sigma_1$. Due to the relatively large number of families of exceptional data listed in Theorem 4.1, this is the instance of the problem which will require the heaviest casework.

Theorem 4.3. *Let $\mathcal{D} = (\Sigma_1; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$.

(2) $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$.

(3) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

(4) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$.

(5) $\mathcal{D} = (\Sigma_1; 2k; [2,\dots,2], [2,\dots,2,3,5], [k,k])$ *with* $k \geq 5$.

Proof. Thanks to Proposition 2.11 we can assume that $2 \leq s \leq d-2$. If $d \leq 16$, a computer-aided search (see the results in Section A.4) shows that the only exceptional cases are:

(1) $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$;

(2) $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$;

(3) $\mathcal{D} = (\Sigma_1; 10; [2,2,2,2,2], [2,3,5], [5,5])$;

(4) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$;

(5) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [2,2,3,5], [6,6])$;

(6) $\mathcal{D} = (\Sigma_1; 14; [2,2,2,2,2,2,2], [2,2,2,3,5], [7,7])$;

(7) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$;

(8) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [2,2,2,2,3,5], [8,8])$.

This is in agreement with the theorem statement, so we can assume that $d \geq 17$. We now analyze several cases.

Case 1. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $2 \in \pi_2$;
- $\pi_2 \neq [2, \dots, 2]$.

Combinatorial move B.2 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-2; \pi_1 \setminus [x] \cup [1, x-3], \pi_2 \setminus [2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless one of the following holds[2].

- $\mathcal{D} = (\Sigma_1; kh+2; [1, \dots, 1, k+4], [2, h, \dots, h], [lh+1, (k-l)h+1])$ with $k \geq 2$, $h \geq 3$, $1 \leq l \leq k-1$. By applying Combinatorial move B.2 we get

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; kh; [1, \dots, 1, 2, k], [h, \dots, h], [lh, (k-l)h]),$$

  which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; kh+2; [1, \dots, 1, 4, k+1], [2, h, \dots, h], [lh+1, (k-l)h+1])$ with $k \geq 2$, $h \geq 3$, $1 \leq l \leq k-1$. By applying Combinatorial move B.4 we get

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; kh; [1, \dots, 1, 2, k+1], [2, h-2, h, \dots, h], [lh, (k-l)h]),$$

  which is realizable by Theorem 4.1.

Case 2. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $y \in \pi_2$ for some $y \geq 4$;
- $2 \notin \pi_1$ and $2 \notin \pi_2$.

Combinatorial move B.4 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-2; \pi_1 \setminus [x] \cup [x-2], \pi_2 \setminus [y] \cup [y-2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless

$$\mathcal{D} = (\Sigma_1; kh+2; [1, \dots, 1, k+3], [h, \dots, h, h+2], [lh+1, (k-l)h+1])$$

for some $k \geq 2$, $h \geq 3$, $1 \leq l \leq k-1$. If this is the case, applying Combinatorial move B.4 yields

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; kh; [1, \dots, 1, k+1], [h-2, h, \dots, h, h+2], [lh, (k-l)h]),$$

which is realizable by Theorem 4.1.

[2]Since this proof is long enough as it is, we will refrain from explaining in detail why candidate data of a certain form are realizable by Theorem 4.1; we leave the tedious yet elementary casework required to the motivated reader. In this situation, for instance, one could simply note that $\pi_1 \setminus [x] \cup [1, x-3]$ and $\pi_2 \setminus [2]$ are both different from $[2, \dots, 2]$, therefore the only exceptional data to consider among those listed in Theorem 4.1 are those belonging to family (6).

Case 3. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $\max(\pi_2) = 3$;
- $2 \notin \pi_2$.

Combinatorial move B.4 gives

$$\mathcal{D} \rightsquigarrow = (\mathbb{S}; d-2; \pi_1 \setminus [x] \cup [x-2], \pi_2 \setminus [3] \cup [1], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless

$$\mathcal{D} = (\Sigma_1; 2h+2; [h, h+2], [1, \ldots, 1, 3, 3], [h+1, h+1])$$

for some $h \geq 8$ (recall that we are assuming $d \geq 17$). If this is the case, applying Combinatorial move B.1 yields

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2h+2; [h, h+2], [1, \ldots, 1, 3], [h+1, h+1]),$$

which is realizable by Theorem 4.1.

Case 4. Assume that:

- $\max(\pi_1) = 3$;
- $\max(\pi_2) \leq 3$;
- $\pi_2 \neq [2, \ldots, 2]$.

We analyze a few sub-cases.

- **Case 4.1:** $[1, 1] \subseteq \pi_1$. Combinatorial move B.1 gives

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; d; \pi_1 \setminus [3] \cup [1, 1, 1], \pi_2, [s, d-s]),$$

  which is realizable by Theorem 4.1.

- **Case 4.2:** $[3, 3] \subseteq \pi_1$ and $[2, 2] \subseteq \pi_2$. If $s = 2$ or $s = d-2$ then $\mathcal{D}$ is realizable by Proposition 3.2. Otherwise, Combinatorial move B.3 gives

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-4; \pi_1 \setminus [3, 3] \cup [1, 1], \pi_2 \setminus [2, 2], [s-2, d-s-2]),$$

  which is realizable by Theorem 4.1.

