Chapter 1

Hurwitz existence problem

1.1 Branched coverings of surfaces

According to standard terminology, a *surface* is simply a topological 2-manifold. We will, however, only be concerned with compact, connected surfaces without boundary. For the sake of conciseness, unless otherwise stated, we will always implicitly assume that the surfaces we mention have these properties.

There is a structure theorem which completely classifies orientable and non-orientable surfaces.

- An orientable surface is (homeomorphic to) a connected sum of $g \geq 0$ tori. We call such a connected sum a *surface of genus* g, and we denote it by Σ_g. By definition, we say that Σ_0 is the 2-sphere S^2; this is consistent with the formula $\chi(\Sigma_g) = 2 - 2g$ for the Euler characteristic.

- A non-orientable surface is (homeomorphic to) a connected sum of $g \geq 1$ real projective planes. We denote such a connected sum by N_g. The Euler characteristic for a non-orientable surfaces is given by $\chi(N_g) = 2 - g$.

Loosely speaking, given two surfaces $\Sigma, \widetilde{\Sigma}$, a *covering map* between them is a continuous function $f\colon \widetilde{\Sigma} \to \Sigma$ which is locally modeled on the identity function $\mathbb{R}^2 \to \mathbb{R}^2$. Sometimes, however, the notion of covering map can be too restrictive. Consider, for instance, the sphere S^2: being simply connected, it does not admit any non-trivial coverings. However, every (non-constant) homolorphic function $S^2 \to S^2$ is *almost* a covering map, in the sense that it is locally modeled on the identity $\mathbb{C} \to \mathbb{C}$, except for a finite number of *branching points*, where it looks like the map

$$\begin{aligned} F_k\colon \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^k \end{aligned}$$

for some $k \geq 2$. In fact, it turns out that every (non-constant) holomorphic function between two complex 1-manifolds (which, topologically, are just oriented surfaces), has this remarkable property. This motivates the following definition.

Definition 1.1. Let $\Sigma, \widetilde{\Sigma}$ be two surfaces. A continuous function $f\colon \widetilde{\Sigma} \to \Sigma$ is a *branched covering map* (or simply a *branched covering*) if the following property holds: for every $x \in \Sigma$, $\widetilde{x} \in f^{-1}(x)$ there exist a positive integer k, open neighborhoods $U, \widetilde{U}$ of $x, \widetilde{x}$ respectively, and homeomorphisms $\varphi\colon U \to \mathbb{C}$, $\widetilde{\varphi}\colon \widetilde{U} \to \mathbb{C}$ such that $\varphi(x) = 0$, $\widetilde{\varphi}(\widetilde{x}) = 0$, $f(\widetilde{U}) = U$ and the diagram

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "f"] \arrow[d, "\widetilde{\varphi}"] & U \arrow[d, "\varphi"] \\
\mathbb{C} \arrow[r, "F_k"] & \mathbb{C}
\end{tikzcd}$$

commutes (F_k is the map defined above). We say that $\widetilde{U}$ is a *trivializing neighborhood* of $\widetilde{x}$.

More informally, we can say that a branched covering is a continuous function between surfaces which is locally modeled on the complex map $z \mapsto z^k$, where $k \geq 1$ depends on the point. Note that, for each point $\widetilde{x} \in \widetilde{\Sigma}$, the integer k is well-defined, independently of the charts φ and $\widetilde{\varphi}$: using the notation from the definition, we have that k is equal to the cardinality of $f^{-1}(y) \cap \widetilde{U}$, where y is any point in $U \smallsetminus \{x\}$. We call this integer the *local degree* of $\widetilde{x}$; to emphasize the dependence on $\widetilde{x}$, we will sometimes denote it by $k(\widetilde{x})$.

A point $\widetilde{x} \in \widetilde{\Sigma}$ is called a *branching point* if $k(\widetilde{x}) > 1$; in other words, if f is *not* a local homeomorphism in a neighborhood of $\widetilde{x}$. We also say that a point $x \in \Sigma$ is a *branching point* if $f^{-1}(x)$ contains at least one branching point; usually, no ambiguity will arise as to which kind of branching point we are referring to.

It is not hard to see that branching points are quite rare. If $\widetilde{x} \in \widetilde{\Sigma}$ is a branching point, then (using again the notation from the definition) every other point in $\widetilde{U}$ is not a branching point; by compactness, it follows that the set of branching points in $\widetilde{\Sigma}$ is finite. As a consequence, the set of branching points in Σ is finite as well.

Given a branched covering $f \colon \widetilde{\Sigma} \to \Sigma$, we denote by $\Sigma^\bullet$ the subspace of Σ containing all the non-branching points; since the set of branching points is finite, $\Sigma^\bullet$ is a non-compact connected surface with finitely many punctures. We also set $\widetilde{\Sigma}^\bullet = f^{-1}(\Sigma^\bullet)$, and we denote by $f^\bullet$ the restriction of f to $\widetilde{\Sigma}^\bullet$. By construction, $f^\bullet \colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ is a covering map and, as such, has a well-defined degree d (the number of preimages of an arbitrary point); we call this integer the *degree* of the branched covering f. The following proposition shows that the notion of degree extends nicely to branching points.

Proposition 1.1. *Let $f \colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d. Then for every point $x \in \Sigma$ we have that the set $f^{-1}(x)$ is finite and*

$$\sum_{\widetilde{x} \in f^{-1}(x)} k(\widetilde{x}) = d.$$

Proof. If x is not a branching point, the conclusion follows immediately, since x has exactly d preimages, all of which have local degree k equal to 1. Assume now that x is a branching point; it is clear from the definition that the set $f^{-1}(x) \subseteq \widetilde{\Sigma}$ is discrete and, hence, finite. Let $f^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_r\}$. Fix disjoint trivializing neighborhoods $\widetilde{U}_1, \ldots, \widetilde{U}_r$ of $\widetilde{x}_1, \ldots, \widetilde{x}_r$ respectively; a routine compactness argument shows that there exists an open neighborhood U of x such that $f^{-1}(U) \subseteq \widetilde{U}_1 \cup \ldots \cup \widetilde{U}_r$. Fix a point $y \in U \smallsetminus \{x\}$: it follows from the discussion above that y is not a branching point and that $\left|f^{-1}(y) \cup \widetilde{U}_i\right| = k(\widetilde{x}_i)$ for every $1 \leq i \leq r$. Since $\left|f^{-1}(y)\right| = d$, we immediately conclude that

$$\sum_{i=1}^{r} k(\widetilde{x}_i) = \sum_{i=1}^{r} \left|f^{-1}(y) \cap \widetilde{U}_i\right| = \left|f^{-1}(y)\right| = d. \qquad \square$$

1.2 Branching data

Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering. For each point $x \in \Sigma$, as we have just seen in proposition 1.1, the sum $k(\widetilde{x}_1)+\ldots k(\widetilde{x}_r)$ of the local degrees of its preimages is equal to the degree d of the branched covering. Since there is no natural ordering on the set $f^{-1}(x)$, the appropriate combinatorial object for representing the collection $k(\widetilde{x}_1), \ldots, k(\widetilde{x}_r)$ is a *partition*.

Definition 1.2. Let d be a positive integer. A *partition* of d is an unordered finite multiset $\pi = [k_1, \ldots, k_r]$, where $k_i > 0$ is an integer for every $1 \leq i \leq r$ and $k_1 + \ldots + k_r = d$.

Given a positive integer d, we denote the set of all partitions of d by $\Pi\,(d)$. If $\pi = [k_1, \ldots, k_r]$ is a partition of d, we call the integer r the *length* (or *size*, or *cardinality*) of π, and we denote it by $\ell(\pi)$. We also say that the *sum* of π, denoted by $\sum \pi$, is $k_1 + \ldots + k_r$ or, in other words, d. Finally, we introduce a new quantity, the *branching number* $v(\pi) = d - \ell(\pi)$, whose purpose will soon become apparent.

For every point $x \in \Sigma$, if $f^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_r\}$, we can define the *associated partition* $\pi(x) = [k(\widetilde{x}_1), \ldots, k(\widetilde{x}_r)] \in \Pi\,(d)$. For all non-branching points, the associated partition will simply be $[1, \ldots, 1]$. On the contrary, if x is a branching point, then $\pi(x) \neq [1, \ldots, 1]$ (or, equivalently, $v(\pi(x)) > 0$).

Definition 1.3. Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d. Let $x_1, \ldots, x_n \in \Sigma$ be the branching points of f. The *branching datum* of f is the tuple

$$\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi(x_1), \ldots, \pi(x_n)),$$

well-defined up to a permutation of the branching points $x_1, \ldots, x_n$.

Branching data are a way to extract some combinatorial information from branched coverings. Even though the exact location of the branching points (both in Σ and in $\widetilde{\Sigma}$) is not encoded in the branching datum, this piece of information is completely irrelevant, since surfaces are *homogeneous*[1]. However, the reader should not be induced to believe that the combinatorial information provided by the branching datum is enough to fully reconstruct the topology of f. In fact, it turns out that this is not the case. More on this later (?).

This entire thesis is devoted to the problem of determining what values can actually be attained by $\mathcal{D}(f)$ as f ranges over all the possible branched covering maps. We start with a very general definition.

Definition 1.4. A *combinatorial datum* is a tuple

$$\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n),$$

where $\Sigma, \widetilde{\Sigma}$ are surfaces, d and n are positive integers, and $\pi_1, \ldots, \pi_n$ are partitions of d different from $[1, \ldots, 1]$.

Again, a combinatorial datum is defined up to a permutation of the partitions $\pi_1, \ldots, \pi_n$. In other words, we will consider two combinatorial data equal if they have the same partitions, irrespective of the ordering.

Technically, we could also allow combinatorial data to have $n = 0$ partitions. However, data of this kind would correspond to standard covering maps, without any branching points. Since covering maps between surfaces are completely understood from a topological point of view, we

[1] By *homogeneous*, we mean that, for each $n \geq 1$, the group of homeomorphisms of a surface Σ acts transitively on the set of n-uples of pairwise distinct points $(y_1, \ldots, y_n)$.

will always assume that $n \geq 1$. For the same reason, when we say "branched covering", we will be implicitly excluding the trivial cases of standard covering maps.

