# Chapter 1

# Hurwitz existence problem

# Chapter 2

# Monodromy

## 2.1 Symmetric group and partitions

## 2.2 Branched covering action of the fundamental group

## 2.3 Monodromy and realizability

**Proposition 2.1.** *Let $\Sigma_g$ be the connected sum of $g \geq 0$ tori, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of $d$. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n \in \mathfrak{S}_g$ such that:*

*(i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;*

*(i) $[\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g] \cdot \alpha_1 \cdots \alpha_n = 1$;*

*(i) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \ldots, d\}$.*

*In this case, $\widetilde{\Sigma}$ is necessarily orientable.*

*Remark* 2.1. Given $\Sigma_g$, $d$ and $\pi_1, \ldots, \pi_n$, there is at most one surface $\widetilde{\Sigma}$ such that $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum. In fact, the Riemann-Hurwitz [??] formula gives

$$\chi(\widetilde{\Sigma}) = d\chi(\Sigma_g) - v(\pi_1) - \ldots - v(\pi_n)$$

which, in turn, uniquely determines the orientable surface $\widetilde{\Sigma}$.

**Proposition 2.2.** *Let $N_g$ be the connected sum of $g \geq 1$ projective planes, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of $d$. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \in \mathfrak{S}_d$ such that:*

*(i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;*

*(i) $\beta_1^2 \cdots \beta_g^2 \cdot \alpha_1 \cdots \alpha_n = 1$;*

*(i) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \ldots, d\}$.*

*In this case, $\widetilde{\Sigma}$ is orientable if and only if ??.*

*Remark* 2.2. Given $N_g$, $d$ and $\pi_1, \ldots, \pi_n$, there are at most two surfaces $\widetilde{\Sigma}$, one orientable and one non-orientable, such that $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum: like in remark 2.1, the Riemann-Hurwitz [??] formula fixes $\chi(\widetilde{\Sigma})$ which, together with orientability, uniquely determines $\widetilde{\Sigma}$.

**Proposition 2.3.** *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots \pi_n)$ be a combinatorial datum, with $\Sigma$ non-orientable and $\widetilde{\Sigma}$ orientable. Then $\mathcal{D}$ is realizable if and only if $d$ is even and there exist partitions $\pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n \in \Pi\,(d/2)$ with $\pi_i = \pi'_i \cup \pi''_i$ for each $1 \leq i \leq n$, such that the combinatorial datum*

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_i, \ldots, \pi''_n)$$

*is realizable, where $\widehat{\Sigma}$ is the orientable double covering of $\Sigma$.*

## 2.4 Non-positive Euler characteristic

**Lemma 2.4.** *Let $\alpha \in \mathfrak{S}_d$ be a permutation. Set $r = d - v(\alpha)$, and let $t \geq 0$ be an integer such that $2t \leq v(\alpha)$. Then $\alpha$ can be written as the product of a $(r + 2t)$-cycle and a $d$-cycle.*

*Proof.* Without loss of generality, assume that

$$\alpha = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{r-1} + 1, \ldots, d_r),$$

where $d_r = d$. Fix

$$\beta_0 = (1, b_1, b_2, \ldots, b_{2t}), \qquad \beta_1 = (1, d_1 + 1, d_2 + 2, \ldots, d_{r-1} + 1).$$

Set

$$\beta = \beta_0\beta_1 = (1, d_1 + 1, d_2 + 1, \ldots, d_{r-1} + 1, b_1, b_2, \ldots, b_{2t}).$$

An easy computation shows that

$$\begin{aligned} \beta\alpha &= \beta_0\beta_1\alpha \\ &= (1, b_1, b_2, \ldots, b_{2t})(1, 2, \ldots, d) \\ &= (1, \ldots, b_1 - 1, b_2, \ldots, b_3 - 1, \ldots, b_4, \ldots, b_{2t-1} - 1, b_{2t}, \ldots, d, b_1, \ldots, b_2 - 1, b_3, \ldots, b_{2t} - 1). \end{aligned}$$

Writing $\alpha = \beta^{-1}(\beta\alpha)$ gives the desired decomposition. □

**Corollary 2.5.** *Let $\alpha \in \mathfrak{A}_d$ be an even permutation. Then $\alpha$ can be written as:*

*(i) a commutator $[\beta, \gamma]$, where $\gamma$ is a $d$-cycle;*

*(i) a product of two squares $\delta^2\epsilon^2$, where $\delta\epsilon$ is a $d$-cycle.*

*Proof.* Since $\alpha$ is an even permutation, its branching number $v(\alpha)$ is even. By lemma 2.4, there exist two $d$-cycles $\tau, \sigma \in \mathfrak{S}_d$ such that $\alpha = \tau\sigma$.

