Chapter 1

Monodromy

1.1 Monodromy and realizability

Proposition 1.1. *Let Σ_g be the connected sum of g tori, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n \in \mathfrak{S}_g$ such that:*

(i) $[\alpha_i] = \pi_i$ *for each* $1 \leq i \leq n$;

(ii) $[\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g] \cdot \alpha_1 \cdots \alpha_n = 1$;

(iii) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_g \rangle \leq \mathfrak{S}_d$ *acts transitively on* $\{1, \ldots, d\}$.

In this case, $\widetilde{\Sigma}$ is necessarily orientable.

Proposition 1.2. *Let N_g be the connected sum of g projective planes, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \in \mathfrak{S}_d$ such that:*

(i) $[\alpha_i] = \pi_i$ *for each* $1 \leq i \leq n$;

(ii) $\beta_1^2 \cdots \beta_g^2 \cdot \alpha_1 \cdots \alpha_n = 1$;

(iii) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \rangle \leq \mathfrak{S}_d$ *acts transitively on* $\{1, \ldots, d\}$.

In this case, $\widetilde{\Sigma}$ is orientable if and only if ??.

1.2 Non-positive Euler characteristic

Lemma 1.3. *Let $\alpha \in \mathfrak{S}_d$ be a permutation. Set $r = d - v(\alpha)$, and let $t \geq 0$ be an integer such that $2t \leq v(\alpha)$. Then α can be written as the product of a $(r + 2t)$-cycle and a d-cycle.*

Corollary 1.4. *Let $\alpha \in \mathfrak{A}_d$ be an even permutation. Then α can be written as:*

(i) a commutator $[\beta, \gamma]$, *where* γ *is a* d*-cycle;*

(ii) a product of two squares $\beta^2\gamma^2$, *where* $\beta\gamma$ *is a* d*-cycle.*

Theorem 1.5. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\chi(\Sigma) \leq 0$, then $\mathcal{D}$ is realizable.*

1.3 Products in symmetric groups

Lemma 1.6. *Let X, Y be finite sets; denote by h the cardinality of Y, and by k the cardinality of $X \cap Y$. Let $\alpha \in \mathfrak{S}(X)$, $\beta \in \mathfrak{S}(Y)$, $\gamma \in \mathfrak{S}(X \cap Y)$. Assume that $\beta = (b_1, \ldots, b_h)$ is a h-cycle, and that γ is a k-cycle of the form $\gamma = (b_{i_1}, \ldots, b_{i_k})$ with $1 \leq i_1 \leq \ldots \leq i_k \leq h$. Then $\alpha \in \mathfrak{S}(X)$ and $\alpha\gamma^{-1}\beta \in \mathfrak{S}(X \cup Y)$ have the same number of cycles.*

Proof. Let $\gamma = (u_1, \ldots, u_k)$, where $u_j = b_{i_j}$. Without loss of generality, assume that $i_1 = 1$. Then we can write

$$\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{k-1}, u_k, \ldots, w_k),$$

possibly with $u_j = w_j$ for some values of j. We immediately get that

$$\gamma^{-1}\beta = (u_1, \ldots, w_1)(u_2, \ldots, w_2) \cdots (u_k, \ldots, w_k).$$

If we denote by $A_1, \ldots, A_r \subseteq X$ the orbits of α, it is then easy to see that the orbits of $\alpha\gamma^{-1}\beta$ are $A'_1, \ldots, A'_r \subseteq X \cup Y$, where

$$A'_j = A_j \cup \bigcup_{u_l \in A_j} \{u_l, \ldots, w_l\}. \qquad \square$$

Proposition 1.7. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) < d$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = v(\alpha) + v(\beta)$.*

Proof. First of all, note that the conclusion is trivial whenever $v(\pi) = 0$ or $v(\rho) = 0$. This already solves the cases $d = 1$ and $d = 2$. We now proceed by induction on $d \geq 3$, assuming that $v(\pi) > 0$ and $v(\rho) > 0$. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$; without loss of generality, assume that $b_1 > 1$. Fix

$$\beta = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where $b_1 = d_1 \geq 2$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Note that

$$d - 1 \geq v(\pi) + v(\rho) = (a_1 - 1) + \ldots + (a_r - 1) + d - s \geq a_1 - 1 + d - s,$$

hence $a_1 \leq s$. Fix $\alpha_1 = (1, d_1 + 1, \ldots, d_{a_1 - 1} + 1)$, and let $A = \{1, d_1 + 1, \ldots, d_{a_1-1} + 1\}$ be the support of α_1. Define $Q = \{1, \ldots, d_{a_1}\} \setminus A$; note that Q_1 is non-empty, since $d_1 + 1 \geq 3$ implies that $2 \in Q_1$. Consider the partitions

$$\pi' = [a_2, a_3, \ldots, a_r], \qquad \rho' = [|Q|, b_{a_1+1}, \ldots, b_s].$$

