1   Notation

Definition 1. Let d be a positive integer. A *partition* of d is an unordered multiset of positive integers $[x_1, \ldots, x_n]$ such that $x_1 + \ldots + x_n = d$.

Definition 2. A *combinatorial datum* is a tuple

$$\mathcal{D} = \left(\widetilde{\Sigma}, \Sigma, d; \pi_1, \ldots, \pi_n \right),$$

where Σ, $\widetilde{\Sigma}$ are closed surfaces, d is a positive integer and $\pi_1, \ldots, \pi_n$ are partitions of d. The datum $\mathcal{D}$ is *proper* if none of the partitions $\pi_1, \ldots, \pi_n$ is equal to $[1, \ldots, 1]$, *improper* otherwise.

A few remarks about combinatorial data.

- A combinatorial datum is only defined up to permutation of its partitions. In other words, if two data have the same partitions but in a different order, they are considered equal.

- When Σ is the 2-sphere, we will sometimes shorten the notation for a combinatorial datum by omitting it:
$$\mathcal{D} = \left(\widetilde{\Sigma}, d; \pi_1, \ldots, \pi_n \right).$$

Definition 3. A combinatorial datum

$$\mathcal{D} = \left(\widetilde{\Sigma}, \Sigma, d; \pi_1, \ldots, \pi_n \right)$$

is a *candidate datum* if it satisfies the *Riemann-Hurwitz* condition

$$\chi(\widetilde{\Sigma}) - (|\pi_1| + \ldots + |\pi_n|) = d \cdot (\chi(\Sigma) - n).$$

Let Σ_g denote the closed orientable surface of genus g. From now on, we will almost always deal with combinatorial data of the form

$$(\Sigma_g, d; \pi_1, \pi_2, [s, d - s]).$$

In this setting, the Riemann-Hurwitz condition becomes

$$|\pi_1| + |\pi_2| = d - 2g. \tag{1}$$

Definition 4. A candidate datum $\mathcal{D}$ is *realizable* if it is the combinatorial datum of a branched covering, and is *exceptional* otherwise.

2   Combinatorial moves

Definition 5. A *combinatorial move* is a pair of candidate data $\mathcal{D}, \mathcal{D}'$ such that $\mathcal{D}$ is realizable provided that $\mathcal{D}'$ is realizable. We use the notation $\mathcal{D} \implies \mathcal{D}'$ to denote the existence of a combinatorial move between $\mathcal{D}$ and $\mathcal{D}'$.

This section is devoted to listing several combinatorial moves, which will be used in the next section to classify the realizable candidate data of the form

$$\left(\Sigma_g, d; \pi_1, \pi_2, [s, d-s]\right).$$

We will make extensive use of the following.

Fact. *A candidate datum* $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, \pi_3)$ *is realizable if and only if it admits a* dessin d'enfant, *i.e. a graph* Γ *embedded in* Σ_g *with the following properties.*

(1) *The vertices of* Γ *are colored with two colors, say black and white; each edge of* Γ *has endpoints of different colors.*

(2) *There are* $|\pi_1|$ *black vertices and* $|\pi_2|$ *white vertices in* Γ; π_1 *is the multiset of the degrees of black vertices; similarly,* π_2 *is the multiset of the degrees of white vertices.*

(3) *The complement* $\Sigma_g \setminus \Gamma$ *is the disjoint union of* $|\pi_3|$ *open disks;* π_3 *is the multiset of the combinatorial perimeters[1] of these disks divided by 2.*

Before we start describing the combinatorial moves, let us introduce procedure that we will employ several times in this section. Consider an uncolored graph Γ embedded in a surface Σ, and fix an edge e with endpoints x, y. The operation of *joining* x and y along e consists in shrinking e to a single point; x and y are merged into a single vertex z, whose degree is $\deg z = \deg x + \deg y - 2$.

Joining x and y leaves the complement $\Sigma \setminus \Gamma$ topologically unchanged. From a combinatorial point of view, if $\Sigma \setminus \Gamma$ was a disjoint union of open disks, then the same holds after joining; the combinatorial perimeter of the two regions touching e decreases by 1 (if the two regions are the same, then it decreases by 2), while the perimeter of the other regions does not change.

We are now ready to introduce the combinatorial moves. Given a candidate datum $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [s, d-s])$ and a *dessin d'enfant* for $\mathcal{D}$, we will conventionally use the color black for the vertices assigned to the partition π_1, and the color white for the other vertices. Moreover, the disk with perimeter $2s$ will be colored orange, and the disk with perimeter $2(d-s)$ will be colored pink.