(i) Since $\tau$ and $\sigma^{-1}$ are conjugated, there exists a permutation $\beta \in \mathfrak{S}_d$ such that $\tau = \beta\sigma^{-1}\beta^{-1}$. Setting $\gamma = \sigma^{-1}$, we immediately get that

$$\alpha = \tau\sigma = \beta\sigma^{-1}\beta^{-1}\sigma = \beta\gamma\beta^{-1}\gamma^{-1} = [\beta, \gamma].$$

(i) Since $\tau$ and $\sigma$ are conjugated, there exists a permutation $\delta \in \mathfrak{S}_d$ such that $\tau = \delta\sigma\delta^{-1}$. Setting $\epsilon = \delta^{-1}\sigma$, we have that

$$\alpha = \tau\sigma = \delta\sigma\delta^{-1}\sigma = \delta^2(\delta^{-1}\sigma)^2 = \delta^2\epsilon^2.$$ □

**Theorem 2.6.** *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\chi(\Sigma) \leq 0$, then $\mathcal{D}$ is realizable.*

*Proof.* Let us first assume that $\Sigma$ is orientable; this means that $\Sigma = \Sigma_g$ is the connected sum of $g \geq 1$ tori, and that $\widetilde{\Sigma}$ is orientable as well. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Since $\mathcal{D}$ is a candidate datum, we have that

$$v(\alpha) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

By corollary 2.5, we can find permutations $\beta_1, \gamma_1 \in \mathfrak{S}_d$ such that $\alpha = [\gamma_1, \beta_1]$ and $\beta_1$ is a $d$-cycle. Set $\beta_2 = \ldots = \beta_g = \gamma_2 = \ldots = \gamma_g = \mathrm{id} \in \mathfrak{S}_d$. All the conditions of proposition 2.1 are satisfied; since $\widetilde{\Sigma}$ is orientable, this implies that $\mathcal{D}$ is realizable (see remark 2.1).

Assume now that $\Sigma$ and $\widetilde{\Sigma}$ are both non-orientable; this means that $\Sigma = N_g$ is the connected sum of $g \geq 2$ projective planes. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Similarly to what we did for the previous case, we can find $\beta_1, \beta_2 \in \mathfrak{S}_d$ such that $\alpha = \beta_2^{-2}\beta_1^{-2}$ and $\beta_2\beta_1$ is a $d$-cycle. By setting $\beta_3 = \ldots = \beta_g = \mathrm{id} \in \mathfrak{S}_d$, proposition 2.2 (together with remark 2.2) implies the realizability of $\mathcal{D}$.

Finally, consider the case where $\Sigma$ is non-orientable and $\widetilde{\Sigma}$ is orientable. Since $\mathcal{D}$ is a candidate datum, $d$ is even and there exist partitions $\pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n \in \Pi\,(d/2)$ with $\pi_i = \pi'_1 \cup \pi''_i$ for each $1 \leq i \leq n$. Let $\widehat{\Sigma}$ be the double orientable covering of $\Sigma$; by the first case we analyzed, the candidate datum

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n)$$

is realizable. By proposition 2.3, $\mathcal{D}$ is realizable as well. □

## 2.5 Products in symmetric groups

**Lemma 2.7.** *Let $X, Y$ be finite sets; denote by $h$ the cardinality of $Y$, and by $k$ the cardinality of $X \cap Y$. Let $\alpha \in \mathfrak{S}(X)$, $\beta \in \mathfrak{S}(Y)$, $\gamma \in \mathfrak{S}(X \cap Y)$. Assume that $\beta = (b_1, \ldots, b_h)$ is a $h$-cycle, and that $\gamma$ is a $k$-cycle of the form $\gamma = (b_{i_1}, \ldots, b_{i_k})$ with $1 \leq i_1 \leq \ldots \leq i_k \leq h$. Then $\alpha \in \mathfrak{S}(X)$ and $\alpha\gamma^{-1}\beta \in \mathfrak{S}(X \cup Y)$ have the same number of cycles.*

*Proof.* Write $\gamma = (u_1, \ldots, u_k)$, where $u_j = b_{i_j}$. Without loss of generality, assume that $i_1 = 1$. Then we can write

$$\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{k-1}, u_k, \ldots, w_k),$$

possibly with $u_j = w_j$ for some values of $j$. We immediately get that

$$\gamma^{-1}\beta = (u_1, \ldots, w_1)(u_2, \ldots, w_2) \cdots (u_k, \ldots, w_k).$$

If we denote by $A_1, \ldots, A_r \subseteq X$ the orbits of $\alpha$, it is then easy to see that the orbits of $\alpha\gamma^{-1}\beta$ are $A'_1, \ldots, A'_r \subseteq X \cup Y$, where

$$A'_j = A_j \cup \bigcup_{u_l \in A_j} \{u_l, \ldots, w_l\}.$$ □

**Proposition 2.8.** *Let $\pi, \rho \in \Pi\,(d)$ be partitions of $d$. Assume that $v(\pi) + v(\rho) < d$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = v(\alpha) + v(\beta)$.*