We have that

$$\sum \pi' = \sum \rho' = d - a_1, \qquad v(\pi') = v(\pi) - a_1 + 1, \qquad v(\rho') = v(\rho) - 1.$$

Since $d - a_1 < d$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - a_1 < d - a_1$, by induction we find $\alpha', \beta' \in \mathfrak{S}(\{1, \ldots, d\} \setminus A)$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $v(\alpha'\beta') = v(\alpha') + v(\beta')$. Up to conjugation, we may assume that

$$\beta' = \beta_1(d_{a_1} + 1, \ldots, d_{a_1+1})(d_{a_1+1} + 1, \ldots, d_{a_1+2}) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where β_1 is the $|Q|$-cycle whose entries are the elements of Q in increasing order. An easy computation shows that

$$\alpha_1\beta = (1, \ldots, d_{a_1})(d_{a_1} + 1, \ldots, d_{a_1+1}) \cdots (d_{s-1} + 1, \ldots, d_s).$$

Therefore, setting $\alpha = \alpha'\alpha_1$, we have that

$$\begin{aligned}\alpha\beta &= \alpha_1\alpha'\beta \\ &= \alpha'(1,\dots,d_{a_1})(d_{a_1}+1,\dots,d_{a_1+1})\cdots(d_{s-1}+1,\dots,d_s) \\ &= \alpha'\beta'\beta_1^{-1}(1,\dots,d_{a_1})\end{aligned}$$

By lemma 1.6, this implies that $\alpha\beta$ has the same number of cycles as $\alpha'\beta'$, so that

$$\begin{aligned}v(\alpha\beta) &= a_1 + v(\alpha'\beta') \\ &= a_1 + v(\alpha') + v(\beta') \\ &= a_1 + (v(\pi) - a_1 + 1) + (v(\rho) - 1) \\ &= v(\pi) + v(\rho).\end{aligned}$$

Since $[\alpha] = \pi$ and $[\beta] = \rho$, the conclusion follows. □

Proposition 1.8. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d$, and let $t = v(\pi) + v(\rho) - d + 1$. Fix an integer $0 \leq k \leq t$ such that $k \equiv t \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = d - 1 - k$ and the action of $\langle\alpha, \beta\rangle$ on $\{1, \dots, d\}$ is transitive.*

Maybe only $k = 0$ needed (in this case, transitivity is trivial)?

Proof. Write $\pi = [a_1, \dots, a_r]$. Since $v(\rho) \leq d-1$ and $v(\pi) + v(\rho) \geq d$, there exists a largest integer $0 \leq i \leq r$ such that $(a_1 - 1) + \dots (a_i - 1) + v(\rho) \leq d - 1$. Define

$$z = d - v(\rho) - (a_1 - 1) - \dots - (a_i - 1).$$

Consider the partition $\pi' = [a_1, \dots, a_i, z, 1, \dots, 1] \in \Pi(d)$. Since by construction $v(\pi') + v(\rho) = d - 1$, thanks to proposition 1.7 we can find permutations $\alpha', \beta \in \mathfrak{S}_d$ with $[\alpha'] = \pi'$ and $[\beta] = \rho$ such that $v(\alpha'\beta) = d - 1$; in other words, $\alpha'\beta$ is a d-cycle. Consider now the partition $\pi'' = [a_{i+1} - z + 1, a_{i+2}, \dots, a_r]$, whose branching number is $v(\pi'') = t$. Let $n = \sum \pi''$; fix an element u_1 of the z-cycle of α', and let $u_2, \dots, u_n$ be the fixed points of α' corresponding to the last ones of π' (it is easy to see that there are exactly $n-1$ such ones). Since $k \leq t = v(\rho'')$ and $k \equiv t \pmod 2$, lemma 1.3 gives permutations $\alpha'', \gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$ such that $[\alpha''] = \rho''$, γ is a n-cycle and $\alpha''\gamma$ is a $(n-k)$-cycle. Up to conjugation, we can assume that $\gamma = (u_1, \dots, u_n)$. Moreover, it is not restrictive to assume that $u_1, \dots, u_n$ appear in this order in the d-cycle $\alpha'\beta$. Therefore, setting $\alpha = \alpha''\alpha'$, we have that

