Chapter 4

Exceptional data with a short partition

4.1 Realizability on the sphere

In this final chapter, we will give a full solution of the Hurwitz existence problem for candidate data containing a partition of length 2; as usual, we will assume that $\Sigma = \mathbb{S}$ and $n \geq 3$. This specific instance of the existence problem had already received some interest in the literature, leading to a few partial results.

- Proposition 2.11 addresses the cases where $\pi_n = [1, d-1]$; the proof was actually borrowed from [**edmonds**].
- **pervova-existence-ii** dealt with the cases where $n = 3$ and $\pi_3 = [2, d-2]$.
- **pakovich** solved the existence problem for $\ell(\pi_n) = 2$ and $\widetilde{\Sigma} = \mathbb{S}$.

In particular, we will consistently exploit the results by **pakovich** as a base case for genus-reducing combinatorial moves. We will now state the relevant theorems, but we decide to omit the proofs: while the core ideas are very ingenious, filling in the details is quite tedious and time-consuming[1]. We refer the interested reader to [**pakovich**].

Theorem 4.1. *Let $\mathcal{D} = (\mathbb{S}; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

(2) $\mathcal{D} = (\mathbb{S}; 2k; [2,\dots,2], [2,\dots,2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

(3) $\mathcal{D} = (\mathbb{S}; 2k; [2,\dots,2], [1,2,\dots,2,3], [k,k])$ *with* $k \geq 2$.

(4) $\mathcal{D} = (\mathbb{S}; 4k+2; [2,\dots,2], [1,\dots,1,k+1,k+2], [2k+1,2k+1])$ *with* $k \geq 1$.

(5) $\mathcal{D} = (\mathbb{S}; 4k; [2,\dots,2], [1,\dots,1,k+1,k+1], [2k-1,2k+1])$ *with* $k \geq 2$.

(6) $\mathcal{D} = (\mathbb{S}; kh; [h,\dots,h], [1,\dots,1,k+1], [lh,(k-l)h])$ *with* $h \geq 2$, $k \geq 2$, $1 \leq l \leq k-1$.

[1] The same could probably be said about the other proofs in this chapter, which cannot be omitted for obvious reasons.

Theorem 4.2. *Let $\mathcal{D} = (\mathbb{S}; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable.*

In the upcoming proofs, we will also make extensive use of the computational results from Appendix A; although delegating work to the computer is never necessary (and in theory we could prove the same results by hand), doing so will save us a lot of effort in dealing with tricky corner-cases, and allow us to focus more on the reduction-oriented part of the proofs.

4.2 Realizability on the torus for $n = 3$

As anticipated in the title, this section deals with the cases where $n = 3$ and $\widetilde{\Sigma} = \Sigma_1$. Due to the relatively large number of families of exceptional data listed in Theorem 4.1, this is the instance of the problem which will require the heaviest casework.

Theorem 4.3. *Let $\mathcal{D} = (\Sigma_1; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_1; 6; [3, 3], [3, 3], [2, 4])$.

(2) $\mathcal{D} = (\Sigma_1; 8; [2, 2, 2, 2], [4, 4], [3, 5])$.

(3) $\mathcal{D} = (\Sigma_1; 12; [2, 2, 2, 2, 2, 2], [3, 3, 3, 3], [5, 7])$.

(4) $\mathcal{D} = (\Sigma_1; 16; [2, 2, 2, 2, 2, 2, 2, 2], [1, 3, 3, 3, 3, 3], [8, 8])$.

(5) $\mathcal{D} = (\Sigma_1; 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k])$ *with* $k \geq 5$.

Proof. Thanks to Proposition 2.11 we can assume that $2 \leq s \leq d-2$. If $d \leq 16$, a computer-aided search (see the results in Section A.4) shows that the only exceptional cases are:

(1) $\mathcal{D} = (\Sigma_1; 6; [3, 3], [3, 3], [2, 4])$;

(2) $\mathcal{D} = (\Sigma_1; 8; [2, 2, 2, 2], [4, 4], [3, 5])$;

(3) $\mathcal{D} = (\Sigma_1; 10; [2, 2, 2, 2, 2], [2, 3, 5], [5, 5])$;

(4) $\mathcal{D} = (\Sigma_1; 12; [2, 2, 2, 2, 2, 2], [3, 3, 3, 3], [5, 7])$;

(5) $\mathcal{D} = (\Sigma_1; 12; [2, 2, 2, 2, 2, 2], [2, 2, 3, 5], [6, 6])$;

(6) $\mathcal{D} = (\Sigma_1; 14; [2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 3, 5], [7, 7])$;

(7) $\mathcal{D} = (\Sigma_1; 16; [2, 2, 2, 2, 2, 2, 2, 2], [1, 3, 3, 3, 3, 3], [8, 8])$;

(8) $\mathcal{D} = (\Sigma_1; 16; [2, 2, 2, 2, 2, 2, 2, 2], [2, 2, 2, 2, 3, 5], [8, 8])$.