- **Case 4.3:** $[2, 2] \not\subseteq \pi_2$. The Riemann-Hurwitz formula immediately implies that $3 \in \pi_2$. Assume by contradiction that $\mathcal{D}$ is not realizable. If this is the case, $[1, 1] \not\subseteq \pi_2$ by case 4.1, but then $[3, 3] \subseteq \pi_2$. It follows (case 4.2) that $[2, 2] \not\subseteq \pi_1$; moreover, case 4.1 also implies that $[1, 1] \not\subseteq \pi_1$. In other words, both π_1 and π_2 can be written as $\rho \cup [3, \ldots, 3]$, where $\rho \subseteq [1, 2]$ (possibly different for π_1 and π_2). As a consequence, we have the inequalities

  $$d \geq 3\ell(\pi_1) - 3, \qquad d \geq 3\ell(\pi_2) - 3,$$

  which contradict the Riemann-Hurwitz formula if $d \geq 13$.

- **Case 4.4:** $[3, 3] \not\subseteq \pi_1$. From the Riemann-Hurwitz formula it follows that $[3, 3] \subseteq \pi_2$, but then $\mathcal{D}$ is realizable by case 4.1.

Case 5. Assume that:

- $\max(\pi_1) \geq 4$;
- $\pi_2 = [2, \ldots, 2]$.

Let $x = \max(\pi_1)$; Combinatorial move B.2 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-2; \pi_1 \setminus [x] \cup [1, x-3], [2, \ldots, 2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless one of the following holds.

- $\mathcal{D} = (\Sigma_1; 2k+2; [2, \ldots, 2, 3, 5], [2, \ldots, 2], [k+1, k+1])$ with $k \geq 7$. This is in fact one of the exceptional data listed in the statement.
- $\mathcal{D} = (\Sigma_1; 2k+2; [2, \ldots, 2, 6], [2, \ldots, 2], [k+1, k+1])$ with $k \geq 7$. By applying Combinatorial move B.2 we get
$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2k; [2, \ldots, 2], [2, \ldots, 2], [k, k]),$$
which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 4k+4; [1, \ldots, 1, k+2, k+4], [2, \ldots, 2], [2k+2, 2k+2])$ with $k \geq 3$. By applying Combinatorial move B.2 we get
$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 4k+2; [1, \ldots, 1, 2, k, k+2], [2, \ldots, 2], [2k+1, 2k+1]),$$
which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 4k+4; [1, \ldots, 1, k+1, k+5], [2, \ldots, 2], [2k+2, 2k+2])$ with $k \geq 3$. By applying Combinatorial move B.2 we get
$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 4k+2; [1, \ldots, 1, 2, k+1, k+1], [2, \ldots, 2], [2k+1, 2k+1]),$$
which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 4k+2; [1, \ldots, 1, k+1, k+4], [2, \ldots, 2], [2k, 2k+2])$ with $k \geq 4$. By applying Combinatorial move B.2 we get
$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 4k; [1, \ldots, 1, 2, k, k+1], [2, \ldots, 2], [2k-1, 2k+1]),$$
which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 2k+2; [1, \ldots, 1, k+4], [2, \ldots, 2], [2l+1, 2(k-l)+1])$ with $k \geq 7$, $1 \leq l \leq k-1$. By applying Combinatorial move B.2 we get
$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2k; [1, \ldots, 1, 2, k], [2, \ldots, 2], [2l, 2(k-l)]),$$
which is realizable by Theorem 4.1.

Case 6. Assume that:

- $\max(\pi_1) = 3$;
- $\pi_2 = [2, \dots, 2]$.

The Riemann-Hurwitz formula immediately implies that $[3, 3, 3, 3] \subseteq \pi_1$. If $s = 2$ or $s = d - 2$ then $\mathcal{D}$ is realizable by Proposition 3.2. Otherwise, Combinatorial move B.3 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-4; \pi_1 \smallsetminus [3,3] \cup [1,1], [2, \dots, 2], [s-2, d-s-2]),$$

which is realizable by Theorem 4.1, since we are assuming that $d \geq 17$.

The cases we have analyzed, up to swapping π_1 and π_2, cover all the candidate data of the form $(\Sigma_1; d; \pi_1, \pi_2, [s, d-s])$. We have shown that every datum which is not listed in the statement is realizable, therefore the proof is complete. □

4.3 Realizability on higher genus surfaces for $n = 3$

It turns out that, for $n = 3$ and $\ell(\pi_3) = 2$, there are no exceptional data on surfaces with genus $g \geq 2$.

Theorem 4.4. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable.*

Proof. Thanks to Proposition 2.11 we can assume that $2 \leq s \leq d - 2$. For $d \leq 18$, a computer-aided search shows that there are no exceptional data (see the results in Section A.4). Therefore, we can further assume that $d \geq 19$. We proceed by induction on $g \geq 2$, analyzing several cases.

Case 1. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $2 \in \pi_2$.

Combinatorial move B.2 gives

$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d-2; \pi_1 \smallsetminus [x] \cup [1, x-3], \pi_2 \smallsetminus [2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

Case 2. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $y \in \pi_2$ for some $y \geq 3$;
- $2 \notin \pi_1$ and $2 \notin \pi_2$.

Combinatorial move B.4 gives

$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d-2; \pi_1 \smallsetminus [x] \cup [x-2], \pi_2 \smallsetminus [y] \cup [y-2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

Case 3. Assume that:

- $\max(\pi_1) = 3$;
- $\max(\pi_2) \leq 3$.

We analyze a few sub-cases.

