Chapter 1

Hurwitz existence problem

1.1   Branched coverings of surfaces

According to standard terminology, a *surface* is simply a topological 2-manifold. We will, however, only be concerned with compact, connected surfaces without boundary. For the sake of conciseness, unless otherwise stated, we will always implicitly assume that the surfaces we mention have these properties.

There is a structure theorem which completely classifies orientable and non-orientable surfaces.

- An orientable surface is (homeomorphic to) a connected sum of $g \geq 0$ tori. We call such a connected sum a *surface of genus g*, and we denote it by Σ_g. By definition, we say that Σ_0 is the 2-sphere S^2; this is consistent with the formula $\chi(\Sigma_g) = 2 - 2g$ for the Euler characteristic.

- A non-orientable surface is (homeomorphic to) a connected sum of $g \geq 1$ real projective planes. We denote such a connected sum by N_g. The Euler characteristic for a non-orientable surfaces is given by $\chi(N_g) = 2 - g$.

Loosely speaking, given two surfaces $\Sigma, \widetilde{\Sigma}$, a *covering map* between them is a continuous function $f \colon \widetilde{\Sigma} \to \Sigma$ which is locally modeled on the identity function $\mathbb{R}^2 \to \mathbb{R}^2$. Sometimes, however, the notion of covering map can be too restrictive. Consider, for instance, the sphere S^2: being simply connected, it does not admit any non-trivial coverings. However, every (non-constant) homolorphic function $S^2 \to S^2$ is *almost* a covering map, in the sense that it is locally modeled on the identity $\mathbb{C} \to \mathbb{C}$, except for a finite number of *branching points*, where it looks like the map

$$F_k \colon \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto z^k$$

for some $k \geq 2$. In fact, it turns out that every (non-constant) holomorphic function between two complex 1-manifolds (which, topologically, are just oriented surfaces), has this remarkable property. This motivates the following definition.

Definition 1.1. Let $\Sigma, \widetilde{\Sigma}$ be two surfaces. A continuous function $f \colon \widetilde{\Sigma} \to \Sigma$ is a *branched covering map* (or simply a *branched covering*) if the following property holds: for every $x \in \Sigma$, $\widetilde{x} \in f^{-1}(x)$ there exist a positive integer k, open neighborhoods $U, \widetilde{U}$ of $x, \widetilde{x}$ respectively, and homeomorphisms $\varphi \colon U \to \mathbb{C}$, $\widetilde{\varphi} \colon \widetilde{U} \to \mathbb{C}$ such that $\varphi(x) = 0$, $\widetilde{\varphi}(\widetilde{x}) = 0$, $f(\widetilde{U}) = U$ and the

diagram

$$
\begin{array}{ccc}
\widetilde{U} & \xrightarrow{\ f\ } & U \\
\Big\downarrow{\scriptstyle \widetilde{\varphi}} & & \Big\downarrow{\scriptstyle \varphi} \\
\mathbb{C} & \xrightarrow{\ F_k\ } & \mathbb{C}
\end{array}
$$

commutes (F_k is the map defined above).

More informally, we can say that a branched covering is a continuous function between surfaces which is locally modeled on the complex map $z \mapsto z^k$, where $k \geq 1$ depends on the point. Note that, for each point $\widetilde{x} \in \widetilde{\Sigma}$, the integer k is well-defined, independently of the charts φ and $\widetilde{\varphi}$: using the notation from the definition, we have that k is equal to the cardinality of $f^{-1}(y) \cap \widetilde{U}$, where y is any point in $U \setminus \{x\}$. We call this integer the *branching degree* of $\widetilde{x}$; to emphasize the dependence on $\widetilde{x}$, we will sometimes denote it by $k(\widetilde{x})$.

A point $\widetilde{x} \in \widetilde{\Sigma}$ is called a *branching point* if $k(\widetilde{x}) > 1$; in other words, if f is *not* a local homeomorphism in a neighborhood of $\widetilde{x}$. We also say that a point $x \in \Sigma$ is a *branching point* if $f^{-1}(x)$ contains at least one branching point; usually, no ambiguity will arise as to which kind of branching point we are referring to.

It is not hard to see that branching points are quite rare. If $\widetilde{x} \in \widetilde{\Sigma}$ is a branching point, then (using again the notation from the definition) every other point in $\widetilde{U}$ is not a branching point; by compactness, it follows that the set of branching points in $\widetilde{\Sigma}$ is finite. As a consequence, the set of branching points in Σ is finite as well.

Given a branched covering $f \colon \widetilde{\Sigma} \to \Sigma$, we denote by $\Sigma^{\bullet}$ the subspace of Σ containing all the non-branching points; since the set of branching points is finite, $\Sigma^{\bullet}$ is a non-compact connected surface with finitely many punctures. We also set $\widetilde{\Sigma}^{\bullet} = f^{-1}(\Sigma^{\bullet})$, and we denote by $f^{\bullet}$ the restriction of f to $\widetilde{\Sigma}^{\bullet}$. By construction, $f^{\bullet} \colon \widetilde{\Sigma}^{\bullet} \to \Sigma^{\bullet}$ is a covering map and, as such, has a well-defined degree d (the number of preimages of an arbitrary point); we call this integer the *degree* of the branched covering f. The following proposition shows that the notion of degree extends nicely to branching points.

Proposition 1.1. *Let $f \colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d. Then for every point $x \in \Sigma$ we have that the set $f^{-1}(x)$ is finite and*

$$
\sum_{\widetilde{x} \in f^{-1}(x)} k(\widetilde{x}) = d.
$$

Proof. If x is not a branching point, the conclusion follows immediately, since x has exactly d preimages, all of which have branching degree k equal to 1. Assume now that x is a branching point; it is clear from the definition that the set $f^{-1}(x) \subseteq \widetilde{\Sigma}$ is discrete and, hence, finite. Let $f^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_r\}$. Fix open neighborhoods $\widetilde{U}_1, \ldots, \widetilde{U}_r$ of $\widetilde{x}_1, \ldots, \widetilde{x}_r$ respectively as in the definition; a routine compactness argument shows that there exists an open neighborhood U of x such that $f^{-1}(U) \subseteq \widetilde{U}_1 \cup \ldots \cup \widetilde{U}_r$. Fix a point $y \in U \setminus \{x\}$: it follows from the discussion above that y is not a branching point and that $\left| f^{-1}(y) \cup \widetilde{U}_i \right| = k(\widetilde{x}_i)$ for every $1 \leq i \leq r$. Since $\left| f^{-1}(y) \right| = d$, we immediately conclude that

$$
\sum_{i=1}^{r} k(\widetilde{x}_i) = \sum_{i=1}^{r} \left| f^{-1}(y) \cap \widetilde{U}_i \right| = \left| f^{-1}(y) \right| = d. \qquad \square
$$

1.2 Branching data

Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering. For each point $x \in \Sigma$, as we have just seen in proposition 1.1, the sum $k(\widetilde{x}_1) + \ldots k(\widetilde{x}_r)$ of the branching degrees of its preimages is equal to the degree d of the branched covering. Since there is no natural ordering on the set $f^{-1}(x)$, the appropriate combinatorial object for representing the collection $k(\widetilde{x}_1), \ldots, k(\widetilde{x}_r)$ is a *partition*.