This is in agreement with the theorem statement, so we can assume that $d \geq 17$. We now analyze several cases.

Case 1. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $2 \in \pi_2$;
- $\pi_2 \neq [2, \ldots, 2]$.

Combinatorial move B.2 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-2; \pi_1 \smallsetminus [x] \cup [1, x-3], \pi_2 \smallsetminus [2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless one of the following holds[2].

- $\mathcal{D} = (\Sigma_1; kh+2; [1, \ldots, 1, k+4], [2, h, \ldots, h], [lh+1, (k-l)h+1])$ with $k \geq 2$, $h \geq 3$, $1 \leq l \leq k-1$. By applying Combinatorial move B.2 we get

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; kh; [1, \ldots, 1, 2, k], [h, \ldots, h], [lh, (k-l)h]),$$

  which is realizable by Theorem 4.1.

- $\mathcal{D} = (\Sigma_1; kh+2; [1, \ldots, 1, 4, k+1], [2, h, \ldots, h], [lh+1, (k-l)h+1])$ with $k \geq 2$, $h \geq 3$, $1 \leq l \leq k-1$. By applying Combinatorial move B.4 we get

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; kh; [1, \ldots, 1, 2, k+1], [2, h-2, h, \ldots, h], [lh, (k-l)h]),$$

  which is realizable by Theorem 4.1.

Case 2. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $y \in \pi_2$ for some $y \geq 4$;
- $2 \notin \pi_1$ and $2 \notin \pi_2$.

Combinatorial move B.4 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-2; \pi_1 \smallsetminus [x] \cup [x-2], \pi_2 \smallsetminus [y] \cup [y-2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless

$$\mathcal{D} = (\Sigma_1; kh+2; [1, \ldots, 1, k+3], [h, \ldots, h, h+2], [lh+1, (k-l)h+1])$$

for some $k \geq 2$, $h \geq 3$, $1 \leq l \leq k-1$. If this is the case, applying Combinatorial move B.4 yields

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; kh; [1, \ldots, 1, k+1], [h-2, h, \ldots, h, h+2], [lh, (k-l)h]),$$

which is realizable by Theorem 4.1.

[2]Since this proof is long enough as it is, we will refrain from explaining in detail why candidate data of a certain form are realizable by Theorem 4.1; we leave the tedious yet elementary casework required to the motivated reader. In this situation, for instance, one could simply note that $\pi_1 \smallsetminus [x] \cup [1, x-3]$ and $\pi_2 \smallsetminus [2]$ are both different from $[2, \ldots, 2]$, therefore the only exceptional data to consider among those listed in Theorem 4.1 are those belonging to family (6).

Case 3. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $\max(\pi_2) = 3$;
- $2 \notin \pi_2$.

Combinatorial move B.4 gives

$$\mathcal{D} \rightsquigarrow = (\mathbb{S}; d-2; \pi_1 \smallsetminus [x] \cup [x-2], \pi_2 \smallsetminus [3] \cup [1], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless

$$\mathcal{D} = (\Sigma_1; 2h+2; [h, h+2], [1, \ldots, 1, 3, 3], [h+1, h+1])$$

for some $h \geq 8$ (recall that we are assuming $d \geq 17$). If this is the case, applying Combinatorial move B.1 yields

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2h+2; [h, h+2], [1, \ldots, 1, 3], [h+1, h+1]),$$

which is realizable by Theorem 4.1.

Case 4. Assume that:

- $\max(\pi_1) = 3$;
- $\max(\pi_2) \leq 3$;
- $\pi_2 \neq [2, \ldots, 2]$.

We analyze a few sub-cases.

- **Case 4.1:** $[1, 1] \subseteq \pi_1$. Combinatorial move B.1 gives

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; d; \pi_1 \smallsetminus [3] \cup [1, 1, 1], \pi_2, [s, d-s]),$$

  which is realizable by Theorem 4.1.

