Chapter 1

Hurwitz existence problem

Chapter 2

Monodromy

2.1 Symmetric group and partitions

2.2 Branched covering action of the fundamental group

2.3 Monodromy and realizability

Proposition 2.1. *Let Σ_g be the connected sum of $g \geq 0$ tori, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n \in \mathfrak{S}_g$ such that:*

(i) $[\alpha_i] = \pi_i$ *for each* $1 \leq i \leq n$*;*

(i) $[\beta_1, \gamma_1] \cdots [\beta_g, \gamma_g] \cdot \alpha_1 \cdots \alpha_n = 1$*;*

(i) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_g \rangle \leq \mathfrak{S}_d$ *acts transitively on* $\{1, \ldots, d\}$*.*

In this case, $\widetilde{\Sigma}$ is necessarily orientable.

Remark 2.1. Given Σ_g, d and $\pi_1, \ldots, \pi_n$, there is at most one surface $\widetilde{\Sigma}$ such that $\mathcal{D} = (\widetilde{\Sigma}, \Sigma_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum. In fact, the **??** formula gives

$$\chi(\widetilde{\Sigma}) = d\chi(\Sigma_g) - v(\pi_1) - \ldots - v(\pi_n)$$

which, in turn, uniquely determines the orientable surface $\widetilde{\Sigma}$.

Proposition 2.2. *Let N_g be the connected sum of $g \geq 1$ projective planes, $d \geq 1$ an integer, $\pi_1, \ldots, \pi_n \in \Pi(d)$ partitions of d. Then there exists a realizable combinatorial datum $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ if and only if there exist permutations $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \in \mathfrak{S}_d$ such that:*

(i) $[\alpha_i] = \pi_i$ *for each* $1 \leq i \leq n$*;*

(i) $\beta_1^2 \cdots \beta_g^2 \cdot \alpha_1 \cdots \alpha_n = 1$*;*

(i) the subgroup $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_g \rangle \leq \mathfrak{S}_d$ *acts transitively on* $\{1, \ldots, d\}$*.*

In this case, $\widetilde{\Sigma}$ is orientable if and only if ??.

Remark 2.2. Given N_g, d and $\pi_1, \ldots, \pi_n$, there are at most two surfaces $\widetilde{\Sigma}$, one orientable and one non-orientable, such that $\mathcal{D} = (\widetilde{\Sigma}, N_g; d; \pi_1, \ldots, \pi_n)$ is a candidate datum: like in remark 2.1, the **??** formula fixes $\chi(\widetilde{\Sigma})$ which, together with orientability, uniquely determines $\widetilde{\Sigma}$.

Proposition 2.3. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots \pi_n)$ be a combinatorial datum, with Σ non-orientable and $\widetilde{\Sigma}$ orientable. Then $\mathcal{D}$ is realizable if and only if d is even and there exist partitions $\pi'_1, \ldots, \pi'_n, \pi''_1, \ldots, \pi''_n \in \Pi\,(d/2)$ with $\pi_i = \pi'_i \cup \pi''_i$ for each $1 \leq i \leq n$, such that the combinatorial datum*

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi'_1, \ldots, \pi'_n, \pi''_i, \ldots, \pi''_n)$$

is realizable, where $\widehat{\Sigma}$ is the orientable double covering of Σ.

2.4 Non-positive Euler characteristic

Lemma 2.4. *Let $\alpha \in \mathfrak{S}_d$ be a permutation. Set $r = d - v(\alpha)$, and let $t \geq 0$ be an integer such that $2t \leq v(\alpha)$. Then α can be written as the product of a $(r+2t)$-cycle and a d-cycle.*

Proof. Without loss of generality, assume that

$$\alpha = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{r-1} + 1, \ldots, d_r),$$

where $d_r = d$. Fix

$$\beta_0 = (1, b_1, b_2, \ldots, b_{2t}), \qquad \beta_1 = (1, d_1 + 1, d_2 + 2, \ldots, d_{r-1} + 1).$$

Set

$$\beta = \beta_0\beta_1 = (1, d_1 + 1, d_2 + 1, \ldots, d_{r-1} + 1, b_1, b_2, \ldots, b_{2t}).$$

An easy computation shows that

$$\begin{aligned}
\beta\alpha &= \beta_0\beta_1\alpha \\
&= (1, b_1, b_2, \ldots, b_{2t})(1, 2, \ldots, d) \\
&= (1, \ldots, b_1 - 1, b_2, \ldots, b_3 - 1, \ldots, b_4, \ldots, b_{2t-1} - 1, b_{2t}, \ldots, d, b_1, \ldots, b_2 - 1, b_3, \ldots, b_{2t} - 1).
\end{aligned}$$

Writing $\alpha = \beta^{-1}(\beta\alpha)$ gives the desired decomposition. ☐

Corollary 2.5. *Let $\alpha \in \mathfrak{A}_d$ be an even permutation. Then α can be written as:*

(i) a commutator $[\beta, \gamma]$, where γ is a d-cycle;

(i) a product of two squares $\delta^2\epsilon^2$, where $\delta\epsilon$ is a d-cycle.