We say that a combinatorial datum $\mathcal{D}$ is *realizable* if $\mathcal{D} = \mathcal{D}(f)$ for some branched covering $f\colon \widetilde{\Sigma} \to \Sigma$, *exceptional* otherwise. A first, naive guess could be that every combinatorial datum can be realized. However, it turns out that there are a few easy and general necessary conditions for a combinatorial datum to be associated to a branched covering. The first one, and arguably the most important, is known as the *Riemann-Hurwitz formula*; we state it in the next proposition.

Proposition 1.2. *Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering, and let $\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be its branching datum. Then we have the equality*

$$d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = v(\pi_1) + \ldots + v(\pi_n).$$

Proof. Using the same notation as above, consider the covering map $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$. Since $f^\bullet$ has degree d, the Euler characteristics of $\Sigma^\bullet$ and $\widetilde{\Sigma}^\bullet$ are related by the formula $d\chi(\Sigma^\bullet) = \chi(\widetilde{\Sigma}^\bullet)$. Note that there are n branching points in Σ, and the number of points in $\widetilde{\Sigma} \smallsetminus \widetilde{\Sigma}^\bullet$ is $\ell(\pi_1) + \ldots + \ell(\pi_n)$; therefore

$$\chi(\Sigma^\bullet) = \chi(\Sigma) - n, \qquad \chi(\widetilde{\Sigma}^\bullet) = \chi(\widetilde{\Sigma}) - \ell(\pi_1) - \ldots - \ell(\pi_n).$$

As a consequence, we have that

$$d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = d\chi(\Sigma^\bullet) + dn - \chi(\widetilde{\Sigma}^\bullet) - \ell(\pi_1) - \ldots - \ell(\pi_n) = v(\pi_1) + \ldots + v(\pi_n). \qquad \square$$

There are three more conditions that every branching datum must satisfy: two of them concern the orientability of Σ and/or $\widetilde{\Sigma}$, and the other one provides an additional constraint for the *total branching number* $v(\pi_1) + \ldots + v(\pi_n)$; we group these four requirements in the following definition.

Definition 1.5. A combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ is a *candidate datum* if it satisfies the following conditions:

(i) $d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = v(\pi_1) + \ldots + v(\pi_n)$;

(ii) $v(\pi_1) + \ldots + v(\pi_n)$ is even;

(iii) if Σ is orientable, then $\widetilde{\Sigma}$ is orientable as well;

(iv) if Σ is non-orientable and $\widetilde{\Sigma}$ is orientable, then d is even, and each π_i can be written as $\pi_i' \cup \pi_i''$, where π_i' and π_i'' are partitions of $d/2$.

Remark 1.1. In proposition 1.10, we will show that conditions (i)–(iv) are necessary for a combinatorial datum to be realizable (actually, the first condition was already proved in proposition 1.2). For now, let us focus on condition (iii), which is perhaps the most obvious one. If Σ is orientable, then so is $\Sigma^\bullet$, since removing a finite number of points does not affect orientability. The non-compact surface $\widetilde{\Sigma}^\bullet$, being a covering space of an orientable manifold, is itself orientable. Finally, this implies that $\widetilde{\Sigma}$ is orientable as well.

It would be natural to ask whether the necessary conditions we have enumerated are also sufficient for a candidate datum to be realizable. We will see in the following chapters that the answer is negative, and that, in general, determining the full list of exceptional data is remarkably difficult; we call this the *Hurwitz existence problem*[2], and we state it in the following deliberately vague fashion.

[2]The question originally posed by Hurwitz was actually even harder: the task was counting *how many* branched coverings (up to a suitable notion of isomorphism) realize a given candidate datum. We will, however, mostly ignore this point of view, considering that the existence problem is hard enough as it is. The interested reader may…

Hurwitz existence problem. Determine necessary and sufficient conditions for a candidate datum $(\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ to be realizable.

In the remainder of this thesis, we will try to address some instances of this problem. We will present a variety of techniques, which will provide us with a full solution for a few classes of candidate data.

1.3 Symmetric group and partitions

The first approach we will describe is based on the monodromy action. Before describing this action and how it relates to branched coverings, we will recall a few elementary facts about permutations and partitions.

Given a set X, we denote by $\mathfrak{S}(X)$ the set of bijective functions $X \to X$. The set $\mathfrak{S}(X)$ is naturally endowed with a group structure, where the product is given by the composition operator $(\circ)$; we call this group the *symmetric group* of X. Elements of $\mathfrak{S}(X)$ are also called *permutations*.

We will sometimes need to work in a setting where the product in $\mathfrak{S}(X)$ is reversed. Given a group G with product $(*)$, we define the *opposite group* G^{op}, which has the same underlying set as G, but the product $(*^{\mathrm{op}})$ is defined as $g_1 *^{\mathrm{op}} g_2 = g_2 * g_1$.

If d is a positive integer, we will employ the notation $\mathfrak{S}_d$ as a shorthand for $\mathfrak{S}(\{1, \ldots, d\})$. Whenever X is a finite (non-empty) set, we have an isomorphism $\mathfrak{S}(X) \simeq \mathfrak{S}_d$, where d is the cardinality of X. We will therefore restrict out attentions to the groups $\mathfrak{S}_d$, keeping in mind that everything we say also holds for symmetric groups of arbitrary finite sets.

Recall that every permutation $\alpha \in \mathfrak{S}_d$ has a *cycle decomposition* (that is, it can be written as a product of disjoint, possibly trivial cycles):

$$\alpha = (x_{1,1}, \ldots, x_{1,k_1})(x_{2,1}, \ldots, x_{2,k_2}) \cdots (x_{r,1}, \ldots, x_{r,k_r}).$$

The natural action of (the subgroup generated by) α on the set $A = \{1, \ldots, d\}$ induces a decomposition $A = A_1 \sqcup \ldots \sqcup A_r$ into orbits, where $A_i = \{x_{i,1}, \ldots, x_{i,k_i}\}$. This decomposition, in turn, gives rise to a partition $\pi = [|A_1|, \ldots, |A_r|] \in \Pi(d)$. We say that α *matches* the partition π, and we use the notation $[\alpha]$ to refer to the (unique) partition matched by α. We also define the *branching number* $v(\alpha)$ as the branching number of $[\alpha]$; note that $v(\alpha) = 0$ if and only if $\alpha = \mathrm{id} \in \mathfrak{S}_d$ is the trivial permutation.

It is well known that two permutations are conjugate if and only if they match the same partition. In other words, there is a natural bijection between conjugacy classes of $\mathfrak{S}_d$ and the set of partitions $\Pi(d)$.

We conclude this section with a very simple result relating the branching number of two permutations to the branching number of their product. Recall that every permutation has a well-defined *sign*: *even* permutations can only be written as a product of an even number of transpositions (2-cycles), while *odd* permutations can only be written as a product of an odd number of transpositions. The *alternating group* of order d is the index-2 subgroup $\mathfrak{A}_d \leq \mathfrak{S}_d$ containing all the even permutations.

Proposition 1.3. *Let $\alpha, \beta \in \mathfrak{S}_d$ be permutations. Then $v(\alpha\beta) \equiv v(\alpha) + v(\beta) \pmod 2$.*

Proof. We show that $v(\alpha)$ is even if and only if α is even; the conclusion will then follow trivially. Fix a cycle decomposition

$$\alpha = (x_{1,1}, \ldots, x_{1,k_1})(x_{2,1}, \ldots, x_{2,k_2}) \cdots (x_{r,1}, \ldots, x_{r,k_r}).$$

Since a k-cycle can be written as a product of $k-1$ transpositions, we have that α is the product of $d - r = v(\alpha)$ transpositions. This shows that $v(\alpha)$ is even if and only if $\alpha \in \mathfrak{A}_d$, which concludes the proof. $\square$

1.4 Branched covering action of the fundamental group

We now introduce the monodromy action in the general setting of topological spaces; we will assume that all the spaces we mention are locally path-connected and locally simply-connected.

Let $f\colon \widetilde{X} \to X$ be a covering map between topological spaces, with X path-connected. Fix a base-point $x_0 \in X$, and let $\widetilde{x}_0 \in \widetilde{X}$ be a point in $f^{-1}(x_0)$. Given a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$, denote by $\operatorname{lift}(\gamma, \widetilde{x}_0)\colon [0,1] \to \widetilde{X}$ the unique lift of γ such that $\operatorname{lift}(\gamma, \widetilde{x}_0)(0) = \widetilde{x}_0$. There is a very natural right action of the fundamental group $\pi_1(X, x_0)$ on the fiber $f^{-1}(x_0)$, defined by[3]

$$\widetilde{x} \bullet \gamma = \operatorname{lift}(\gamma, \widetilde{x})(1) \qquad \text{for } \widetilde{x} \in f^{-1}(x_0),\ \gamma \in \pi_1(X, x_0).$$

We call this the *monodromy action* of the covering map. It is easy to see that $\widetilde{X}$ is path-connected if and only if the monodromy action on $f^{-1}(x_0)$ is transitive.

There is a relation between monodromy and the fundamental group of $\widetilde{X}$. Fix a base-point $\widetilde{x}_0 \in f^{-1}(x_0) \subseteq \widetilde{X}$. By elementary properties of covering spaces, we have that

$$\begin{aligned} f_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) &= \{\gamma \in \pi_1(X, x_0) : \operatorname{lift}(\gamma, \widetilde{x}_0) = \widetilde{x}_0\} \\ &= \{\gamma \in \pi_1(X, x_0) : \widetilde{x}_0 \bullet \gamma = \widetilde{x}_0\} \\ &= \operatorname{Stab}_{\pi_1(X, x_0)}(\widetilde{x}_0). \end{aligned}$$

The monodromy action induces a group homomorphism $\mathfrak{m}\colon \pi_1(X, x_0) \to \mathfrak{S}(f^{-1}(x_0))^{\mathrm{op}}$. If the covering map has a finite degree d, we will sometimes implicitly fix a bijection between $f^{-1}(x_0)$ and $\{1, \dots, d\}$, and consider the homomorphism $\mathfrak{m}\colon \pi_1(X, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ instead; of course, this map is well-defined up to a conjugation in $\mathfrak{S}_d^{\mathrm{op}}$. The following existence theorem shows that every group homomorphism of this kind is induced by the monodromy action of some covering space.