Definition 1.2. Let d be a positive integer. A *partition* of d is an unordered finite multiset $\pi = [k_1, \ldots, k_r]$, where $k_i > 0$ is an integer for every $1 \le i \le r$ and $k_1 + \ldots + k_r = d$.

Given a positive integer d, we denote the set of all partitions of d by $\Pi(d)$. If $\pi = [k_1, \ldots, k_r]$ is a partition of d, we call the integer r the *length* (or *size*, or *cardinality*) of π, and we denote it by $\ell(\pi)$. We also introduce a new quantity, the *branching number* $v(\pi) = d - \ell(\pi)$, whose purpose will soon become apparent.

For every point $x \in \Sigma$, if $f^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_r\}$, we can define the *associated partition* $\pi(x) = [k(\widetilde{x}_1), \ldots, k(\widetilde{x}_r)] \in \Pi(d)$. For all non-branching points, the associated partition will simply be $[1, \ldots, 1]$. On the contrary, if x is a branching point, then $\pi(x) \ne [1, \ldots, 1]$ (or, equivalently, $v(\pi(x)) > 0$).

Definition 1.3. Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a branched covering of degree d. Let $x_1, \ldots, x_n \in \Sigma$ be the branching points of f. The *branching datum* of f is the tuple

$$\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi(x_1), \ldots, \pi(x_n)),$$

well-defined up to a permutation of the branching points $x_1, \ldots, x_n$.

Branching data are a way to extract some combinatorial information from branched coverings. Even though the exact location of the branching points (both in Σ and in $\widetilde{\Sigma}$) is not encoded in the branching datum, this piece of information is completely irrelevant, since surfaces are *homogeneous*[1]. However, the reader should not be induced to believe that the combinatorial information provided by the branching datum is enough to fully reconstruct the topology of f. In fact, it turns out that this is not the case. More on this later (?).

This entire thesis is devoted to the problem of determining what values can actually be attained by $\mathcal{D}(f)$ as f ranges over all the possible branched covering maps. We start with a very general definition.

Definition 1.4. A *combinatorial datum* is a tuple

$$\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n),$$

where $\Sigma, \widetilde{\Sigma}$ are surfaces, d is a positive integer, and $\pi_1, \ldots, \pi_n$ are partitions of d different from $[1, \ldots, 1]$.

Again, a combinatorial datum is defined up to a permutation of the partitions $\pi_1, \ldots, \pi_n$. In other words, we will consider two combinatorial data equal if they have the same partitions, irrespective of the ordering.

A first, naive guess could be that every combinatorial datum can be realized. However, it turns out that there are a few easy and general necessary conditions for a combinatorial datum to be associated to a branched covering. The first one, and arguably the most important, is known as the *Riemann-Hurwitz formula*; we state it in the next proposition.

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[1] By *homogeneous*, we mean that, for each $n \ge 1$, the group of homeomorphisms of a surface Σ acts transitively on the set of n-uples of points $(y_1, \ldots, y_n) \in \Sigma \times \ldots \times \Sigma$.

Proposition 1.2. *Let $f \colon \widetilde{\Sigma} \to \Sigma$ be a branched covering, and let $\mathcal{D}(f) = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be its branching datum. Then we have the equality*

$$d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = v(\pi_1) + \ldots + v(\pi_n).$$

Proof. Using the same notation as above, consider the covering map $f^{\bullet} \colon \widetilde{\Sigma}^{\bullet} \to \Sigma^{\bullet}$. Since $f^{\bullet}$ has degree d, the Euler characteristics of $\Sigma^{\bullet}$ and $\widetilde{\Sigma}^{\bullet}$ are related by the formula $d\chi(\Sigma^{\bullet}) = \chi(\widetilde{\Sigma}^{\bullet})$. Note that there are n branching points in Σ, and $\ell(\pi_1) + \ldots + \ell(\pi_n)$ branching points in $\widetilde{\Sigma}$; therefore

$$\chi(\Sigma^{\bullet}) = \chi(\Sigma) - n, \qquad\qquad \chi(\widetilde{\Sigma}^{\bullet}) = \chi(\widetilde{\Sigma}) - \ell(\pi_1) - \ldots - \ell(\pi_n).$$

As a consequence, we have that

$$d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = d\chi(\Sigma^{\bullet}) + dn - \chi(\widetilde{\Sigma}^{\bullet}) - \ell(\pi_1) - \ldots - \ell(\pi_n) = v(\pi_1) + \ldots + v(\pi_n). \qquad \square$$

There are three more conditions that every branching datum must satisfy: two of them concern the orientability of Σ and/or $\widetilde{\Sigma}$, and the other one provides an additional constraint for the *total branching number* $v(\pi_1) + \ldots + v(\pi_n)$; we group these four requirements in the following definition.

Definition 1.5. A combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ is a *candidate datum* if it satisfies the following conditions:

  (i) $d\chi(\Sigma) - \chi(\widetilde{\Sigma}) = v(\pi_1) + \ldots + v(\pi_n)$;

  (ii) $v(\pi_1) + \ldots + v(\pi_n)$ is even;

  (iii) if Σ is orientable, then $\widetilde{\Sigma}$ is orientable as well;

  (iv) if Σ is non-orientable and $\widetilde{\Sigma}$ is orientable, then d is even, and each π_i can be written as $\pi_i' \cup \pi_i''$, where π_i' and π_i'' are partitions of $d/2$.

We say that $\mathcal{D}$ is *realizable* if $\mathcal{D} = \mathcal{D}(f)$ for some branched covering $f \colon \widetilde{\Sigma} \to \Sigma$, *exceptional* otherwise.

Remark 1.1. In **??**, we will show that conditions (i)–(iv) are necessary for a combinatorial datum to be realizable (actually, the first condition was already proved in proposition 1.2). For now, let us focus on condition (iii), which is perhaps the most obvious one. If Σ is orientable, then so is $\Sigma^{\bullet}$, since removing a finite number of points does not affect orientability. The surface $\widetilde{\Sigma}^{\bullet}$, being a covering space of an orientable manifold, is itself orientable. Finally, this implies that $\widetilde{\Sigma}$ is orientable as well.

It would be natural to ask whether the necessary conditions we have enumerated are also sufficient for a candidate datum to be realizable.