Proof. Since α is an even permutation, its branching number $v(\alpha)$ is even. By lemma 2.4, there exist two d-cycles $\tau, \sigma \in \mathfrak{S}_d$ such that $\alpha = \tau\sigma$.

(i) Since τ and σ^{-1} are conjugated, there exists a permutation $\beta \in \mathfrak{S}_d$ such that $\tau = \beta\sigma^{-1}\beta^{-1}$. Setting $\gamma = \sigma^{-1}$, we immediately get that

$$\alpha = \tau\sigma = \beta\sigma^{-1}\beta^{-1}\sigma = \beta\gamma\beta^{-1}\gamma^{-1} = [\beta, \gamma].$$

(i) Since τ and σ are conjugated, there exists a permutation $\delta \in \mathfrak{S}_d$ such that $\tau = \delta\sigma\delta^{-1}$. Setting $\epsilon = \delta^{-1}\sigma$, we have that

$$\alpha = \tau\sigma = \delta\sigma\delta^{-1}\sigma = \delta^2(\delta^{-1}\sigma)^2 = \delta^2\epsilon^2.$$

☐

Theorem 2.6. *Let $\mathcal{D} = (\widetilde{\Sigma}, \Sigma; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\chi(\Sigma) \leq 0$, then $\mathcal{D}$ is realizable.*

Proof. Let us first assume that Σ is orientable; this means that $\Sigma = \Sigma_g$ is the connected sum of $g \geq 1$ tori, and that $\widetilde{\Sigma}$ is orientable as well. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Since $\mathcal{D}$ is a candidate datum, we have that

$$v(\alpha) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

By corollary 2.5, we can find permutations $\beta_1, \gamma_1 \in \mathfrak{S}_d$ such that $\alpha = [\gamma_1, \beta_1]$ and β_1 is a d-cycle. Set $\beta_2 = \ldots = \beta_g = \gamma_2 = \ldots = \gamma_g = \mathrm{id} \in \mathfrak{S}_d$. All the conditions of proposition 2.1 are satisfied; since $\widetilde{\Sigma}$ is orientable, this implies that $\mathcal{D}$ is realizable (see remark 2.1).

Assume now that Σ and $\widetilde{\Sigma}$ are both non-orientable; this means that $\Sigma = N_g$ is the connected sum of $g \geq 2$ projective planes. Choose permutations $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for each $1 \leq i \leq n$. Let $\alpha = \alpha_1 \cdots \alpha_n$. Similarly to what we did for the previous case, we can find $\beta_1, \beta_2 \in \mathfrak{S}_d$ such that $\alpha = \beta_2^{-2}\beta_1^{-2}$ and $\beta_2\beta_1$ is a d-cycle. By setting $\beta_3 = \ldots = \beta_g = \mathrm{id} \in \mathfrak{S}_d$, proposition 2.2 (together with remark 2.2) implies the realizability of $\mathcal{D}$.

Finally, consider the case where Σ is non-orientable and $\widetilde{\Sigma}$ is orientable. Since $\mathcal{D}$ is a candidate datum, d is even and there exist partitions $\pi_1', \ldots, \pi_n', \pi_1'', \ldots, \pi_n'' \in \Pi\,(d/2)$ with $\pi_i = \pi_1' \cup \pi_i''$ for each $1 \leq i \leq n$. Let $\widehat{\Sigma}$ be the double orientable covering of Σ; by the first case we analyzed, the candidate datum

$$\mathcal{D}' = (\widetilde{\Sigma}, \widehat{\Sigma}; d/2; \pi_1', \ldots, \pi_n', \pi_1'', \ldots, \pi_n'')$$

is realizable. By proposition 2.3, $\mathcal{D}$ is realizable as well. □

2.5 Products in symmetric groups

Lemma 2.7. *Let X, Y be finite sets; denote by h the cardinality of Y, and by k the cardinality of $X \cap Y$. Let $\alpha \in \mathfrak{S}(X)$, $\beta \in \mathfrak{S}(Y)$, $\gamma \in \mathfrak{S}(X \cap Y)$. Assume that $\beta = (b_1, \ldots, b_h)$ is a h-cycle, and that γ is a k-cycle of the form $\gamma = (b_{i_1}, \ldots, b_{i_k})$ with $1 \leq i_1 \leq \ldots \leq i_k \leq h$. Then $\alpha \in \mathfrak{S}(X)$ and $\alpha\gamma^{-1}\beta \in \mathfrak{S}(X \cup Y)$ have the same number of cycles.*