Combinatorial move 1. *Let* $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [1, d-1])$ *be a proper candidate datum. Assume that:*

(1) $g \geq 1$;

(2) *there is an element* $x \in \pi_1$ *such that* $x \geq 3$.

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[1] The *combinatorial perimeter* of a disk D in the complement $\Sigma_g \setminus \Gamma$ is the number of edges of Γ that make up the perimeter of D; edges that are traversed twice are counted accordingly.

Let x_1, x_2, x_3 be positive integers whose sum equals x, and consider the partitions

$$\pi_1' = \pi_1 \smallsetminus [x] \cup [x_1, x_2, x_3], \qquad\qquad \pi_2' = \pi_2.$$

Then

$$\mathcal{D}' = (\Sigma_{g-1}, d; \pi_1', \pi_2', [1, d-1])$$

is a proper candidate datum and $\mathcal{D} \implies \mathcal{D}'$.

Proof. Consider a *dessin d'enfant* Γ' realizing $\mathcal{D}'$. Since the orange disk has perimeter 2, every vertex of Γ' must touch the pink disk. We perform the following operations.

(1) Attach a tube to Σ_{g-1} with both endpoints in the pink disk (effectively increasing the genus by 1).

(2) Connect the black vertices of Γ' labeled x_1, x_2, x_3 to a dummy black vertex v as shown in the picture.

(3) Join x_1, x_2, x_3 and v along the drawn edges.
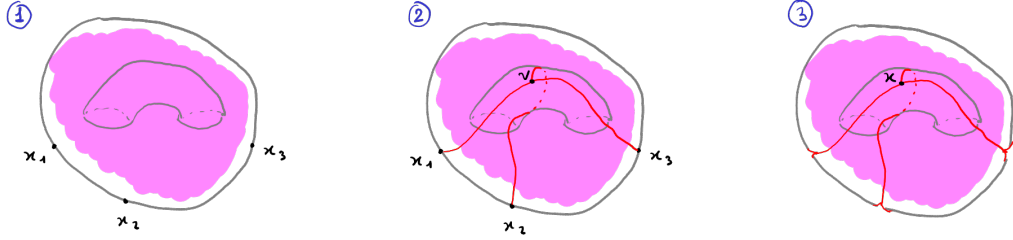
After these operations, we get a new *dessin d'enfant* Γ. It's easy to check that Γ realizes the datum $\mathcal{D}$. $\qquad\square$

Combinatorial move 2. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [2, d-2])$ be a proper candidate datum. Assume that:*

(1) $[x, y] \subseteq \pi_1$ for some $x \geq 2$, $y \geq 3$;

(2) $[2, 2] \subseteq \pi_2$.

Consider the partitions

$$\pi_1' = \pi_1 \smallsetminus [x, y] \cup [x + y - 4], \qquad\qquad \pi_2' = \pi_2 \smallsetminus [2, 2].$$

Then

$$\mathcal{D}' = (\Sigma_g, d - 4; \pi_1', \pi_2', [d - 4])$$

is a proper candidate datum and $\mathcal{D} \implies \mathcal{D}'$. In particular, $\mathcal{D}$ is always realizable.

Proof. Consider a *dessin d'enfant* Γ' realizing $\mathcal{D}'$. We perform the following operations.

(1) Take the black vertex of Γ' labeled $x + y - 4$ and split it into two vertices with degrees $x - 2$ and $y - 2$.

(2) Add two white vertices as shown in the picture.

① $x+y-4$ $\rightsquigarrow$ $x-2$ $y-2$ ② x 2 2 y

After these operations, we get a new *dessin d'enfant* Γ. It's easy to check that Γ realizes the datum $\mathcal{D}$. □

Combinatorial move 3. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [s, d - s])$ be a proper candidate datum. Assume that:*

(1) $g \geq 1$;

(2) $[1, 1, 3] \subseteq \pi_1$.

Consider the partitions

$$\pi_1' = \pi_1 \smallsetminus [3] \cup [1, 1, 1], \qquad\qquad \pi_2' = \pi_2.$$

Then

$$\mathcal{D}' = (\Sigma_{g-1}, d; \pi_1', \pi_2', [s, d - s])$$

is a proper candidate datum and $\mathcal{D} \implies \mathcal{D}'$.