Chapter 2

Realizablility by monodromy

2.1 Symmetric group and partitions

2.2 Branched covering action of the fundamental group

2.3 Monodromy and realizability

Proposition 2.1. *Let Σ_g be the connected sum of $g \geq 0$ tori, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n \in \mathfrak{S}_g$ such that:*

   (i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;

   (ii) $[\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g] \cdot \alpha_1 \cdots \alpha_n = 1$;

   (iii) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \ldots, d\}$.

In this case, $\widetilde{\Sigma}$ is necessarily orientable.

Remark 2.1. Given Σ_g, d and $\pi_1, \ldots, \pi_n$, there is at most one surface $\widetilde{\Sigma}$ such that $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum. In fact, the Riemann-Hurwitz [??] formula gives

$$\chi(\widetilde{\Sigma}) = d\chi(\Sigma_g) - v(\pi_1) - \ldots - v(\pi_n)$$

which, in turn, uniquely determines the orientable surface $\widetilde{\Sigma}$.

Proposition 2.2. *Let N_g be the connected sum of $g \geq 1$ projective planes, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \in \mathfrak{S}_d$ such that:*

   (i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;

   (ii) $\beta_1^2 \cdots \beta_g^2 \cdot \alpha_1 \cdots \alpha_n = 1$;

   (iii) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \ldots, d\}$.

In this case, $\widetilde{\Sigma}$ is orientable if and only if ??.

Remark 2.2. Given N_g, d and $\pi_1, \ldots, \pi_n$, there are at most two surfaces $\widetilde{\Sigma}$, one orientable and one non-orientable, such that $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum: like in remark 2.1, the Riemann-Hurwitz [??] formula fixes $\chi(\widetilde{\Sigma})$ which, together with orientability, uniquely determines $\widetilde{\Sigma}$.

Proposition 2.3. *Let* $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots \pi_n)$ *be a combinatorial datum, with* Σ *non-orientable and* $\widetilde{\Sigma}$ *orientable. Then* $\mathcal{D}$ *is realizable if and only if* d *is even and there exist partitions* $\pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n \in \Pi(d/2)$ *with* $\pi_i = \pi'_i \cup \pi''_i$ *for each* $1 \le i \le n$, *such that the combinatorial datum*

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_i, \ldots, \pi''_n)$$

is realizable, where $\widehat{\Sigma}$ *is the orientable double covering of* Σ.

2.4   Non-positive Euler characteristic

Lemma 2.4. *Let* $\alpha \in \mathfrak{S}_d$ *be a permutation. Set* $r = d - v(\alpha)$, *and let* $t \ge 0$ *be an integer such that* $2t \le v(\alpha)$. *Then* α *can be written as the product of a* $(r + 2t)$-*cycle and a* d-*cycle.*

Proof. Without loss of generality, assume that

$$\alpha = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{r-1} + 1, \ldots, d_r),$$

where $d_r = d$. Fix

$$\beta_0 = (1, b_1, b_2, \ldots, b_{2t}), \qquad\qquad \beta_1 = (1, d_1 + 1, d_2 + 2, \ldots, d_{r-1} + 1).$$

Set

$$\beta = \beta_0 \beta_1 = (1, d_1 + 1, d_2 + 1, \ldots, d_{r-1} + 1, b_1, b_2, \ldots, b_{2t}).$$

An easy computation shows that

$$\begin{aligned} \beta\alpha &= \beta_0 \beta_1 \alpha \\ &= (1, b_1, b_2, \ldots, b_{2t})(1, 2, \ldots, d) \\ &= (1, \ldots, b_1 - 1, b_2, \ldots, b_3 - 1, \ldots, b_4, \ldots, b_{2t-1} - 1, b_{2t}, \ldots, d, b_1, \ldots, b_2 - 1, b_3, \ldots, b_{2t} - 1). \end{aligned}$$

Writing $\alpha = \beta^{-1}(\beta\alpha)$ gives the desired decomposition. $\square$

Corollary 2.5. *Let* $\alpha \in \mathfrak{A}_d$ *be an even permutation. Then* α *can be written as:*

  (i) a commutator $[\beta, \gamma]$, *where* γ *is a* d-*cycle;*

  (ii) a product of two squares $\delta^2 \epsilon^2$, *where* $\delta\epsilon$ *is a* d-*cycle.*

Proof. Since α is an even permutation, its branching number $v(\alpha)$ is even. By lemma 2.4, there exist two d-cycles $\tau, \sigma \in \mathfrak{S}_d$ such that $\alpha = \tau\sigma$.

  (i) Since τ and σ^{-1} are conjugated, there exists a permutation $\beta \in \mathfrak{S}_d$ such that $\tau = \beta\sigma^{-1}\beta^{-1}$. Setting $\gamma = \sigma^{-1}$, we immediately get that

$$\alpha = \tau\sigma = \beta\sigma^{-1}\beta^{-1}\sigma = \beta\gamma\beta^{-1}\gamma^{-1} = [\beta, \gamma].$$

  (ii) Since τ and σ are conjugated, there exists a permutation $\delta \in \mathfrak{S}_d$ such that $\tau = \delta\sigma\delta^{-1}$. Setting $\epsilon = \delta^{-1}\sigma$, we have that

$$\alpha = \tau\sigma = \delta\sigma\delta^{-1}\sigma = \delta^2(\delta^{-1}\sigma)^2 = \delta^2\epsilon^2. \qquad\square$$

Theorem 2.6. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\chi(\Sigma) \leq 0$, then $\mathcal{D}$ is realizable.*

Proof. Let us first assume that Σ is orientable; this means that $\Sigma = \Sigma_g$ is the connected sum of $g \geq 1$ tori, and that $\widetilde{\Sigma}$ is orientable as well. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Since $\mathcal{D}$ is a candidate datum, we have that

$$v(\alpha) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod{2}.$$

By corollary 2.5, we can find permutations $\beta_1, \gamma_1 \in \mathfrak{S}_d$ such that $\alpha = [\gamma_1, \beta_1]$ and β_1 is a d-cycle. Set $\beta_2 = \ldots = \beta_g = \gamma_2 = \ldots = \gamma_g = \mathrm{id} \in \mathfrak{S}_d$. All the conditions of proposition 2.1 are satisfied; since $\widetilde{\Sigma}$ is orientable, this implies that $\mathcal{D}$ is realizable (see remark 2.1).

Assume now that Σ and $\widetilde{\Sigma}$ are both non-orientable; this means that $\Sigma = N_g$ is the connected sum of $g \geq 2$ projective planes. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Similarly to what we did for the previous case, we can find $\beta_1, \beta_2 \in \mathfrak{S}_d$ such that $\alpha = \beta_2^{-2}\beta_1^{-2}$ and $\beta_2\beta_1$ is a d-cycle. By setting $\beta_3 = \ldots = \beta_g = \mathrm{id} \in \mathfrak{S}_d$, proposition 2.2 (together with remark 2.2) implies the realizability of $\mathcal{D}$.