Proof. Write $\gamma = (u_1, \ldots, u_k)$, where $u_j = b_{i_j}$. Without loss of generality, assume that $i_1 = 1$. Then we can write

$$\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{k-1}, u_k, \ldots, w_k),$$

possibly with $u_j = w_j$ for some values of j. We immediately get that

$$\gamma^{-1}\beta = (u_1, \ldots, w_1)(u_2, \ldots, w_2) \cdots (u_k, \ldots, w_k).$$

If we denote by $A_1, \ldots, A_r \subseteq X$ the orbits of α, it is then easy to see that the orbits of $\alpha\gamma^{-1}\beta$ are $A_1', \ldots, A_r' \subseteq X \cup Y$, where

$$A_j' = A_j \cup \bigcup_{u_l \in A_j} \{u_l, \ldots, w_l\}.$$ □

Proposition 2.8. *Let $\pi, \rho \in \Pi\,(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) < d$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = v(\alpha) + v(\beta)$.*

Proof. First of all, note that the conclusion is trivial whenever $v(\pi) = 0$ or $v(\rho) = 0$. This already solves the cases $d = 1$ and $d = 2$. We now proceed by induction on $d \geq 3$, assuming that $v(\pi) > 0$ and $v(\rho) > 0$. Write $\pi = [a_1, \dots, a_r]$, $\rho = [b_1, \dots, b_s]$; without loss of generality, assume that $b_1 > 1$. Fix

$$\beta = (1, \dots, d_1)(d_1 + 1, \dots, d_2) \cdots (d_{s-1} + 1, \dots, d_s),$$

where $b_1 = d_1 \geq 2$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Note that

$$d - 1 \geq v(\pi) + v(\rho) = (a_1 - 1) + \ldots + (a_r - 1) + d - s \geq a_1 - 1 + d - s,$$

hence $a_1 \leq s$. Fix $\alpha_1 = (1, d_1 + 1, \dots, d_{a_1 - 1} + 1)$, and let $A = \{1, d_1 + 1, \dots, d_{a_1 - 1} + 1\}$ be the support of α_1. Define $Q = \{1, \dots, d_{a_1}\} \setminus A$; note that Q_1 is non-empty, since $d_1 + 1 \geq 3$ implies that $2 \in Q_1$. Consider the partitions

$$\pi' = [a_2, a_3, \dots, a_r], \qquad \rho' = [|Q|, b_{a_1+1}, \dots, b_s].$$

We have that

$$\sum \pi' = \sum \rho' = d - a_1, \qquad v(\pi') = v(\pi) - a_1 + 1, \qquad v(\rho') = v(\rho) - 1.$$

Since $d - a_1 < d$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - a_1 < d - a_1$, by induction we find $\alpha', \beta' \in \mathfrak{S}(\{1, \dots, d\} \setminus A)$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $v(\alpha'\beta') = v(\alpha') + v(\beta')$. Up to conjugation, we may assume that

$$\beta' = \beta_1(d_{a_1} + 1, \dots, d_{a_1+1})(d_{a_1+1} + 1, \dots, d_{a_1+2}) \cdots (d_{s-1} + 1, \dots, d_s),$$

where β_1 is the $|Q|$-cycle whose entries are the elements of Q in increasing order. An easy computation shows that

$$\alpha_1\beta = (1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s).$$

Therefore, setting $\alpha = \alpha'\alpha_1$, we have that

$$\begin{aligned}
\alpha\beta &= \alpha_1\alpha'\beta \\
&= \alpha'(1, \dots, d_{a_1})(d_{a_1} + 1, \dots, d_{a_1+1}) \cdots (d_{s-1} + 1, \dots, d_s) \\
&= \alpha'\beta'\beta_1^{-1}(1, \dots, d_{a_1})
\end{aligned}$$

By lemma 2.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha'\beta'$, so that

$$\begin{aligned}
v(\alpha\beta) &= a_1 + v(\alpha'\beta') \\
&= a_1 + v(\alpha') + v(\beta') \\
&= a_1 + (v(\pi) - a_1 + 1) + (v(\rho) - 1) \\
&= v(\pi) + v(\rho).
\end{aligned}$$

Since $[\alpha] = \pi$ and $[\beta] = \rho$, the conclusion follows. $\square$

Proposition 2.9. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d$, and let $t = v(\pi) + v(\rho) - d + 1$. Fix an integer $0 \leq k \leq t$ such that $k \equiv t \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $v(\alpha\beta) = d - 1 - k$ and the action of $\langle \alpha, \beta \rangle$ on $\{1, \dots, d\}$ is transitive.*

aybe only $k = 0$
eded (in this case,
ansitivity is trivial)?