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$$\alpha\beta = (\alpha''\gamma)\gamma^{-1}(\alpha'\beta);$$

by lemma 1.6, this implies that $\alpha\beta$ has the same number of cycles as $\alpha''\gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$, that is $v(\alpha\beta) = d - (k+1)$. Since $[\alpha] = \pi$ and $[\beta] = \rho$, the only thing left to show is that the action of $\langle\alpha, \beta\rangle$ on $\{1, \dots, d\}$ is transitive. Write

$$\alpha'\beta = (u_1, \dots, w_1, u_2, \dots, w_2, u_3, \dots, w_{n-1}, u_n, \dots, w_n)$$

as in the proof of lemma 1.6, where an explicit description of the orbits of $\alpha\beta = \alpha''\gamma\gamma^{-1}\alpha'\beta$ is given; from that description, it is clear that for each $1 \leq j \leq n$ the elements $u_j, \dots, w_j$ all belong to the same orbit. Moreover, since $\beta = (\alpha')^{-1}(u_1, \dots, w_1, u_2, \dots, w_n)$ and α' fixes $u_2, \dots, u_n$, it follows that w_j and u_{j+1} belong to the same orbit for each $1 \leq j \leq n-1$; this completes the proof. □

Corollary 1.9. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d - 1$ and $v(\pi) + v(\rho) \equiv d - 1 \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\alpha\beta$ is a d-cycle.*

Proof. The conclusion immediately follows from proposition 1.7 if $v(\pi)+v(\rho)=d-1$, or from proposition 1.8 if $v(\pi)+v(\rho)\geq d$. $\square$

Remark 1.1. Write $\pi=[a_1,\dots,a_r]$, $\rho=[b_1,\dots,b_s]$; assume that $b_1\geq 2$. By directly examining the proof of proposition 1.7, we see that the proposed construction yields permutations $\alpha,\beta\in\mathfrak{S}_d$ such that 1 belongs to the a_1-the cycle of α and to the b_1-cycle of β. It is not hard to see, once again by inspecting the proof, that the same can be said for proposition 1.8. As a consequence, the statement of corollary 1.9 can be enhanced by adding the following line: *α and β can be chosen in such a way that 1 belongs to the a_1-the cycle of α and to the b_1-cycle of β, provided that $b_1\geq 2$.* We will need this improvement for the upcoming proof.

aybe explain why?

Proposition 1.10. *Let $\pi,\rho\in\Pi(d)$ be partitions of d. Assume that $v(\pi)+v(\rho)\geq d$ and $v(\pi)+v(\rho)\equiv d \pmod 2$. Then there exist permutations $\alpha,\beta\in\mathfrak{S}_d$ with $[\alpha]=\pi$ and $[\beta]=\rho$ such that $\langle\alpha,\beta\rangle$ acts transitively on $\{1,\dots,d\}$ and*

$$[\alpha\beta]=\begin{cases}[d/2,d/2] & \textit{if } \pi=\rho=[2,\dots,2],\\ [1,d-1] & \textit{otherwise.}\end{cases}$$

Proof. Assume first that $\pi=\rho=[2,\dots,d]$. We can choose

$$\alpha=(2,3)(4,5)\cdots(d,1) \qquad\qquad \beta=(1,2)(3,4)\cdots(d-1,d).$$

The action of $\langle\alpha,\beta\rangle$ is obviously transitive, and

$$\alpha\beta=(1,3,\dots,d-1)(2,4,\dots,2).$$

Otherwise, since $v(\pi)+v(\rho)\geq d$, at least one of π and ρ has an entry which is greater than 2; without loss of generality, we can assume it is ρ. Write $\pi=[a_1,\dots,a_r]$, $\rho=[b_1,\dots,b_s]$ with $a_1\geq 2$ (since $v(\pi)\geq 1$) and $b_1\geq 3$. Fix

$$\beta=(1,\dots,d_1)(d_1+1,\dots,d_2)\cdots(d_{s-1}+1,\dots,d_s),$$

where $b_1=d_1\geq 3$, $b_i=d_i-d_{i-1}$ for $2\leq i\leq s$ (in particular, $d_s=d$). Consider the partitions

$$\pi'=[a_1-1,\dots,a_r], \qquad\qquad \rho'=[b_1-1,\dots,b_s].$$

Since $\sum\pi'=\sum\rho'=d-1$ and $v(\pi')+v(\rho')=v(\pi)+v(\rho)-2$, by corollary 1.9 we can find permutations $\alpha',\beta'\in\mathfrak{S}(\{2,\dots,d\})$ with $[\alpha']=\pi'$ and $[\beta']=\rho'$ such that $\alpha'\beta'$ is a $(d-1)$-cycle. Up to conjugation, we can assume that