Finally, consider the case where Σ is non-orientable and $\widetilde{\Sigma}$ is orientable. Since $\mathcal{D}$ is a candidate datum, d is even and there exist partitions $\pi_1', \ldots, \pi_n', \pi_1'', \ldots, \pi_n'' \in \Pi(d/2)$ with $\pi_i = \pi_i' \cup \pi_i''$ for each $1 \leq i \leq n$. Let $\widehat{\Sigma}$ be the double orientable covering of Σ; by the first case we analyzed, the candidate datum

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi_1', \ldots, \pi_n', \pi_1'', \ldots, \pi_n'')$$

is realizable. By proposition 2.3, $\mathcal{D}$ is realizable as well. $\qquad \square$

2.5 Products in symmetric groups

Lemma 2.7. *Let X, Y be finite sets; denote by h the cardinality of Y, and by k the cardinality of $X \cap Y$. Let $\alpha \in \mathfrak{S}(X)$, $\beta \in \mathfrak{S}(Y)$, $\gamma \in \mathfrak{S}(X \cap Y)$. Assume that $\beta = (b_1, \ldots, b_h)$ is a h-cycle, and that γ is a k-cycle of the form $\gamma = (b_{i_1}, \ldots, b_{i_k})$ with $1 \leq i_1 \leq \ldots \leq i_k \leq h$. Then $\alpha \in \mathfrak{S}(X)$ and $\alpha\gamma^{-1}\beta \in \mathfrak{S}(X \cup Y)$ have the same number of cycles.*

Proof. Write $\gamma = (u_1, \ldots, u_k)$, where $u_j = b_{i_j}$. Without loss of generality, assume that $i_1 = 1$. Then we can write

$$\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{k-1}, u_k, \ldots, w_k),$$

possibly with $u_j = w_j$ for some values of j. We immediately get that

$$\gamma^{-1}\beta = (u_1, \ldots, w_1)(u_2, \ldots, w_2) \cdots (u_k, \ldots, w_k).$$

If we denote by $A_1, \ldots, A_r \subseteq X$ the orbits of α, it is then easy to see that the orbits of $\alpha\gamma^{-1}\beta$ are $A_1', \ldots, A_r' \subseteq X \cup Y$, where

$$A_j' = A_j \cup \bigcup_{u_l \in A_j} \{u_l, \ldots, w_l\}. \qquad \square$$

Proposition 2.8. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) < d$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = v(\alpha) + v(\beta)$.*

Proof. First of all, note that the conclusion is trivial whenever $v(\pi) = 0$ or $v(\rho) = 0$. This already solves the cases $d = 1$ and $d = 2$. We now proceed by induction on $d \geq 3$, assuming that $v(\pi) > 0$ and $v(\rho) > 0$. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$; without loss of generality, assume that $b_1 > 1$. Fix

$$\beta = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where $b_1 = d_1 \geq 2$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Note that

$$d - 1 \geq v(\pi) + v(\rho) = (a_1 - 1) + \ldots + (a_r - 1) + d - s \geq a_1 - 1 + d - s,$$

hence $a_1 \leq s$. Fix $\alpha_1 = (1, d_1 + 1, \ldots, d_{a_1-1} + 1)$, and let $A = \{1, d_1 + 1, \ldots, d_{a_1-1} + 1\}$ be the support of α_1. Define $Q = \{1, \ldots, d_{a_1}\} \setminus A$; note that Q_1 is non-empty, since $d_1 + 1 \geq 3$ implies that $2 \in Q_1$. Consider the partitions

$$\pi' = [a_2, a_3, \ldots, a_r], \qquad\qquad \rho' = [|Q|, b_{a_1+1}, \ldots, b_s].$$

We have that

$$\sum \pi' = \sum \rho' = d - a_1, \qquad v(\pi') = v(\pi) - a_1 + 1, \qquad v(\rho') = v(\rho) - 1.$$

Since $d - a_1 < d$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - a_1 < d - a_1$, by induction we find $\alpha', \beta' \in \mathfrak{S}(\{1, \ldots, d\} \setminus A)$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $v(\alpha'\beta') = v(\alpha') + v(\beta')$. Up to conjugation, we may assume that

$$\beta' = \beta_1(d_{a_1} + 1, \ldots, d_{a_1+1})(d_{a_1+1} + 1, \ldots, d_{a_1+2}) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where β_1 is the $|Q|$-cycle whose entries are the elements of Q in increasing order. An easy computation shows that

$$\alpha_1 \beta = (1, \ldots, d_{a_1})(d_{a_1} + 1, \ldots, d_{a_1+1}) \cdots (d_{s-1} + 1, \ldots, d_s).$$

Therefore, setting $\alpha = \alpha'\alpha_1$, we have that

$$
\begin{aligned}
\alpha\beta &= \alpha_1\alpha'\beta \\
&= \alpha'(1, \ldots, d_{a_1})(d_{a_1} + 1, \ldots, d_{a_1+1}) \cdots (d_{s-1} + 1, \ldots, d_s) \\
&= \alpha'\beta'\beta_1^{-1}(1, \ldots, d_{a_1})
\end{aligned}
$$

By lemma 2.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha'\beta'$, so that

$$
\begin{aligned}
v(\alpha\beta) &= a_1 + v(\alpha'\beta') \\
&= a_1 + v(\alpha') + v(\beta') \\
&= a_1 + (v(\pi) - a_1 + 1) + (v(\rho) - 1) \\
&= v(\pi) + v(\rho).
\end{aligned}
$$

Since $[\alpha] = \pi$ and $[\beta] = \rho$, the conclusion follows. $\square$

Proposition 2.9. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d$, and let $t = v(\pi) + v(\rho) - d + 1$. Fix an integer $0 \leq k \leq t$ such that $k \equiv t \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = d - 1 - k$ and the action of $\langle \alpha, \beta \rangle$ on $\{1, \ldots, d\}$ is transitive.*

aybe only $k = 0$
eded (in this case,
ansitivity is trivial)?