Proof. Consider a *dessin d'enfant* Γ' realizing $\mathcal{D}'$. Since $[1, 1, 1, 1, 1] \subseteq \pi_1'$, we can assume without loss of generality that there are at least three black vertices labeled 1 touching the pink disk. Then we can follow the strategy presented in the proof of Combinatorial move 1 (with $x_1 = x_2 = x_3 = 1$) to obtain a new *dessin d'enfant* Γ which realizes $\mathcal{D}$. □

Combinatorial move 4. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [s, d - s])$ be a proper candidate datum. Assume that:*

(1) $2 \leq s \leq d - 2$;

(2) $g \geq 1$;

(3) there is an element $x \in \pi_1$ such that $x \geq 4$;

(4) $2 \in \pi_2$.

Let x_1, x_2 be positive integers whose sum equals $x - 2$, and consider the partitions

$$\pi_1' = \pi_1 \smallsetminus [x] \cup [x_1, x_2], \qquad\qquad \pi_2' = \pi_2 \smallsetminus [2].$$
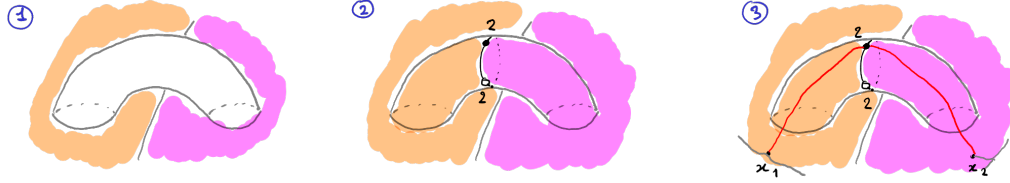
Then

$$\mathcal{D}' = (\Sigma_{g-1}, d - 2; \pi_1', \pi_2', [s - 1, d - s - 1])$$

is a proper candidate datum and $\mathcal{D} \implies \mathcal{D}'$.

Proof. Consider a *dessin d'enfant* Γ' realizing $\mathcal{D}'$. There are two cases.

Case 1: the vertex labeled x_1 touches the orange disk and the vertex labeled x_2 touches the pink disk (or vice versa). Then we perform the following operations.

(1) Attach a tube to Σ_{g-1} with one endpoint in the orange disk and the other in the pink one.

(2) Add one black vertex and one white vertex as shown in the picture.

(3) Perform the joining operation along the red edges.

$\square$

Combinatorial move 5. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [s, d-s])$ be a proper candidate datum. Assume that:*

(1) $2 \le s \le d - 2$;

(2) $g \ge 1$;

(3) there is an element $x \in \pi_1$ such that $x \ge 4$;

(4) there is an element $y \in \pi_2$ such that $x \ge 3$.

Consider the partitions

$$\pi_1' = \pi_1 \smallsetminus [x] \cup [x-2], \qquad\qquad \pi_2' = \pi_2 \smallsetminus [y] \cup [y-2].$$

Then

$$\mathcal{D}' = (\Sigma_{g-1}, d-2; \pi_1', \pi_2', [s-1, d-s-1])$$

is a proper candidate datum and $\mathcal{D} \implies \mathcal{D}'$.

Combinatorial move 6. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [s, d-s])$ be a proper candidate datum. Assume that:*

(1) $3 \le s \le d - 3$;

(2) $g \ge 1$;

(3) $[x, x] \subseteq \pi_1$ for some $x \ge 3$;

(4) $[2, 2] \subseteq \pi_2$.

Consider the partitions

$$\pi_1' = \pi_1 \smallsetminus [x, x] \cup [x-2, x-2], \qquad\qquad \pi_2' = \pi_2 \smallsetminus [2, 2].$$

Then

$$\mathcal{D}' = (\Sigma_{g-1}, d-4; \pi_1', \pi_2', [s-2, d-s-2])$$

is a proper candidate datum and $\mathcal{D} \implies \mathcal{D}'$.

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3 Exceptional data

In this section, we will give a complete list of the proper exceptional candidate data of the form

$$(\Sigma_g, d; \pi_1, \pi_2, [s, d - s])\,.$$

Theorem 1. *Let $\mathcal{D} = (\Sigma_0, d; \pi_1, \pi_2, [s, d - s])$ be a proper candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_0, 12; [2, 2, 2, 2, 2, 2], [1, 1, 1, 3, 3, 3], [6, 6])$.