- **Case 3.1:** $[1,1] \subseteq \pi_1$. Combinatorial move B.1 gives
$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d; \pi_1 \smallsetminus [3] \cup [1,1,1], \pi_2, [s, d-s]),$$
which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

- **Case 3.2:** $[3,3] \subseteq \pi_1$ and $[2,2] \subseteq \pi_2$. If $s = 2$ or $s = d-2$ then $\mathcal{D}$ is realizable by Proposition 3.2. Otherwise, Combinatorial move B.3 gives
$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d-4; \pi_1 \smallsetminus [3,3] \cup [1,1], \pi_2 \smallsetminus [2,2], [s-2, d-s-2]),$$
which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

- **Case 3.3:** $[2,2] \not\subseteq \pi_1$. The Riemann-Hurwitz formula immediately implies that $[3,3] \subseteq \pi_2$. Assume by contradiction that $\mathcal{D}$ is not realizable. Then $[1,1] \not\subseteq \pi_1$ and $[1,1] \not\subseteq \pi_2$ (case 3.1), and moreover $[2,2] \not\subseteq \pi_1$ (case 3.2). In other words, both π_1 and π_2 can be written as $\rho \cup [3, \ldots, 3]$, where $\rho \subseteq [1,2]$ (possibly different for π_1 and π_2). It is then easy to see that $\mathcal{D}$ belongs to one of the families listed in Proposition 3.3 and, therefore, is realizable.

- **Case 3.4:** $[3,3] \not\subseteq \pi_1$. From the Riemann-Hurwitz formula it follows that $[3,3] \subseteq \pi_2$, but then $\mathcal{D}$ is realizable by cases 3.2 and 3.3.

The cases we have analyzed, up to swapping π_1 and π_2, cover all the candidate data of the form $(\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ with $g \geq 2$. We have shown that every datum is realizable, therefore the proof is complete. □

4.4 Realizability for $n \geq 4$

After solving the existence problem for $n = 3$ and $\ell(\pi_3) = 2$, we turn to candidate data with $n \geq 4$ partitions. As we are about to see, exceptional data in this setting are very rare: there is an infinite family with $d = 4$, which we already encountered in Proposition 2.14, and a single datum with $n = 4$ and $d = 8$.

Theorem 4.5. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

(2) $\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$.

Proof. If $d \leq 16$, a computer-aided search (see the results in Section A.4) shows that the only exceptional cases are:

(1) $\mathcal{D} = (\Sigma_1; 4; [2,2], [2,2], [2,2], [1,3])$;

(2) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

This is in agreement with the statement, so we can assume that $d \geq 17$. Moreover, every candidate datum with $g = 0$ is realizable by Theorem 4.2; as a consequence, we only have to consider the cases where $g \geq 1$.

We will proceed by induction on n. We start with the base case $n = 4$, which requires the heaviest casework. Fix a candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3, [s, d-s])$.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq 3$; up to reindexing, we can assume that $v(\pi_1) + v(\pi_2) < d$. Combinatorial move A.1 gives

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_g; d; \pi_1', \pi_3, [s, d-s])$$

for a suitable candidate datum $\mathcal{D}'$, where $v(\pi_1') = v(\pi_1) + v(\pi_2)$. If $g \geq 2$, then $\mathcal{D}'$ is realizable by Theorem 4.4. If instead $g = 1$, then $\mathcal{D}'$ is realizable by Theorem 4.3 unless

$$\mathcal{D}' = (\Sigma_1; 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k]) \text{ with } 2k = d.$$

If this is the case, then $\pi_4 = [k, k]$ and $\{\pi_1', \pi_3\} = \{[2, \ldots, 2], [2, \ldots, 2, 3, 5]\}$. Some more casework is required to show that $\mathcal{D}'$ can actually be chosen to be realizable; since $\mathcal{D} \rightsquigarrow \mathcal{D}'$, this will imply that $\mathcal{D}$ is realizable as well.

  - If $\pi_1' = [2, \ldots, 2]$, then $v(\pi_1') = k$ and $v(\pi_3) = k + 2$. Assume without loss of generality that $v(\pi_1) \leq k/2$. We have that

$$k + 2 < 1 + k + 2 \leq v(\pi_1) + v(\pi_3) \leq \frac{k}{2} + k + 2 < d.$$

  Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \ldots, 2]$ and $v(\pi_1) \notin \{2, k, k+1\}$ then $v(\pi_1) + v(\pi_3) < d$ and $v(\pi_1) + v(\pi_3) \notin \{k, k+2\}$. Therefore, repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \ldots, 2]$, $v(\pi_1) = 2$ and $\pi_2 \neq [2, \ldots, 2]$, then repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_2 = \pi_3 = [2, \ldots, 2]$ and $\pi_1 \neq [1, \ldots, 1, 3]$, we follow a different approach. By applying Combinatorial move A.2 to the partitions π_2 and π_3 we get

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2k; \pi_1, [k, k], [k, k]),$$

  which is realizable by Theorem 4.1.
  - Finally, if $\pi_1 = [1, \ldots, 1, 3]$ and $\pi_2 = \pi_3 = [2, \ldots, 2]$, we have to work explicitly with permutations. Consider

$$\alpha_1 = (1, 3, 5), \qquad \alpha_2 = (1, 2)(3, 4) \cdots (2k-1, 2k).$$

  Clearly $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$; moreover,

$$\alpha_1 \alpha_2 = (1, 2, 3, 4, 5, 6)(7, 8) \cdots (2k-1, 2k),$$

  so $[\alpha_1 \alpha_2] = [2, \ldots, 2, 6]$. Note that

$$v(\alpha_1) + v(\alpha_2) = 2 + k = v(\alpha_1 \alpha_2),$$

  so Remark 2.2 gives

$$\mathcal{D} \rightsquigarrow (\Sigma_1; 2k; [2, \ldots, 2, 6], [2, \ldots, 2], [k, k]),$$

  which is realizable by Theorem 4.3.