$$\beta'=(2,\dots,d_1)(d_1+1,\dots,d_2)\cdots(d_{s-1}+1,\dots,d_s)$$

or, in other words, $\beta=(1,2)\beta'$; moreover, as explained in remark 1.1, we can choose α' in such a way that its (a_1-1)-cycle contains 2. By setting $\alpha=\alpha'(1,2)$, we immediately get that $\alpha\beta=\alpha'\beta'$ is a $(d-1)$-cycle fixing 1. Finally, the action of $\langle\alpha,\beta\rangle$ is transitive since α does not fix 1. $\square$

Corollary 1.11. *Let $\pi,\rho\in\Pi(d)$ be partitions of d. Assume that $v(\pi)+v(\rho)\geq d-1$. Then there exist permutations $\alpha,\beta\in\mathfrak{S}_d$ with $[\alpha]=\pi$, $[\beta]=\rho$ such that $\langle\alpha,\beta\rangle$ acts transitively on $\{1,\dots,d\}$ and*

$$[\alpha\beta]\in\{[d],[1,d-1],[d/2,d/2]\}.$$

Proof. The conclusion follows immediately from proposition 1.8 or proposition 1.10 depending on the parity of $v(\pi)+v(\rho)+d$. $\square$

1.4 Sphere and projective plane

Theorem 1.12. *Let $\mathcal{D} = (\widetilde{\Sigma}, \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\widetilde{\Sigma}$ is non-orientable, then $\mathcal{D}$ is realizable.*

Proof. First of all, since $\mathcal{D}$ is a candidate datum, the **??** formula implies that

$$v(\pi_1) + \ldots + v(\pi_n) = d \cdot \chi(\mathbb{RP}^2) - \chi(\widetilde{\Sigma}) \geq d - 1$$

(recall that $\widetilde{\Sigma}$ is non-orientable, so $\chi(\widetilde{\Sigma}) \leq 1$). Moreover, the total branching $v(\pi_1) + \ldots + v(\pi_n)$ is even. In order to make use of proposition 1.2, we will now inductively define representatives $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$, satisfying the following invariant: for every $0 \leq i \leq n$, either

$$v(\alpha_1 \cdots \alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$$

or

$$[\alpha_1 \cdots \alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\} \text{ and } \langle \alpha_1, \ldots, \alpha_i \rangle \text{ acts transitively.}$$

Assume we have already defined $\alpha_1, \ldots, \alpha_{i-1}$; we want to suitably choose α_i. Let $\alpha = \alpha_1 \cdots \alpha_{i-1}$; there are two cases.

- If $v(\alpha) + v(\pi_i) < d$, by proposition 1.7 we can find $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $v(\alpha\alpha_i) = v(\alpha) + v(\alpha_i)$. The invariant is still satisfied: if $v(\alpha) = v(\pi_1) + \ldots + v(\pi_{i-1})$ then obviously $v(\alpha\alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$. If instead $[\alpha] \in \{[d], [1, d-1], [d/2, d/2]\}$, then either α_i is the identity or $\alpha\alpha_i$ is a d-cycle; either way, $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$. Note that if the action of $\langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ is transitive, then the action of $\langle \alpha_1, \ldots, \alpha_i \rangle$ is transitive as well.

- If $v(\alpha) + v(\alpha_i) \geq d$, corollary 1.11 gives a permutation $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$ and the action of $\langle \alpha, \alpha_i \rangle$ is transitive. The invariant is obviously satisfied.

By induction, we can find $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that

$$[\alpha_1 \cdots \alpha_n] \in \{[d], [1, d-1], [d/2, d/2]\}$$

and $\langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$ (the first condition of the invariant also implies that $[\alpha_1, \ldots, \alpha_n] = [d]$). Note that

$$v(\alpha_1 \cdots \alpha_n) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

We now prove that $\alpha = \alpha_1 \cdots \alpha_n$ is a square.

- If $[\alpha] = [d]$, then d is odd, so α is the square of $\alpha^{(d+1)/2}$.
- If $[\alpha] = [1, d-2]$, then d is even, so α is the square of $\alpha^{d/2}$.
- If $[\alpha] = [d/2, d/2]$, then d is even, and it is easy to see that α is the square of a d-cycle.

□

1.5 Prime-degree conjecture