

Chapter 1

Monodromy

1.1 Symmetric group and partitions

1.2 Branched covering action of the fundamental group

1.3 Monodromy and realizability

Proposition 1.1. *Let Σ_g be the connected sum of $g \geq 0$ tori, $d \geq 1$ an integer, $\pi_1, \dots, \pi_n \in \Pi(d)$ partitions of d . Then there exists a realizable combinatorial datum $\mathcal{D} = (\tilde{\Sigma}, \Sigma_g; d; \pi_1, \dots, \pi_n)$ if and only if there exist permutations $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n \in \mathfrak{S}_d$ such that:*

- (i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;
- (ii) $[\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g] \cdot \alpha_1 \cdots \alpha_n = 1$;
- (iii) the subgroup $\langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \dots, d\}$.

In this case, $\tilde{\Sigma}$ is necessarily orientable.

Remark 1.1. Given Σ_g , d and π_1, \dots, π_n , there is at most one surface $\tilde{\Sigma}$ such that $\mathcal{D} = (\tilde{\Sigma}, \Sigma_g; d; \pi_1, \dots, \pi_n)$ is a candidate datum. In fact, the ?? formula gives

$$\chi(\tilde{\Sigma}) = d\chi(\Sigma_g) - v(\pi_1) - \dots - v(\pi_n)$$

which, in turn, uniquely determines the orientable surface $\tilde{\Sigma}$.

Proposition 1.2. *Let N_g be the connected sum of $g \geq 1$ projective planes, $d \geq 1$ an integer, $\pi_1, \dots, \pi_n \in \Pi(d)$ partitions of d . Then there exists a realizable combinatorial datum $\mathcal{D} = (\tilde{\Sigma}, N_g; d; \pi_1, \dots, \pi_n)$ if and only if there exist permutations $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_g \in \mathfrak{S}_d$ such that:*

- (i) $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$;
- (ii) $\beta_1^2 \cdots \beta_g^2 \cdot \alpha_1 \cdots \alpha_n = 1$;
- (iii) the subgroup $\langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_g \rangle \leq \mathfrak{S}_d$ acts transitively on $\{1, \dots, d\}$.

In this case, $\tilde{\Sigma}$ is orientable if and only if ??.

Remark 1.2. Given N_g , d and π_1, \dots, π_n , there are at most two surfaces $\tilde{\Sigma}$, one orientable and one non-orientable, such that $\mathcal{D} = (\tilde{\Sigma}, N_g; d; \pi_1, \dots, \pi_n)$ is a candidate datum: like in remark 1.1, the ?? formula fixes $\chi(\tilde{\Sigma})$ which, together with orientability, uniquely determines $\tilde{\Sigma}$.

Proposition 1.3. *Let $\mathcal{D} = (\tilde{\Sigma}, \Sigma; d; \pi_1, \dots, \pi_n)$ be a combinatorial datum, with Σ non-orientable and $\tilde{\Sigma}$ orientable. Then \mathcal{D} is realizable if and only if d is even and there exist partitions $\pi'_1, \dots, \pi'_n, \pi''_1, \dots, \pi''_n \in \Pi(d/2)$ with $\pi_i = \pi'_i \cup \pi''_i$ for each $1 \leq i \leq n$, such that the combinatorial datum*

$$\mathcal{D}' = (\tilde{\Sigma}, \hat{\Sigma}; d/2; \pi'_1, \dots, \pi'_n, \pi''_1, \dots, \pi''_n)$$

is realizable, where $\hat{\Sigma}$ is the orientable double covering of Σ .