Up to swapping π_1 and π_2, this analysis covers all the possible cases.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq 3$. In particular, up to reindexing, we can assume that $v(\pi_3) \geq d/2$. Note that $v(\pi_1) + v(\pi_2) \geq d$ and $v(\pi_3) + v([s, d-s]) \geq 1 + d - 2 = d - 1$, so Combinatorial move A.2 gives

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_{g'}; d; \pi_1', \pi_3, [s, d-s])$$

for a suitable candidate datum $\mathcal{D}'$ with $v(\pi_1') \geq d - 2$. We can actually compute

$$g' = \frac{1}{2}(v(\pi_1') + v(\pi_3) + v([s, d-s]) - d + 1 \geq \frac{1}{2}\left(d - 2 + \frac{d}{2} + d - 2\right) - d + 1 = \frac{d}{4} - 1 \geq 2.$$

Therefore $\mathcal{D}'$ is realizable by Theorem 4.4, and $\mathcal{D}$ is realizable as well.

We now turn to the case $n \geq 5$; we show by induction that every candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ different from $(\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$ is realizable. The case $d = 4$ is addressed by Proposition 2.14. If $n = 5$ and $d = 8$, the computational results in Section A.4 imply that $\mathcal{D}$ is realizable. Otherwise, the routine reduction argument relying on Combinatorial moves A.1 and A.2 shows that $\mathcal{D}$ is realizable. □

4.5 Exceptionality

This section is devoted to showing that the candidate data listed in the statements of Theorems 4.1, 4.3 and 4.5 are in fact exceptional. First of all, we address the special cases with small degree ($d \leq 16$). The computational results in Section A.4 show that the following candidate data are exceptional.

(1) $\mathcal{D} = (\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

(2) $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$.

(3) $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$.

(4) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

(5) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$.

(6) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

We now turn to infinite families of exceptional data.

Proposition 4.6. *Let $n \geq 3$ be a positive integer. Then the candidate datum*

$$\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$$

is exceptional.

Proof. In order to show the exceptionality of $\mathcal{D}$, we will employ the monodromy approach. Let $\alpha_1, \ldots, \alpha_{n-1} \in \mathfrak{S}_4$ be permutations matching $[2,2]$. It is easy to see that the three permutations matching $[2,2]$, together with the identity, form a subgroup

$$\{\mathrm{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \leq \mathfrak{S}_4,$$

which does not contain any 3-cycles. As a consequence, the product $\alpha_1 \cdots \alpha_{n-1}$ cannot match $[1,3]$; by Corollary 1.7, this implies that $\mathcal{D}$ is exceptional. □

For the other families of exceptional data, we will instead make use of dessins d'enfant.

Proposition 4.7. *Let $h \geq 2$, $k \geq 2$, $1 \leq l \leq k-1$ be integers. Then the candidate datum*

$$\mathcal{D} = (\mathbb{S}; kh; [h, \ldots, h], [1, \ldots, 1, k+1], [lh, (k-l)h])$$

is exceptional.

Proof. Assume by contradiction that there is a dessin d'enfant $\Gamma \subseteq \mathbb{S}$ realizing $\mathcal{D}$. Let v be the white vertex of degree $k+1$. Since Γ is connected, each of the k black vertices must have an edge connecting it to v. Therefore there are exactly $k-1$ black vertices with one edge between them and v, and one black vertex with two edges between it and v; we call this special black vertex u. The two edges connecting u and v define the two complementary disks of Γ. As shown in the picture below, there are (excluding u) a black vertices inside D_1 and $k-1-a$ black vertices inside D_2, for some integer $0 \leq a \leq k-1$; of the $h-2$ white vertices of degree 1 connected to u, b lie inside D_1, while $h-2-b$ lie inside D_2, for some $0 \leq b \leq h-2$. It is easy to compute the perimeters:

$$\frac{1}{2}|\partial D_1| = ah + b + 1, \qquad \frac{1}{2}|\partial D_2| = (k-1-a)h + (h-2-b) + 1.$$

In particular, we see that $|\partial D_1|/2$ and $|\partial D_2|/2$ are not divisible by h, hence Γ cannot be a dessin d'enfant realizing $\mathcal{D}$. □

The following result will allow us to deal with candidate data of the form $(\Sigma_g; d; [2, \ldots, 2], \pi_2, \pi_3)$.

Lemma 4.8. *Let $\mathcal{D} = (\Sigma_g; d; [2, \ldots, 2], \pi_2, \pi_3)$ be a combinatorial datum. Then $\mathcal{D}$ is realizable if and only if there exists a graph Γ embedded in Σ_g such that:*

- *$\pi_2 = [k(v_1), \ldots, k(v_r)]$, where $v_1, \ldots, v_r$ are the vertices of Γ;*
- *the complementary regions of Γ are topological disks $D_1, \ldots, D_h$;*
- *$\pi_3 = [|\partial D_1|, \ldots, |\partial D_h|]$.*

Proof. The key idea is that vertices of degree 2 do not contribute to the topology of a graph in any meaningful way.

If are given a dessin d'enfant $\Gamma' \subseteq \Sigma_g$ realizing $\mathcal{D}$, we can obtain a suitable graph Γ simply by removing all the black vertices, and merging the two edges adjacent to each one of them into a single edge.

Conversely, if we are given a graph Γ, we can recover a dessin d'enfant Γ' by coloring the vertices of Γ with the color white, and inserting a black vertex in the middle of every edge. □

This lemma suggests an approach by enumeration for showing the exceptionality of candidate data of the form $(\Sigma_g; d; [2, \ldots, 2], \pi_2, \pi_3)$. In fact, if the number of fat graphs whose degrees are the entries of π_2 is small, we can check them one by one to see if the associated embedded graph satisfies the conditions of Lemma 4.8.