Proof. Write $\pi = [a_1, \ldots, a_r]$. Since $v(\rho) \leq d-1$ and $v(\pi) + v(\rho) \geq d$, there exists a largest integer $0 \leq i \leq r$ such that $(a_1 - 1) + \ldots (a_i - 1) + v(\rho) \leq d - 1$. Define

$$z = d - v(\rho) - (a_1 - 1) - \ldots - (a_i - 1).$$

Consider the partition $\pi' = [a_1, \ldots, a_i, z, 1, \ldots, 1] \in \Pi(d)$. Since by construction $v(\pi') + v(\rho) = d - 1$, thanks to proposition 2.8 we can find permutations $\alpha', \beta \in \mathfrak{S}_d$ with $[\alpha'] = \pi'$ and $[\beta] = \rho$ such that $v(\alpha'\beta) = d - 1$; in other words, $\alpha'\beta$ is a d-cycle. Consider now the partition $\pi'' = [a_{i+1} - z + 1, a_{i+2}, \ldots, a_r]$, whose branching number is $v(\pi'') = t$. Let $n = \sum \pi''$; fix an element u_1 of the z-cycle of α', and let $u_2, \ldots, u_n$ be the fixed points of α' corresponding to the last ones of π' (it is easy to see that there are exactly $n-1$ such ones). Since $k \leq t = v(\rho'')$ and $k \equiv t \pmod 2$, lemma 2.4 gives permutations $\alpha'', \gamma \in \mathfrak{S}(\{u_1, \ldots, u_n\})$ such that $[\alpha''] = \rho''$, γ is a n-cycle and $\alpha''\gamma$ is a $(n-k)$-cycle. Up to conjugation, we can assume that $\gamma = (u_1, \ldots, u_n)$. Moreover, it is not restrictive to assume that $u_1, \ldots, u_n$ appear in this order in the d-cycle $\alpha'\beta$. Therefore, setting $\alpha = \alpha''\alpha'$, we have that

Terminology?

$$\alpha\beta = (\alpha''\gamma)\gamma^{-1}(\alpha'\beta);$$

by lemma 2.7, this implies that $\alpha\beta$ has the same number of cycles as $\alpha''\gamma \in \mathfrak{S}(\{u_1, \ldots, u_n\})$, that is $v(\alpha\beta) = d - (k+1)$. Since $[\alpha] = \pi$ and $[\beta] = \rho$, the only thing left to show is that the action of $\langle \alpha, \beta \rangle$ on $\{1, \ldots, d\}$ is transitive. Write

$$\alpha'\beta = (u_1, \ldots, w_1, u_2, \ldots, w_2, u_3, \ldots, w_{n-1}, u_n, \ldots, w_n)$$

as in the proof of lemma 2.7, where an explicit description of the orbits of $\alpha\beta = \alpha''\gamma\gamma^{-1}\alpha'\beta$ is given; from that description, it is clear that for each $1 \leq j \leq n$ the elements $u_j, \ldots, w_j$ all belong to the same orbit. Moreover, since $\beta = (\alpha')^{-1}(u_1, \ldots, w_1, u_2, \ldots, w_n)$ and α' fixes $u_2, \ldots, u_n$, it follows that w_j and u_{j+1} belong to the same orbit for each $1 \leq j \leq n-1$; this completes the proof. □

Corollary 2.10. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d - 1$ and $v(\pi) + v(\rho) \equiv d - 1 \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\alpha\beta$ is a d-cycle.*

Proof. The conclusion immediately follows from proposition 2.8 if $v(\pi) + v(\rho) = d - 1$, or from proposition 2.9 if $v(\pi) + v(\rho) \geq d$. □

Remark 2.3. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$; assume that $b_1 \geq 2$. By directly examining the proof of proposition 2.8, we can see that the proposed construction yields permutations $\alpha, \beta \in \mathfrak{S}_d$ such that 1 belongs to the a_1-the cycle of α and to the b_1-cycle of β. It is not hard to see, once again by inspecting the proof, that the same can be said for proposition 2.9. As a consequence, the statement of corollary 2.10 can be enhanced by adding the following line: *α and β can be chosen in such a way that 1 belongs to the a_1-the cycle of α and to the b_1-cycle of β, provided that $b_1 \geq 2$.* We will need this improvement for the upcoming proof.