1.4 Non-positive Euler characteristic

Lemma 1.4. *Let $\alpha \in \mathfrak{S}_d$ be a permutation. Set $r = d - v(\alpha)$, and let $t \geq 0$ be an integer such that $2t \leq v(\alpha)$. Then α can be written as the product of a $(r + 2t)$ -cycle and a d -cycle.*

Proof. Without loss of generality, assume that

$$\alpha = (1, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{r-1} + 1, \dots, d_r),$$

where $d_r = d$. Fix

$$\beta_0 = (1, b_1, b_2, \dots, b_{2t}), \quad \beta_1 = (1, d_1 + 1, d_2 + 2, \dots, d_{r-1} + 1).$$

Set

$$\beta = \beta_0 \beta_1 = (1, d_1 + 1, d_2 + 1, \dots, d_{r-1} + 1, b_1, b_2, \dots, b_{2t}).$$

An easy computation shows that

$$\begin{aligned} \beta \alpha &= \beta_0 \beta_1 \alpha \\ &= (1, b_1, b_2, \dots, b_{2t})(1, 2, \dots, d) \\ &= (1, \dots, b_1 - 1, b_2, \dots, b_3 - 1, \dots, b_4, \dots, b_{2t-1} - 1, b_{2t}, \dots, d, b_1, \dots, b_2 - 1, b_3, \dots, b_{2t} - 1). \end{aligned}$$

Writing $\alpha = \beta^{-1}(\beta \alpha)$ gives the desired decomposition. \square

Corollary 1.5. *Let $\alpha \in \mathfrak{A}_d$ be an even permutation. Then α can be written as:*

- (i) *a commutator $[\beta, \gamma]$, where γ is a d -cycle;*
- (ii) *a product of two squares $\delta^2 \epsilon^2$, where $\delta \epsilon$ is a d -cycle.*

Proof. Since α is an even permutation, its branching number $v(\alpha)$ is even. By lemma 1.4, there exist two d -cycles $\tau, \sigma \in \mathfrak{S}_d$ such that $\alpha = \tau \sigma$.

- (i) Since τ and σ^{-1} are conjugated, there exists a permutation $\beta \in \mathfrak{S}_d$ such that $\tau = \beta \sigma^{-1} \beta^{-1}$. Setting $\gamma = \sigma^{-1}$, we immediately get that

$$\alpha = \tau \sigma = \beta \sigma^{-1} \beta^{-1} \sigma = \beta \gamma \beta^{-1} \gamma^{-1} = [\beta, \gamma].$$

- (ii) Since τ and σ are conjugated, there exists a permutation $\delta \in \mathfrak{S}_d$ such that $\tau = \delta \sigma \delta^{-1}$. Setting $\epsilon = \delta^{-1} \sigma$, we have that

$$\alpha = \tau \sigma = \delta \sigma \delta^{-1} \sigma = \delta^2 (\delta^{-1} \sigma)^2 = \delta^2 \epsilon^2. \quad \square$$

Theorem 1.6. *Let $\mathcal{D} = (\tilde{\Sigma}, \Sigma; d; \pi_1, \dots, \pi_n)$ be a candidate datum. If $\chi(\Sigma) \leq 0$, then \mathcal{D} is realizable.*

Proof. Let us first assume that Σ is orientable; this means that $\Sigma = \Sigma_g$ is the connected sum of $g \geq 1$ tori, and that $\tilde{\Sigma}$ is orientable as well. Choose permutations $\alpha_1, \dots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Since \mathcal{D} is a candidate datum, we have that

$$v(\alpha) \equiv v(\alpha_1) + \dots + v(\alpha_n) \equiv 0 \pmod{2}.$$

By corollary 1.5, we can find permutations $\beta_1, \gamma_1 \in \mathfrak{S}_d$ such that $\alpha = [\gamma_1, \beta_1]$ and β_1 is a d -cycle. Set $\beta_2 = \dots = \beta_g = \gamma_2 = \dots = \gamma_g = \text{id} \in \mathfrak{S}_d$. All the conditions of proposition 1.1 are satisfied; since $\tilde{\Sigma}$ is orientable, this implies that \mathcal{D} is realizable (see remark 1.1).