Proposition 4.9. *The following families of candidate data are exceptional.*

(1) $\mathcal{D} = (\mathbb{S}; 2k; [2, \ldots, 2], [2, \ldots, 2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

(2) $\mathcal{D} = (\mathbb{S}; 2k; [2, \ldots, 2], [1, 2, \ldots, 2, 3], [k, k])$ *with* $k \geq 2$.

(3) $\mathcal{D} = (\mathbb{S}; 4k+2; [2, \ldots, 2], [1, \ldots, 1, k+1, k+2], [2k+1, 2k+1])$ *with* $k \geq 1$.

(4) $\mathcal{D} = (\mathbb{S}; 4k; [2, \dots, 2], [1, \dots, 1, k+1, k+1], [2k-1, 2k+1])$ *with* $k \geq 2$.

Proof. For each family listed in the statement, we follow the approach by enumeration presented above: we draw all the possible graphs embedded in $\mathbb{S}$ whose vertices have the entries of π_2 as degrees and whose complementary regions are two disks, and then we compute the perimeters of these disks, showing that they cannot be equal to the entries of π_3.

(1) There is only one graph embedded in $\mathbb{S}$ whose vertices have degrees $[2, \dots, 2]$.

By Lemma 4.8, it follows that $\mathcal{D}$ is exceptional unless $s = k$.

(2) A graph embedded in $\mathbb{S}$ whose vertices have degrees $[1, 2, \dots, 2, 3]$ can be represented by a diagram like the following, where a denotes the number of blue edges.

The perimeters of the complementary disks are easily computed as

$$|\partial D_1| = k - a - 1, \qquad |\partial D_2| = k + a + 1,$$

hence $\mathcal{D}$ is exceptional by Lemma 4.8.

(3) There are only three kinds of graphs embedded in $\mathbb{S}$ whose vertices have degrees $[1, \dots, 1, k+1, k+2]$. First case:

Probably a table?

$$|\partial D_1| = 4k - 2a - 2b, \qquad |\partial D_2| = 2a + 2b + 2.$$

Second case:

$$|\partial D_1| = 2k - 2b - 1, \qquad |\partial D_2| = 2k + 2b + 3.$$

Third case:

$$|\partial D_1| = 2k - 2a + 1, \qquad |\partial D_2| = 2k + 2a + 1.$$

(4) A graph embedded in $\mathbb{S}$ whose vertices have degrees $[1, \dots, 1, k+1, k+2]$ can be represented by a diagram like the following, where a denotes the number of blue edges connected to the vertex of degree $k+1$, and b denotes the number of blue edges connected to the vertex of degree $k+2$.

The perimeters of the complementary disks are easily computed as

$$|\partial D_1| = k - a - 1, \qquad |\partial D_2| = k + a + 1,$$

hence $\mathcal{D}$ is exceptional by Lemma 4.8.

□

Proposition 4.10. *Let* $k \geq 5$ *be an integer. Then the candidate datum*

$$\mathcal{D} = (\Sigma_1; 2k; [2, \dots, 2], [2, \dots, 2, 3, 5], [k, k])$$

is exceptional.

Proof. □

4.6 Final result

By combining all the results in this chapter, we are finally able to compile the full list of exceptional data with a partition of length 2.

Solution of the Hurwitz existence problem for $\ell(\pi_n) = 2$. *Let*

$$\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$$

be a candidate datum with $n \geq 3$. Then $\mathcal{D}$ is exceptional if and only if one of the following holds.

(1) $\mathcal{D} = (\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

(2) $\mathcal{D} = (\mathbb{S}; 2k; [2,\ldots,2], [2,\ldots,2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

(3) $\mathcal{D} = (\mathbb{S}; 2k; [2,\ldots,2], [1,2,\ldots,2,3], [k,k])$ *with* $k \geq 2$.

(4) $\mathcal{D} = (\mathbb{S}; 4k+2; [2,\ldots,2], [1,\ldots,1,k+1,k+2], [2k+1,2k+1])$ *with* $k \geq 1$.

(5) $\mathcal{D} = (\mathbb{S}; 4k; [2,\ldots,2], [1,\ldots,1,k+1,k+1], [2k-1,2k+1])$ *with* $k \geq 2$.

(6) $\mathcal{D} = (\mathbb{S}; kh; [h,\ldots,h], [1,\ldots,1,k+1], [lh,(k-l)h])$ *with* $h \geq 2$, $k \geq 2$, $1 \leq l \leq k-1$.

(7) $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$.

(8) $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$.

(9) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

(10) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$.

(11) $\mathcal{D} = (\Sigma_1; 2k; [2,\ldots,2], [2,\ldots,2,3,5], [k,k])$ *with* $k \geq 5$.

(12) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

(13) $\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$.

Appendix A

Computational results

A.1 Zheng's formula

This chapter is devoted to the presentation of a computational approach to the Hurwitz existence problem, which was first introduced by Zheng in [17], and to the discussion of some experimental results we have obtained by exploiting it. In this short section, we will merely report the formula used for the computation, without any explanation. For an in-depth derivation of said formula, we refer the reader to Zheng's article.

We will require some basic notions about the representation theory of $\mathfrak{S}_d$; all the material we will need is extensively covered in [15]. Let d be a positive integer. Given a partition $\pi \in \Pi(d)$, we denote by ρ^π, ζ^π and C_π the irreducible representation of $\mathfrak{S}_d$, the character and the conjugacy class of $\mathfrak{S}_d$ associated to π.

A *secondary partition* of d is an unordered multiset $\mu = [\pi_1, \ldots, \pi_k]$, where $\pi_1, \ldots, \pi_k$ are partitions such that $\sum \pi_1 + \ldots + \sum \pi_k = d$. We denote the set of all secondary partitions of d by $\Pi^2(d)$.