Maybe explain why?

Proposition 2.11. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d$ and $v(\pi) + v(\rho) \equiv d \pmod 2$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$ and $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \ldots, d\}$ and*

$$[\alpha\beta] = \begin{cases} [d/2, d/2] & \text{if } \pi = \rho = [2, \ldots, 2], \\ [1, d-1] & \text{otherwise.} \end{cases}$$

Proof. Assume first that $\pi = \rho = [2, \ldots, d]$. We can choose

$$\alpha = (2,3)(4,5)\cdots(d,1), \qquad \beta = (1,2)(3,4)\cdots(d-1,d).$$

The action of $\langle \alpha, \beta \rangle$ is obviously transitive, and

$$\alpha\beta = (1,3,\ldots,d-1)(2,4,\ldots,2).$$

Otherwise, since $v(\pi) + v(\rho) \geq d$, at least one of π and ρ has an entry which is greater than 2; without loss of generality, we can assume it is ρ. Write $\pi = [a_1, \ldots, a_r]$, $\rho = [b_1, \ldots, b_s]$ with $a_1 \geq 2$ (since $v(\pi) \geq 1$) and $b_1 \geq 3$. Fix

$$\beta = (1, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{s-1} + 1, \ldots, d_s),$$

where $b_1 = d_1 \geq 3$, $b_i = d_i - d_{i-1}$ for $2 \leq i \leq s$ (in particular, $d_s = d$). Consider the partitions

$$\pi' = [a_1 - 1, \ldots, a_r], \qquad \rho' = [b_1 - 1, \ldots, b_s].$$

Since $\sum \pi' = \sum \rho' = d - 1$ and $v(\pi') + v(\rho') = v(\pi) + v(\rho) - 2$, by corollary 2.10 we can find permutations $\alpha', \beta' \in \mathfrak{S}(\{2, \ldots, d\})$ with $[\alpha'] = \pi'$ and $[\beta'] = \rho'$ such that $\alpha'\beta'$ is a $(d-1)$-cycle. Up to conjugation, we can assume that

$$\beta' = (2, \ldots, d_1)(d_1 + 1, \ldots, d_2) \cdots (d_{s-1} + 1, \ldots, d_s)$$

or, in other words, $\beta = (1,2)\beta'$; moreover, as explained in remark 2.3, we can choose α' in such a way that its $(a_1 - 1)$-cycle contains 2. By setting $\alpha = \alpha'(1,2)$, we immediately get that $\alpha\beta = \alpha'\beta'$ is a $(d-1)$-cycle fixing 1. Finally, the action of $\langle \alpha, \beta \rangle$ is transitive since α does not fix 1. □

Corollary 2.12. *Let $\pi, \rho \in \Pi(d)$ be partitions of d. Assume that $v(\pi) + v(\rho) \geq d - 1$. Then there exist permutations $\alpha, \beta \in \mathfrak{S}_d$ with $[\alpha] = \pi$, $[\beta] = \rho$ such that $\langle \alpha, \beta \rangle$ acts transitively on $\{1, \ldots, d\}$ and*

$$[\alpha\beta] \in \{[d], [1, d-1], [d/2, d/2]\}.$$

Proof. The conclusion follows immediately from proposition 2.9 or proposition 2.11 depending on the parity of $v(\pi) + v(\rho) + d$. □

2.6 Projective plane and sphere

Theorem 2.13. *Let $\mathcal{D} = (\widetilde{\Sigma}, \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. If $\widetilde{\Sigma}$ is non-orientable, then $\mathcal{D}$ is realizable.*

Proof. First of all, since $\mathcal{D}$ is a candidate datum, the **??** formula implies that

$$v(\pi_1) + \ldots + v(\pi_n) = d\chi(\mathbb{RP}^2) - \chi(\widetilde{\Sigma}) \geq d - 1$$

(recall that $\widetilde{\Sigma}$ is non-orientable, so $\chi(\widetilde{\Sigma}) \leq 1$). Moreover, the total branching $v(\pi_1) + \ldots + v(\pi_n)$ is even. In order to apply proposition 2.2, we will now inductively define representatives $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$, satisfying the following invariant: for every $0 \leq i \leq n$, either

$$v(\alpha_1 \cdots \alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$$

or

$$[\alpha_1 \cdots \alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\} \text{ and } \langle \alpha_1, \ldots, \alpha_i \rangle \text{ acts transitively.}$$

Assume we have already defined $\alpha_1, \ldots, \alpha_{i-1}$; we want to suitably choose α_i. Let $\alpha = \alpha_1 \cdots \alpha_{i-1}$; there are two cases.