Assume now that Σ and $\tilde{\Sigma}$ are both non-orientable; this means that $\Sigma = N_g$ is the connected sum of $g \geq 2$ projective planes. Choose permutations $\alpha_1, \dots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Similarly to what we did for the previous case, we can find $\beta_1, \beta_2 \in \mathfrak{S}_d$ such that $\alpha = \beta_2^{-2} \beta_1^{-2}$ and $\beta_2 \beta_1$ is a d -cycle. By setting $\beta_3 = \dots = \beta_g = \text{id} \in \mathfrak{S}_d$, proposition 1.2 (together with remark 1.2) implies the realizability of \mathcal{D} .

Finally, consider the case where Σ is non-orientable and $\tilde{\Sigma}$ is orientable. Since \mathcal{D} is a candidate datum, d is even and there exist partitions $\pi'_1, \dots, \pi'_n, \pi''_1, \dots, \pi''_n \in \Pi(d/2)$ with $\pi_i = \pi'_1 \cup \pi''_i$ for each $1 \leq i \leq n$. Let $\hat{\Sigma}$ be the double orientable covering of Σ ; by the first case we analyzed, the candidate datum

$$\mathcal{D}' = (\tilde{\Sigma}, \hat{\Sigma}; d/2; \pi'_1, \dots, \pi'_n, \pi''_1, \dots, \pi''_n)$$

is realizable. By proposition 1.3, \mathcal{D} is realizable as well. \square

1.5 Products in symmetric groups

Lemma 1.7. *Let X, Y be finite sets; denote by h the cardinality of Y , and by k the cardinality of $X \cap Y$. Let $\alpha \in \mathfrak{S}(X)$, $\beta \in \mathfrak{S}(Y)$, $\gamma \in \mathfrak{S}(X \cap Y)$. Assume that $\beta = (b_1, \dots, b_h)$ is a h -cycle, and that γ is a k -cycle of the form $\gamma = (b_{i_1}, \dots, b_{i_k})$ with $1 \leq i_1 \leq \dots \leq i_k \leq h$. Then $\alpha \in \mathfrak{S}(X)$ and $\alpha\gamma^{-1}\beta \in \mathfrak{S}(X \cup Y)$ have the same number of cycles.*

Proof. Write $\gamma = (u_1, \dots, u_k)$, where $u_j = b_{i_j}$. Without loss of generality, assume that $i_1 = 1$. Then we can write

$$\beta = (u_1, \dots, w_1, u_2, \dots, w_2, u_3, \dots, w_{k-1}, u_k, \dots, w_k),$$

possibly with $u_j = w_j$ for some values of j . We immediately get that

$$\gamma^{-1}\beta = (u_1, \dots, w_1)(u_2, \dots, w_2) \cdots (u_k, \dots, w_k).$$

If we denote by $A_1, \dots, A_r \subseteq X$ the orbits of α , it is then easy to see that the orbits of $\alpha\gamma^{-1}\beta$ are $A'_1, \dots, A'_r \subseteq X \cup Y$, where

$$A'_j = A_j \cup \bigcup_{u_l \in A_j} \{u_l, \dots, w_l\}. \quad \square$$

Proposition 1.8. *Let $\pi, \rho \in \Pi(d)$ be partitions of d . Assume that $v(\pi) + v(\rho) < d$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = v(\alpha) + v(\beta)$.*

Proof. First of all, note that the conclusion is trivial whenever $v(\pi) = 0$ or $v(\rho) = 0$. This already solves the cases $d = 1$ and $d = 2$. We now proceed by induction on $d \geq 3$, assuming that $v(\pi) > 0$ and $v(\rho) > 0$. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$; without loss of generality, assume that $b_1 > 1$. Fix

$$\beta = (1, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{s-1} + 1, \dots, d_s),$$

where $b_1 = d_1 \geq 2$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Note that

$$d - 1 \geq v(\pi) + v(\rho) = (a_1 - 1) + \dots + (a_r - 1) + d - s \geq a_1 - 1 + d - s,$$