*Proof.* First of all, note that the conclusion is trivial whenever $v(\pi) = 0$ or $v(\rho) = 0$. This already solves the cases $d = 1$ and $d = 2$. We now proceed by induction on $d \geq 3$, assuming that $v(\pi) > 0$ and $v(\rho) > 0$. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$; without loss of generality, assume that $b_1 > 1$. Fix

$$\beta = (1, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{s-1} + 1, \dots, d_s),$$

where $b_1 = d_1 \geq 2$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Note that

$$d - 1 \geq v(\pi) + v(\rho) = (a_1 - 1) + \ldots + (a_r - 1) + d - s \geq a_1 - 1 + d - s,$$

hence $a_1 \leq s$. Fix $\alpha_1 = (1, d_1 + 1, \dots, d_{a_1 - 1} + 1)$, and let $A = \{1, d_1 + 1, \dots, d_{a_1-1} + 1\}$ be the support of $\alpha_1$. Define $Q = \{1, \dots, d_{a_1}\} \setminus A$; note that $Q_1$ is non-empty, since $d_1 + 1 \geq 3$ implies that $2 \in Q_1$. Consider the partitions

$$\pi' = [a_2, a_3, \dots, a_r], \qquad \rho' = [|Q|, b_{a_1+1}, \dots, b_s].$$

We have that

$$\sum \pi' = \sum \rho' = d - a_1, \qquad v(\pi') = v(\pi) - a_1 + 1, \qquad v(\rho') = v(\rho) - 1.$$

Since $d - a_1 < d$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - a_1 < d - a_1$, by induction we find $\alpha', \beta' \in \mathfrak{S}(\{1, \dots, d\} \setminus A)$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $v(\alpha'\beta') = v(\alpha') + v(\beta')$. Up to conjugation, we may assume that

$$\beta' = \beta_1 (d_{a_1} + 1, \dots, d_{a_1+1})(d_{a_1+1} + 1, \dots, d_{a_1+2}) \cdots (d_{s-1} + 1, \dots, d_s),$$

where $\beta_1$ is the $|Q|$-cycle whose entries are the elements of $Q$ in increasing order. An easy computation shows that

$$\alpha_1 \beta = (1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s).$$

Therefore, setting $\alpha = \alpha'\alpha_1$, we have that

$$\begin{aligned}
\alpha\beta &= \alpha_1 \alpha' \beta \\
&= \alpha'(1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s) \\
&= \alpha' \beta' \beta_1^{-1} (1, \dots, d_{a_1})
\end{aligned}$$

By lemma 2.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha'\beta'$, so that

$$\begin{aligned}
v(\alpha\beta) &= a_1 + v(\alpha'\beta') \\
&= a_1 + v(\alpha') + v(\beta') \\
&= a_1 + (v(\pi) - a_1 + 1) + (v(\rho) - 1) \\
&= v(\pi) + v(\rho).
\end{aligned}$$

Since $[\alpha] = \pi$ and $[\beta] = \rho$, the conclusion follows. $\square$

**Proposition 2.9.** *Let $\pi, \rho \in \Pi(d)$ be partitions of $d$. Assume that $v(\pi) + v(\rho) \geq d$, and let $t = v(\pi) + v(\rho) - d + 1$. Fix an integer $0 \leq k \leq t$ such that $k \equiv t \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = d - 1 - k$ and the action of $\langle \alpha, \beta \rangle$ on $\{1, \dots, d\}$ is transitive.*

aybe only $k = 0$ eded (in this case, ansitivity is trivial)?

*Proof.* Write $\pi = [a_1, \dots, a_r]$. Since $v(\rho) \le d-1$ and $v(\pi) + v(\rho) \ge d$, there exists a largest integer $0 \le i \le r$ such that $(a_1 - 1) + \dots (a_i - 1) + v(\rho) \le d - 1$. Define

$$z = d - v(\rho) - (a_1 - 1) - \dots - (a_i - 1).$$

Consider the partition $\pi' = [a_1, \dots, a_i, z, 1, \dots, 1] \in \Pi(d)$. Since by construction $v(\pi') + v(\rho) = d - 1$, thanks to proposition 2.8 we can find permutations $\alpha', \beta \in \mathfrak{S}_d$ with $[\alpha'] = \pi'$ and $[\beta] = \rho$ such that $v(\alpha'\beta) = d - 1$; in other words, $\alpha'\beta$ is a $d$-cycle. Consider now the partition $\pi'' = [a_{i+1} - z + 1, a_{i+2}, \dots, a_r]$, whose branching number is $v(\pi'') = t$. Let $n = \sum \pi''$; fix an element $u_1$ of the $z$-cycle of $\alpha'$, and let $u_2, \dots, u_n$ be the fixed points of $\alpha'$ corresponding to the last ones of $\pi'$ (it is easy to see that there are exactly $n-1$ such ones). Since $k \le t = v(\rho'')$ and $k \equiv t \pmod 2$, lemma 2.4 gives permutations $\alpha'', \gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$ such that $[\alpha''] = \rho''$, $\gamma$ is a $n$-cycle and $\alpha''\gamma$ is a $(n-k)$-cycle. Up to conjugation, we can assume that $\gamma = (u_1, \dots, u_n)$. Moreover, it is not restrictive to assume that $u_1, \dots, u_n$ appear in this order in the $d$-cycle $\alpha'\beta$. Therefore, setting $\alpha = \alpha''\alpha'$, we have that

Terminology?