- If $v(\alpha) + v(\pi_i) < d$, by proposition 2.8 we can find $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $v(\alpha\alpha_i) = v(\alpha) + v(\alpha_i)$. The invariant is still satisfied: if $v(\alpha) = v(\pi_1) + \ldots + v(\pi_{i-1})$ then obviously $v(\alpha\alpha_i) = v(\pi_1) + \ldots + v(\pi_i)$. If instead $[\alpha] \in \{[d], [1, d-1], [d/2, d/2]\}$, then either α_i is the identity, or $\alpha\alpha_i$ is a d-cycle; either way, $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$. Note that if the action of $\langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ is transitive, then the action of $\langle \alpha_1, \ldots, \alpha_i \rangle$ is transitive as well.
- If $v(\alpha) + v(\alpha_i) \geq d$, corollary 2.12 gives a permutation $\alpha_i \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that $[\alpha\alpha_i] \in \{[d], [1, d-1], [d/2, d/2]\}$ and the action of $\langle \alpha, \alpha_i \rangle$ is transitive. The invariant is obviously satisfied.

By induction, we can find $\alpha_1, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ such that

$$[\alpha_1 \cdots \alpha_n] \in \{[d], [1, d-1], [d/2, d/2]\}$$

and $\langle \alpha_1, \ldots, \alpha_n \rangle$ acts transitively on $\{1, \ldots, d\}$ (note that $v(\alpha_1 \cdots \alpha_n) = v(\alpha_1) + \ldots + v(\alpha_n)$ also implies that $[\alpha_1 \cdots \alpha_n] = [d]$). Note that

Show that $v(\alpha\beta) \equiv v(\alpha) + v(\beta) \pmod 2$.

$$v(\alpha_1 \cdots \alpha_n) \equiv v(\alpha_1) + \ldots + v(\alpha_n) \equiv 0 \pmod 2.$$

We now prove that $\alpha = \alpha_1 \cdots \alpha_n$ is a square.

- If $[\alpha] = [d]$, then d is odd, so α is the square of $\alpha^{(d+1)/2}$.
- If $[\alpha] = [1, d-2]$, then d is even, so α is the square of $\alpha^{d/2}$.
- If $[\alpha] = [d/2, d/2]$, then d is even, and it is easy to see that α is the square of a d-cycle.

By proposition 2.2, this implies that there exists a realizable candidate datum $\mathcal{D}' = (\widetilde{\Sigma}', \mathbb{RP}^2; d; \pi_1, \ldots, \pi_n)$. To see that $\widetilde{\Sigma}'$ is non-orientable (and, therefore, equal to $\widetilde{\Sigma}$, as shown in remark 2.2), simply note that ??. □

2.7 First results on the sphere

Assume $n \geq 3$.

Proposition 2.14. *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$ be a candidate datum. Then $\mathcal{D}$ is realizable.*

Proof. We proceed by induction on n, starting with the base case $n = 3$. If $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \pi_2, [d])$ is a candidate datum, by the **??** formula we have that

$$v(\pi_1) + v(\pi_2) = 2d - 2 + 2g - (d-1) \geq d - 1.$$

Moreover, the total branching number $v(\pi_1) + v(\pi_2) + d - 1$ is even. Corollary 2.10 then gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $[\alpha_1\alpha_2] = [d]$. By proposition 2.1, $\mathcal{D}$ is realizable (see remark 2.1).

Assume now that the statement holds for candidate data with at most $n-1$ partitions, and fix $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [d])$. There are two cases.

- Assume that $v(\pi_1) + v(\pi_2) \leq d - 1$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [d])$. By induction, $\mathcal{D}'$ is realizable; proposition 2.1 then gives permutations $\alpha_3, \ldots, \alpha_n \in \mathfrak{S}_d$ with $[\alpha_i] = \pi_i$ for $3 \leq i \leq n-1$ and $[\alpha_n] = [d]$ such that $(\alpha_1\alpha_2)\alpha_3 \cdots \alpha_n = 1$. It is easy to see that the permutations $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ imply the realizability of $\mathcal{D}$, again by proposition 2.1 (together, as usual, with remark 2.1).

- Otherwise, we have that $v(\pi_1) + v(\pi_2) \geq d$. By corollary 2.12, we can find permutations α_1, α_2 with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 1) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + 1 + d - 1) - d + 1 \geq 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$ is a candidate datum, so it is realizable by induction. Similarly to the previous case, this implies that $\mathcal{D}$ is realizable. □

Proposition 2.15. *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_{n-1}, [1, d-1])$ be a proper candidate datum. Then $\mathcal{D}$ is not realizable if and only if it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3])$.