hence $a_1 \leq s$. Fix $\alpha_1 = (1, d_1 + 1, \dots, d_{a_1-1} + 1)$, and let $A = \{1, d_1 + 1, \dots, d_{a_1-1} + 1\}$ be the support of α_1 . Define $Q = \{1, \dots, d_{a_1}\} \setminus A$; note that Q_1 is non-empty, since $d_1 + 1 \geq 3$ implies that $2 \in Q_1$. Consider the partitions

$$\pi' = [a_2, a_3, \dots, a_r], \quad \rho' = [|Q|, b_{a_1+1}, \dots, b_s].$$

We have that

$$\sum \pi' = \sum \rho' = d - a_1, \quad v(\pi') = v(\pi) - a_1 + 1, \quad v(\rho') = v(\rho) - 1.$$

Since $d - a_1 < d$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - a_1 < d - a_1$, by induction we find $\alpha', \beta' \in \mathfrak{S}(\{1, \dots, d\} \setminus A)$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $v(\alpha'\beta') = v(\alpha') + v(\beta')$. Up to conjugation, we may assume that

$$\beta' = \beta_1(d_{a_1} + 1, \dots, d_{a_1+1})(d_{a_1+1} + 1, \dots, d_{a_1+2}) \cdots (d_{s-1} + 1, \dots, d_s),$$

where β_1 is the $|Q|$ -cycle whose entries are the elements of Q in increasing order. An easy computation shows that

$$\alpha_1\beta = (1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s).$$

Therefore, setting $\alpha = \alpha'\alpha_1$, we have that

$$\begin{aligned} \alpha\beta &= \alpha_1\alpha'\beta \\ &= \alpha'(1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s) \\ &= \alpha'\beta'\beta_1^{-1}(1, \dots, d_{a_1}) \end{aligned}$$

By lemma 1.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha'\beta'$, so that

$$\begin{aligned} v(\alpha\beta) &= a_1 + v(\alpha'\beta') \\ &= a_1 + v(\alpha') + v(\beta') \\ &= a_1 + (v(\pi) - a_1 + 1) + (v(\rho) - 1) \\ &= v(\pi) + v(\rho). \end{aligned}$$

Since $[\alpha] = \pi$ and $[\beta] = \rho$, the conclusion follows. \square

Proposition 1.9. *Let $\pi, \rho \in \Pi(d)$ be partitions of d . Assume that $v(\pi) + v(\rho) \geq d$, and let $t = v(\pi) + v(\rho) - d + 1$. Fix an integer $0 \leq k \leq t$ such that $k \equiv t \pmod{2}$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = d - 1 - k$ and the action of $\langle \alpha, \beta \rangle$ on $\{1, \dots, d\}$ is transitive.*

Maybe only $k = 0$ is needed (in this case, transitivity is trivial)?

Proof. Write $\pi = [a_1, \dots, a_r]$. Since $v(\rho) \leq d - 1$ and $v(\pi) + v(\rho) \geq d$, there exists a largest integer $0 \leq i \leq r$ such that $(a_1 - 1) + \dots + (a_i - 1) + v(\rho) \leq d - 1$. Define