Theorem 1.4. *Let X be a path-connected topological space, $x_0 \in X$ a point. Let d be a positive integer, and let $\psi\colon \pi_1(X, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ be a group homomorphism. Then there exists a covering map $f\colon \widetilde{X} \to X$ with $\mathfrak{m} = \psi$ (up to conjugation).*

After introducing monodromy for general topological spaces, we turn our attention to the case of branched coverings of surfaces. Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering. As we have seen, by removing the branching points from Σ, we get an associated covering map $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$. Fix a base-point $x_0 \in \Sigma^\bullet$, and let $x \in \Sigma$ be any point. There is a path $\gamma \in \pi_1(\Sigma^\bullet, x_0)$ which "goes around x once"; to be more precise, we can construct it as follows. Pick a small neighborhood $U \subseteq \Sigma$ of x which is homeomorphic to $\mathbb{R}^2$ and does not contain any other branching points; here we can define a loop $\beta\colon [0,1] \to U \smallsetminus \{x\}$ that goes around x once (we do not care about the orientation). Finally, pick a path $\alpha\colon [0,1] \to \Sigma^\bullet$ connecting x_0 to $\beta(0) = \beta(1)$, and define $\gamma = \alpha * \beta * \iota(\alpha)$, where $(*)$ is the concatenation of paths and $\iota(\alpha)$ denotes the inverse path $\iota(\alpha)(t) = \alpha(1-t)$. Of course, the homotopy class of $\gamma \in \pi_1(\Sigma^\bullet, x_0)$ depends on α and even β (we could have chosen $\iota(\beta)$ instead); we will say that any path constructed with the procedure we have described is a *loop around x*.

We are finally ready to prove the connection between branched coverings and the monodromy of the associated covering map.

Proposition 1.5. *Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d, and let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ be the associated covering map. Fix a base-point $x_0 \in \Sigma^\bullet$ and a point $y \in \Sigma$. Let γ be a loop around y.*

[3] Since the lifting operation is, in some sense, invariant up to homotopy, we will often use the same symbol (γ, in this case) to interchangeably represent a class in the fundamental group $\pi_1(X, x_0)$ and a representative of that class.

Consider the monodromy homomorphism $\mathfrak{m}\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}(f^{-1}(x_0))^{\mathrm{op}}$. *Then the permutation* $\mathfrak{m}(\gamma)$ *matches* $\pi(y)$.

Proof. Let the preimages $\widetilde{y}_1, \ldots, \widetilde{y}_r$ of y have local degrees $k_1, \ldots, k_r$. Fix disjoint trivializing neighborhoods $\widetilde{U}_1, \ldots, \widetilde{U}_r$ of $\widetilde{y}_1, \ldots, \widetilde{y}_r$ respectively. Let $U \subseteq \Sigma$ be an open neighborhood of x which is homeomorphic to $\mathbb{R}^2$ and such that $f^{-1}(U) \subseteq \widetilde{U}_1 \cup \ldots \cup \widetilde{U}_r$. Since γ is a loop around y, we can write $\gamma = \alpha * \beta * \iota(\alpha)$ as described above. Up to homotopy, we can assume that $\beta(t) \in U$ for every $t \in [0,1]$. Let $z = \alpha(1) = \beta(0) = \beta(1) \in U$; note that every preimage of z lies exactly in one $\widetilde{U}_l$. Finally, let $\widetilde{x}_1, \ldots, \widetilde{x}_d$ be the preimages of x_0. We will show that $\widetilde{x}_i$ and $\widetilde{x}_j$ belong to the same orbit of $\mathfrak{m}(\gamma)$ if and only if $\mathrm{lift}(\alpha, \widetilde{x}_i)(1)$ and $\mathrm{lift}(\alpha, \widetilde{x}_j)(1)$ lie in the same $\widetilde{U}_l$. Since $f^{-1}(z) \cap \widetilde{U}_l$ has cardinality k_l and $\pi(y) = [k_1, \ldots, k_r]$, this completes the proof.

Let $\widetilde{x}_i \in f^{-1}(x_0)$, $\widetilde{z}_1 = \mathrm{lift}(\alpha, \widetilde{x}_i)(1)$. Let l be the unique index such that $\widetilde{z}_1 \in \widetilde{U}_l$. For each $m \geq 1$, inductively define $\widetilde{z}_{m+1} = \mathrm{lift}(\beta, \widetilde{z}_m)(1)$. Since the support of β lies entirely in U, we have that the support of $\mathrm{lift}(\beta, \widetilde{z}_m)$ lies entirely in $\widetilde{U}_l$ and, therefore, $\widetilde{z}_{m+1} \in \widetilde{U}_l$ as well. Since $\widetilde{U}_l$ is a trivializing neighborhood of $\widetilde{y}_l$, which has local degree k_l, it is easy to see that the sequence $\widetilde{z}_1, \widetilde{z}_2, \ldots$ is periodic of period k_l, and that $f^{-1}(z) \cap \widetilde{U}_l = \{\widetilde{z}_1, \ldots \widetilde{z}_{k_l}\}$. It is also clear that

$$\mathfrak{m}(\gamma)(\widetilde{x}_i) = \mathrm{lift}(\iota(\alpha), \widetilde{z}_2)(1)$$

and, by induction, that

$$\mathfrak{m}(\gamma)^s(\widetilde{x}_i) = \mathrm{lift}(\iota(\alpha), \widetilde{z}_{s+1})(1) \qquad \text{for every } s \geq 1.$$

This shows that, if $\widetilde{x}_j = \mathfrak{m}(\gamma)^s(\widetilde{x}_i)$, then $\mathrm{lift}(\alpha, \widetilde{x}_j)(1) = \widetilde{z}_{s+1} \in \widetilde{U}_l$.

Conversely, assume that $\mathrm{lift}(\alpha, \widetilde{x}_j)(1) \in \widetilde{U}_l$. This implies that $\mathrm{lift}(\alpha, \widetilde{x}_j)(1) = \widetilde{z}_s$ for some $s \geq 1$. But then

$$\widetilde{x}_j = \mathrm{lift}(\iota(\alpha), \widetilde{z}_s)(1) = \mathfrak{m}(\gamma)^{s-1}(\widetilde{x}_i),$$

so $\widetilde{x}_i$ and $\widetilde{x}_j$ belong to the same orbit of $\mathfrak{m}(\gamma)$. □

1.5 Monodromy and realizability

From Proposition 1.5 we can derive a group-theoretic criterion for the realizability of a given combinatorial datum.

Proposition 1.6. *Let* Σ *be a surface,* $d \geq 1$ *an integer,* $\pi_1, \ldots, \pi_n \in \Pi(d)$ *partitions of* d. *Let* $x_1, \ldots x_n \in \Sigma$ *be distinct points; define* $\Sigma^\bullet = \Sigma \setminus \{x_1, \ldots, x_n\}$. *Fix a base-point* $x_0 \in \Sigma^\bullet$ *and loops* $\gamma_1, \ldots, \gamma_n \in \pi_1(\Sigma^\bullet, x_0)$ *around* $x_1, \ldots, x_n$ *respectively. Then the following are equivalent.*

(i) There exist a surface $\widetilde{\Sigma}$ *and a branched covering* $f\colon \widetilde{\Sigma} \to \Sigma$ *with branching datum* $\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$.

(ii) There exists a group homomorphism $\psi\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ *such that* $\mathrm{im}(\psi) \leq \mathfrak{S}_d^{\mathrm{op}}$ *acts transitively on* $\{1, \ldots, d\}$ *and* $[\psi(\gamma_i)] = \pi_i$ *for every* $1 \leq i \leq n$.

Proof. Assume we are given a branched covering $f\colon \widetilde{\Sigma} \to \Sigma$ with $\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$. Since surfaces are homogeneous, we can assume that the branching points are exactly $x_1, \ldots, x_n$, with associated partitions $\pi_1, \ldots, \pi_n$ respectively. Let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ be the induced covering map, and let $\mathfrak{m}\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$ be the monodromy homomorphism. By proposition 1.5 we have $[\mathfrak{m}(\gamma_i)] = \pi_i$ for every $1 \leq i \leq n$, and the action of $\mathfrak{m}(\pi_1(\Sigma^\bullet, x_0))$ on $\{1, \ldots, d\}$ is transitive since $\widetilde{\Sigma}^\bullet$ is connected. Therefore setting $\psi = \mathfrak{m}$ gives the desired homomorphism.

Assume conversely that we are given a group homomorphism $\psi\colon \pi_1(\Sigma^\bullet, x_0) \to \mathfrak{S}_d^{\mathrm{op}}$. Theorem 1.4 gives a covering map $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ of degree d with monodromy $\mathfrak{m} = \psi$. The non-compact surface $\widetilde{\Sigma}^\bullet$ is connected by the transitivity hypothesis. We can "fill the holes" in $\Sigma^\bullet$ and $\widetilde{\Sigma}^\bullet$ to get a branched covering between compact surfaces; here are the details. Consider one of the points x_i, and fix an open neighborhood $U \subseteq \Sigma$ of x_i homeomorphic to $\mathbb{R}^2$. Now $(f^\bullet)^{-1}(U \smallsetminus \{x_i\})$ is a covering space of $U \smallsetminus \{x_i\} \simeq \mathbb{R}^2 \smallsetminus \{0\}$. By the classification of covering spaces, we get that $(f^\bullet)^{-1}(U \smallsetminus \{x_i\})$ is a disjoint union of punctured disks $\widetilde{V}_1 \sqcup \ldots \sqcup \widetilde{V}_r$; moreover, for each $1 \le j \le r$ we have charts $\varphi_j\colon U \smallsetminus \{x_i\} \to \mathbb{C} \smallsetminus \{0\}$, $\widetilde{\varphi}_j\colon \widetilde{V}_j \to \mathbb{C} \smallsetminus \{0\}$ and a positive integer k_j such that the diagram

$$\begin{tikzcd} \widetilde{V}_j \arrow[r, "f^\bullet"] \arrow[d, "\widetilde{\varphi}_j"] & U \smallsetminus \{x_i\} \arrow[d, "\varphi_j"] \\ \mathbb{C} \smallsetminus \{0\} \arrow[r, "F_{k_j}"] & \mathbb{C} \smallsetminus \{0\} \end{tikzcd}$$

commutes, where $F_{k_j}(z) = z^{k_j}$. We can "fill the hole" in $\widetilde{V}_j$ by considering the surface $\widetilde{\Sigma}_1^\bullet = \widetilde{\Sigma}^\bullet \sqcup \mathbb{C}/\sim$, where $y \sim \widetilde{\varphi}_j(y)$ for each $y \in \widetilde{V}_j$. It is clear that $f^\bullet$ extends to a continuous function $\widetilde{\Sigma}_1^\bullet \to \Sigma$ by sending $0 \in \mathbb{C}$ to x_i; this map is locally modeled on $z \mapsto z^{k_j}$ near 0. If we "fill the hole" in $\widetilde{V}_j$ for $1 \le j \le r$, and then repeat the process for each x_i, we end up with a surface $\widetilde{\Sigma}$ and a map $f\colon \widetilde{\Sigma} \to \Sigma$. It is easy to see from the construction that $\widetilde{\Sigma}$ is compact and connected, and that f is a branched covering. Since the associated covering map is exactly $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$, proposition 1.5 implies that $\pi(x_i)$ matches $\mathfrak{m}(\gamma_i) = \psi(\gamma_i)$, so $\pi(x_i) = \pi_i$. $\square$

By combining proposition 1.6 with the classification of surfaces, we obtain the following criteria for realizability.