Proof. Write $\pi = [a_1, \ldots, a_r]$. Since $v(\rho) \le d - 1$ and $v(\pi) + v(\rho) \ge d$, there exists a largest integer $0 \le i \le r$ such that $(a_1 - 1) + \ldots (a_i - 1) + v(\rho) \le d - 1$. Define

$$z = d - v(\rho) - (a_1 - 1) - \ldots - (a_i - 1).$$

Consider the partition $\pi' = [a_1, \ldots, a_i, z, 1, \ldots, 1] \in \Pi(d)$. Since by construction $v(\pi') + v(\rho) = d - 1$, thanks to proposition 2.8 we can find permutations $\alpha', \beta \in \mathfrak{S}_d$ with $[\alpha'] = \pi'$ and $[\beta] = \rho$ such that $v(\alpha'\beta) = d - 1$; in other words, $\alpha'\beta$ is a d-cycle. Consider now the partition $\pi'' = [a_{i+1} - z + 1, a_{i+2}, \ldots, a_r]$, whose branching number is $v(\pi'') = t$. Let $n = \sum \pi''$; fix an element u_1 of the z-cycle of α', and let $u_2, \ldots, u_n$ be the fixed points of α' corresponding to the last ones of π' (it is easy to see that there are exactly $n - 1$ such ones). Since $k \le t = v(\rho'')$ and $k \equiv t \pmod 2$, lemma 2.4 gives permutations $\alpha'', \gamma \in \mathfrak{S}(\{u_1, \ldots, u_n\})$ such that $[\alpha''] = \rho''$, γ is a n-cycle and $\alpha''\gamma$ is a $(n - k)$-cycle. Up to conjugation, we can assume that $\gamma = (u_1, \ldots, u_n)$. Moreover, it is not restrictive to assume that $u_1, \ldots, u_n$ appear in this order in the d-cycle $\alpha'\beta$. Therefore, setting $\alpha = \alpha''\alpha'$, we have that

$$\alpha\beta = (\alpha''\gamma)\gamma^{-1}(\alpha'\beta);$$

by lemma 2.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha''\gamma \in \mathfrak{S}(\{u_1, \ldots, u_n\})$, that is $v(\alpha\beta) = d - (k + 1)$. Since $[\alpha] = \pi$ and $[\beta] = \rho$, the only thing left to show is that the action of $\langle \alpha, \beta \rangle$ on $\{1, \ldots, d\}$ is transitive. Write

$$\alpha'\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{n-1}, u_n, \ldots, w_n)$$

as in the proof of lemma 2.7, where an explicit description of the orbits of $\alpha\beta = \alpha''\gamma\gamma^{-1}\alpha'\beta$ is given; from that description, it is clear that for each $1 \le j \le n$ the elements $u_j, \ldots, w_j$ all belong to the same orbit. Moreover, since $\beta = (\alpha')^{-1}(u_1, \ldots, w_1, u_2, \ldots, w_n)$ and α' fixes $u_2, \ldots, u_n$, it follows that w_j and u_{j+1} belong to the same orbit for each $1 \le j \le n - 1$; this completes the proof. $\qquad \square$

Corollary 2.10. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \ge d - 1$ and $v(\pi) + v(\rho) \equiv d - 1 \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\alpha\beta$ is a d-cycle.*

Proof. The conclusion immediately follows from proposition 2.8 if $v(\pi) + v(\rho) = d - 1$, or from proposition 2.9 if $v(\pi) + v(\rho) \ge d$. $\qquad \square$

Remark 2.3. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$; assume that $b_1 \ge 2$. By directly examining the proof of proposition 2.8, we can see that the proposed construction yields permutations $\alpha, \beta \in \mathfrak{S}_d$ such that 1 belongs to the a_1-the cycle of α and to the b_1-cycle of β. It is not hard to see, once again by inspecting the proof, that the same can be said for proposition 2.9. As a consequence, the statement of corollary 2.10 can be enhanced by adding the following line: α *and* β *can be chosen in such a way that* 1 *belongs to the a_1-the cycle of α and to the b_1-cycle of β, provided that $b_1 \ge 2$.* We will need this improvement for the upcoming proof.

Proposition 2.11. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \ge d$ and $v(\pi) + v(\rho) \equiv d \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \ldots, d\}$ and*

$$[\alpha\beta] = \begin{cases} [d/2, d/2] & \text{if } \pi = \rho = [2, \ldots, 2], \\ [1, d - 1] & \text{otherwise.} \end{cases}$$

Proof. Assume first that $\pi = \rho = [2, \ldots, d]$. We can choose

$$\alpha = (2, 3)(4, 5) \cdots (d, 1), \qquad\qquad \beta = (1, 2)(3, 4) \cdots (d - 1, d).$$

The action of $\langle \alpha, \beta \rangle$ is obviously transitive, and

$$\alpha\beta = (1, 3, \ldots, d - 1)(2, 4, \ldots, 2).$$

Otherwise, since $v(\pi) + v(\rho) \geq d$, at least one of π and ρ has an entry which is greater than 2; without loss of generality, we can assume it is ρ. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$ with $a_1 \geq 2$ (since $v(\pi) \geq 1$) and $b_1 \geq 3$. Fix

$$\beta = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where $b_1 = d_1 \geq 3$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Consider the partitions

$$\pi' = [a_1 - 1, \ldots, a_r], \qquad\qquad \rho' = [b_1 - 1, \ldots, b_s].$$

Since $\sum \pi' = \sum \rho' = d - 1$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - 2$, by corollary 2.10 we can find permutations $\alpha', \beta' \in \mathfrak{S}(\{2, \ldots, d\})$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $\alpha'\beta'$ is a $(d - 1)$-cycle. Up to conjugation, we can assume that

$$\beta' = (2, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{s-1} + 1, \ldots, d_s)$$

or, in other words, $\beta = (1, 2)\beta'$; moreover, as explained in remark 2.3, we can choose α' in such a way that its $(a_1 - 1)$-cycle contains 2. By setting $\alpha = \alpha'(1, 2)$, we immediately get that $\alpha\beta = \alpha'\beta'$ is a $(d - 1)$-cycle fixing 1. Finally, the action of $\langle \alpha, \beta \rangle$ is transitive since α does not fix 1. $\qquad\square$

Corollary 2.12. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d - 1$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$, $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \ldots, d\}$ and*

$$[\alpha\beta] \in \{[d], [1, d - 1], [d/2, d/2]\}.$$

Proof. The conclusion follows immediately from proposition 2.9 or proposition 2.11 depending on the parity of $v(\pi) + v(\rho) + d$. $\qquad\square$

2.6 Projective plane

Theorem 2.13. *Let $\mathcal{D} = (\widetilde{\Sigma}, \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\widetilde{\Sigma}$ is non-orientable, then $\mathcal{D}$ is realizable.*

Proof. First of all, since $\mathcal{D}$ is a candidate datum, the Riemann-Hurwitz [??] formula implies that

$$v(\pi_1) + \ldots + v(\pi_n) = d\chi(\mathbb{RP}^2) - \chi(\widetilde{\Sigma}) \geq d - 1$$

(recall that $\widetilde{\Sigma}$ is non-orientable, so $\chi(\widetilde{\Sigma}) \leq 1$). Moreover, the total branching $v(\pi_1) + \ldots + v(\pi_n)$ is even. In order to apply proposition 2.2, we will now inductively define representatives $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$, satisfying the following invariant: for every $0 \leq i \leq n$, either

$$v(\alpha_1 \cdots \alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$$

or

$$[\alpha_1 \cdots \alpha_i] \in \{[d], [1, d - 1], [d/2, d/2]\} \text{ and } \langle \alpha_1, \ldots, \alpha_i \rangle \text{ acts transitively.}$$

Assume we have already defined $\alpha_1, \ldots, \alpha_{i-1}$; we want to suitably choose α_i. Let $\alpha = \alpha_1 \cdots \alpha_{i-1}$; there are two cases.