$$z = d - v(\rho) - (a_1 - 1) - \dots - (a_i - 1).$$

Consider the partition $\pi' = [a_1, \dots, a_i, z, 1, \dots, 1] \in \Pi(d)$. Since by construction $v(\pi') + v(\rho) = d - 1$, thanks to proposition 1.8 we can find permutations $\alpha', \beta \in \mathfrak{S}_d$ with $[\alpha'] = \pi'$ and $[\beta] = \rho$ such that $v(\alpha'\beta) = d - 1$; in other words, $\alpha'\beta$ is a d -cycle. Consider now the partition $\pi'' = [a_{i+1} - z + 1, a_{i+2}, \dots, a_r]$, whose branching number is $v(\pi'') = t$. Let $n = \sum \pi''$; fix an element u_1 of the z -cycle of α' , and let u_2, \dots, u_n be the fixed points of α' corresponding to the last ones of π' (it is easy to see that there are exactly $n - 1$ such ones). Since $k \leq t = v(\rho'')$ and $k \equiv t \pmod{2}$, lemma 1.4 gives permutations $\alpha'', \gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$ such that $[\alpha''] = \rho''$, γ is a n -cycle and $\alpha''\gamma$ is a $(n - k)$ -cycle. Up to conjugation, we can assume that $\gamma = (u_1, \dots, u_n)$. Moreover, it is not restrictive to assume that u_1, \dots, u_n appear in this order in the d -cycle $\alpha'\beta$. Therefore, setting $\alpha = \alpha''\alpha'$, we have that

$$\alpha\beta = (\alpha''\gamma)\gamma^{-1}(\alpha'\beta);$$

by lemma 1.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha''\gamma \in \mathfrak{S}(\{u_1, \dots, u_n\})$, that is $v(\alpha\beta) = d - (k + 1)$. Since $[\alpha] = \pi$ and $[\beta] = \rho$, the only thing left to show is that the action of $\langle \alpha, \beta \rangle$ on $\{1, \dots, d\}$ is transitive. Write

$$\alpha'\beta = (u_1, \dots, w_1, u_2, \dots, w_2, u_3, \dots, w_{n-1}, u_n, \dots, w_n)$$

as in the proof of lemma 1.7, where an explicit description of the orbits of $\alpha\beta = \alpha''\gamma\gamma^{-1}\alpha'\beta$ is given; from that description, it is clear that for each $1 \leq j \leq n$ the elements u_j, \dots, w_j all belong to the same orbit. Moreover, since $\beta = (\alpha')^{-1}(u_1, \dots, w_1, u_2, \dots, w_n)$ and α' fixes u_2, \dots, u_n , it follows that w_j and u_{j+1} belong to the same orbit for each $1 \leq j \leq n - 1$; this completes the proof. \square

Corollary 1.10. *Let $\pi, \rho \in \Pi(d)$ be partitions of d . Assume that $v(\pi) + v(\rho) \geq d - 1$ and $v(\pi) + v(\rho) \equiv d - 1 \pmod{2}$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\alpha\beta$ is a d -cycle.*

Proof. The conclusion immediately follows from proposition 1.8 if $v(\pi) + v(\rho) = d - 1$, or from proposition 1.9 if $v(\pi) + v(\rho) \geq d$. \square

Remark 1.3. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$; assume that $b_1 \geq 2$. By directly examining the proof of proposition 1.8, we see that the proposed construction yields permutations $\alpha, \beta \in \mathfrak{S}_d$ such that 1 belongs to the a_1 -the cycle of α and to the b_1 -cycle of β . It is not hard to see, once again by inspecting the proof, that the same can be said for proposition 1.9. As a consequence, the statement of corollary 1.10 can be enhanced by adding the following line: α and β can be chosen in such a way that 1 belongs to the a_1 -the cycle of α and to the b_1 -cycle of β , provided that $b_1 \geq 2$. We will need this improvement for the upcoming proof.

Proposition 1.11. *Let $\pi, \rho \in \Pi(d)$ be partitions of d . Assume that $v(\pi) + v(\rho) \geq d$ and $v(\pi) + v(\rho) \equiv d \pmod{2}$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \dots, d\}$ and*

$$[\alpha\beta] = \begin{cases} [d/2, d/2] & \text{if } \pi = \rho = [2, \dots, 2], \\ [1, d - 1] & \text{otherwise.} \end{cases}$$

Terminology?

Maybe explain why?