(2) $\mathcal{D} = (\Sigma_0, 2k; [2, \ldots, 2], [2, \ldots, 2], [s, 2k - s])$ *with $k \geq 2$, $s \neq k$.*

(3) $\mathcal{D} = (\Sigma_0, 2k; [2, \ldots, 2], [1, 2, \ldots, 2, 3], [k, k])$ *with $k \geq 2$.*

(4) $\mathcal{D} = (\Sigma_0, 4k + 2; [2, \ldots, 2], [1, \ldots, 1, k + 1, k + 2], [2k + 1, 2k + 1])$ *with $k \geq 1$.*

(5) $\mathcal{D} = (\Sigma_0, 4k; [2, \ldots, 2], [1, \ldots, 1, k + 1, k + 1], [2k - 1, 2k + 1])$ *with $k \geq 2$.*

(6) $\mathcal{D} = (\Sigma_0, kh; [h, \ldots, h], [1, \ldots, 1, k + 1], [lk, (h - l)k])$ *with $h \geq 2$, $k \geq 2$, $1 \leq l < h$.*

Lemma 2. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [1, d - 1])$ be a proper candidate datum. Then $\mathcal{D}$ is realizable unless*

$$\mathcal{D} = (\Sigma_0, 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k - 1])$$

for some $k \geq 2$.

Proof. If $g = 0$ the claim is a corollary of Theorem 1. We then proceed by induction on $g \geq 1$. The Riemann-Hurwitz condition (1) immediately implies that one of the partitions, say π_1, contains an element $x \geq 3$. By applying Combinatorial move 1, we can reduce the problem to the realizability of the candidate datum

$$\mathcal{D}' = (\Sigma_{g-1}, d; \pi_1 \smallsetminus [x] \cup [1, 1, x - 2], \pi_2, [1, d - 1])\,,$$

which is realizable by induction. $\qquad\square$

Theorem 3. *Let $\mathcal{D} = (\Sigma_1, d; \pi_1, \pi_2, [s, d - s])$ be a proper candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_1, 6; [3, 3], [3, 3], [2, 4])$.

(2) $\mathcal{D} = (\Sigma_1, 8; [2, 2, 2, 2], [4, 4], [3, 5])$.

(3) $\mathcal{D} = (\Sigma_1, 12; [2, 2, 2, 2, 2, 2], [3, 3, 3, 3], [5, 7])$.

(4) $\mathcal{D} = (\Sigma_1, 16; [2, 2, 2, 2, 2, 2, 2, 2], [1, 3, 3, 3, 3, 3], [8, 8])$.

(5) $\mathcal{D} = (\Sigma_1, 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k])$ *with $k \geq 5$.*

Proof. Thanks to Lemma 2 we can assume that $2 \leq s \leq d - 2$. We analyze several cases.

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Case 0. Assume that $d \leq 14$. Using Zheng's algorithm, a list of all the proper exceptional candidate data has been computed. It turns out that $\mathcal{D}$ is realizable unless one of the following holds.

(1) $\mathcal{D} = (\Sigma_1, 6; [3,3], [3,3], [2,4])$.

(2) $\mathcal{D} = (\Sigma_1, 8; [2,2,2,2], [4,4], [3,5])$.

(3) $\mathcal{D} = (\Sigma_1, 10; [2,2,2,2,2], [2,3,5], [5,5])$.

(4) $\mathcal{D} = (\Sigma_1, 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

(5) $\mathcal{D} = (\Sigma_1, 12; [2,2,2,2,2,2], [2,2,3,5], [6,6])$.

(6) $\mathcal{D} = (\Sigma_1, 14; [2,2,2,2,2,2,2], [2,2,2,3,5], [7,7])$.

This is in agreement with the theorem statement.

Case 1. Assume that:

(1) there is an element $x \in \pi_1$ such that $x \geq 4$;

(2) $2 \in \pi_2$;

(3) $\pi_2 \neq [2, \ldots, 2]$.

By applying Combinatorial move 4, we can reduce the problem to the realizablility of the candidate datum
$$\mathcal{D}' = (\Sigma_0, d-2; \pi_1', \pi_2', [s-1, d-s-1]),$$
where $\pi_1' = \pi_1 \smallsetminus [x] \cup [1, x-3]$ and $\pi_2' = \pi_2 \smallsetminus [2]$. Notice that π_1' and π_2' are both different from $[2, \ldots, 2]$. Therefore, by Theorem 1, $\mathcal{D}'$ is realizable unless

$$\pi_1' = [1, \ldots, 1, k+1], \qquad\qquad \pi_2' = [h, \ldots, h]$$

for some $k \geq 2$, $h \geq 3$. If this is the case, then one of the following must hold.