Corollary 1.7. *Let Σ_g be the connected sum of $g \ge 0$ tori, $d \ge 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_g \in \mathfrak{S}_d$ such that:*

(i) $[\alpha_i] = \pi_i$ *for each* $1 \le i \le n$*;*

(ii) $[\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g] \cdot \alpha_1 \cdots \alpha_n = 1$*;*

(iii) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_g \rangle \le \mathfrak{S}_d$ *acts transitively on* $\{1, \ldots, d\}$.

In this case, $\widetilde{\Sigma}$ is necessarily orientable.

Proof. Let $\Sigma_g^\bullet$ be the non-compact surface obtained by removing n points $x_1, \ldots, x_n$ from Σ_g. Fix a base-point $x_0 \in \Sigma_g^\bullet$. Once we observe that $\pi_1(\Sigma_g^\bullet, x_0)$ has a presentation

$$\pi_1(\Sigma_g^\bullet, x_0) = \langle a_1, \ldots, a_n, b_1, \ldots, b_g, c_1, \ldots, c_g \mid [b_1, c_1] \cdots [b_g, c_g] \cdot a_1 \cdots a_n \rangle,$$

where a_i is a loop around x_i, the criterion follows immediately from proposition 1.6. The orientability of $\widetilde{\Sigma}$ was already addressed in remark 1.1. $\square$

Remark 1.2. Given Σ_g, d and $\pi_1, \ldots, \pi_n$, there is at most one surface $\widetilde{\Sigma}$ such that $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum. In fact, the Riemann-Hurwitz formula gives

$$\chi(\widetilde{\Sigma}) = d\chi(\Sigma_g) - v(\pi_1) - \ldots - v(\pi_n)$$

which, in turn, uniquely determines the orientable surface $\widetilde{\Sigma}$. As an application, assume that we have a candidate datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$, and we find permutations in $\mathfrak{S}_d$ satisfying conditions (i)–(iii). This implies that $\mathcal{D}$ is realizable.

Before stating the monodromy criterion for non-orientable surfaces, we recall a few basic facts about covering maps and orientability.

Given a non-orientable topological manifold M, let $\omega\colon \widehat{M} \to M$ be its orientable double covering. Fix a base-point $x_0 \in M$, and consider the monodromy homomorphism

$$w\colon \pi_1(M, x_0) \longrightarrow \mathfrak{S}(\omega^{-1}(x_0)) \simeq \mathbb{Z}/2.$$

We denote the kernel of w (which is a subgroup of $\pi_1(M, x_0)$ of index 2) by $W(M, x_0)$. Note that $W = \omega_*(\pi_1(\widehat{M}, \widehat{x}_0))$ for every base-point $\widehat{x}_0 \in \omega^{-1}(x_0)$. Given a connected topological manifold N and a covering map $f\colon N \to M$, elementary properties of covering spaces imply that the following are equivalent:

(i) N is orientable;

(ii) $f_*(\pi_1(N, y_0)) \leq W(M, x_0)$ for every base-point $y_0 \in f^{-1}(x_0)$;

(iii) there exists a covering map $\widehat{f}\colon N \to \widehat{M}$ such that f factors as $f = \omega \circ \widehat{f}$.

Corollary 1.8. *Let N_g be the connected sum of $g \geq 1$ projective planes, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \in \mathfrak{S}_d$ such that:*

(i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;

(ii) $\beta_1^2 \cdots \beta_g^2 \cdot \alpha_1 \cdots \alpha_n = 1$;

(iii) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \ldots, d\}$.

In this case, $\widetilde{\Sigma}$ is non-orientable if and only if there exists a permutation $\gamma \in \mathfrak{S}_d$ which has a fixed point and can be written as a product of an odd number of β_j and any number of α_i.

Proof. The proof of the criterion is identical to that of corollary 1.7: simply let $N_g^\bullet = N_g \smallsetminus \{x_1, \ldots, x_n\}$ and observe that there is a presentation

$$\pi_1(N_g^\bullet, x_0) = \langle a_1, \ldots, a_n, b_1, \ldots, b_g \mid b_1^2 \cdots b_g^2 \cdot a_1 \cdots a_n \rangle,$$

where a_i is a loop around x_i. As far as the orientability of $\widetilde{\Sigma}$ is concerned, note that $w(a_1) = \ldots = w(a_n) = 0$, while $w(b_1) = \ldots = w(b_g) = 1$. Let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to N_g^\bullet$ be the covering map associated to the branched covering, and let $\mathfrak{m}\colon \pi_1(N_g^\bullet, x_0) \to \mathfrak{S}_d$ be the monodromy homomorphism. Fix a base-point $\widetilde{x}_0 \in f^{-1}(x_0)$; we have that $\widetilde{\Sigma}^\bullet$ is orientable if and only if $f_*(\pi_1(\widetilde{\Sigma}^\bullet, \widetilde{x}_0)) \leq W(N_g^\bullet, x_0)$. Since $f_*(\pi_1(\widetilde{\Sigma}^\bullet, \widetilde{x}_0)) = \mathrm{Stab}_{\pi_1(N_g^\bullet, x_0)}(\widetilde{x}_0)$, it follows that $\widetilde{\Sigma}^\bullet$ is non-orientable if and only if there exists a loop $c \in \pi_1(N_g^\bullet, x_0)$ such that $w(c) = 1$ and $\widetilde{x}_0 \bullet c = \widetilde{x}_0$. By applying $\mathfrak{m}$, we see that this is equivalent to the existence of a permutation $\gamma \in \mathfrak{S}_d$ which has a fixed point and can be written as a product of an odd number of β_j and an arbitrary number of α_i; note that the exact fixed point does not matter, since the subgroup generated by the α_i and β_j acts transitively on $\{1, \ldots, d\}$. Finally, observe that $\widetilde{\Sigma}^\bullet$ is orientable if and only if $\widetilde{\Sigma}$ is. $\square$

Remark 1.3. Given N_g, d and $\pi_1, \ldots, \pi_n$, there are at most two surfaces $\widetilde{\Sigma}$, one orientable and one non-orientable, such that $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum: like in remark 1.2, the Riemann-Hurwitz formula fixes $\chi(\widetilde{\Sigma})$ which, together with orientability, uniquely determines the surface $\widetilde{\Sigma}$.

Moreover, we can show that realizability in the case where Σ is non-orientable and $\widetilde{\Sigma}$ is orientable can be reduced to a situation where both surfaces are orientable.

Proposition 1.9. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \dots \pi_n)$ be a combinatorial datum, with Σ non-orientable and $\widetilde{\Sigma}$ orientable. Then $\mathcal{D}$ is realizable if and only if d is even and there exist partitions $\pi'_1, \dots, \pi'_n, \pi''_1, \dots, \pi''_n \in \Pi(d/2)$ with $\pi_i = \pi'_i \cup \pi''_i$ for each $1 \le i \le n$, such that the combinatorial datum*

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \dots, \pi'_n, \pi''_i, \dots, \pi''_n)$$

is realizable, where $\widehat{\Sigma}$ is the orientable double covering of Σ.

Proof. Assume that $\mathcal{D}$ is realized by a branched covering $f\colon \widetilde{\Sigma} \to \Sigma$. Let $f^\bullet\colon \widetilde{\Sigma}^\bullet \to \Sigma^\bullet$ be the associated covering map, and let $\omega\colon \widehat{\Sigma} \to \Sigma$ be the orientable double covering. Define $\widehat{\Sigma}^\bullet = \omega^{-1}(\Sigma^\bullet)$, and denote by $\omega^\bullet$ the restriction of ω to $\widetilde{\Sigma}^\bullet$; clearly, $\omega^\bullet\colon \widehat{\Sigma}^\bullet \to \Sigma^\bullet$ is the orientable double covering of $\Sigma^\bullet$. Since $\widetilde{\Sigma}^\bullet$ is orientable while $\Sigma^\bullet$ is not, $f^\bullet$ factors as $f^\bullet = \widehat{f}^\bullet \circ \omega^\bullet$ for some covering map $\widehat{f}^\bullet\colon \widetilde{\Sigma}^\bullet \to \widehat{\Sigma}^\bullet$; in particular, d is even and $\widehat{f}^\bullet$ has degree $d/2$. The conclusion follows from a routine topological argument; for the sake of completeness, we will now report the details. Fix a branching point $x \in \Sigma \setminus \Sigma^\bullet$; let $f^{-1}(x) = \{\widetilde{x}_1, \dots, \widetilde{x}_r\}$ and $\omega^{-1}(x) = \{\widehat{x}_1, \widehat{x}_2\}$. There exists a small open neighborhood $U \subseteq \Sigma$ of x homeomorphic to $\mathbb{R}^2$ such that:

- $f^{-1}(U) = \widetilde{U}_1 \sqcup \ldots \sqcup \widetilde{U}_r$, where $\widetilde{U}_i$ is a trivializing neighborhood of $\widetilde{x}_i$;
- $\omega^{-1}(U) = \widehat{U}_1 \sqcup \widehat{U}_2$, where $\widehat{U}_i$ is a trivializing neighborhood of $\widehat{x}_i$.