- **Case 4.2:** $[3, 3] \subseteq \pi_1$ and $[2, 2] \subseteq \pi_2$. If $s = 2$ or $s = d-2$ then $\mathcal{D}$ is realizable by Proposition 3.2. Otherwise, Combinatorial move B.3 gives

  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-4; \pi_1 \smallsetminus [3, 3] \cup [1, 1], \pi_2 \smallsetminus [2, 2], [s-2, d-s-2]),$$

  which is realizable by Theorem 4.1.

- **Case 4.3:** $[2, 2] \not\subseteq \pi_2$. The Riemann-Hurwitz formula immediately implies that $3 \in \pi_2$. Assume by contradiction that $\mathcal{D}$ is not realizable. If this is the case, $[1, 1] \not\subseteq \pi_2$ by case 4.1, but then $[3, 3] \subseteq \pi_2$. It follows (case 4.2) that $[2, 2] \not\subseteq \pi_1$; moreover, case 4.1 also implies that $[1, 1] \not\subseteq \pi_1$. In other words, both π_1 and π_2 can be written as $\rho \cup [3, \ldots, 3]$, where $\rho \subseteq [1, 2]$ (possibly different for π_1 and π_2). As a consequence, we have the inequalities

  $$d \geq 3\ell(\pi_1) - 3, \qquad d \geq 3\ell(\pi_2) - 3,$$

  which contradict the Riemann-Hurwitz formula if $d \geq 13$.

- **Case 4.4:** $[3, 3] \not\subseteq \pi_1$. From the Riemann-Hurwitz formula it follows that $[3, 3] \subseteq \pi_2$, but then $\mathcal{D}$ is realizable by case 4.1.

Case 5. Assume that:

- $\max(\pi_1) \geq 4$;
- $\pi_2 = [2, \ldots, 2]$.

Let $x = \max(\pi_1)$; Combinatorial move B.2 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-2; \pi_1 \setminus [x] \cup [1, x-3], [2, \ldots, 2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.1 unless one of the following holds.

- $\mathcal{D} = (\Sigma_1; 2k+2; [2, \ldots, 2, 3, 5], [2, \ldots, 2], [k+1, k+1])$ with $k \geq 7$. This is in fact one of the exceptional data listed in the statement.
- $\mathcal{D} = (\Sigma_1; 2k+2; [2, \ldots, 2, 6], [2, \ldots, 2], [k+1, k+1])$ with $k \geq 7$. By applying Combinatorial move B.2 we get
  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2k; [2, \ldots, 2], [2, \ldots, 2], [k, k]),$$
  which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 4k+4; [1, \ldots, 1, k+2, k+4], [2, \ldots, 2], [2k+2, 2k+2])$ with $k \geq 3$. By applying Combinatorial move B.2 we get
  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; 4k+2; [1, \ldots, 1, 2, k, k+2], [2, \ldots, 2], [2k+1, 2k+1]),$$
  which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 4k+4; [1, \ldots, 1, k+1, k+5], [2, \ldots, 2], [2k+2, 2k+2])$ with $k \geq 3$. By applying Combinatorial move B.2 we get
  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; 4k+2; [1, \ldots, 1, 2, k+1, k+1], [2, \ldots, 2], [2k+1, 2k+1]),$$
  which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 4k+2; [1, \ldots, 1, k+1, k+4], [2, \ldots, 2], [2k, 2k+2])$ with $k \geq 4$. By applying Combinatorial move B.2 we get
  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; 4k; [1, \ldots, 1, 2, k, k+1], [2, \ldots, 2], [2k-1, 2k+1]),$$
  which is realizable by Theorem 4.1.
- $\mathcal{D} = (\Sigma_1; 2k+2; [1, \ldots, 1, k+4], [2, \ldots, 2], [2l+1, 2(k-l)+1])$ with $k \geq 7$, $1 \leq l \leq k-1$. By applying Combinatorial move B.2 we get
  $$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2k; [1, \ldots, 1, 2, k], [2, \ldots, 2], [2l, 2(k-l)]),$$
  which is realizable by Theorem 4.1.

Case 6. Assume that:

- $\max(\pi_1) = 3$;
- $\pi_2 = [2, \dots, 2]$.