- If $v(\alpha) + v(\pi_i) < d$, by proposition 2.8 we can find $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $v(\alpha\alpha_i) = v(\alpha) + v(\alpha_i)$. The invariant is still satisfied: if $v(\alpha) = v(\pi_1) + \ldots + v(\pi_{i-1})$ then obviously $v(\alpha\alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$. If instead $[\alpha] \in \{[d], [1, d-1], [d/2, d/2]\}$, then either α_i is the identity, or $\alpha\alpha_i$ is a d-cycle; either way, $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$. Note that if the action of $\langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ is transitive, then the action of $\langle \alpha_1, \ldots, \alpha_i \rangle$ is transitive as well.

- If $v(\alpha) + v(\alpha_i) \geq d$, corollary 2.12 gives a permutation $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$ and the action of $\langle \alpha, \alpha_i \rangle$ is transitive. The invariant is obviously satisfied.

By induction, we can find $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that

$$[\alpha_1 \cdots \alpha_n] \in \{[d], [1, d-1], [d/2, d/2]\}$$

and $\langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$ (note that $v(\alpha_1 \cdots \alpha_n) = v(\alpha_1) + \ldots + v(\alpha_n)$ also implies that $[\alpha_1 \cdots \alpha_n] = [d]$). Note that

$$v(\alpha_1 \cdots \alpha_n) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

> Show that $v(\alpha\beta) \equiv v(\alpha) + v(\beta) \pmod 2$.

We now prove that $\alpha = \alpha_1 \cdots \alpha_n$ is a square.

- If $[\alpha] = [d]$, then d is odd, so α is the square of $\alpha^{(d+1)/2}$.

- If $[\alpha] = [1, d-2]$, then d is even, so α is the square of $\alpha^{d/2}$.

- If $[\alpha] = [d/2, d/2]$, then d is even, and it is easy to see that α is the square of a d-cycle.

By proposition 2.2, this implies that there exists a realizable candidate datum $\mathcal{D}' = (\widetilde{\Sigma}', \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$. To see that $\widetilde{\Sigma}'$ is non-orientable (and, therefore, equal to $\widetilde{\Sigma}$, as shown in remark 2.2), simply note that ??. $\qquad\square$

2.7   Reduction technique on the sphere

> Assume $n \geq 3$.

Proposition 2.14. *Let* $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$ *be a candidate datum. Then* $\mathcal{D}$ *is realizable.*

Proof. We proceed by induction on n, starting with the base case $n = 3$. If $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \pi_2, [d])$ is a candidate datum, by the Riemann-Hurwitz [??] formula we have that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d-1) \geq d - 1.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 1$ is even. Corollary 2.10 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1 \alpha_2] = [d]$. By proposition 2.1, $\mathcal{D}$ is realizable (see remark 2.1).

We now turn to the case $n \geq 4$. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$; there are two cases.

- Assume that $v(\pi_1) + v(\pi_2) \leq d - 1$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [d])$. By induction, $\mathcal{D}'$ is realizable; proposition 2.1 then gives permutations $\alpha_3, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for $3 \leq i \leq n - 1$ and $[\alpha_n] = [d]$ such that $(\alpha_1\alpha_2)\alpha_3 \cdots \alpha_n = 1$. It is easy to see that the permutations $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ imply the realizability of $\mathcal{D}$, again by proposition 2.1 (together, as usual, with remark 2.1).

- Otherwise, we have that $v(\pi_1) + v(\pi_2) \geq d$. By corollary 2.12, we can find permutations α_1, α_2 with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1 \alpha_2) \geq d - 2$. Let

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 1) - d + 1$$
$$\geq \frac{1}{2}(d - 2 + 1 + d - 1) - d + 1 \geq 0.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$ is a candidate datum, so it is realizable by induction. Similarly to the previous case, this implies that $\mathcal{D}$ is realizable. $\square$

Proposition 2.15. *Let* $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [1, d-1])$ *be a candidate datum. Then* $\mathcal{D}$ *is non-realizable if and only if it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3])$.

(2) $\mathcal{D} = (S^2, S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k-1])$ *with* $k \geq 2$.

Proof. We proceed by induction on n, starting from the base case $n = 3$. Let

$$\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \pi_2, [1, d-1]) \neq (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$$

be a candidate datum. The Riemann-Hurwitz [??] formula implies that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d - 2) \geq d.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 2$ is even. Proposition 2.11 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1 \alpha_2] = [1, d-1]$ and the action of $\langle \alpha_1, \alpha_2 \rangle$ is transitive. As usual, proposition 2.1 implies that $\mathcal{D}$ is realizable.

We now turn to the case $n \geq 4$; we will employ a reduction technique. Fix a candidate datum

$$\mathcal{D} = (\Sigma_{n-3}, S^2; d; \pi_1, \ldots, \pi_{n-1}, [1, d-1]) \neq (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3]).$$

There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n-1$; up to reindexing, we can assume that $i = 1$ and $j = 2$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1 \alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [1, d-1])$. By induction, $\mathcal{D}'$ is realizable unless one of the following happens.

  - $\mathcal{D}' = (S^2, S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k-1])$ with $2k = d$. If this is the case, then $n = 4$ and $v(\pi_1) + v(\pi_2) = v(\pi_3) = k$. This implies that

    $$k < 1 + k \leq v(\pi_1) + v(\pi_3) < 2k = d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - $\mathcal{D}' = (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3])$. This implies that $\pi_1 = \pi_2 = [1, 1, 2]$ and $\pi_3 = \ldots = \pi_{n-1} = [2, 2]$. Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

  Therefore we can assume that $\mathcal{D}'$ is realizable. A reduction argument shows that $\mathcal{D}$ is realizable as well.

- Otherwise, the inequality $v(\pi_1) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n-1$. Corollary 2.12 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d-2$. If $d = 4$ we can assume that $\pi_1 \neq [2,2]$, so that we can choose $[\alpha_1\alpha_2] \neq [2,2]$ (see proposition 2.11). Let

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 2) - d + 1$$

$$\geq \frac{1}{2}(d - 2 + 1 + d - 2) - d + 1 \geq -\frac{1}{2}.$$

Actually, since the total branching number of $\mathcal{D}$ is even, g' must be an integer, therefore $g' \geq 0$. It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [1, d-1])$ is a candidate datum. Moreover, $\mathcal{D}'$ is realizable by induction: the case $d = 4$ was addressed above, and $\mathcal{D}' = (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-2])$ is impossible for $d \geq 6$, since

$$v(\alpha_1\alpha_2) \geq d - 2 > \frac{d}{2} = v([2, \ldots, 2]).$$

The usual reduction argument implies that $\mathcal{D}$ is realizable as well. $\qquad\square$

Proposition 2.16. *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. Assume that $d \neq 4$ and that $2g \geq d - 1$. Then $\mathcal{D}$ is realizable.*

Proof. Note that the the condition $2g \geq d - 1$ is equivalent to

$$v(\pi_1) + \ldots + v(\pi_n) \geq 3d - 3$$

by the Riemann-Hurwitz [??] formula. Moreover, the cases where $d = 2$ are trivial; therefore, assume that $d \geq 3$.