Fix a positive integer n. We will work in the polynomial ring $\mathbb{Q}[\mathcal{T}]$, where $\mathcal{T}$ is the infinite set of variables $\mathcal{T} = \{t_{i,j} \colon 1 \leq i \leq n, j \geq 1\}$. Given an integer $1 \leq i \leq n$ and a partition $\pi = [a_1, \ldots, a_p] \in \Pi(d)$, we define $\mathbf{t}_i^\pi = t_{i,a_1} \cdots t_{i,a_p}$.

Consider now a secondary partition $\mu = [\pi_1, \ldots, \pi_k] \in \Pi^2(d)$; let $d_i = \sum \pi_i$ for $1 \leq i \leq k$, and let $m_1, \ldots, m_l$ be the multiplicities of the elements of $[\pi_1, \ldots, \pi_k]$ (in particular, $m_1 + \ldots + m_l = k$). We define:

- the rational number $r(\mu) = (-1)^{k-1} \dfrac{(k-1)!}{m_1! \cdots m_l!} \displaystyle\prod_{j=1}^{k} \left(\frac{\dim \rho^{\pi_j}}{d_j!} \right)^2$;
- the polynomials $s(\mu)(t_{i,1}, \ldots, t_{i,d}) = \displaystyle\prod_{j=1}^{k} \sum_{\nu \in \Pi(d_j)} \frac{\zeta^{\pi_j}(C_\nu) \cdot |C_\nu|}{\dim \rho^{\pi_j}} \cdot \mathbf{t}_j^\nu$ for $1 \leq i \leq n$.

Finally, for fixed positive integers n and d, we define the polynomial

$$P_{n,d}(t_{1,1}, \ldots, t_{1,d}, t_{2,1}, \ldots, t_{n,d}) = \sum_{\mu \in \Pi^2(d)} r(\mu) \prod_{i=1}^{k} s(\mu)(t_{i,1}, \ldots, t_{i,d}).$$

As shown in [17], given a candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_n)$, the coefficient of the monomial $\mathbf{t}_1^{\pi_1} \cdots \mathbf{t}_n^{\pi_n}$ in $P_{n,d}$ is non-zero if and only if $\mathcal{D}$ is realizable. This property gives a computationally efficient method for finding all the exceptional data for given values of n and d.

A.2 Implementation

We wrote a C++ program for efficiently listing the exceptional data using Zheng's formula. Here are a few implementation details.

- For a given partition $\pi = [a_1, \ldots, a_p] \in \Pi(d)$, computing $|C_\pi|$ is very easy: we have the formula
$$|C_\pi| = \frac{d!}{a_1! \cdots a_p!}.$$
- The characters ζ^π can be recursively computed by means of the Murnaghan–Nakayama formula (see [15, Theorem 4.10.2]).
- In order to compute the dimension of ρ^π, we can exploit the fact that $\dim \rho^\pi = \zeta^\pi(\mathrm{id})$, since we need the characters anyway.
- Instead of carrying out the exact computation in $\mathbb{Q}$, which is quite expensive, we can work in a finite field $\mathbb{F}_q$ for some prime q. If a candidate datum is realizable according the the reduced computation in $\mathbb{F}_q$, then it is realizable; if it is exceptional for sufficiently many primes q, then it is exceptional.
- The computation of $P_{n,d}$ is very well-suited to parallelization, since it involves a summation whose terms can be evaluated independently.

A.3 Results

To the best of the author's knowledge, the only available computational results date back to 2006, when Zheng managed to compute the polynomial $P_{n,d}$ for $n = 3$ and $d \leq 20$. Thanks to the resources kindly provided by the University of Pisa (and the undeniable advancement in hardware technology in the past 15 years), we were able to carry out the computation up to $d = 29$ which, very conveniently, is a prime number. The results are displayed in the following table (for obvious reasons, we cannot include the full list).

Number of exceptional data with $n = 3$, $d \leq 29$			
d	$\widetilde{\Sigma} = \mathbb{S}$	$\widetilde{\Sigma} = \Sigma_1$	Total
2	–	–	0
3	–	–	0
4	1	–	1
5	–	–	0
6	5	1	6
7	–	–	0
8	13	1	14
9	6	1	7
10	34	1	35
11	–	–	0
12	91	4	95
13	–	–	0
14	148	1	149
15	38	2	40
16	307	7	314

17	–	–	0
18	653	8	661
19	–	–	0
20	1068	4	1072
21	148	2	150
22	2064	1	2065
23	–	–	0
24	3747	9	3756
25	27	–	27
26	6359	1	6360
27	531	1	532
28	10 627	3	10 630
29	–	–	0

Remark A.1. For the sake of efficiency, as explained in the previous section, the computation was carried out in the finite fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$, where $q_1 = 1\,000\,000\,007$ and $q_2 = 1\,000\,000\,009$. As a consequence, the numbers reported in the table are merely an upper bound, rather than exact values. However, heuristic arguments suggests that, at least for $d \leq 29$, working in these two finite fields is enough to avoid any false positives.

A few considerations are in order. First of all, one of the most prominent goals of the computer-assisted approach to the Hurwitz existence problem is providing experimental evidence supporting the prime-degree conjecture. Thanks to Proposition 2.15, the verification can be reduced to the $n = 3$ case. Therefore, our results prove that the conjecture holds for prime numbers up to 29 (recall that the algorithm is able to prove realizability even when working in finite fields).