It is now clear that $\widehat{f}^\bullet\colon \widetilde{\Sigma}^\bullet \to \widehat{\Sigma}^\bullet$ extends to a branched covering $\widehat{f}\colon \widetilde{\Sigma} \to \widehat{\Sigma}$: for each $1 \le i \le r$, we simply set $\widehat{f}(\widetilde{x}_i) = \widehat{x}_1$ if $\widehat{f}^\bullet(\widetilde{U}_i \setminus \{\widetilde{x}_i\}) = \widehat{U}_1 \setminus \{\widehat{x}_1\}$, and $\widehat{f}(\widetilde{x}_i) = \widehat{x}_2$ otherwise; obviously, we must repeat this process for every branching point x. The branched covering $\widehat{f}$ has degree $d/2$; moreover, it is easy to see from the construction that $\pi_f(x) = \pi_{\widehat{f}}(\widehat{x}_1) \cup \pi_{\widehat{f}}(\widehat{x}_2)$ (we use the subscript in order to clarify which branched covering we are referring to). Therefore the realizable combinatorial datum $\mathcal{D}(\widehat{f})$ has the required form.

Conversely, assume that d is even and we are given a branched covering $\widehat{f}\colon \widetilde{\Sigma} \to \widehat{\Sigma}$ with branching datum

$$\mathcal{D}(\widehat{f}) = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \dots, \pi'_n, \pi''_1, \dots, \pi''_n),$$

where $\pi_i = \pi'_i \cup \pi''_i$ for every $1 \le i \le n$. Let $x'_1, \dots, x'_n, x''_1, \dots, x''_n \in \widehat{\Sigma}$ be the branching points corresponding to the similarly named partitions of $\mathcal{D}(\widehat{f})$. Since surfaces are homogeneous, we can assume that $\omega(x'_i) = \omega(x''_i)$ for each $1 \le i \le n$, where $\omega\colon \widehat{\Sigma} \to \Sigma$ is the covering map. It is now easy to see that

$$\mathcal{D}(\omega \circ \widehat{f}) = (\widetilde{\Sigma}, \Sigma; d; \pi'_1 \cup \pi''_1, \dots, \pi'_n \cup \pi''_n). \qquad \square$$

We conclude this introductory chapter by showing that the four conditions described in definition 1.5 are actually necessary for a combinatorial datum to be realizable.

Proposition 1.10. *Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering. Then its branching datum $\mathcal{D}(f)$ is a candidate datum.*

Proof. Conditions (i) and (iii) of definition 1.5 were already addressed, respectively, in proposition 1.2 and remark 1.1. Condition (iv) follows immediately from proposition 1.9, so we only have to show that the total branching number $v(\pi_1) + \ldots + v(\pi_n)$ is even. Using the notations of corollaries 1.7 and 1.8, by proposition 1.3 we have that

$$v(\alpha_1 \cdots \alpha_n) \equiv \begin{cases} v([\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g]) & \text{if } \Sigma \text{ is orientable,} \\ v(\beta_1^2 \cdots \beta_g^2) & \text{if } \Sigma \text{ is non-orientable} \end{cases} \pmod 2.$$

But commutators and squares are even permutations, therefore

$$v(\pi_1) + \ldots + v(\pi_n) = v(\alpha_1) + \ldots v(\alpha_n) \equiv v(\alpha_1 \cdots \alpha_n) \equiv 0 \pmod 2.$$

$\square$

Chapter 2

Realizablility by monodromy

2.1 Non-positive Euler characteristic

The monodromy approach developed in the previous chapter will allow us to completely solve the Hurwitz existence problem for a large class of candidate data. With the following elementary result about symmetric groups, we will already be able to address the cases where $\chi(\Sigma) \leq 0$.

Lemma 2.1. *Let $\alpha \in \mathfrak{S}_d$ be a permutation. Set $r = d - v(\alpha)$, and let $t \geq 0$ be an integer such that $2t \leq v(\alpha)$. Then α can be written as the product of a $(r+2t)$-cycle and a d-cycle.*

Proof. Without loss of generality, assume that

$$\alpha = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{r-1} + 1, \ldots, d_r),$$

where $d_r = d$. Fix

$$\beta_0 = (1, b_1, b_2, \ldots, b_{2t}), \qquad \beta_1 = (1, d_1 + 1, d_2 + 1, \ldots, d_{r-1} + 1),$$

where $b_1 < b_2 < \ldots < b_{2t}$ are elements of $\{1, \ldots, d\} \smallsetminus \{d_1 + 1, \ldots, d_{r-1} + 1\}$. Set

$$\beta = \beta_0 \beta_1 = (1, d_1 + 1, d_2 + 1, \ldots, d_{r-1} + 1, b_1, b_2, \ldots, b_{2t}).$$

An easy computation shows that

$$\begin{aligned}
\beta\alpha &= \beta_0\beta_1\alpha \\
&= (1, b_1, b_2, \ldots, b_{2t})(1, 2, \ldots, d) \\
&= (1, \ldots, b_1 - 1, b_2, \ldots, b_3 - 1, \ldots, b_4, \ldots, b_{2t-1} - 1, b_{2t}, \ldots, d, b_1, \ldots, b_2 - 1, b_3, \ldots, b_{2t} - 1).
\end{aligned}$$

Writing $\alpha = \beta^{-1}(\beta\alpha)$ gives the desired decomposition. $\square$

Corollary 2.2. *Let $\alpha \in \mathfrak{A}_d$ be an even permutation. Then α can be written as:*

(i) a commutator $[\beta, \gamma]$, where γ is a d-cycle;

(ii) a product of two squares $\delta^2\epsilon^2$, where $\delta\epsilon$ is a d-cycle.

Proof. Since α is an even permutation, its branching number $v(\alpha)$ is even. By lemma 2.1, there exist two d-cycles $\tau, \sigma \in \mathfrak{S}_d$ such that $\alpha = \tau\sigma$.

(i) Since τ and σ^{-1} are conjugated, there exists a permutation $\beta \in \mathfrak{S}_d$ such that $\tau = \beta\sigma^{-1}\beta^{-1}$. Setting $\gamma = \sigma^{-1}$, we immediately get that

$$\alpha = \tau\sigma = \beta\sigma^{-1}\beta^{-1}\sigma = \beta\gamma\beta^{-1}\gamma^{-1} = [\beta, \gamma].$$

(ii) Since τ and σ are conjugated, there exists a permutation $\delta \in \mathfrak{S}_d$ such that $\tau = \delta\sigma\delta^{-1}$. Setting $\epsilon = \delta^{-1}\sigma$, we have that

$$\alpha = \tau\sigma = \delta\sigma\delta^{-1}\sigma = \delta^2(\delta^{-1}\sigma)^2 = \delta^2\epsilon^2. \qquad \square$$

Theorem 2.3. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\chi(\Sigma) \leq 0$, then $\mathcal{D}$ is realizable.*

Proof. Let us first assume that Σ is orientable; this means that $\Sigma = \Sigma_g$ is the connected sum of $g \geq 1$ tori, and that $\widetilde{\Sigma}$ is orientable as well. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Since $\mathcal{D}$ is a candidate datum, we have that

$$v(\alpha) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

By corollary 2.2, we can find permutations $\beta_1, \gamma_1 \in \mathfrak{S}_d$ such that $\alpha = [\gamma_1, \beta_1]$ and β_1 is a d-cycle. Set $\beta_2 = \ldots = \beta_g = \gamma_2 = \ldots = \gamma_g = \mathrm{id} \in \mathfrak{S}_d$. All the conditions of corollary 1.7 are satisfied; since $\widetilde{\Sigma}$ is orientable, this implies that $\mathcal{D}$ is realizable (see remark 1.2).

Assume now that Σ and $\widetilde{\Sigma}$ are both non-orientable; this means that $\Sigma = N_g$ is the connected sum of $g \geq 2$ projective planes. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Similarly to what we did for the previous case, we can find $\beta_1, \beta_2 \in \mathfrak{S}_d$ such that $\alpha = \beta_2^{-2}\beta_1^{-2}$ and $\beta_1\beta_2$ is a d-cycle. Note that, since $\beta_1\beta_2$ is a d-cycle, there exists an integer $m \geq 0$ such that $\beta_1(\beta_1\beta_2)^m$ has a fixed point. By setting $\beta_3 = \ldots = \beta_g = \mathrm{id} \in \mathfrak{S}_d$, corollary 1.8 (together with remark 1.3) implies the realizability of $\mathcal{D}$.

Finally, consider the case where Σ is non-orientable and $\widetilde{\Sigma}$ is orientable. Since $\mathcal{D}$ is a candidate datum, d is even and there exist partitions $\pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n \in \Pi\,(d/2)$ with $\pi_i = \pi'_1 \cup \pi''_i$ for each $1 \leq i \leq n$. Let $\widehat{\Sigma}$ be the double orientable covering of Σ; by the first case we analyzed, the candidate datum

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n)$$

is realizable. By proposition 1.9, $\mathcal{D}$ is realizable as well. $\square$

2.2 Products in symmetric groups

The monodromy approach can prove to be quite fruitful also in the cases where $\chi(\Sigma) > 0$ (namely, when Σ is either the sphere S^2 or the projective plain $\mathbb{RP}^2$). In order to deal with this instances of the Hurwitz existence problem, however, we will need a few more technical results about products of permutations.