(2) $\mathcal{D} = (S^2, S^2; 2k; [2, \ldots, 2], [2, \ldots, 2], [1, 2k-1])$ *with* $k \geq 2$.

Proposition 2.16. *Let $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$ be a candidate datum. Assume that $d \neq 4$ and that $2g \geq d - 1$. Then $\mathcal{D}$ is realizable.*

Corollary 2.17. *Let d be a positive integer with $d \neq 4$. Then there exist at most finitely many non-realizable candidate data of the form $(\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$.*

Proposition 2.18. *Let $\mathcal{D} = (\Sigma_g, S^2; 4; \pi_1, \ldots, \pi_n)$ be a proper candidate datum. Then $\mathcal{D}$ is realizable if and only if*

$$\mathcal{D} \neq (\Sigma_{n-3}, S^2; 4; [2,2], \ldots, [2,2], [1,3]).$$

2.8 Prime-degree conjecture

Proposition 2.19. *Let d be a positive integer. Assume that every candidate datum $(\Sigma_g, S^2; d; \pi_1, \pi_2, \pi_3)$ is realizable. Then every candidate datum $(\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$ with $n \geq 3$ is realizable.*

Proof. We proceed by induction on $n \geq 4$. Assume that every candidate datum with at most $n - 1$ partitions is realizable. Fix a candidate datum $\mathcal{D} = (\Sigma_g, S^2; d; \pi_1, \ldots, \pi_n)$; there are two cases.

- Assume that there are two indices $1 \leq i < j \leq n$ such that $v(\pi_i) + v(\pi_j) \leq d - 1$; without loss of generality, consider $i = 1$ and $j = 2$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum $\mathcal{D}' = (\Sigma_g, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$. By induction, $\mathcal{D}'$ is realizable; a reduction argument shows that $\mathcal{D}$ is realizable as well[1].

- Otherwise, $v(\pi_i) + v(\pi_j) \geq d$ for every $1 \leq i < j \leq n$. By corollary 2.12, we can find permutations α_1, α_2 with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

ere it is crucial that ≥ 4.

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_n)) - d + 1 \geq \frac{1}{2}(d - 2 + d) - d + 1 \geq 0.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}, S^2; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_n)$ is a candidate datum, so it is realizable by induction. Similarly to the previous case, this implies that $\mathcal{D}$ is realizable. □

[1]

Chapter 3

Exceptional data with a short partition

Always assume $n \geq 3$

.

3.1 Realizability on the sphere

Theorem 3.1. *Let $\mathcal{D} = (S^2; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (S^2; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

(2) $\mathcal{D} = (S^2; 2k; [2,\dots,2], [2,\dots,2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

(3) $\mathcal{D} = (S^2; 2k; [2,\dots,2], [1,2,\dots,2,3], [k,k])$ *with* $k \geq 2$.

(4) $\mathcal{D} = (S^2; 4k+2; [2,\dots,2], [1,\dots,1,k+1,k+2], [2k+1, 2k+1])$ *with* $k \geq 1$.

(5) $\mathcal{D} = (S^2; 4k; [2,\dots,2], [1,\dots,1,k+1,k+1], [2k-1, 2k+1])$ *with* $k \geq 2$.

(6) $\mathcal{D} = (S^2; kh; [h,\dots,h], [1,\dots,1,k+1], [lk, (h-l)k])$ *with* $h \geq 2$, $k \geq 2$, $1 \leq l < h$.

Theorem 3.2. *Let $\mathcal{D} = (S^2; d; \pi_1, \dots, \pi_{n-1}, [s, d-s])$ be a proper candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable.*

3.2 Realizability on the torus for $n = 3$

Theorem 3.3. *Let $\mathcal{D} = (\Sigma_1; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$.

(2) $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$.

(3) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

(4) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$.

(5) $\mathcal{D} = (\Sigma_1; 2k; [2,\dots,2], [2,\dots,2,3,5], [k,k])$ *with* $k \geq 5$.

3.3 Realizability on higher genus surfaces for $n = 3$

Theorem 3.4. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable.*

3.4 Realizability for $n \geq 4$

Theorem 3.5. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \dots, \pi_{n-1}, [s, d-s])$ be a proper candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \dots, [2,2], [1,3])$.

(2) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

Proof. We will proceed by induction on n. We start with the base case $n = 4$, which requires the heaviest casework. Fix a proper candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3, [s, d-s])$.