The Riemann-Hurwitz formula immediately implies that $[3, 3, 3, 3] \subseteq \pi_1$. If $s = 2$ or $s = d - 2$ then $\mathcal{D}$ is realizable by Proposition 3.2. Otherwise, Combinatorial move B.3 gives

$$\mathcal{D} \rightsquigarrow (\mathbb{S}; d-4; \pi_1 \smallsetminus [3,3] \cup [1,1], [2, \dots, 2], [s-2, d-s-2]),$$

which is realizable by Theorem 4.1, since we are assuming that $d \geq 17$.

The cases we have analyzed, up to swapping π_1 and π_2, cover all the candidate data of the form $(\Sigma_1; d; \pi_1, \pi_2, [s, d-s])$. We have shown that every datum which is not listed in the statement is realizable, therefore the proof is complete. □

4.3 Realizability on higher genus surfaces for $n = 3$

It turns out that, for $n = 3$ and $\ell(\pi_3) = 2$, there are no exceptional data on surfaces with genus $g \geq 2$.

Theorem 4.4. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ be a candidate datum with $g \geq 2$. Then $\mathcal{D}$ is realizable.*

Proof. Thanks to Proposition 2.11 we can assume that $2 \leq s \leq d - 2$. For $d \leq 18$, a computer-aided search shows that there are no exceptional data (see the results in Section A.4). Therefore, we can further assume that $d \geq 19$. We proceed by induction on $g \geq 2$, analyzing several cases.

Case 1. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $2 \in \pi_2$.

Combinatorial move B.2 gives

$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d-2; \pi_1 \smallsetminus [x] \cup [1, x-3], \pi_2 \smallsetminus [2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

Case 2. Assume that:

- $x \in \pi_1$ for some $x \geq 4$;
- $y \in \pi_2$ for some $y \geq 3$;
- $2 \notin \pi_1$ and $2 \notin \pi_2$.

Combinatorial move B.4 gives

$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d-2; \pi_1 \smallsetminus [x] \cup [x-2], \pi_2 \smallsetminus [y] \cup [y-2], [s-1, d-s-1]),$$

which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

Case 3. Assume that:

- $\max(\pi_1) = 3$;
- $\max(\pi_2) \leq 3$.

We analyze a few sub-cases.

- **Case 3.1:** $[1,1] \subseteq \pi_1$. Combinatorial move B.1 gives
$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d; \pi_1 \setminus [3] \cup [1,1,1], \pi_2, [s, d-s]),$$
which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

- **Case 3.2:** $[3,3] \subseteq \pi_1$ and $[2,2] \subseteq \pi_2$. If $s = 2$ or $s = d-2$ then $\mathcal{D}$ is realizable by Proposition 3.2. Otherwise, Combinatorial move B.3 gives
$$\mathcal{D} \rightsquigarrow (\Sigma_{g-1}; d-4; \pi_1 \setminus [3,3] \cup [1,1], \pi_2 \setminus [2,2], [s-2, d-s-2]),$$
which is realizable by Theorem 4.3 if $g = 2$, or by induction if $g \geq 3$.

- **Case 3.3:** $[2,2] \not\subseteq \pi_1$. The Riemann-Hurwitz formula immediately implies that $[3,3] \subseteq \pi_2$. Assume by contradiction that $\mathcal{D}$ is not realizable. Then $[1,1] \not\subseteq \pi_1$ and $[1,1] \not\subseteq \pi_2$ (case 3.1), and moreover $[2,2] \not\subseteq \pi_1$ (case 3.2). In other words, both π_1 and π_2 can be written as $\rho \cup [3, \ldots, 3]$, where $\rho \subseteq [1,2]$ (possibly different for π_1 and π_2). It is then easy to see that $\mathcal{D}$ belongs to one of the families listed in Proposition 3.3 and, therefore, is realizable.

- **Case 3.4:** $[3,3] \not\subseteq \pi_1$. From the Riemann-Hurwitz formula it follows that $[3,3] \subseteq \pi_2$, but then $\mathcal{D}$ is realizable by cases 3.2 and 3.3.

The cases we have analyzed, up to swapping π_1 and π_2, cover all the candidate data of the form $(\Sigma_g; d; \pi_1, \pi_2, [s, d-s])$ with $g \geq 2$. We have shown that every datum is realizable, therefore the proof is complete. □

4.4 Realizability for $n \geq 4$

After solving the existence problem for $n = 3$ and $\ell(\pi_3) = 2$, we turn to candidate data with $n \geq 4$ partitions. As we are about to see, exceptional data in this setting are very rare: there is an infinite family with $d = 4$, which we already encountered in Proposition 2.14, and a single datum with $n = 4$ and $d = 8$.