$$\alpha\beta = (\alpha''\gamma)\gamma^{-1}(\alpha'\beta);$$

by lemma 2.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha''\gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$, that is $v(\alpha\beta) = d - (k+1)$. Since $[\alpha] = \pi$ and $[\beta] = \rho$, the only thing left to show is that the action of $\langle \alpha, \beta \rangle$ on $\{1, \dots, d\}$ is transitive. Write

$$\alpha'\beta = (u_1, \dots, w_1, u_2, \dots, w_2, u_3, \dots, w_{n-1}, u_n, \dots, w_n)$$

as in the proof of lemma 2.7, where an explicit description of the orbits of $\alpha\beta = \alpha''\gamma\gamma^{-1}\alpha'\beta$ is given; from that description, it is clear that for each $1 \le j \le n$ the elements $u_j, \dots, w_j$ all belong to the same orbit. Moreover, since $\beta = (\alpha')^{-1}(u_1, \dots, w_1, u_2, \dots, w_n)$ and $\alpha'$ fixes $u_2, \dots, u_n$, it follows that $w_j$ and $u_{j+1}$ belong to the same orbit for each $1 \le j \le n-1$; this completes the proof. □

**Corollary 2.10.** *Let $\pi, \rho \in \Pi(d)$ be partitions of $d$. Assume that $v(\pi) + v(\rho) \ge d - 1$ and $v(\pi) + v(\rho) \equiv d - 1 \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\alpha\beta$ is a $d$-cycle.*

*Proof.* The conclusion immediately follows from proposition 2.8 if $v(\pi) + v(\rho) = d - 1$, or from proposition 2.9 if $v(\pi) + v(\rho) \ge d$. □

*Remark* 2.3. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$; assume that $b_1 \ge 2$. By directly examining the proof of proposition 2.8, we can see that the proposed construction yields permutations $\alpha, \beta \in \mathfrak{S}_d$ such that 1 belongs to the $a_1$-the cycle of $\alpha$ and to the $b_1$-cycle of $\beta$. It is not hard to see, once again by inspecting the proof, that the same can be said for proposition 2.9. As a consequence, the statement of corollary 2.10 can be enhanced by adding the following line: *$\alpha$ and $\beta$ can be chosen in such a way that 1 belongs to the $a_1$-the cycle of $\alpha$ and to the $b_1$-cycle of $\beta$, provided that $b_1 \ge 2$.* We will need this improvement for the upcoming proof.

Maybe explain why?

**Proposition 2.11.** *Let $\pi, \rho \in \Pi(d)$ be partitions of $d$. Assume that $v(\pi) + v(\rho) \ge d$ and $v(\pi) + v(\rho) \equiv d \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \dots, d\}$ and*

$$[\alpha\beta] = \begin{cases} [d/2, d/2] & \text{if } \pi = \rho = [2, \dots, 2], \\ [1, d-1] & \text{otherwise.} \end{cases}$$

*Proof.* Assume first that $\pi = \rho = [2, \ldots, d]$. We can choose

$$\alpha = (2,3)(4,5)\cdots(d,1), \qquad \beta = (1,2)(3,4)\cdots(d-1,d).$$

The action of $\langle \alpha, \beta \rangle$ is obviously transitive, and

$$\alpha\beta = (1,3,\ldots,d-1)(2,4,\ldots,2).$$

Otherwise, since $v(\pi) + v(\rho) \geq d$, at least one of $\pi$ and $\rho$ has an entry which is greater than 2; without loss of generality, we can assume it is $\rho$. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$ with $a_1 \geq 2$ (since $v(\pi) \geq 1$) and $b_1 \geq 3$. Fix

$$\beta = (1,\ldots,d_1)(d_1+1,\ldots,d_2)\cdots(d_{s-1}+1,\ldots,d_s),$$

where $b_1 = d_1 \geq 3$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Consider the partitions

$$\pi' = [a_1 - 1, \ldots, a_r], \qquad \rho' = [b_1 - 1, \ldots, b_s].$$

Since $\sum \pi' = \sum \rho' = d - 1$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - 2$, by corollary 2.10 we can find permutations $\alpha', \beta' \in \mathfrak{S}(\{2, \ldots, d\})$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $\alpha'\beta'$ is a $(d-1)$-cycle. Up to conjugation, we can assume that

$$\beta' = (2,\ldots,d_1)(d_1+1,\ldots,d_2)\cdots(d_{s-1}+1,\ldots,d_s)$$

or, in other words, $\beta = (1,2)\beta'$; moreover, as explained in remark 2.3, we can choose $\alpha'$ in such a way that its $(a_1-1)$-cycle contains 2. By setting $\alpha = \alpha'(1,2)$, we immediately get that $\alpha\beta = \alpha'\beta'$ is a $(d-1)$-cycle fixing 1. Finally, the action of $\langle \alpha, \beta \rangle$ is transitive since $\alpha$ does not fix 1. $\square$

**Corollary 2.12.** *Let $\pi, \rho \in \Pi(d)$ be partitions of $d$. Assume that $v(\pi) + v(\rho) \geq d - 1$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$, $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \ldots, d\}$ and*

$$[\alpha\beta] \in \{[d], [1, d-1], [d/2, d/2]\}.$$

*Proof.* The conclusion follows immediately from proposition 2.9 or proposition 2.11 depending on the parity of $v(\pi) + v(\rho) + d$. $\square$