We proceed by induction, starting from the base case $n = 3$. If $n = 3$, the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 3d - 3$ implies that $\pi_1 = \pi_2 = \pi_3 = [d]$; by proposition 2.14, $\mathcal{D}$ is realizable.

We now turn to the case $n \geq 4$; we will employ a reduction technique. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$. We can assume that $\pi_i \neq [d]$ for every $1 \leq i \leq n$, otherwise $\mathcal{D}$ is immediately realizable by proposition 2.14. There are two cases.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n$. The standard reduction argument shows that $\mathcal{D}$ is realizable in this case.

- Otherwise, we have $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. We consider two sub-cases.

  - Assume first that there is a partition, say π_1, which is different from $[2, \ldots, 2]$. By corollary 2.10 and proposition 2.11, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] \in \{[d], [1, d-1]\}$. Let

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_n)) - d + 1$$

$$\geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0.$$

Consider the candidate datum $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$. By propositions 2.14 and 2.15, $\mathcal{D}'$ is realizable unless $\mathcal{D}' = (S^2, S^2; d; [2, \ldots, 2], [2, \ldots, 2], [1, d-1])$. If this were the case, we would have $n = 4$ and

$$v(\pi_1) + v(\pi_2) + v(\pi_3) + v(\pi_4) \leq d - 2 + d - 2 + \frac{d}{2} + \frac{d}{2} = 3d - 4,$$

which contradicts the hypothesis. Therefore $\mathcal{D}'$ is realizable; by the usual reduction argument, $\mathcal{D}$ is realizable as well.

– Finally, consider the case where $\pi_1 = \ldots = \pi_n = [2, \ldots, 2]$. In this situation $d = 2k$ is even and

$$v(\pi_1) + \ldots + v(\pi_n) = nk > 6k - 3$$

(the inequality is strict since the total branching number is even, while $6k - 3$ is odd). Since $k \geq 3$, this immediately implies that $n \geq 6$. Applying proposition 2.11 once and then corollary 2.10 or proposition 2.11 again yields permutations $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$, $[\alpha_2] = \pi_2$ and $[\alpha_3] = \pi_3$ such that $[\alpha_1 \alpha_2 \alpha_3] \in \{[d], [1, 2k - 1]\}$. Let

$$g' = \frac{1}{2}(v(\alpha_1 \alpha_2 \alpha_3) + v(\pi_4) + \ldots + v(\pi_n)) - 2k + 1$$
$$\geq \frac{1}{2}(2k - 2 + (n - 3)k) - 2k + 1 = \frac{n - 5}{2} \cdot k > 0.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; 2k; [\alpha_1 \alpha_2 \alpha_3], \pi_4, \ldots, \pi_n)$ is a candidate datum. By propositions 2.14 and 2.15, $\mathcal{D}'$ is realizable; a reduction argument shows that $\mathcal{D}$ is realizable as well. □

Corollary 2.17. *Let d be a positive integer with $d \neq 4$. Then there exist at most finitely many non-realizable candidate data of the form $(\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$.*

Proof. By proposition 2.16, every candidate datum with $n \geq 3d - 3$ is realizable, and there are a finite number of candidate data with $n < 3d - 3$. □

Proposition 2.18. *Let $\mathcal{D} = (\Sigma_g, S^2; 4; \pi_1, \ldots, \pi_n)$ be a candidate datum. Then $\mathcal{D}$ is realizable if and only if*

$$\mathcal{D} \neq (\Sigma_{n-3}, S^2; 4; [2, 2], \ldots, [2, 2], [1, 3]).$$

Proof. We proceed by induction, starting from the base case $n = 3$. Note that if any partition is either $[4]$ or $[1, 3]$, we immediately conclude by propositions 2.14 and 2.15. By the Riemann-Hurwitz [??] formula, we have the inequality $v(\pi_1) + v(\pi_2) + v(\pi_3) \geq 6$; therefore, the only candidate datum left to consider is $(S^2, S^2; 4; [2, 2], [2, 2], [2, 2])$. This datum is realizable, for instance, by choosing

$$\alpha_1 = (1, 2)(3, 4), \qquad \alpha_2 = (1, 3)(2, 4), \qquad \alpha_3 = (1, 4)(2, 3).$$

We now turn to the case $n \geq 4$; once again, we can assume that all the partitions are either $[1, 1, 2]$ or $[2, 2]$. There are three cases.

• If all the partitions are equal to $[2, 2]$, then we can apply proposition 2.11 to combine two partitions into a single $[2, 2]$, and conclude by a reduction argument.

• If all the partitions are equal to $[1, 1, 2]$, then by the Riemann-Hurwitz [??] formula we have $n \geq 6$. We can apply proposition 2.8 twice to combine three partitions into a single $[d]$, and conclude by reduction.

• Otherwise, there is at least one $[2, 2]$ and one $[1, 1, 2]$; using proposition 2.8, we can combine them into a single $[d]$ and conclude by reduction. □

2.8 Prime-degree conjecture

Proposition 2.19. *Let d be a positive integer. Assume that every candidate datum $(\Sigma_g, S^2; d; \pi_1, \pi_2, \pi_3)$ is realizable. Then every candidate datum $(\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$ with $n \geq 3$ is realizable.*

Proof. We proceed by induction on $n \geq 4$. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$; there are two cases.

- If there are two indices $1 \leq i < j \leq n$ such that $v(\pi_i) + v(\pi_j) \leq d - 1$, then a routine reduction argument shows that $\mathcal{D}$ is realizable.

- Otherwise, $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. By corollary 2.12, we can find permutations α_1, α_2 with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1 \alpha_2) \geq d - 2$. Let

$$
\begin{aligned}
g' &= \frac{1}{2}(v(\alpha_1 \alpha_2) + v(\pi_3) + \ldots + v(\pi_n)) - d + 1 \\
&\geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0.
\end{aligned}
$$

  It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1 \alpha_2], \pi_3, \ldots, \pi_n)$ is a candidate datum, so it is realizable by induction. A reduction argument shows that $\mathcal{D}$ is realizable as well. $\qquad\square$

Chapter 3

Exceptional data with a short partition

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3.1 Realizability on the sphere

Theorem 3.1. *Let $\mathcal{D} = (S^2; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (S^2; 12; [2, 2, 2, 2, 2, 2], [1, 1, 1, 3, 3, 3], [6, 6])$.