Theorem 4.5. *Let $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ be a candidate datum with $n \geq 4$. Then $\mathcal{D}$ is realizable unless it satisfies one of the following.*

(1) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

(2) $\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$.

Proof. If $d \leq 16$, a computer-aided search (see the results in Section A.4) shows that the only exceptional cases are:

(1) $\mathcal{D} = (\Sigma_1; 4; [2,2], [2,2], [2,2], [1,3])$;

(2) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

This is in agreement with the statement, so we can assume that $d \geq 17$. Moreover, every candidate datum with $g = 0$ is realizable by Theorem 4.2; as a consequence, we only have to consider the cases where $g \geq 1$.

We will proceed by induction on n. We start with the base case $n = 4$, which requires the heaviest casework. Fix a candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \pi_2, \pi_3, [s, d-s])$.

- Assume that the inequality $v(\pi_i) + v(\pi_j) < d$ holds for a pair of indices $1 \leq i < j \leq 3$; up to reindexing, we can assume that $v(\pi_1) + v(\pi_2) < d$. Combinatorial move A.1 gives

  $$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_g; d; \pi_1', \pi_3, [s, d-s])$$

  for a suitable candidate datum $\mathcal{D}'$, where $v(\pi_1') = v(\pi_1) + v(\pi_2)$. If $g \geq 2$, then $\mathcal{D}'$ is realizable by Theorem 4.4. If instead $g = 1$, then $\mathcal{D}'$ is realizable by Theorem 4.3 unless

  $$\mathcal{D}' = (\Sigma_1; 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k]) \text{ with } 2k = d.$$

  If this is the case, then $\pi_4 = [k, k]$ and $\{\pi_1', \pi_3\} = \{[2, \ldots, 2], [2, \ldots, 2, 3, 5]\}$. Some more casework is required to show that $\mathcal{D}'$ can actually be chosen to be realizable; since $\mathcal{D} \rightsquigarrow \mathcal{D}'$, this will imply that $\mathcal{D}$ is realizable as well.

  - If $\pi_1' = [2, \ldots, 2]$, then $v(\pi_1') = k$ and $v(\pi_3) = k + 2$. Assume without loss of generality that $v(\pi_1) \leq k/2$. We have that

    $$k + 2 < 1 + k + 2 \leq v(\pi_1) + v(\pi_3) \leq \frac{k}{2} + k + 2 < d.$$

    Repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \ldots, 2]$ and $v(\pi_1) \notin \{2, k, k+1\}$ then $v(\pi_1) + v(\pi_3) < d$ and $v(\pi_1) + v(\pi_3) \notin \{k, k+2\}$. Therefore, repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_3 = [2, \ldots, 2]$, $v(\pi_1) = 2$ and $\pi_2 \neq [2, \ldots, 2]$, then repeating the construction with $i = 1$ and $j = 3$ will yield a realizable $\mathcal{D}'$.
  - If $\pi_2 = \pi_3 = [2, \ldots, 2]$ and $\pi_1 \neq [1, \ldots, 1, 3]$, we follow a different approach. By applying Combinatorial move A.2 to the partitions π_2 and π_3 we get

    $$\mathcal{D} \rightsquigarrow (\mathbb{S}; 2k; \pi_1, [k, k], [k, k]),$$

    which is realizable by Theorem 4.1.
  - Finally, if $\pi_1 = [1, \ldots, 1, 3]$ and $\pi_2 = \pi_3 = [2, \ldots, 2]$, we have to work explicitly with permutations. Consider

    $$\alpha_1 = (1, 3, 5), \qquad \alpha_2 = (1, 2)(3, 4) \cdots (2k-1, 2k).$$

    Clearly $[\alpha_1] = \pi_1$ and $[\alpha_2] = \pi_2$; moreover,

    $$\alpha_1 \alpha_2 = (1, 2, 3, 4, 5, 6)(7, 8) \cdots (2k-1, 2k),$$

    so $[\alpha_1 \alpha_2] = [2, \ldots, 2, 6]$. Note that

    $$v(\alpha_1) + v(\alpha_2) = 2 + k = v(\alpha_1 \alpha_2),$$

    so Remark 2.2 gives

    $$\mathcal{D} \rightsquigarrow (\Sigma_1; 2k; [2, \ldots, 2, 6], [2, \ldots, 2], [k, k]),$$

    which is realizable by Theorem 4.3.