## 2.6 Projective plane

**Theorem 2.13.** *Let $\mathcal{D} = (\widetilde{\Sigma}, \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\widetilde{\Sigma}$ is non-orientable, then $\mathcal{D}$ is realizable.*

*Proof.* First of all, since $\mathcal{D}$ is a candidate datum, the Riemann-Hurwitz [??] formula implies that

$$v(\pi_1) + \ldots + v(\pi_n) = d\chi(\mathbb{RP}^2) - \chi(\widetilde{\Sigma}) \geq d - 1$$

(recall that $\widetilde{\Sigma}$ is non-orientable, so $\chi(\widetilde{\Sigma}) \leq 1$). Moreover, the total branching $v(\pi_1) + \ldots + v(\pi_n)$ is even. In order to apply proposition 2.2, we will now inductively define representatives $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$, satisfying the following invariant: for every $0 \leq i \leq n$, either

$$v(\alpha_1 \cdots \alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$$

or

$$[\alpha_1 \cdots \alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\} \text{ and } \langle \alpha_1, \ldots, \alpha_i \rangle \text{ acts transitively.}$$

Assume we have already defined $\alpha_1, \ldots, \alpha_{i-1}$; we want to suitably choose $\alpha_i$. Let $\alpha = \alpha_1 \cdots \alpha_{i-1}$; there are two cases.

- If $v(\alpha) + v(\pi_i) < d$, by proposition 2.8 we can find $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $v(\alpha\alpha_i) = v(\alpha) + v(\alpha_i)$. The invariant is still satisfied: if $v(\alpha) = v(\pi_1) + \ldots + v(\pi_{i-1})$ then obviously $v(\alpha\alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$. If instead $[\alpha] \in \{[d], [1, d-1], [d/2, d/2]\}$, then either $\alpha_i$ is the identity, or $\alpha\alpha_i$ is a $d$-cycle; either way, $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$. Note that if the action of $\langle\alpha_1, \ldots, \alpha_{i-1}\rangle$ is transitive, then the action of $\langle\alpha_1, \ldots, \alpha_i\rangle$ is transitive as well.
- If $v(\alpha) + v(\alpha_i) \geq d$, corollary 2.12 gives a permutation $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$ and the action of $\langle\alpha, \alpha_i\rangle$ is transitive. The invariant is obviously satisfied.

By induction, we can find $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that

$$[\alpha_1 \cdots \alpha_n] \in \{[d], [1, d-1], [d/2, d/2]\}$$

and $\langle\alpha_1, \ldots, \alpha_n\rangle$ acts transitively on $\{1, \ldots, d\}$ (note that $v(\alpha_1 \cdots \alpha_n) = v(\alpha_1) + \ldots + v(\alpha_n)$ also implies that $[\alpha_1 \cdots \alpha_n] = [d]$). Note that

Show that $v(\alpha\beta) \equiv v(\alpha) + v(\beta) \pmod 2$.

$$v(\alpha_1 \cdots \alpha_n) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

We now prove that $\alpha = \alpha_1 \cdots \alpha_n$ is a square.

- If $[\alpha] = [d]$, then $d$ is odd, so $\alpha$ is the square of $\alpha^{(d+1)/2}$.
- If $[\alpha] = [1, d-2]$, then $d$ is even, so $\alpha$ is the square of $\alpha^{d/2}$.
- If $[\alpha] = [d/2, d/2]$, then $d$ is even, and it is easy to see that $\alpha$ is the square of a $d$-cycle.

By proposition 2.2, this implies that there exists a realizable candidate datum $\mathcal{D}' = (\widetilde{\Sigma}', \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$. To see that $\widetilde{\Sigma}'$ is non-orientable (and, therefore, equal to $\widetilde{\Sigma}$, as shown in remark 2.2), simply note that ??. □

## 2.7 Reduction technique on the sphere

Assume $n \geq 3$.

**Proposition 2.14.** *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$ be a candidate datum. Then $\mathcal{D}$ is realizable.*

*Proof.* We proceed by induction on $n$, starting with the base case $n = 3$. If $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \pi_2, [d])$ is a candidate datum, by the Riemann-Hurwitz [??] formula we have that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d-1) \geq d - 1.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 1$ is even. Corollary 2.10 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] = [d]$. By proposition 2.1, $\mathcal{D}$ is realizable (see remark 2.1).

We now turn to the case $n \geq 4$. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$; there are two cases.

- Assume that $v(\pi_1) + v(\pi_2) \leq d - 1$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [d])$. By induction, $\mathcal{D}'$ is realizable; proposition 2.1 then gives permutations $\alpha_3, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for $3 \leq i \leq n-1$ and $[\alpha_n] = [d]$ such that $(\alpha_1\alpha_2)\alpha_3 \cdots \alpha_n = 1$. It is easy to see that the permutations $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ imply the realizability of $\mathcal{D}$, again by proposition 2.1 (together, as usual, with remark 2.1).

- Otherwise, we have that $v(\pi_1) + v(\pi_2) \geq d$. By corollary 2.12, we can find permutations $\alpha_1, \alpha_2$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 1) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + 1 + d - 1) - d + 1 \geq 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$ is a candidate datum, so it is realizable by induction. Similarly to the previous case, this implies that $\mathcal{D}$ is realizable. $\square$

xplain the reduction chnique, with transi- ity.