- $\pi_1 = [1, \ldots, 1, k+4]$ and $\pi_2 = [2, h, \ldots, h]$. Then we can apply Combinatorial move 4 and reduce to
$$\mathcal{D} \implies (\Sigma_0, d-2; [1, \ldots, 1, 2, k], [h, \ldots, h], [s-1, d-s-1]),$$
which is realizable by Theorem 1.

- $\pi_1 = [1, \ldots, 1, 4, k+1]$ and $\pi_2 = [2, h, \ldots, h]$. Then we can apply Combinatorial move 5 and reduce to

$$\mathcal{D} \implies (\Sigma_0, d-2; [1, \ldots, 1, 2, k+1], [2, h-2, h, \ldots, h], [s-1, d-s-1]),$$

which is realizable by Theorem 1.

Case 2. Assume that:

(1) there is an element $x \in \pi_1$ such that $x \geq 4$;

(2) there is an element $y \in \pi_2$ such that $y \geq 4$.

We can further assume that $2 \notin \pi_1$, $2 \notin \pi_2$, otherwise we are done by case 1. By applying Combinatorial move 5, we can reduce the problem to the realizability of the candidate datum

$$\mathcal{D}' = (\Sigma_0, d - 2; \pi_1', \pi_2', [s - 1, d - s - 2]),$$

where $\pi_1' = \pi_1 \smallsetminus [x] \cup [x - 2]$ and $\pi_2' = \pi_2 \smallsetminus [y] \cup [y - 2]$. Notice that π_1' and π_2' are bot different from $[2, \ldots, 2]$ (unless one of them is equal to $[2]$, but then $\mathcal{D}'$ would be realizable anyway). Therefore, by Theorem 1, $\mathcal{D}'$ is realizable, unless, without loss of generality,

$$\pi_1' = [1, \ldots, 1, k + 1], \qquad\qquad \pi_2' = [h, \ldots, h]$$

for some $k \geq 2$, $h \geq 3$. If this is the case, then $\pi_1 = [1, \ldots, 1, k + 3]$ and $\pi_2 = [h, \ldots, h + 2]$, but then we can apply Combinatorial move 5 and reduce to

$$\mathcal{D} \implies (\Sigma_2, d - 2; [1, \ldots, 1, k + 1], [h - 2, h, \ldots, h, h + 2], [s - 1, d - s - 1]),$$

which is realizable by Theorem 1.

Case 3. Assume that:

(1) there is an element $x \in \pi_1$ such that $x \geq 4$;

(2) $3 \in \pi_2$.

We can further assume that $2 \notin \pi_2$, otherwise we are done by case 1, and that $\max(\pi_2) = 3$, otherwise we are done by case 2. By applying Combinatorial move 5, we can reduce the problem to the realizability of the candidate datum

$$\mathcal{D}' = (\Sigma_0, d - 2; \pi_1', \pi_2', [s - 1, d - s - 1]),$$

where $\pi_1' = \pi_1 \smallsetminus [x] \cup [x - 2]$ and $\pi_2' = \pi_2 \smallsetminus [3] \cup [1]$. By Theorem 1, $\mathcal{D}'$ is realizable unless one of the following holds.

- $\mathcal{D}' = (\Sigma_0, 12; [2, 2, 2, 2, 2, 2], [1, 1, 1, 3, 3, 3], [6, 6])$. In this case $d = 14$, so we are done by case 0.

- $\mathcal{D}' = (\Sigma_0, 4; [2, 2], [1, 3], [2, 2])$. In this case $d = 6$, so we are done by case 0.

- $\mathcal{D}' = (\Sigma_0, 8; [2, 2, 2, 2], [1, 1, 3, 3], [3, 5])$. In this case $d = 10$, so we are done by case 0.

- $\mathcal{D}' = (\Sigma_0, 2h; [h, h], [1, \ldots, 1, 3], [2l, 2(h - l)])$ for some $h \geq 2$, $1 \leq l < h$. In this case $\pi_1 = [h, h + 2]$ and $\pi_2 = [1, \ldots, 1, 3, 3]$. If $d \leq 7$ we are done by case 0. Otherwise $[1, 1, 3] \subseteq \pi_2$, therefore we can apply Combinatorial move 3 and reduce to

$$\mathcal{D} \implies (\Sigma_0, 2h + 2; [h, h + 2], [1, \ldots, 1, 3], [s, 2h + 2 - s]),$$

which is realizable by Theorem 1.