Moreover, the results reveal that most of the exceptional data have $\widetilde{\Sigma} = \mathbb{S}$; there are a handful with $\widetilde{\Sigma} = \Sigma_1$, and none with $\widetilde{\Sigma} = \Sigma_g$ for $g \geq 2$. This curious pattern calls for a more in-depth investigation: the following tables show the results we have obtained by running Zheng's algorithm for $n = 4$ and $n = 5$.

Number of exceptional data with $n = 4$, $d \leq 20$				
d	$\widetilde{\Sigma} = \mathbb{S}$	$\widetilde{\Sigma} = \Sigma_1$	$\widetilde{\Sigma} = \Sigma_2$	Total
2	–	–	–	0
3	–	–	–	0
4	–	1	–	1
5	–	–	–	0
6	2	1	–	3
7	–	–	–	0
8	7	2	1	10
9	1	–	–	1
10	30	2	1	33
11	–	–	–	0
12	75	2	1	78
13	–	–	–	0
14	213	2	1	216
15	21	–	–	21
16	454	2	2	458
17	–	–	–	0

18	1184	2	1	1187
19	–	–	–	0
20	2185	2	1	2188

Number of exceptional data with $n = 5$, $d \leq 15$				
d	$\widetilde{\Sigma} = \mathbb{S}$	$\widetilde{\Sigma} = \Sigma_1$	$\widetilde{\Sigma} = \Sigma_2$	Total
2	–	–	–	0
3	–	–	–	0
4	–	–	1	1
5	–	–	–	0
6	–	–	–	0
7	–	–	–	0
8	1	–	–	1
9	–	–	–	0
10	9	–	–	9
11	–	–	–	0
12	25	–	–	25
13	–	–	–	0
14	107	–	–	107
15	5	–	–	5

From the experimental results it looks like, even for $n \geq 4$, the majority of the exceptional data occurs when $\widetilde{\Sigma} = \mathbb{S}$. We note that, unlike in the $n = 3$ case, when $n \geq 4$ we also find a few exceptional data on surfaces of genus $g \geq 2$; however, all of them are covered by the following families:

(1) $\mathcal{D} = (\Sigma_2; 16; [2, \dots, 2], [2, \dots, 2], [2, \dots, 2], [1, 3, 3, 3, 3, 3])$;

(2) $\mathcal{D} = (\Sigma_2; 2k; [2, \dots, 2], [2, \dots, 2], [2, \dots, 2], [2, \dots, 2, 3, 5])$ with $k \geq 4$;

(3) $\mathcal{D} = (\Sigma_{n-3}; 4; [2, 2], \dots, [2, 2], [1, 3])$ with $n \geq 5$.

This observation leads to the following conjecture, which was already proposed in [17].

Conjecture (Zheng)**.** *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \dots, \pi_n)$ be a candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable unless it belongs to one of the aforementioned families.*

According to Zheng, the statement can be reduced to the $n = 3$ case, just like the prime-degree conjecture. Therefore, our computation shows that Zheng's conjecture holds at least for $d \leq 29$.

A.4 Lists of exceptional data with a short partition

We conclude this appendix by enumerating all the exceptional data $\mathcal{D} = (\Sigma_g; d; \pi_1, \dots, \pi_n)$ with $\ell(\pi_n) = 2$ for small values of n and d. These lists, which were computed by our C++ program, were used in the proofs of Theorems 4.3 to 4.5 to dramatically reduce the number of cases we had to handle. In the unlikely (but not impossible) event that running the algorithm in finite fields yields some false positives, the results in this section were double-checked by carrying out the exact computation in the field $\mathbb{Q}$ of rational numbers.

Exceptional data with $n = 3$, $\ell(\pi_3) = 2$, $\widetilde{\Sigma} = \mathbb{S}$ and $d \leq 12$
$(\mathbb{S}; 4; [2,2], [2,2], [1,3])$
$(\mathbb{S}; 6; [2,2,2], [2,2,2], [2,4])$
$(\mathbb{S}; 6; [2,2,2], [2,2,2], [1,5])$
$(\mathbb{S}; 6; [2,2,2], [1,2,3], [3,3])$
$(\mathbb{S}; 6; [2,2,2], [1,1,4], [2,4])$
$(\mathbb{S}; 6; [1,1,1,3], [3,3], [3,3])$
$(\mathbb{S}; 8; [2,2,2,2], [2,2,2,2], [3,5])$
$(\mathbb{S}; 8; [2,2,2,2], [2,2,2,2], [2,6])$
$(\mathbb{S}; 8; [2,2,2,2], [2,2,2,2], [1,7])$
$(\mathbb{S}; 8; [2,2,2,2], [1,2,2,3], [4,4])$
$(\mathbb{S}; 8; [2,2,2,2], [1,1,3,3], [3,5])$
$(\mathbb{S}; 8; [2,2,2,2], [1,1,1,5], [4,4])$
$(\mathbb{S}; 8; [2,2,2,2], [1,1,1,5], [2,6])$
$(\mathbb{S}; 8; [1,1,1,1,1,3], [4,4], [4,4])$
$(\mathbb{S}; 9; [1,1,1,1,1,4], [3,3,3], [3,6])$
$(\mathbb{S}; 10; [2,2,2,2,2], [2,2,2,2,2], [4,6])$
$(\mathbb{S}; 10; [2,2,2,2,2], [2,2,2,2,2], [3,7])$
$(\mathbb{S}; 10; [2,2,2,2,2], [2,2,2,2,2], [2,8])$
$(\mathbb{S}; 10; [2,2,2,2,2], [2,2,2,2,2], [1,9])$
$(\mathbb{S}; 10; [2,2,2,2,2], [1,2,2,2,3], [5,5])$
$(\mathbb{S}; 10; [2,2,2,2,2], [1,1,1,3,4], [5,5])$
$(\mathbb{S}; 10; [2,2,2,2,2], [1,1,1,1,6], [4,6])$
$(\mathbb{S}; 10; [2,2,2,2,2], [1,1,1,1,6], [2,8])$
$(\mathbb{S}; 10; [1,1,1,1,1,1,1,3], [5,5], [5,5])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [2,2,2,2,2,2], [5,7])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [2,2,2,2,2,2], [4,8])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [2,2,2,2,2,2], [3,9])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [2,2,2,2,2,2], [2,10])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [2,2,2,2,2,2], [1,11])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [1,2,2,2,2,3], [6,6])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,1,4,4], [5,7])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,1,1,7], [6,6])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,1,1,7], [4,8])$
$(\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,1,1,7], [2,10])$
$(\mathbb{S}; 12; [1,1,1,1,1,1,1,1,1,3], [6,6], [6,6])$
$(\mathbb{S}; 12; [1,1,1,1,1,1,1,5], [3,3,3,3], [6,6])$
$(\mathbb{S}; 12; [1,1,1,1,1,1,1,5], [3,3,3,3], [3,9])$
$(\mathbb{S}; 12; [1,1,1,1,1,1,1,1,4], [4,4,4], [4,8])$