**Proposition 2.15.** *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [1, d-1])$ be a proper candidate datum. Then $\mathcal{D}$ is non-realizable if and only if it satisfies one of the following.*

*(1)* $\mathcal{D} = (\Sigma_{n-3}, S^2; 4; [2, 2], \ldots, [2, 2], [1, 3])$.

*(2)* $\mathcal{D} = (S^2, S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k-1])$ *with* $k \geq 2$.

e non-realizability the listed candidate ta will be addressed some point.

*Proof.* We proceed by induction on $n$, starting from the base case $n = 3$. Let

$$\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \pi_2, [1, d-1]) \neq (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$$

be a candidate datum. The Riemann-Hurwitz [??] formula implies that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d - 2) \geq d.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 2$ is even. Proposition 2.11 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] = [1, d-1]$ and the action of $\langle \alpha_1, \alpha_2 \rangle$ is transitive. As usual, proposition 2.1 implies that $\mathcal{D}$ is realizable.

We now turn to the case $n \geq 4$; we will employ a reduction technique. Fix a candidate datum

$$\mathcal{D} = (\Sigma_{n-3}, S^2; d; \pi_1, \ldots, \pi_{n-1}, [1, d-1]) \neq (\Sigma_{n-3}, S^2; 4; [2, 2], \ldots, [2, 2], [1, 3]).$$

There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n-1$; up to reindexing, we can assume that $i = 1$ and $j = 2$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [1, d-1])$. By induction, $\mathcal{D}'$ is realizable unless one of the following happens.
  - $\mathcal{D}' = (S^2, S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k-1])$ with $2k = d$. If this is the case, then $n = 4$ and $v(\pi_1) + v(\pi_2) = v(\pi_3) = k$. This implies that

    $$k < 1 + k \leq v(\pi_1) + v(\pi_3) < 2k = d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - $\mathcal{D}' = (\Sigma_{n-3}, S^2; 4; [2, 2], \ldots, [2, 2], [1, 3])$. This implies that $\pi_1 = \pi_2 = [1, 1, 2]$ and $\pi_3 = \ldots = \pi_{n-1} = [2, 2]$. Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

  Therefore we can assume that $\mathcal{D}'$ is realizable. A reduction argument shows that $\mathcal{D}$ is realizable as well.

- Otherwise, the inequality $v(\pi_1)+v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n-1$. Corollary 2.12 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d-2$. If $d = 4$ we can assume that $\pi_1 \neq [2,2]$, so that we can choose $[\alpha_1\alpha_2] \neq [2,2]$ (see proposition 2.11). Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 2) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + 1 + d - 2) - d + 1 \geq -\frac{1}{2}. \end{aligned}$$

Actually, since the total branching number of $\mathcal{D}$ is even, $g'$ must be an integer, therefore $g' \geq 0$. It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [1, d-1])$ is a candidate datum. Moreover, $\mathcal{D}'$ is realizable by induction: the case $d = 4$ was addressed above, and $\mathcal{D}' = (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-2])$ is impossible for $d \geq 6$, since

$$v(\alpha_1\alpha_2) \geq d - 2 > \frac{d}{2} = v([2, \ldots, 2]).$$

The usual reduction argument implies that $\mathcal{D}$ is realizable as well. □

**Proposition 2.16.** *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. Assume that $d \neq 4$ and that $2g \geq d - 1$. Then $\mathcal{D}$ is realizable.*

*Proof.* Note that the the condition $2g \geq d - 1$ is equivalent to

$$v(\pi_1) + \ldots + v(\pi_n) \geq 3d - 3$$

by the Riemann-Hurwitz [??] formula. Moreover, the cases where $d = 2$ are trivial; therefore, assume that $d \geq 3$.

We proceed by induction, starting from the base case $n = 3$. If $n = 3$, the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 3d - 3$ implies that $\pi_1 = \pi_2 = \pi_3 = [d]$; by proposition 2.14, $\mathcal{D}$ is realizable.

We now turn to the case $n \geq 4$; we will employ a reduction technique. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$. We can assume that $\pi_i \neq [d]$ for every $1 \leq i \leq n$, otherwise $\mathcal{D}$ is immediately realizable by proposition 2.14. There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n$. The standard reduction argument shows that $\mathcal{D}$ is realizable in this case.

- Otherwise, we have $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. We consider two sub-cases.

  – Assume first that there is a partition, say $\pi_1$, which is different from $[2, \ldots, 2]$. By corollary 2.10 and proposition 2.11, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] \in \{[d], [1, d-1]\}$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_n)) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0. \end{aligned}$$

  Consider the candidate datum $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$. By propositions 2.14 and 2.15, $\mathcal{D}'$ is realizable unless $\mathcal{D}' = (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$. If this were the case, we would have $n = 4$ and

$$v(\pi_1) + v(\pi_2) + v(\pi_3) + v(\pi_4) \leq d - 2 + d - 2 + \frac{d}{2} + \frac{d}{2} = 3d - 4,$$

which contradicts the hypothesis. Therefore $\mathcal{D}'$ is realizable; by the usual reduction argument, $\mathcal{D}$ is realizable as well.