Lemma 2.4. *Let X, Y be finite sets; denote by h the cardinality of Y, and by k the cardinality of $X \cap Y$. Let $\alpha \in \mathfrak{S}(X)$, $\beta \in \mathfrak{S}(Y)$, $\gamma \in \mathfrak{S}(X \cap Y)$. Assume that $\beta = (b_1, \ldots, b_h)$ is a h-cycle, and that γ is a k-cycle of the form $\gamma = (b_{i_1}, \ldots, b_{i_k})$ with $1 \leq i_1 \leq \ldots \leq i_k \leq h$. Then $\alpha \in \mathfrak{S}(X)$ and $\alpha\gamma^{-1}\beta \in \mathfrak{S}(X \cup Y)$ have the same number of cycles.*

Proof. Write $\gamma = (u_1, \ldots, u_k)$, where $u_j = b_{i_j}$. Without loss of generality, assume that $i_1 = 1$. Then we can write

$$\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{k-1}, u_k, \ldots, w_k),$$

possibly with $u_j = w_j$ for some values of j. We immediately get that

$$\gamma^{-1}\beta = (u_1, \dots, w_1)(u_2, \dots, w_2) \cdots (u_k, \dots, w_k).$$

If we denote by $A_1, \dots, A_r \subseteq X$ the orbits of α, it is then easy to see that the orbits of $\alpha\gamma^{-1}\beta$ are $A'_1, \dots, A'_r \subseteq X \cup Y$, where

$$A'_j = A_j \cup \bigcup_{u_l \in A_j} \{u_l, \dots, w_l\}. \qquad \square$$

Proposition 2.5. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) < d$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = v(\alpha) + v(\beta)$.*

Proof. First of all, note that the conclusion is trivial whenever $v(\pi) = 0$ or $v(\rho) = 0$. This already solves the cases $d = 1$ and $d = 2$. We now proceed by induction on $d \geq 3$, assuming that $v(\pi) > 0$ and $v(\rho) > 0$. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$; without loss of generality, assume that $b_1 > 1$. Fix

$$\beta = (1, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{s-1} + 1, \dots, d_s),$$

where $b_1 = d_1 \geq 2$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Note that

$$d - 1 \geq v(\pi) + v(\rho) = (a_1 - 1) + \ldots + (a_r - 1) + d - s \geq a_1 - 1 + d - s,$$

hence $a_1 \leq s$. Fix $\alpha_1 = (1, d_1 + 1, \dots, d_{a_1 - 1} + 1)$, and let $A = \{1, d_1 + 1, \dots, d_{a_1 - 1} + 1\}$ be the support of α_1. Define $Q = \{1, \dots, d_{a_1}\} \smallsetminus A$; note that Q_1 is non-empty, since $d_1 + 1 \geq 3$ implies that $2 \in Q_1$. Consider the partitions

$$\pi' = [a_2, a_3, \dots, a_r], \qquad \rho' = [|Q|, b_{a_1+1}, \dots, b_s].$$

We have that

$$\sum \pi' = \sum \rho' = d - a_1, \qquad v(\pi') = v(\pi) - a_1 + 1, \qquad v(\rho') = v(\rho) - 1.$$

Since $d - a_1 < d$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - a_1 < d - a_1$, by induction we find $\alpha', \beta' \in \mathfrak{S}(\{1, \dots, d\} \smallsetminus A)$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $v(\alpha'\beta') = v(\alpha') + v(\beta')$. Up to conjugation, we may assume that

$$\beta' = \beta_1(d_{a_1} + 1, \dots, d_{a_1+1})(d_{a_1+1} + 1, \dots, d_{a_1+2}) \cdots (d_{s-1} + 1, \dots, d_s),$$

where β_1 is the $|Q|$-cycle whose entries are the elements of Q in increasing order. An easy computation shows that

$$\alpha_1\beta = (1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s).$$

Therefore, setting $\alpha = \alpha'\alpha_1$, we have that

$$\begin{aligned}\alpha\beta &= \alpha_1\alpha'\beta \\ &= \alpha'(1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s) \\ &= \alpha'\beta'\beta_1^{-1}(1, \dots, d_{a_1})\end{aligned}$$

By lemma 2.4, this implies that $\alpha\beta$ has the same number of cycles as $\alpha'\beta'$, so that

$$\begin{aligned}v(\alpha\beta) &= a_1 + v(\alpha'\beta') \\ &= a_1 + v(\alpha') + v(\beta') \\ &= a_1 + (v(\pi) - a_1 + 1) + (v(\rho) - 1) \\ &= v(\pi) + v(\rho).\end{aligned}$$

Since $[\alpha] = \pi$ and $[\beta] = \rho$, the conclusion follows. $\square$

Proposition 2.6. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi)+v(\rho) \geq d-1$ and $v(\pi)+v(\rho) \equiv d-1 \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha]=\pi$ and $[\beta]=\rho$ such that $\alpha\beta$ is a d-cycle.*

Proof. If $v(\pi)+v(\rho)=d-1$, the conclusion follows immediately from proposition 2.5. Therefore, we can assume that $v(\pi)+v(\rho) \geq d$.

Write $\pi=[a_1,\ldots,a_r]$. Since $v(\rho) \leq d-1$ and $v(\pi)+v(\rho) \geq d$, there exists a largest integer $0 \leq i \leq r$ such that $(a_1-1)+\ldots(a_i-1)+v(\rho) \leq d-1$. Define

$$z = d - v(\rho) - (a_1-1) - \ldots - (a_i - 1).$$

Consider the partition $\pi'=[a_1,\ldots,a_i,z,1,\ldots,1] \in \Pi(d)$. Since by construction $v(\pi')+v(\rho)=d-1$, thanks to proposition 2.5 we can find permutations $\alpha', \beta \in \mathfrak{S}_d$ with $[\alpha']=\pi'$ and $[\beta]=\rho$ such that $v(\alpha'\beta)=d-1$; in other words, $\alpha'\beta$ is a d-cycle. Consider now the partition $\pi''=[a_{i+1}-z+1, a_{i+2},\ldots,a_r]$, whose branching number is $v(\pi'')=v(\pi)+v(\rho)-d+1$. Let $n=\sum\pi''$; fix an element u_1 of the z-cycle of α', and let $u_2,\ldots,u_n$ be the fixed points of α' corresponding to the last ones of π' (it is easy to see that there are exactly $n-1$ such ones). Since $v(\pi'')$ is even, lemma 2.1 gives permutations $\alpha'', \gamma \in \mathfrak{S}(\{u_1,\ldots,u_n\})$ such that $[\alpha'']=\pi''$ and γ and $\alpha''\gamma$ are n-cycles. Up to conjugation, we can assume that $\gamma=(u_1,\ldots,u_n)$. Moreover, it is not restrictive to assume that $u_1,\ldots,u_n$ appear in this order in the d-cycle $\alpha'\beta$. Therefore, setting $\alpha=\alpha''\alpha'$, we have that

$$\alpha\beta = (\alpha''\gamma)\gamma^{-1}(\alpha'\beta);$$

by lemma 2.4, this implies that $\alpha\beta$ has the same number of cycles as $\alpha''\gamma \in \mathfrak{S}(\{u_1,\ldots,u_n\})$, that is $v(\alpha\beta)=d-(k+1)$. Since $[\alpha]=\pi$ and $[\beta]=\rho$, the proof is complete. □

Remark 2.1. Write $\pi=[a_1,\ldots,a_r]$, $\rho=[b_1,\ldots,b_s]$; assume that $b_1 \geq 2$. By directly examining the proof of proposition 2.6. As a consequence, its statement can be enhanced by adding the following line: *α and β can be chosen in such a way that* 1 *belongs to the a_1-cycle of α and to the b_1-cycle of β, provided that* $b_1 \geq 2$. We will need this improvement for the upcoming proof.

Proposition 2.7. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi)+v(\rho) \geq d$ and $v(\pi)+v(\rho) \equiv d \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha]=\pi$ and $[\beta]=\rho$ such that $\langle\alpha,\beta\rangle$ acts transitively on $\{1,\ldots,d\}$ and*

$$[\alpha\beta] = \begin{cases} [d/2, d/2] & \text{if } \pi=\rho=[2,\ldots,2], \\ [1, d-1] & \text{otherwise.} \end{cases}$$

Proof. Assume first that $\pi=\rho=[2,\ldots,d]$. We can choose

$$\alpha = (2,3)(4,5)\cdots(d,1), \qquad \beta = (1,2)(3,4)\cdots(d-1,d).$$

The action of $\langle\alpha,\beta\rangle$ is obviously transitive, and

$$\alpha\beta = (1,3,\ldots,d-1)(2,4,\ldots,2).$$

Otherwise, since $v(\pi)+v(\rho) \geq d$, at least one of π and ρ has an entry which is greater than 2; without loss of generality, we can assume it is ρ. Write $\pi=[a_1,\ldots,a_r]$, $\rho=[b_1,\ldots,b_s]$ with $a_1 \geq 2$ (since $v(\pi) \geq 1$) and $b_1 \geq 3$. Fix

$$\beta = (1,\ldots,d_1)(d_1+1,\ldots,d_2)\cdots(d_{s-1}+1,\ldots,d_s),$$

where $b_1 = d_1 \geq 3$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Consider the partitions

$$\pi' = [a_1 - 1, \dots, a_r], \qquad \rho' = [b_1 - 1, \dots, b_s].$$

Since $\sum \pi' = \sum \rho' = d - 1$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - 2$, by proposition 2.6 we can find permutations $\alpha', \beta' \in \mathfrak{S}(\{2, \dots, d\})$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $\alpha'\beta'$ is a $(d-1)$-cycle. Up to conjugation, we can assume that

$$\beta' = (2, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{s-1} + 1, \dots, d_s)$$

or, in other words, $\beta = (1, 2)\beta'$; moreover, as explained in remark 2.1, we can choose α' in such a way that its $(a_1 - 1)$-cycle contains 2. By setting $\alpha = \alpha'(1, 2)$, we immediately get that $\alpha\beta = \alpha'\beta'$ is a $(d-1)$-cycle fixing 1. Finally, the action of $\langle \alpha, \beta \rangle$ is transitive since α does not fix 1. □

The following result sums up the statements of propositions 2.6 and 2.7, at the cost of some generality. However, for many of the upcoming applications, it will be more than enough.

Corollary 2.8. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d - 1$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$, $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \dots, d\}$ and*

$$[\alpha\beta] \in \{[d], [1, d-1], [d/2, d/2]\}.$$

Proof. The conclusion follows immediately from proposition 2.6 or proposition 2.7 depending on the parity of $v(\pi) + v(\rho) + d$. □

2.3 Combinatorial moves

Combinatorial move A.1. Let d be a positive integers, and let be $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ permutations. Let

$$\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; [\alpha_1], [\alpha_2], \pi_3, \dots, \pi_n), \qquad \mathcal{D}' = (\widetilde{\Sigma}', \Sigma; d; [\alpha_1\alpha_2], \pi_3, \dots, \pi_n)$$

be candidate data, where Σ, $\widetilde{\Sigma}$ and $\widetilde{\Sigma}'$ are surfaces which are either all orientable or all non-orientable. Then there is a combinatorial move $\mathcal{D} \rightsquigarrow \mathcal{D}'$.

2.4 Projective plane

As a first application of the results established in the previous section, we fully solve the Hurwitz existence problem in the cases where $\Sigma = \mathbb{RP}^2$ and $\widetilde{\Sigma}$ is non-orientable.