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Case 4. Assume that:

(1) $3 \in \pi_1$;

(2) $\pi_2 \neq [2, \ldots, 2]$.

We can further assume that $\max(\pi_1) = 3$ and $\max(\pi_2) \leq 3$, otherwise the situation is already covered by cases 1,2 and 3. We analyze different sub-cases.

Case 4.1: $[1, 1] \subseteq \pi_1$. By Combinatorial move 3 we have

$$\mathcal{D} \implies \left(\Sigma_0, d; \pi_1 \smallsetminus [3] \cup [1, 1, 1], \pi_2, [s, d - s] \right),$$

which is realizable by Theorem 1.

Case 4.2: $[3, 3] \subseteq \pi_1$ and $[2, 2] \subseteq \pi_2$. If $s = 2$ or $s = d - 2$ then $\mathcal{D}$ is realizable by Combinatorial move 2. Otherwise, by Combinatorial move 6 we have

$$\mathcal{D} \implies \left(\Sigma_0, d - 4; \pi_1 \smallsetminus [3, 3] \cup [1, 1], \pi_2 \smallsetminus [2, 2], [s - 2, d - s - 2] \right),$$

which is realizable by Theorem 1 unless $\pi_1 = [1, 3, 3, 3]$ and $\pi_2 = [2, 2, 3, 3]$. If this is the case, then we are done by case 0, since $d = 10$.

Case 4.3: $[2, 2] \not\subseteq \pi_2$. The Riemann-Hurwitz condition (1) immediately implies that $3 \in \pi_2$. We can assume that $d \geq 13$, otherwise we are done by case 0. Assume by contradiction that $\mathcal{D}$ is not realizable. Then $[1, 1] \not\subseteq \pi_2$ (case 4.1), but then $[3, 3] \subseteq \pi_2$. It follows (case 4.2) that $[2, 2] \not\subseteq \pi_1$; moreover, case 4.1 also implies that $[1, 1] \not\subseteq \pi_1$. In other words, both π_1 and π_2 can be written as $\rho \cup [3, \ldots, 3]$, where $\rho \subseteq [1, 2]$ (possibly different for π_1 and π_2). As a consequence, we have the inequalities

$$d \geq 3 \, |\pi_1| - 3, \qquad\qquad d \geq 3 \, |\pi_2| - 3,$$

which contradict the Riemann-Hurwitz condition (1) if $d \geq 13$.

Case 4.4: $[3, 3] \not\subseteq \pi_1$. From the Riemann-Hurwitz condition (1) it follows that $[3, 3] \subseteq \pi_2$, but then $\mathcal{D}$ is realizable by cases 4.2 and 4.3.

Case 5. Assume that:

(1) $\max(\pi_1) \geq 4$;

(2) $\pi_2 = [2, \ldots, 2]$.

Let $x = \max(\pi_1)$. By applying Combinatorial move 4, we can reduce the problem to the realizability of the candidate datum

$$\mathcal{D}' = \left(\Sigma_0, d - 2; \pi_1 \smallsetminus [x] \cup [1, x - 3], [2, \ldots, 2], [s - 1, d - s - 1] \right),$$

which is realizable by Theorem 1 unless one of the following holds.

- $\mathcal{D} = (\Sigma_1, 14; \pi_1, [2, \ldots, 2], [s, 14 - s])$. In this case $d = 14$, so we are done by case 0.

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- $\mathcal{D} = (\Sigma_1, 2k+2; [2, \ldots, 2, 3, 5], [2, \ldots, 2], [k+1, k+1])$ with $k \geq 2$. If $k = 2$ or $k = 3$ then we are done by case 0, otherwise this datum is in fact one of the exceptions listed in the statement of the theorem.

- $\mathcal{D} = (\Sigma_1, 2k+2; [2, \ldots, 2, 6], [2, \ldots, 2], [k+1, k+1])$ with $k \geq 2$. By Combinatorial move 4 we have
$$\mathcal{D} \implies (\Sigma_0, 2k; [2, \ldots, 2], [2, \ldots, 2], [k, k]),$$
which is realizable by Theorem 1.

- $\mathcal{D} = (\Sigma_1, 4k+4; [1, \ldots, 1, k+1, k+2, 4], [2, \ldots, 2], [2k+2, 2k+2])$ with $k \in \{1, 2\}$. In this case $d \leq 12$, so we are done by case 0.