Proof. Assume first that $\pi = \rho = [2, \dots, d]$. We can choose

$$\alpha = (2, 3)(4, 5) \cdots (d, 1) \quad \beta = (1, 2)(3, 4) \cdots (d-1, d).$$

The action of $\langle \alpha, \beta \rangle$ is obviously transitive, and

$$\alpha\beta = (1, 3, \dots, d-1)(2, 4, \dots, 2).$$

Otherwise, since $v(\pi) + v(\rho) \geq d$, at least one of π and ρ has an entry which is greater than 2; without loss of generality, we can assume it is ρ . Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$ with $a_1 \geq 2$ (since $v(\pi) \geq 1$) and $b_1 \geq 3$. Fix

$$\beta = (1, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{s-1} + 1, \dots, d_s),$$

where $b_1 = d_1 \geq 3$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Consider the partitions

$$\pi' = [a_1 - 1, \dots, a_r], \quad \rho' = [b_1 - 1, \dots, b_s].$$

Since $\sum \pi' = \sum \rho' = d - 1$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - 2$, by corollary 1.10 we can find permutations $\alpha', \beta' \in \mathfrak{S}(\{2, \dots, d\})$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $\alpha'\beta'$ is a $(d-1)$ -cycle. Up to conjugation, we can assume that

$$\beta' = (2, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{s-1} + 1, \dots, d_s)$$

or, in other words, $\beta = (1, 2)\beta'$; moreover, as explained in remark 1.3, we can choose α' in such a way that its $(a_1 - 1)$ -cycle contains 2. By setting $\alpha = \alpha'(1, 2)$, we immediately get that $\alpha\beta = \alpha'\beta'$ is a $(d-1)$ -cycle fixing 1. Finally, the action of $\langle \alpha, \beta \rangle$ is transitive since α does not fix 1. \square

Corollary 1.12. *Let $\pi, \rho \in \Pi(d)$ be partitions of d . Assume that $v(\pi) + v(\rho) \geq d - 1$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$, $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \dots, d\}$ and*

$$[\alpha\beta] \in \{[d], [1, d-1], [d/2, d/2]\}.$$

Proof. The conclusion follows immediately from proposition 1.9 or proposition 1.11 depending on the parity of $v(\pi) + v(\rho) + d$. \square

1.6 Projective plane

Theorem 1.13. *Let $\mathcal{D} = (\tilde{\Sigma}, \mathbb{RP}^2; d; \pi_1, \dots, \pi_n)$ be a candidate datum. If $\tilde{\Sigma}$ is non-orientable, then \mathcal{D} is realizable.*

Proof. First of all, since \mathcal{D} is a candidate datum, the ?? formula implies that

$$v(\pi_1) + \dots + v(\pi_n) = d\chi(\mathbb{RP}^2) - \chi(\tilde{\Sigma}) \geq d - 1$$

(recall that $\tilde{\Sigma}$ is non-orientable, so $\chi(\tilde{\Sigma}) \leq 1$). Moreover, the total branching $v(\pi_1) + \dots + v(\pi_n)$ is even. In order to apply proposition 1.2, we will now inductively define representatives $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$, satisfying the following invariant: for every $0 \leq i \leq n$, either

$$v(\alpha_1 \cdots \alpha_i) = v(\pi_1) + \dots + v(\pi_i)$$

or

$$[\alpha_1 \cdots \alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\} \text{ and } \langle \alpha_1, \dots, \alpha_i \rangle \text{ acts transitively.}$$

Assume we have already defined $\alpha_1, \dots, \alpha_{i-1}$; we want to suitably choose α_i . Let $\alpha = \alpha_1 \cdots \alpha_{i-1}$; there are two cases.