Theorem 2.9. *Let $\mathcal{D} = (\widetilde{\Sigma}, \mathbb{RP}^2; d; \pi_1, \dots, \pi_n)$ be a candidate datum. If $\widetilde{\Sigma}$ is non-orientable, then $\mathcal{D}$ is realizable.*

Proof. First of all, since $\mathcal{D}$ is a candidate datum, the Riemann-Hurwitz formula implies that

$$v(\pi_1) + \ldots + v(\pi_n) = d\chi(\mathbb{RP}^2) - \chi(\widetilde{\Sigma}) \geq d - 1$$

(recall that $\widetilde{\Sigma}$ is non-orientable, so $\chi(\widetilde{\Sigma}) \leq 1$). Moreover, the total branching $v(\pi_1) + \ldots + v(\pi_n)$ is even. In order to apply corollary 1.8, we will now inductively define permutations $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$, satisfying the following invariant: for every $0 \leq i \leq n$, either

$$v(\alpha_1 \cdots \alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$$

or

$$[\alpha_1 \cdots \alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\} \text{ and } \langle \alpha_1, \ldots, \alpha_i \rangle \text{ acts transitively.}$$

Assume we have already defined $\alpha_1, \ldots, \alpha_{i-1}$; we want to suitably choose α_i. Let $\alpha = \alpha_1 \cdots \alpha_{i-1}$; there are two cases.

- If $v(\alpha) + v(\pi_i) < d$, by proposition 2.5 we can find $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $v(\alpha\alpha_i) = v(\alpha) + v(\alpha_i)$. The invariant is still satisfied: if $v(\alpha) = v(\pi_1) + \ldots + v(\pi_{i-1})$ then obviously $v(\alpha\alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$. If instead $[\alpha] \in \{[d], [1, d-1], [d/2, d/2]\}$, then either α_i is the identity, or $\alpha\alpha_i$ is a d-cycle; either way, $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$. Note that if the action of $\langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ is transitive, then the action of $\langle \alpha_1, \ldots, \alpha_i \rangle$ is transitive as well.

- If $v(\alpha) + v(\alpha_i) \geq d$, corollary 2.8 gives a permutation $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$ and the action of $\langle \alpha, \alpha_i \rangle$ is transitive. The invariant is obviously satisfied.

By induction, we can find $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that

$$[\alpha_1 \cdots \alpha_n] \in \{[d], [1, d-1], [d/2, d/2]\}$$

and $\langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$ (note that $v(\alpha_1 \cdots \alpha_n) = v(\alpha_1) + \ldots + v(\alpha_n)$ also implies that $[\alpha_1 \cdots \alpha_n] = [d]$). By proposition 1.3 we have that

$$v(\alpha_1 \cdots \alpha_n) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

We now prove that $\alpha = \alpha_1 \cdots \alpha_n$ is a square.

- If $[\alpha] = [d]$, then d is odd, so α is the square of $\alpha^{(d+1)/2}$.
- If $[\alpha] = [1, d-1]$, then d is even, so α is the square of $\alpha^{d/2}$.
- If $[\alpha] = [d/2, d/2]$, then d is even, and it is easy to see that α is the square of a d-cycle.

In any case, we obtain a permutations $\beta_1 \in \mathfrak{S}_d$ such that $\alpha = \beta_1^{-2}$ By corollary 1.8, this implies that there exists a realizable candidate datum $\mathcal{D}' = (\widetilde{\Sigma}', \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$. To see that $\widetilde{\Sigma}'$ is non-orientable (and, therefore, equal to $\widetilde{\Sigma}$, as shown in remark 1.3), simply note that there exists a permutation $\gamma \in \langle \alpha_1, \ldots, \alpha_n \rangle$ such that $\gamma\beta_1$ has a fixed point, since $\gamma \in \langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$. □

2.5 Reduction technique on the sphere

sume $n \geq 3$.

Proposition 2.10. *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$ be a candidate datum. Then $\mathcal{D}$ is realizable.*

Proof. We proceed by induction on n, starting with the base case $n = 3$. If $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \pi_2, [d])$ is a candidate datum, by the Riemann-Hurwitz formula we have that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d-1) \geq d - 1.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 1$ is even. Proposition 2.6 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] = [d]$. By corollary 1.7, $\mathcal{D}$ is realizable (see remark 1.2).

We now turn to the case $n \geq 4$. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$; there are two cases.

- Assume that $v(\pi_1)+v(\pi_2) \leq d-1$. By proposition 2.5, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [d])$. By induction, $\mathcal{D}'$ is realizable; corollary 1.7 then gives permutations $\alpha_3, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for $3 \leq i \leq n-1$ and $[\alpha_n] = [d]$ such that $(\alpha_1\alpha_2)\alpha_3 \cdots \alpha_n = 1$. It is easy to see that the permutations $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ imply the realizability of $\mathcal{D}$, again by corollary 1.7 (together, as usual, with remark 1.2).

- Otherwise, we have that $v(\pi_1) + v(\pi_2) \geq d$. By corollary 2.8, we can find permutations α_1, α_2 with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d-2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 1) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + 1 + d - 1) - d + 1 \geq 0. \end{aligned}$$

  It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$ is a candidate datum, so it is realizable by induction. Similarly to the previous case, this implies that $\mathcal{D}$ is realizable. □

Explain the reduction technique, with transitivity.

Proposition 2.11. *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [1, d-1])$ be a candidate datum. Then $\mathcal{D}$ is non-realizable if and only if it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3])$.

(2) $\mathcal{D} = (S^2, S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k-1])$ *with* $k \geq 2$.

Proof. We proceed by induction on n, starting from the base case $n = 3$. Let

The non-realizability of the listed candidate data will be addressed at some point.

$$\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \pi_2, [1, d-1]) \neq (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$$

be a candidate datum. The Riemann-Hurwitz formula implies that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d-2) \geq d.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 2$ is even. Proposition 2.7 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] = [1, d-1]$ and the action of $\langle \alpha_1, \alpha_2 \rangle$ is transitive. As usual, corollary 1.7 implies that $\mathcal{D}$ is realizable.

We now turn to the case $n \geq 4$; we will employ a reduction technique. Fix a candidate datum

$$\mathcal{D} = (\Sigma_{n-3}, S^2; d; \pi_1, \ldots, \pi_{n-1}, [1, d-1]) \neq (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3]).$$

There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n-1$; up to reindexing, we can assume that $i = 1$ and $j = 2$. By proposition 2.5, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [1, d-1])$. By induction, $\mathcal{D}'$ is realizable unless one of the following happens.

  - $\mathcal{D}' = (S^2, S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k-1])$ with $2k = d$. If this is the case, then $n = 4$ and $v(\pi_1) + v(\pi_2) = v(\pi_3) = k$. This implies that

$$k < 1 + k \leq v(\pi_1) + v(\pi_3) < 2k = d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

- $\mathcal{D}' = (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3])$. This implies that $\pi_1 = \pi_2 = [1,1,2]$ and $\pi_3 = \ldots = \pi_{n-1} = [2,2]$. Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

Therefore we can assume that $\mathcal{D}'$ is realizable. A reduction argument shows that $\mathcal{D}$ is realizable as well.

- Otherwise, the inequality $v(\pi_1) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n-1$. Corollary 2.8 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d-2$. If $d = 4$ we can assume that $\pi_1 \neq [2,2]$, so that we can choose $[\alpha_1\alpha_2] \neq [2,2]$ (see proposition 2.7). Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 2) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + 1 + d - 2) - d + 1 \geq -\frac{1}{2}. \end{aligned}$$

Actually, since the total branching number of $\mathcal{D}$ is even, g' must be an integer, therefore $g' \geq 0$. It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [1, d-1])$ is a candidate datum. Moreover, $\mathcal{D}'$ is realizable by induction: the case $d = 4$ was addressed above, and $\mathcal{D}' = (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-2])$ is impossible for $d \geq 6$, since

$$v(\alpha_1\alpha_2) \geq d - 2 > \frac{d}{2} = v([2, \ldots, 2]).$$

The usual reduction argument implies that $\mathcal{D}$ is realizable as well. □

Proposition 2.12. *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. Assume that $d \neq 4$ and that $2g \geq d - 1$. Then $\mathcal{D}$ is realizable.*

Proof. Note that the the condition $2g \geq d - 1$ is equivalent to

$$v(\pi_1) + \ldots + v(\pi_n) \geq 3d - 3$$

by the Riemann-Hurwitz formula. Moreover, the cases where $d = 2$ are trivial; therefore, assume that $d \geq 3$.

We proceed by induction, starting from the base case $n = 3$. If $n = 3$, the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 3d - 3$ implies that $\pi_1 = \pi_2 = \pi_3 = [d]$; by proposition 2.10, $\mathcal{D}$ is realizable.

We now turn to the case $n \geq 4$; we will employ a reduction technique. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$. We can assume that $\pi_i \neq [d]$ for every $1 \leq i \leq n$, otherwise $\mathcal{D}$ is immediately realizable by proposition 2.10. There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n$. The standard reduction argument shows that $\mathcal{D}$ is realizable in this case.

- Otherwise, we have $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. We consider two sub-cases.

  - Assume first that there is a partition, say π_1, which is different from $[2, \ldots, 2]$. By propositions 2.6 and 2.7, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] \in \{[d], [1, d-1]\}$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_n)) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0. \end{aligned}$$

Consider the candidate datum $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$. By propositions 2.10 and 2.11, $\mathcal{D}'$ is realizable unless $\mathcal{D}' = (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$. If this were the case, we would have $n = 4$ and

$$v(\pi_1) + v(\pi_2) + v(\pi_3) + v(\pi_4) \leq d - 2 + d - 2 + \frac{d}{2} + \frac{d}{2} = 3d - 4,$$

which contradicts the hypothesis. Therefore $\mathcal{D}'$ is realizable; by the usual reduction argument, $\mathcal{D}$ is realizable as well.