– Finally, consider the case where $\pi_1 = \ldots = \pi_n = [2, \ldots, 2]$. In this situation $d = 2k$ is even and

$$v(\pi_1) + \ldots + v(\pi_n) = nk > 6k - 3$$

(the inequality is strict since the total branching number is even, while $6k - 3$ is odd). Since $k \geq 3$, this immediately implies that $n \geq 6$. Applying proposition 2.11 once and then corollary 2.10 or proposition 2.11 again yields permutations $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$, $[\alpha_2] = \pi_2$ and $[\alpha_3] = \pi_3$ such that $[\alpha_1\alpha_2\alpha_3] \in \{[d], [1, 2k - 1]\}$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2\alpha_3) + v(\pi_4) + \ldots + v(\pi_n)) - 2k + 1 \\ &\geq \frac{1}{2}(2k - 2 + (n - 3)k) - 2k + 1 = \frac{n - 5}{2} \cdot k > 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; 2k; [\alpha_1\alpha_2\alpha_3], \pi_4, \ldots, \pi_n)$ is a candidate datum. By propositions 2.14 and 2.15, $\mathcal{D}'$ is realizable; a reduction argument shows that $\mathcal{D}$ is realizable as well. □

**Corollary 2.17.** *Let $d$ be a positive integer with $d \neq 4$. Then there exist at most finitely many non-realizable proper candidate data of the form $(\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$.*

*Proof.* By proposition 2.16, every proper candidate datum with $n \geq 3d - 3$ is realizable, and there are a finite number of proper candidate data with $n < 3d - 3$. □

**Proposition 2.18.** *Let $\mathcal{D} = (\Sigma_g, S^2; 4; \pi_1, \ldots, \pi_n)$ be a proper candidate datum. Then $\mathcal{D}$ is realizable if and only if*

$$\mathcal{D} \neq (\Sigma_{n-3}, S^2; 4; [2, 2], \ldots, [2, 2], [1, 3]).$$

on-realizability will addressed at some int.

*Proof.* We proceed by induction, starting from the base case $n = 3$. Note that if any partition is either $[4]$ or $[1, 3]$, we immediately conclude by propositions 2.14 and 2.15. By the Riemann-Hurwitz [??] formula, we have the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 6$; therefore, the only candidate datum left to consider is $(S^2, S^2; 4; [2, 2], [2, 2], [2, 2])$. This datum is realizable, for instance, by choosing

$$\alpha_1 = (1, 2)(3, 4), \qquad \alpha_2 = (1, 3)(2, 4), \qquad \alpha_3 = (1, 4)(2, 3).$$

We now turn to the case $n \geq 4$; once again, we can assume that all the partitions are either $[1, 1, 2]$ or $[2, 2]$. There are three cases.

- If all the partitions are equal to $[2, 2]$, then we can apply proposition 2.11 to combine two partitions into a single $[2, 2]$, and conclude by a reduction argument.
- If all the partitions are equal to $[1, 1, 2]$, then by the Riemann-Hurwitz [??] formula we have $n \geq 6$. We can apply proposition 2.8 twice to combine three partitions into a single $[d]$, and conclude by reduction.
- Otherwise, there is at least one $[2, 2]$ and one $[1, 1, 2]$; using proposition 2.8, we can combine them into a single $[d]$ and conclude by reduction. □

## 2.8 Prime-degree conjecture

**Proposition 2.19.** *Let $d$ be a positive integer. Assume that every candidate datum $(\Sigma_g, S^2; d; \pi_1, \pi_2, \pi_3)$ is realizable. Then every candidate datum $(\Sigma_g, S^2; d; \pi_1, \dots, \pi_n)$ with $n \geq 3$ is realizable.*

*Proof.* We proceed by induction on $n \geq 4$. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \dots, \pi_n)$; there are two cases.

- If there are two indices $1 \leq i < j \leq n$ such that $v(\pi_i) + v(\pi_j) \leq d - 1$, then a routine reduction argument shows that $\mathcal{D}$ is realizable.

- Otherwise, $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. By corollary 2.12, we can find permutations $\alpha_1, \alpha_2$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_n)) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0. \end{aligned}$$

  It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \dots, \pi_n)$ is a candidate datum, so it is realizable by induction. A reduction argument shows that $\mathcal{D}$ is realizable as well. $\square$

# Chapter 3

# Exceptional data with a short partition

Always assume $n \geq 3$

.

## 3.1 Realizability on the sphere

**Theorem 3.1.** *Let $\mathcal{D} = (S^2; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

*(1)* $\mathcal{D} = (S^2; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

*(2)* $\mathcal{D} = (S^2; 2k; [2,\ldots,2], [2,\ldots,2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

*(3)* $\mathcal{D} = (S^2; 2k; [2,\ldots,2], [1,2,\ldots,2,3], [k,k])$ *with* $k \geq 2$.

*(4)* $\mathcal{D} = (S^2; 4k+2; [2,\ldots,2], [1,\ldots,1,k+1,k+2], [2k+1,2k+1])$ *with* $k \geq 1$.

*(5)* $\mathcal{D} = (S^2; 4k; [2,\ldots,2], [1,\ldots,1,k+1,k+1], [2k-1,2k+1])$ *with* $k \geq 2$.