Up to swapping π_1 and π_2, this analysis covers all the possible cases.

- Otherwise, the inequality $v(\pi_i) + v(\pi_j) \geq d$ holds for every $1 \leq i < j \leq 3$. In particular, up to reindexing, we can assume that $v(\pi_3) \geq d/2$. Note that $v(\pi_1) + v(\pi_2) \geq d$ and $v(\pi_3) + v([s, d-s]) \geq 1 + d - 2 = d - 1$, so Combinatorial move A.2 gives

$$\mathcal{D} \rightsquigarrow \mathcal{D}' = (\Sigma_{g'}; d; \pi_1', \pi_3, [s, d-s])$$

for a suitable candidate datum $\mathcal{D}'$ with $v(\pi_1') \geq d - 2$. We can actually compute

$$g' = \frac{1}{2}(v(\pi_1') + v(\pi_3) + v([s, d-s]) - d + 1 \geq \frac{1}{2}\left(d - 2 + \frac{d}{2} + d - 2\right) - d + 1 = \frac{d}{4} - 1 \geq 2.$$

Therefore $\mathcal{D}'$ is realizable by Theorem 4.4, and $\mathcal{D}$ is realizable as well.

We now turn to the case $n \geq 5$; we show by induction that every candidate datum $\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$ different from $(\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$ is realizable. The case $d = 4$ is addressed by Proposition 2.14. If $n = 5$ and $d = 8$, the computational results in Section A.4 imply that $\mathcal{D}$ is realizable. Otherwise, the routine reduction argument relying on Combinatorial moves A.1 and A.2 shows that $\mathcal{D}$ is realizable. □

4.5 Exceptionality

This section is devoted to showing that the candidate data listed in the statements of Theorems 4.1, 4.3 and 4.5 are in fact exceptional. First of all, we address the special cases with small degree ($d \leq 16$). The computational results in Section A.4 show that the following candidate data are exceptional.

(1) $\mathcal{D} = (\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

(2) $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$.

(3) $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$.

(4) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

(5) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$.

(6) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

We now turn to infinite families of exceptional data.

Proposition 4.6. *Let $n \geq 3$ be a positive integer. Then the candidate datum*

$$\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$$

is exceptional.

Proof. In order to show the exceptionality of $\mathcal{D}$, we will employ the monodromy approach. Let $\alpha_1, \ldots, \alpha_{n-1} \in \mathfrak{S}_4$ be permutations matching $[2,2]$. It is easy to see that the three permutations matching $[2,2]$, together with the identity, form a subgroup

$$\{\mathrm{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \leq \mathfrak{S}_4,$$

which does not contain any 3-cycles. As a consequence, the product $\alpha_1 \cdots \alpha_{n-1}$ cannot match $[1,3]$; by Corollary 1.7, this implies that $\mathcal{D}$ is exceptional. □

For the other families of exceptional data, we will instead make use of dessins d'enfant.

Proposition 4.7. *Let $h \geq 2$, $k \geq 2$, $1 \leq l \leq k-1$ be integers. Then the candidate datum*

$$\mathcal{D} = (\mathbb{S}; kh; [h, \ldots, h], [1, \ldots, 1, k+1], [lh, (k-l)h])$$

is exceptional.

Proof. Assume by contradiction that there is a dessin d'enfant $\Gamma \subseteq \mathbb{S}$ realizing $\mathcal{D}$. Let v be the white vertex of degree $k+1$. Since Γ is connected, each of the k black vertices must have an edge connecting it to v. Therefore there are exactly $k-1$ black vertices with one edge between them and v, and one black vertex with two edges between it and v; we call this special black vertex u. The two edges connecting u and v define the two complementary disks of Γ. As shown in the picture below, there are (excluding u) a black vertices inside D_1 and $k-1-a$ black vertices inside D_2, for some integer $0 \leq a \leq k-1$; of the $h-2$ white vertices of degree 1 connected to u, b lie inside D_1, while $h-2-b$ lie inside D_2, for some $0 \leq b \leq h-2$. It is easy to compute the perimeters:

$$\frac{1}{2}|\partial D_1| = ah + b + 1, \qquad \frac{1}{2}|\partial D_2| = (k-1-a)h + (h-2-b) + 1.$$

In particular, we see that $|\partial D_1|/2$ and $|\partial D_2|/2$ are not divisible by h, hence Γ cannot be a dessin d'enfant realizing $\mathcal{D}$. □

The following result will allow us to deal with candidate data of the form $(\Sigma_g; d; [2, \ldots, 2], \pi_2, \pi_3)$.