- $\mathcal{D} = (\Sigma_1, 4k+4; [1, \ldots, 1, k+2, k+4], [2, \ldots, 2], [2k+2, 2k+2])$ with $k \geq 1$. If $k = 1$ then $d = 8$, so we are done by case 0. Otherwise, by Combinatorial move 4 we have
$$\mathcal{D} \implies (\Sigma_0, 4k+2; [1, \ldots, 1, 2, k, k+2], [2, \ldots, 2], [2k+1, 2k+1]),$$
which is realizable for $k \geq 2$ by Theorem 1.

- $\mathcal{D} = (\Sigma_1, 4k+4; [1, \ldots, 1, k+1, k+5], [2, \ldots, 2], [2k+2, 2k+2])$ with $k \geq 1$. By Combinatorial move 4 we have
$$\mathcal{D} \implies (\Sigma_0, 4k+2; [1, \ldots, 1, 2, k+1, k+1], [2, \ldots, 2], [2k+1, 2k+1]),$$
which is realizable by Theorem 1 (even if $k = 1$).

- $\mathcal{D} = (\Sigma_1, 4k+2; [1, \ldots, 1, k+1, k+1, 4], [2, \ldots, 2], [2k, 2k+2])$ with $k \in \{2, 3\}$. In this case $d \leq 14$, so we are done by case 0.

- $\mathcal{D} = (\Sigma_1, 4k+2; [1, \ldots, 1, k+1, k+4], [2, \ldots, 2], [2k, 2k+2])$ with $k \geq 2$. If $k = 2$ then $d = 10$, so we are done by case 0. Otherwise, by Combinatorial move 4 we have
$$\mathcal{D} \implies (\Sigma_0, 4k; [1, \ldots, 1, 2, k, k+1], [2, \ldots, 2], [2k-1, 2k+1]),$$
which is realizable for $k \geq 3$ by Theorem 1.

- $\mathcal{D} = (\Sigma_1, 2k+2; [1, \ldots, 1, k+1, 4], [2, \ldots, 2], [k+1, k+1])$ with $k \in \{2, 3\}$. In this case $d \leq 8$, so we are done by case 0.

- $\mathcal{D} = (\Sigma_1, 2k+2; [1, \ldots, 1, k+4], [2, \ldots, 2], [k+1, k+1])$ with $k \geq 2$. If $k = 2$ or $k = 3$ then $d \leq 8$, so we are done by case 0. Otherwise, by Combinatorial move 4 we have
$$\mathcal{D} \implies (\Sigma_0, 2k; [1, \ldots, 1, 2, k], [2, \ldots, 2], [k, k]),$$
which is realizable for $k \geq 4$ by Theorem 1.

Case 6. Assume that:

(1) $\max(\pi_1) = 3$;

(2) $\pi_2 = [2, \ldots, 2]$.

The Riemann-Hurwitz condition (1) immediately implies that $[3, 3, 3, 3] \subseteq \pi_1$. If $s = 2$ or $s = d - 2$ then $\mathcal{D}$ is realizable by Combinatorial move 2. Otherwise, by Combinatorial move 6 we have

$$\mathcal{D} \implies (\Sigma_0, d - 4; \pi_1 \smallsetminus [3, 3] \cup [1, 1], [2, \ldots, 2], [s - 2, d - s - 2]),$$

which is realizable by Theorem 1 unless one of the following holds.

- $\mathcal{D} = (\Sigma_1, 16; [1, 3, 3, 3, 3, 3], [2, 2, 2, 2, 2, 2, 2, 2], [8, 8])$. This is in fact one of the exceptions listed in the statement.

- $\mathcal{D} = (\Sigma_1, 12; [3, 3, 3, 3, 3], [2, 2, 2, 2, 2, 2], [5, 7])$. This is in fact one of the exceptions listed in the statement.

Conclusion. The cases we have analyzed (up to swapping π_1 and π_2) cover all the proper candidate data of the form $(\Sigma_1, d; \pi_1, \pi_2, [s, d - s])$. We have showed that every datum that is not listed in the statement is realizable, therefore the proof is complete. □

Lemma 4. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [s, d - s])$ be a proper candidate datum. Assume that:*

(1) $g \geq 2$;

(2) $\pi_1 = \rho_1 \cup [3, \ldots, 3]$ for some $\rho_1 \subseteq [1, 2]$.

(3) $\pi_2 = \rho_2 \cup [3, \ldots, 3]$ for some $\rho_2 \subseteq [1, 2]$.

Then $\mathcal{D}$ is realizable.