- If $v(\alpha) + v(\pi_i) < d$, by proposition 1.8 we can find $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $v(\alpha\alpha_i) = v(\alpha) + v(\alpha_i)$. The invariant is still satisfied: if $v(\alpha) = v(\pi_1) + \dots + v(\pi_{i-1})$ then obviously $v(\alpha\alpha_i) = v(\pi_1) + \dots + v(\pi_i)$. If instead $[\alpha] \in \{[d], [1, d-1], [d/2, d/2]\}$, then either α_i is the identity, or $\alpha\alpha_i$ is a d -cycle; either way, $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$. Note that if the action of $\langle \alpha_1, \dots, \alpha_{i-1} \rangle$ is transitive, then the action of $\langle \alpha_1, \dots, \alpha_i \rangle$ is transitive as well.
- If $v(\alpha) + v(\alpha_i) \geq d$, corollary 1.12 gives a permutation $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$ and the action of $\langle \alpha, \alpha_i \rangle$ is transitive. The invariant is obviously satisfied.

By induction, we can find $\alpha_1, \dots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that

$$[\alpha_1 \cdots \alpha_n] \in \{[d], [1, d-1], [d/2, d/2]\}$$

and $\langle \alpha_1, \dots, \alpha_n \rangle$ acts transitively on $\{1, \dots, d\}$ (note that $v(\alpha_1 \cdots \alpha_n) = v(\alpha_1) + \dots + v(\alpha_n)$ also implies that $[\alpha_1 \cdots \alpha_n] = [d]$). Note that

$$v(\alpha_1 \cdots \alpha_n) \equiv v(\alpha_1) + \dots + v(\alpha_n) \equiv 0 \pmod{2}.$$

Show that $v(\alpha\beta) \equiv v(\alpha) + v(\beta) \pmod{2}$.

We now prove that $\alpha = \alpha_1 \cdots \alpha_n$ is a square.

- If $[\alpha] = [d]$, then d is odd, so α is the square of $\alpha^{(d+1)/2}$.
- If $[\alpha] = [1, d-2]$, then d is even, so α is the square of $\alpha^{d/2}$.
- If $[\alpha] = [d/2, d/2]$, then d is even, and it is easy to see that α is the square of a d -cycle.

By proposition 1.2, this implies that there exists a realizable candidate datum $\mathcal{D}' = (\tilde{\Sigma}', \mathbb{RP}^2; d; \pi_1, \dots, \pi_n)$.

To see that $\tilde{\Sigma}'$ is non-orientable (and, therefore, equal to $\tilde{\Sigma}$, as shown in remark 1.2), simply note that ??.

□

1.7 Prime-degree conjecture

Proposition 1.14. *Let d be a positive integer. Assume that every candidate datum $(\Sigma_g, S^2; d; \pi_1, \pi_2, \pi_3)$ is realizable. Then every candidate datum $(\Sigma_g, S^2; d; \pi_1, \dots, \pi_n)$ with $n \geq 3$ is realizable.*

Proof. We proceed by induction on $n \geq 4$. Assume that every candidate datum with at most $n-1$ partitions is realizable. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \dots, \pi_n)$; there are two cases.

- Assume that there are two indices $1 \leq i < j \leq n$ such that $v(\pi_i) + v(\pi_j) \leq d-1$; without loss of generality, consider $i = 1$ and $j = 2$. By proposition 1.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \dots, \pi_n)$. By induction, \mathcal{D}' is realizable; proposition 1.1 then gives permutations $\alpha_3, \dots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $(\alpha_1\alpha_2)\alpha_3 \cdots \alpha_n = 1$ and the action of $\langle \alpha_1\alpha_2, \alpha_2, \dots, \alpha_n \rangle$ on $\{1, \dots, d\}$ is transitive. It is trivial to see that the permutations $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ imply the realizability of \mathcal{D} , again by proposition 1.1.
- Otherwise, $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. By corollary 1.12, we can find permutations α_1, α_2 with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d-2$. Let

Here it is crucial that $n \geq 4$.

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\alpha_3) + \dots + v(\alpha_n)) - d + 1 \geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \dots, \pi_n)$ is a candidate datum, so it is realizable by induction. Similarly to the case above, this implies that \mathcal{D} is realizable.

□