*(6)* $\mathcal{D} = (S^2; kh; [h,\ldots,h], [1,\ldots,1,k+1], [lk,(h-l)k])$ *with* $h \geq 2$, $k \geq 2$, $1 \leq l < h$.

**Theorem 3.2.** *Let $\mathcal{D} = (S^2; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ be a proper candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable.*

## 3.2 Realizability on the torus for $n = 3$

**Theorem 3.3.** *Let $\mathcal{D} = (\Sigma_1; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

*(1)* $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$.

*(2)* $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$.

*(3)* $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

*(4)* $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$.

*(5)* $\mathcal{D} = (\Sigma_1; 2k; [2,\ldots,2], [2,\ldots,2,3,5], [k,k])$ *with* $k \geq 5$.

## 3.3 Realizability on higher genus surfaces for $n = 3$

**Theorem 3.4.** *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable.*

## 3.4 Realizability for $n \geq 4$

**Theorem 3.5.** *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ be a proper candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

*(1)* $\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$.

*(2)* $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

*Proof.* We will proceed by induction on $n$. We start with the base case $n = 4$, which requires the heaviest casework. Fix a proper candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3, [s, d-s])$.

- If $d \leq 16$, realizability can be checked by a computer. The only exceptional data are:
  (1) $\mathcal{D} = (\Sigma_1; 4; [2,2], [2,2], [2,2], [1,3])$;
  (2) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.
- If $g = 0$, then $\mathcal{D}$ is realizable by theorem 3.2.
- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq 3$; up to reindexing, we can assume that $v(\pi_1) + v(\pi_2) < d$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum

  $$\mathcal{D}' = (\Sigma_g; d; [\alpha_1\alpha_2], \pi_3, [s, d-s]).$$

  A standard reduction argument implies that $\mathcal{D}$ is realizable provided that $\mathcal{D}'$ is. If $g \geq 2$, then $\mathcal{D}'$ is realizable by theorem 3.4. If instead $g = 1$, then $\mathcal{D}'$ is realizable unless

  $$\mathcal{D}' = (\Sigma_1; 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k]) \text{ with } 2k = d.$$

  If this is the case, then $s = k$ and $\{[\alpha_1\alpha_2], \pi_3\} = \{[2, \ldots, 2], [2, \ldots, 2, 3, 5]\}$. Some more casework is required to show that $\mathcal{D}'$ can actually be chosen to be realizable.

  - If $[\alpha_1\alpha_2] = [2, \ldots, 2]$, then $v(\alpha_1\alpha_2) = k$ and $v(\pi_3) = k+2$. Assume without loss of generality that $v(\pi_1) \leq k/2$. We have that

    $$k + 2 < 1 + k + 2 \leq v(\pi_1) + v(\pi_3) \leq \frac{k}{2} + k + 2 < d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \ldots, 2]$ and $v(\pi_1) \notin \{2, k, k+1\}$ then $v(\pi_1) + v(\pi_3) < d$ and $v(\pi_1) + v(\pi_3) \notin \{k, k+2\}$. Therefore, repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \ldots, 2]$, $v(\pi_1) = 2$ and $\pi_2 \neq [2, \ldots, 2]$, then repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

- If $\pi_2 = \pi_3 = [2, \ldots, 2]$, we follow a different approach. By proposition 2.11, we can find permutations $\beta_2, \beta_3 \in \mathfrak{S}_d$ with $[\beta_2] = [\beta_3] = [2, \ldots, 2]$ such that $[\beta_2\beta_3] = [k, k]$. It is easy to see that $\mathcal{D}'' = (S^2; 2k; \pi_1, [k, k], [k, k])$ is a candidate datum, and it is realizable by theorem 3.1. A reduction argument implies that $\mathcal{D}$ is realizable as well.

Up to swapping $\pi_1$ and $\pi_2$, this analysis covers all the possible cases.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq 3$. In particular, up to reindexing, we can assume that $v(\pi_3) \geq d/2$. Corollary 2.12 gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + d - 2) - d + 1 \geq \frac{1}{2}\left(d - 2 + \frac{d}{2} + d - 2\right) - d + 1 = \frac{d}{4} - 1 \geq 2.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1, \alpha_2], \pi_3, [s, d - s])$ is candidate datum, and it is realizable by theorem 3.4. A reduction argument implies that $\mathcal{D}$ is realizable as well.

We now turn to the case $n \geq 5$; we will show by induction that every candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ different from $(\Sigma_{n-3}; 4; [2, 2], \ldots, [2, 2], [1, 3])$ is realizable. The case $d = 4$ is addressed by proposition 2.18. If $n = 5$ and $d = 8$, a computer-aided search shows that $\mathcal{D}$ is realizable. Otherwise, we once again employ a reduction argument.

- If the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n - 1$, it is now routine to show that $\mathcal{D}$ is realizable.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n - 1$. By corollary 2.12, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 2) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d + d - 2) - d + 1 \\ &= \frac{d}{2} - 1 \geq 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [s, d - s])$ is a candidate datum, and it is realizable by induction. The usual reduction argument implies that $\mathcal{D}$ is realizable as well. $\square$