- If $d \leq 16$, realizability can be checked by a computer. The only exceptional data are:
  (1) $\mathcal{D} = (\Sigma_1; 4; [2,2], [2,2], [2,2], [1,3])$;
  (2) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.
- If $g = 0$, then $\mathcal{D}$ is realizable by theorem 3.2.
- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq 3$; up to reindexing, we can assume that $v(\pi_1) + v(\pi_2) < d$. By proposition 2.8, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$. Consider the candidate datum

  $$\mathcal{D}' = (\Sigma_g; d; [\alpha_1\alpha_2], \pi_3, [s, d-s]).$$

  The standard reduction argument implies that $\mathcal{D}$ is realizable provided that $\mathcal{D}'$ is. If $g \geq 2$, then $\mathcal{D}'$ is realizable by theorem 3.4. If instead $g = 1$, then $\mathcal{D}'$ is realizable unless

  $$\mathcal{D}' = (\Sigma_1; 2k; [2, \dots, 2], [2, \dots, 2, 3, 5], [k, k]) \text{ with } 2k = d.$$

  If this is the case, then $s = k$ and $\{[\alpha_1\alpha_2], \pi_3\} = \{[2, \dots, 2], [2, \dots, 2, 3, 5]\}$. Some more casework is required to show that $\mathcal{D}'$ can actually be chosen to be realizable.

  - If $[\alpha_1\alpha_2] = [2, \dots, 2]$, then $v(\alpha_1\alpha_2) = k$ and $v(\pi_3) = k + 2$. Assume without loss of generality that $v(\pi_1) \leq k/2$. We have that

    $$k + 2 < 1 + k + 2 \leq v(\pi_1) + v(\pi_3) \leq \frac{k}{2} + k + 2 < d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \dots, 2]$ and $v(\pi_1) \notin \{2, k, k+1\}$ then $v(\pi_1) + v(\pi_3) < d$ and $v(\pi_1) + v(\pi_3) \notin \{k, k+2\}$. Therefore, repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \dots, 2]$, $v(\pi_1) = 2$ and $\pi_2 \neq [2, \dots, 2]$, then repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.

– If $\pi_2 = \pi_3 = [2, \ldots, 2]$, we follow a different approach. By proposition 2.11, we can find permutations $\beta_2, \beta_3 \in \mathfrak{S}_d$ with $[\beta_2] = [\beta_3] = [2, \ldots, 2]$ such that $[\beta_2\beta_3] = [k, k]$. It is easy to see that $\mathcal{D}'' = (S^2; 2k; \pi_1, [k, k], [k, k])$ is a candidate datum, and it is realizable by theorem 3.1. The standard reduction argument implies that $\mathcal{D}$ is realizable as well.

Up to swapping π_1 and π_2, this analysis covers all the possible cases.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq 3$. In particular, up to reindexing, we can assume that $v(\pi_3) \geq d/2$. Corollary 2.12 gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$g' = \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + d - 2) - d + 1 \geq \frac{1}{2}\left(d - 2 + \frac{d}{2} + d - 2\right) - d + 1 = \frac{d}{4} - 1 \geq 2.$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1, \alpha_2], \pi_3, [s, d - s])$ is candidate datum, and it is realizable by theorem 3.4. The standard reduction argument implies that $\mathcal{D}$ is realizable as well.

We now turn to the case $n \geq 5$; we will show by induction that every candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ different from $(\Sigma_{n-3}; 4; [2, 2], \ldots, [2, 2], [1, 3])$ is realizable. The case $d = 4$ is covered by **??**. If $n = 5$ and $d = 8$, a computer-aided search shows that $\mathcal{D}$ is realizable. Otherwise, we once again employ a reduction argument.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq n - 1$; up to reindexing, we can assume that $v(\pi_1) + v(\pi_2) < d$. Proposition 2.8 gives permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) = v(\alpha_1\alpha_2)$. By induction the candidate datum
$$\mathcal{D}' = (\Sigma_g; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [s, d - s])$$
is realizable, therefore $\mathcal{D}$ is realizable as well.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq n - 1$. By corollary 2.12, we can find permutations $\alpha_1, \alpha_2 \in \mathfrak{S}_d$ with $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$ such that $v(\alpha_1\alpha_2) \geq d - 2$. Let

$$\begin{aligned} g' &= \frac{1}{2}(v(\alpha_1\alpha_2) + v(\pi_3) + \ldots + v(\pi_{n-1}) + d - 2) - d + 1 \\ &\geq \frac{1}{2}(d - 2 + d + d - 2) - d + 1 \\ &= \frac{d}{2} - 1 \geq 0. \end{aligned}$$

It is easy to see that $\mathcal{D}' = (\Sigma_{g'}; d; [\alpha_1\alpha_2], \pi_3, \ldots, \pi_{n-1}, [s, d - s])$ is a candidate datum, and it is realizable by induction. The standard reduction argument implies that $\mathcal{D}$ is realizable as well.

□