- Finally, consider the case where $\pi_1 = \ldots = \pi_n = [2, \ldots, 2]$. In this situation $d = 2k$ is even and

$$v(\pi_1) + \ldots + v(\pi_n) = nk > 6k - 3$$

(the inequality is strict since the total branching number is even, while $6k - 3$ is odd). Since $k \geq 3$, this immediately implies that $n \geq 6$. Applying proposition 2.7 once and then proposition 2.6 or proposition 2.7 again yields permutations $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$, $[\alpha_2] = \pi_2$ and $[\alpha_3] = \pi_3$ such that $[\alpha_1\alpha_2\alpha_3] \in \{[d], [1, 2k-1]\}$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2\alpha_3) + v(\pi_4) + \ldots + v(\pi_n)) - 2k + 1 \\ &\geq \frac{1}{2}(2k - 2 + (n-3)k) - 2k + 1 = \frac{n-5}{2} \cdot k > 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; 2k; [\alpha_1\alpha_2\alpha_3], \pi_4, \ldots, \pi_n)$ is a candidate datum. By propositions 2.10 and 2.11, $\mathcal{D}'$ is realizable; a reduction argument shows that $\mathcal{D}$ is realizable as well. □

Corollary 2.13. *Let d be a positive integer with $d \neq 4$. Then there exist at most finitely many non-realizable candidate data of the form $(\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$.*

Proof. By proposition 2.12, every candidate datum with $n \geq 3d - 3$ is realizable, and there are a finite number of candidate data with $n < 3d - 3$. □

Proposition 2.14. *Let $\mathcal{D} = (\Sigma_g, S^2; 4; \pi_1, \ldots, \pi_n)$ be a candidate datum. Then $\mathcal{D}$ is realizable if and only if*

$$\mathcal{D} \neq (\Sigma_{n-3}, S^2; 4; [2, 2], \ldots, [2, 2], [1, 3]).$$

Proof. We proceed by induction, starting from the base case $n = 3$. Note that if any partition is either $[4]$ or $[1, 3]$, we immediately conclude by propositions 2.10 and 2.11. By the Riemann-Hurwitz formula, we have the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 6$; therefore, the only candidate datum left to consider is $(S^2, S^2; 4; [2, 2], [2, 2], [2, 2])$. This datum is realizable, for instance, by choosing

Non-realizability will be addressed at some point.

$$\alpha_1 = (1, 2)(3, 4), \qquad \alpha_2 = (1, 3)(2, 4), \qquad \alpha_3 = (1, 4)(2, 3).$$

We now turn to the case $n \geq 4$; once again, we can assume that all the partitions are either $[1, 1, 2]$ or $[2, 2]$. There are three cases.

- If all the partitions are equal to $[2, 2]$, then we can apply proposition 2.7 to combine two partitions into a single $[2, 2]$, and conclude by a reduction argument.
- If all the partitions are equal to $[1, 1, 2]$, then by the Riemann-Hurwitz formula we have $n \geq 6$. We can apply proposition 2.5 twice to combine three partitions into a single $[4]$, and conclude by reduction.
- Otherwise, there is at least one $[2, 2]$ and one $[1, 1, 2]$; using proposition 2.5, we can combine them into a single $[4]$ and conclude by reduction. □

2.6 Prime-degree conjecture

Proposition 2.15. *Let d be a positive integer. Assume that every candidate datum $(\Sigma_g, S^2; d; \pi_1, \pi_2, \pi_3)$ is realizable. Then every candidate datum $(\Sigma_g, S^2; d; \pi_1, \dots, \pi_n)$ with $n \geq 3$ is realizable.*

Proof. We proceed by induction on $n \geq 4$. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \dots, \pi_n)$; there are two cases.

- If there are two indices $1 \leq i < j \leq n$ such that $v(\pi_i) + v(\pi_j) \leq d - 1$, then a routine reduction argument shows that $\mathcal{D}$ is realizable.
- Otherwise, $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. By corollary 2.8, we can find permutations α_1, α_2 with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_n)) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0. \end{aligned}$$

  It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \dots, \pi_n)$ is a candidate datum, so it is realizable by induction. A reduction argument shows that $\mathcal{D}$ is realizable as well. $\square$

Chapter 3

Exceptional data with a short partition

Always assume $n \geq 3$

.

3.1 Realizability on the sphere

Theorem 3.1. *Let $\mathcal{D} = (S^2; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (S^2; 12; [2, 2, 2, 2, 2, 2], [1, 1, 1, 3, 3, 3], [6, 6])$.

(2) $\mathcal{D} = (S^2; 2k; [2, \dots, 2], [2, \dots, 2], [s, 2k - s])$ *with* $k \geq 2$, $s \neq k$.

(3) $\mathcal{D} = (S^2; 2k; [2, \dots, 2], [1, 2, \dots, 2, 3], [k, k])$ *with* $k \geq 2$.

(4) $\mathcal{D} = (S^2; 4k + 2; [2, \dots, 2], [1, \dots, 1, k + 1, k + 2], [2k + 1, 2k + 1])$ *with* $k \geq 1$.

(5) $\mathcal{D} = (S^2; 4k; [2, \dots, 2], [1, \dots, 1, k + 1, k + 1], [2k - 1, 2k + 1])$ *with* $k \geq 2$.

(6) $\mathcal{D} = (S^2; kh; [h, \dots, h], [1, \dots, 1, k + 1], [lk, (h - l)k])$ *with* $h \geq 2$, $k \geq 2$, $1 \leq l < h$.

Theorem 3.2. *Let $\mathcal{D} = (S^2; d; \pi_1, \dots, \pi_{n-1}, [s, d - s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable.*

3.2 Realizability on the torus for $n = 3$

Theorem 3.3. *Let $\mathcal{D} = (\Sigma_1; d; \pi_1, \pi_2, [s, d - s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_1; 6; [3, 3], [3, 3], [2, 4])$.

(2) $\mathcal{D} = (\Sigma_1; 8; [2, 2, 2, 2], [4, 4], [3, 5])$.

(3) $\mathcal{D} = (\Sigma_1; 12; [2, 2, 2, 2, 2, 2], [3, 3, 3, 3], [5, 7])$.

(4) $\mathcal{D} = (\Sigma_1; 16; [2, 2, 2, 2, 2, 2, 2, 2], [1, 3, 3, 3, 3, 3], [8, 8])$.

(5) $\mathcal{D} = (\Sigma_1; 2k; [2, \dots, 2], [2, \dots, 2, 3, 5], [k, k])$ *with* $k \geq 5$.

3.3 Realizability on higher genus surfaces for $n = 3$

Theorem 3.4. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable.*

3.4 Realizability for $n \geq 4$

Theorem 3.5. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \dots, \pi_{n-1}, [s, d-s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_{n-3}; 4; [2, 2], \dots, [2, 2], [1, 3])$.

(2) $\mathcal{D} = (\Sigma_2; 8; [2, 2, 2, 2], [2, 2, 2, 2], [2, 2, 2, 2], [3, 5])$.

Proof. We will proceed by induction on n. We start with the base case $n = 4$, which requires the heaviest casework. Fix a candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3, [s, d-s])$.

- If $d \leq 16$, realizability can be checked by a computer. The only exceptional data are:
  (1) $\mathcal{D} = (\Sigma_1; 4; [2, 2], [2, 2], [2, 2], [1, 3])$;
  (2) $\mathcal{D} = (\Sigma_2; 8; [2, 2, 2, 2], [2, 2, 2, 2], [2, 2, 2, 2], [3, 5])$.
- If $g = 0$, then $\mathcal{D}$ is realizable by theorem 3.2.
- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq 3$; up to reindexing, we can assume that $v(\pi_1) + v(\pi_2) < d$. By proposition 2.5, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum

$$\mathcal{D}' = (\Sigma_g; d; [\alpha_1\alpha_2], \pi_3, [s, d-s]).$$

  A standard reduction argument implies that $\mathcal{D}$ is realizable provided that $\mathcal{D}'$ is. If $g \geq 2$, then $\mathcal{D}'$ is realizable by theorem 3.4. If instead $g = 1$, then $\mathcal{D}'$ is realizable unless

$$\mathcal{D}' = (\Sigma_1; 2k; [2, \dots, 2], [2, \dots, 2, 3, 5], [k, k]) \text{ with } 2k = d.$$

  If this is the case, then $s = k$ and $\{[\alpha_1\alpha_2], \pi_3\} = \{[2, \dots, 2], [2, \dots, 2, 3, 5]\}$. Some more casework is required to show that $\mathcal{D}'$ can actually be chosen to be realizable.

  - If $[\alpha_1\alpha_2] = [2, \dots, 2]$, then $v(\alpha_1\alpha_2) = k$ and $v(\pi_3) = k + 2$. Assume without loss of generality that $v(\pi_1) \leq k/2$. We have that

$$k + 2 < 1 + k + 2 \leq v(\pi_1) + v(\pi_3) \leq \frac{k}{2} + k + 2 < d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \dots, 2]$ and $v(\pi_1) \notin \{2, k, k+1\}$ then $v(\pi_1) + v(\pi_3) < d$ and $v(\pi_1) + v(\pi_3) \notin \{k, k+2\}$. Therefore, repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \dots, 2]$, $v(\pi_1) = 2$ and $\pi_2 \neq [2, \dots, 2]$, then repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

- If $\pi_2 = \pi_3 = [2, \ldots, 2]$, we follow a different approach. By proposition 2.7, we can find permutations $\beta_2, \beta_3 \in \mathfrak{S}_d$ with $[\beta_2] = [\beta_3] = [2, \ldots, 2]$ such that $[\beta_2\beta_3] = [k, k]$. It is easy to see that $\mathcal{D}'' = (S^2; 2k; \pi_1, [k, k], [k, k])$ is a candidate datum, and it is realizable by theorem 3.1. A reduction argument implies that $\mathcal{D}$ is realizable as well.

Up to swapping π_1 and π_2, this analysis covers all the possible cases.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq 3$. In particular, up to reindexing, we can assume that $v(\pi_3) \geq d/2$. Corollary 2.8 gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + d - 2) - d + 1 \geq \frac{1}{2}\left(d - 2 + \frac{d}{2} + d - 2\right) - d + 1 = \frac{d}{4} - 1 \geq 2.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1, \alpha_2], \pi_3, [s, d - s])$ is candidate datum, and it is realizable by theorem 3.4. A reduction argument implies that $\mathcal{D}$ is realizable as well.

We now turn to the case $n \geq 5$; we will show by induction that every candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ different from $(\Sigma_{n-3}; 4; [2, 2], \ldots, [2, 2], [1, 3])$ is realizable. The case $d = 4$ is addressed by proposition 2.14. If $n = 5$ and $d = 8$, a computer-aided search shows that $\mathcal{D}$ is realizable. Otherwise, we once again employ a reduction argument.

- If the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n - 1$, it is now routine to show that $\mathcal{D}$ is realizable.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n - 1$. By corollary 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 2) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d + d - 2) - d + 1 \\ &= \frac{d}{2} - 1 \geq 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [s, d - s])$ is a candidate datum, and it is realizable by induction. The usual reduction argument implies that $\mathcal{D}$ is realizable as well. □