Lemma 4.8. *Let $\mathcal{D} = (\Sigma_g; d; [2, \ldots, 2], \pi_2, \pi_3)$ be a combinatorial datum. Then $\mathcal{D}$ is realizable if and only if there exists a graph Γ embedded in Σ_g such that:*

- $\pi_2 = [k(v_1), \ldots, k(v_r)]$, *where $v_1, \ldots, v_r$ are the vertices of Γ;*
- *the complementary regions of Γ are topological disks $D_1, \ldots, D_h$;*
- $\pi_3 = [|\partial D_1|, \ldots, |\partial D_h|]$.

Proof. The key idea is that vertices of degree 2 do not contribute to the topology of a graph in any meaningful way.

If are given a dessin d'enfant $\Gamma' \subseteq \Sigma_g$ realizing $\mathcal{D}$, we can obtain a suitable graph Γ simply by removing all the black vertices, and merging the two edges adjacent to each one of them into a single edge.

Conversely, if we are given a graph $\Gamma \subseteq \Sigma_g$, we can recover a dessin d'enfant Γ' by coloring the vertices of Γ with the color white, and inserting a black vertex in the middle of every edge. □

This lemma suggests an approach by enumeration for showing the exceptionality of candidate data of the form $(\Sigma_g; d; [2, \ldots, 2], \pi_2, \pi_3)$. In fact, if the number of fat graphs whose degrees are the entries of π_2 is small, we can check them one by one to see if the associated embedded graph satisfies the conditions of Lemma 4.8.

Proposition 4.9. *The following families of candidate data are exceptional.*

(1) $\mathcal{D} = (\mathbb{S}; 2k; [2, \ldots, 2], [2, \ldots, 2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

(2) $\mathcal{D} = (\mathbb{S}; 2k; [2, \ldots, 2], [1, 2, \ldots, 2, 3], [k, k])$ *with* $k \geq 2$.

(3) $\mathcal{D} = (\mathbb{S}; 4k+2; [2, \ldots, 2], [1, \ldots, 1, k+1, k+2], [2k+1, 2k+1])$ *with* $k \geq 1$.

(4) $\mathcal{D} = (\mathbb{S}; 4k; [2, \ldots, 2], [1, \ldots, 1, k+1, k+1], [2k-1, 2k+1])$ *with* $k \geq 2$.

Proof. For each family listed in the statement, we follow the approach by enumeration presented above: we draw all the possible graphs embedded in $\mathbb{S}$ whose vertices have the entries of π_2 as degrees and whose complementary regions are two disks, and then we compute the perimeters of these disks, showing that they cannot be equal to the entries of π_3.

(1) There is only one graph embedded in $\mathbb{S}$ whose vertices have degrees $[2, \ldots, 2]$.

By Lemma 4.8, it follows that $\mathcal{D}$ is exceptional unless $s = k$.

(2) A graph embedded in $\mathbb{S}$ whose vertices have degrees $[1, 2, \ldots, 2, 3]$ can be represented by a diagram like the following, where a denotes the number of blue edges.

The perimeters of the complementary disks are easily computed as

$$|\partial D_1| = k - a - 1, \qquad |\partial D_2| = k + a + 1,$$

hence $\mathcal{D}$ is exceptional by Lemma 4.8.

(3) There are only three kinds of graphs embedded in $\mathbb{S}$ whose vertices have degrees $[1, \ldots, 1, k+1, k+2]$. First case:

Probably a table?

$$|\partial D_1| = 4k - 2a - 2b, \qquad |\partial D_2| = 2a + 2b + 2.$$

Second case:

$$|\partial D_1| = 2k - 2b - 1, \qquad |\partial D_2| = 2k + 2b + 3.$$

Third case:

$$|\partial D_1| = 2k - 2a + 1, \qquad |\partial D_2| = 2k + 2a + 1.$$

(4) A graph embedded in $\mathbb{S}$ whose vertices have degrees $[1, \ldots, 1, k+1, k+2]$ can be represented by a diagram like the following, where a denotes the number of blue edges connected to the vertex of degree $k+1$, and b denotes the number of blue edges connected to the vertex of degree $k+2$.