Exceptional data with $n = 3$ $\ell(\pi_3) = 2$, $g \geq 1$ and $d \leq 18$
$(\Sigma_1; 6; [3,3], [3,3], [2,4])$
$(\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$
$(\Sigma_1; 10; [2,2,2,2,2], [2,3,5], [5,5])$
$(\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$

$(\Sigma_1; 12; [2,2,2,2,2,2], [2,2,3,5], [6,6])$
$(\Sigma_1; 14; [2,2,2,2,2,2,2], [2,2,2,3,5], [7,7])$
$(\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$
$(\Sigma_1; 16; [2,2,2,2,2,2,2,2], [2,2,2,2,3,5], [8,8])$
$(\Sigma_1; 18; [2,2,2,2,2,2,2,2,2], [2,2,2,2,2,3,5], [9,9])$

Exceptional data with $n = 4$, $\ell(\pi_4) = 2$ and $d \leq 16$
$(\Sigma_1; 4; [2,2], [2,2], [2,2], [1,3])$
$(\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$

Exceptional data with $n = 5$, $\ell(\pi_5) = 2$ and $d \leq 8$
$(\Sigma_2; 4; [2,2], [2,2], [2,2], [2,2], [1,3])$

Bibliography

[1] Allan L. Edmonds, Ravi S. Kulkarni, and Robert E. Stong. "Realizability of branched coverings of surfaces". In: *Trans. Amer. Math. Soc.* 282.2 (1984), pp. 773–790.

[2] Alexandre Grothendieck. "Esquisse d'un programme". In: *Geometric Galois actions, 1.* Vol. 242. London Math. Soc. Lecture Note Ser. With an English translation on pp. 243–283. Cambridge Univ. Press, Cambridge, 1997, pp. 5–48.

[3] A. Hurwitz. "Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten". In: *Math. Ann.* 39.1 (1891), pp. 1–60.

[4] Jin Ho Kwak and Alexander Mednykh. "Enumerating branched coverings over surfaces with boundaries". In: *European J. Combin.* 25.1 (2004), pp. 23–34.

[5] W. S. Massey. "Finite covering spaces of 2-manifolds with boundary". In: *Duke Math. J.* 41 (1974), pp. 875–887.

[6] Stefano Monni, Jun S. Song, and Yun S. Song. "The Hurwitz enumeration problem of branched covers and Hodge integrals". In: *J. Geom. Phys.* 50.1-4 (2004), pp. 223–256.

[7] James R. Munkres. *Topology.* Second edition. Prentice Hall, Inc., Upper Saddle River, NJ, 2000, pp. xvi+537.

[8] F. Pakovich. "Solution of the Hurwitz problem for Laurent polynomials". In: *J. Knot Theory Ramifications* 18.2 (2009), pp. 271–302.

[9] Ekaterina Pervova and Carlo Petronio. "On the existence of branched coverings between surfaces with prescribed branch data. II". In: *J. Knot Theory Ramifications* 17.7 (2008), pp. 787–816.

[10] Ekaterina Pervova and Carlo Petronio. "Realizability and exceptionality of candidate surface branched covers: methods and results". In: *Geometry Seminars. 2005–2009 (Italian).* Univ. Stud. Bologna, Bologna, 2010, pp. 105–120.

[11] Carlo Petronio. "Explicit computation of some families of Hurwitz numbers". In: *European J. Combin.* 75 (2019), pp. 136–151.

[12] Carlo Petronio. "Explicit computation of some families of Hurwitz numbers, II". In: *Adv. Geom.* 20.4 (2020), pp. 483–498.

[13] Carlo Petronio. "Realizations of certain odd-degree surface branch data". In: *Rend. Istit. Mat. Univ. Trieste* 52 (2020), pp. 355–379.

[14] Carlo Petronio and Filippo Sarti. "Counting surface branched covers". In: *Studia Sci. Math. Hungar.* 56.3 (2019), pp. 309–322.

[15] Bruce E. Sagan. *The symmetric group.* Second. Vol. 203. Graduate Texts in Mathematics. Representations, combinatorial algorithms, and symmetric functions. Springer-Verlag, New York, 2001, pp. xvi+238.

[16] Tamás Szamuely. *Galois groups and fundamental groups*. Vol. 117. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2009, pp. x+270.

[17] Hao Zheng. "Realizability of branched coverings of S^2". In: *Topology Appl.* 153.12 (2006), pp. 2124–2134.