Theorem 5. *Let $\mathcal{D} = (\Sigma_g, d; \pi_1, \pi_2, [s, d - s])$ be a proper candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable.*

Proof. Thanks to Lemma 2 we can assume that $2 \leq s \leq d - 2$. We then proceed by induction on $g \geq 2$.

Case 0. Assume that $d \leq 18$. Using Zheng's algorithm, a list of all the proper exceptional candidate data has been computed. It turns out that $\mathcal{D}$ is always realizable.

Case 1. Assume that

(1) there is an element $x \in \pi_1$ such that $x \geq 4$;

(2) $2 \in \pi_2$.

By applying Combinatorial move 4, we can reduce the problem to the realizability of the candidate datum

$$\mathcal{D}' = (\Sigma_{g-1}, d - 2; \pi_1 \smallsetminus [x] \cup [1, x - 3], \pi_2 \smallsetminus [2], [s - 1, d - s - 1]),$$

which is realizable by Theorem 3 if $g = 2$ (by case 0 we can assume that $d - 2 \geq 17$), or by induction if $g \geq 3$.

Case 2. Assume that:

(1) there is an element $x \in \pi_1$ such that $x \geq 4$;

(2) there is an element $y \in \pi_2$ such that $y \geq 4$.

We can further assume that $2 \notin \pi_2$ and $2 \notin \pi_2$, otherwise we are done by case 1. By applying Combinatorial move 5, we can reduce the problem to the realizability of the candidate datum

$$\mathcal{D}' = \left(\Sigma_{g-1}, d - 2; \pi_1 \smallsetminus [x] \cup [x - 2], \pi_2 \smallsetminus [y] \cup [y - 2], [s - 1, d - s - 1]\right),$$

which is realizable by Theorem 3 if $g = 2$, or by induction if $g \geq 3$.

Case 3. Assume that:

(1) there is an element $x \in \pi_1$ such that $x \geq 4$;

(2) $3 \in \pi_2$.

By applying Combinatorial move 5, we can reduce the problem to the realizability of the candidate datum

$$\mathcal{D}' = \left(\Sigma_{g-1}, d - 2; \pi_1 \smallsetminus [x] \cup [x - 2], \pi_2 \smallsetminus [3] \cup [1], [s - 1, d - s - 1]\right),$$

which is realizable by Theorem 3 if $g = 2$ (by case 0 we can assume that $d - 2 \geq 17$), or by induction if $g \geq 3$.

Case 4. Assume that $3 \in \pi_1$. We can further assume that $\max(\pi_1) = 3$ and $\max(\pi_2) \leq 3$, otherwise the situation is already covered by cases 1,2 and 3. We analyze different sub-cases.

Case 4.1: $[1,1] \subseteq \pi_1$. By Combinatorial move 3 we have

$$\mathcal{D} \implies \left(\Sigma_{g-1}, d; \pi_1 \smallsetminus [3] \cup [1, 1, 1], \pi_2, [s, d - s]\right),$$

which is realizable by Theorem 3 if $g = 2$, or by induction if $g \geq 3$.

Case 4.2: $[3,3] \subseteq \pi_1$ and $[2,2] \subseteq \pi_2$. If $s = 2$ or $s = d-2$ then $\mathcal{D}$ is realizable by Combinatorial move 2. Otherwise, by Combinatorial move 6 we have

$$\mathcal{D} \implies \left(\Sigma_{g-1}, d - 4; \pi_1 \smallsetminus [3, 3] \cup [1, 1], \pi_2 \smallsetminus [2, 2], [s - 2, d - s - 2]\right),$$

which is realizable by Theorem 3 if $g = 2$, or by induction if $g \geq 3$.

Case 4.3: $[2,2] \not\subseteq \pi_2$. The Riemann-Hurwitz condition (1) immediately implies that $[3,3] \subseteq \pi_2$. Assume by contradiction that $\mathcal{D}$ is not realizable. Then $[1,1] \not\subseteq \pi_1$ and $[1,1] \not\subseteq \pi_2$ (case 4.1), and moreover $[2,2] \not\subseteq \pi_1$ (case 4.2). In other words, both π_1 and π_2 can be written as $\rho \cup [3, \ldots, 3]$, where $\rho \subseteq [1, 2]$ (possibly different for π_1 and π_2). As a consequence, by Lemma 4, $\mathcal{D}$ is realizable.

Case 4.4: $[3,3] \not\subseteq \pi_1$. From the Riemann-Hurwitz condition (1) it follows that $[3, 3] \subseteq \pi_2$, but then $\mathcal{D}$ is realizable by cases 4.2 and 4.3.

$\square$