(2) $\mathcal{D} = (S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [s, 2k-s])$ with $k \geq 2$, $s \neq k$.

(3) $\mathcal{D} = (S^2; 2k; [2, \ldots, 2], [1, 2, \ldots, 2, 3], [k, k])$ with $k \geq 2$.

(4) $\mathcal{D} = (S^2; 4k+2; [2, \ldots, 2], [1, \ldots, 1, k+1, k+2], [2k+1, 2k+1])$ with $k \geq 1$.

(5) $\mathcal{D} = (S^2; 4k; [2, \ldots, 2], [1, \ldots, 1, k+1, k+1], [2k-1, 2k+1])$ with $k \geq 2$.

(6) $\mathcal{D} = (S^2; kh; [h, \ldots, h], [1, \ldots, 1, k+1], [lk, (h-l)k])$ with $h \geq 2$, $k \geq 2$, $1 \leq l < h$.

Theorem 3.2. *Let $\mathcal{D} = (S^2; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable.*

3.2 Realizability on the torus for $n = 3$

Theorem 3.3. *Let $\mathcal{D} = (\Sigma_1; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_1; 6; [3, 3], [3, 3], [2, 4])$.

(2) $\mathcal{D} = (\Sigma_1; 8; [2, 2, 2, 2], [4, 4], [3, 5])$.

(3) $\mathcal{D} = (\Sigma_1; 12; [2, 2, 2, 2, 2, 2], [3, 3, 3, 3], [5, 7])$.

(4) $\mathcal{D} = (\Sigma_1; 16; [2, 2, 2, 2, 2, 2, 2, 2], [1, 3, 3, 3, 3, 3], [8, 8])$.

(5) $\mathcal{D} = (\Sigma_1; 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k])$ with $k \geq 5$.

3.3   Realizability on higher genus surfaces for $n = 3$

Theorem 3.4. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d - s])$ be a candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable.*

3.4   Realizability for $n \geq 4$

Theorem 3.5. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d - s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_{n-3}; 4; [2, 2], \ldots, [2, 2], [1, 3])$.

(2) $\mathcal{D} = (\Sigma_2; 8; [2, 2, 2, 2], [2, 2, 2, 2], [2, 2, 2, 2], [3, 5])$.

Proof. We will proceed by induction on n. We start with the base case $n = 4$, which requires the heaviest casework. Fix a candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3, [s, d - s])$.

- If $d \leq 16$, realizability can be checked by a computer. The only exceptional data are:

  (1) $\mathcal{D} = (\Sigma_1; 4; [2, 2], [2, 2], [2, 2], [1, 3])$;

  (2) $\mathcal{D} = (\Sigma_2; 8; [2, 2, 2, 2], [2, 2, 2, 2], [2, 2, 2, 2], [3, 5])$.

- If $g = 0$, then $\mathcal{D}$ is realizable by theorem 3.2.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq 3$; up to reindexing, we can assume that $v(\pi_1) + v(\pi_2) < d$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1 \alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum

$$\mathcal{D}' = (\Sigma_g; d; [\alpha_1 \alpha_2], \pi_3, [s, d - s]).$$

A standard reduction argument implies that $\mathcal{D}$ is realizable provided that $\mathcal{D}'$ is. If $g \geq 2$, then $\mathcal{D}'$ is realizable by theorem 3.4. If instead $g = 1$, then $\mathcal{D}'$ is realizable unless

$$\mathcal{D}' = (\Sigma_1; 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k]) \text{ with } 2k = d.$$

If this is the case, then $s = k$ and $\{[\alpha_1 \alpha_2], \pi_3\} = \{[2, \ldots, 2], [2, \ldots, 2, 3, 5]\}$. Some more casework is required to show that $\mathcal{D}'$ can actually be chosen to be realizable.

  - If $[\alpha_1 \alpha_2] = [2, \ldots, 2]$, then $v(\alpha_1 \alpha_2) = k$ and $v(\pi_3) = k + 2$. Assume without loss of generality that $v(\pi_1) \leq k/2$. We have that

$$k + 2 < 1 + k + 2 \leq v(\pi_1) + v(\pi_3) \leq \frac{k}{2} + k + 2 < d.$$

  Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

  - If $\pi_3 = [2, \ldots, 2]$ and $v(\pi_1) \notin \{2, k, k + 1\}$ then $v(\pi_1) + v(\pi_3) < d$ and $v(\pi_1) + v(\pi_3) \notin \{k, k + 2\}$. Therefore, repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

  - If $\pi_3 = [2, \ldots, 2]$, $v(\pi_1) = 2$ and $\pi_2 \neq [2, \ldots, 2]$, then repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

– If $\pi_2 = \pi_3 = [2, \ldots, 2]$, we follow a different approach. By proposition 2.11, we can find permutations $\beta_2, \beta_3 \in \mathfrak{S}_d$ with $[\beta_2] = [\beta_3] = [2, \ldots, 2]$ such that $[\beta_2 \beta_3] = [k, k]$. It is easy to see that $\mathcal{D}'' = (S^2; 2k; \pi_1, [k, k], [k, k])$ is a candidate datum, and it is realizable by theorem 3.1. A reduction argument implies that $\mathcal{D}$ is realizable as well.

Up to swapping π_1 and π_2, this analysis covers all the possible cases.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq 3$. In particular, up to reindexing, we can assume that $v(\pi_3) \geq d/2$. Corollary 2.12 gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1 \alpha_2) \geq d - 2$. Let

$$g' = \frac{1}{2}(v(\alpha_1 \alpha_2) + v(\pi_3) + d - 2) - d + 1 \geq \frac{1}{2}\left(d - 2 + \frac{d}{2} + d - 2\right) - d + 1 = \frac{d}{4} - 1 \geq 2.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1, \alpha_2], \pi_3, [s, d - s])$ is candidate datum, and it is realizable by theorem 3.4. A reduction argument implies that $\mathcal{D}$ is realizable as well.

We now turn to the case $n \geq 5$; we will show by induction that every candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d - s])$ different from $(\Sigma_{n-3}; 4; [2, 2], \ldots, [2, 2], [1, 3])$ is realizable. The case $d = 4$ is addressed by proposition 2.18. If $n = 5$ and $d = 8$, a computer-aided search shows that $\mathcal{D}$ is realizable. Otherwise, we once again employ a reduction argument.

- If the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n - 1$, it is now routine to show that $\mathcal{D}$ is realizable.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n - 1$. By corollary 2.12, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1 \alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1 \alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 2) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d + d - 2) - d + 1 \\ &= \frac{d}{2} - 1 \geq 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1 \alpha_2], \pi_3, \ldots, \pi_{n-1}, [s, d - s])$ is a candidate datum, and it is realizable by induction. The usual reduction argument implies that $\mathcal{D}$ is realizable as well. $\qquad \square$