The perimeters of the complementary disks are easily computed as

$$|\partial D_1| = k - a - 1, \qquad |\partial D_2| = k + a + 1,$$

hence $\mathcal{D}$ is exceptional by Lemma 4.8.

$\square$

Proposition 4.10. *Let* $k \geq 5$ *be an integer. Then the candidate datum*

$$\mathcal{D} = (\Sigma_1; 2k; [2, \ldots, 2], [2, \ldots, 2, 3, 5], [k, k])$$

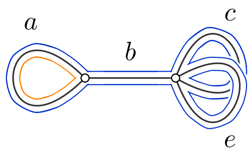
is exceptional.

Proof. We start by enumerating all the abstract graphs whose vertices have degrees $[2, \ldots, 2, 3, 5]$, without considering the embedding in Σ_1. Once again, we observe that vertices of degree 2 do not contribute to the topology of the graph, so we can reduce the number of cases to two.

(1) (2)

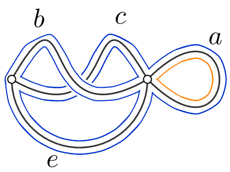
In the pictures above, the arcs are labeled with a letter representing their length (in terms of edges) or, in other words, the number of degree-2 vertices that lie on them plus 1. In particular, $a+b+c+e=k$. We now analyze the two cases separately.

(1) There is only one embedding of this graph in Σ_1 whose complementary regions are two discs.

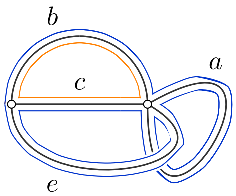


$$a, b, c, e \geq 1$$
$$a+b+c+e=k$$
$$|\partial D_1| = a < k$$

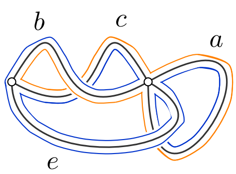
(2) There are three topologically different embedding of this graph in Σ_1 whose complementary regions are two disks.



$$a, b, c, e \geq 1$$
$$a+b+c+e=k$$
$$|\partial D_1| = a < k$$



$$a, b, c, e \geq 1$$
$$a+b+c+e=k$$
$$|\partial D_1| = b+c < k$$



$$a, b, c, e \geq 1$$
$$a+b+c+e=k$$
$$|\partial D_1| = a+b+c < k$$

In all of the cases, we see that $|\partial D_1| \neq k$; if follows that there is no dessin d'enfant $\Gamma \subseteq \Sigma_1$ such that $\mathcal{D}(\Gamma) = \mathcal{D}$. $\square$

4.6 Final result

By combining all the results in this chapter, we are finally able to compile the full list of exceptional data with a partition of length 2.

Solution of the Hurwitz existence problem for $\ell(\pi_n) = 2$. *Let*

$$\mathcal{D} = (\Sigma_g; d; \pi_1, \ldots, \pi_{n-1}, [s, d-s])$$

be a candidate datum with $n \geq 3$. Then $\mathcal{D}$ is exceptional if and only if one of the following holds.

(1) $\mathcal{D} = (\mathbb{S}; 12; [2,2,2,2,2,2], [1,1,1,3,3,3], [6,6])$.

(2) $\mathcal{D} = (\mathbb{S}; 2k; [2,\ldots,2], [2,\ldots,2], [s, 2k-s])$ *with* $k \geq 2$, $s \neq k$.

(3) $\mathcal{D} = (\mathbb{S}; 2k; [2,\ldots,2], [1,2,\ldots,2,3], [k,k])$ *with* $k \geq 2$.

(4) $\mathcal{D} = (\mathbb{S}; 4k+2; [2,\ldots,2], [1,\ldots,1,k+1,k+2], [2k+1, 2k+1])$ *with* $k \geq 1$.

(5) $\mathcal{D} = (\mathbb{S}; 4k; [2,\ldots,2], [1,\ldots,1,k+1,k+1], [2k-1, 2k+1])$ *with* $k \geq 2$.

(6) $\mathcal{D} = (\mathbb{S}; kh; [h,\ldots,h], [1,\ldots,1,k+1], [lh, (k-l)h])$ *with* $h \geq 2$, $k \geq 2$, $1 \leq l \leq k-1$.

(7) $\mathcal{D} = (\Sigma_1; 6; [3,3], [3,3], [2,4])$.

(8) $\mathcal{D} = (\Sigma_1; 8; [2,2,2,2], [4,4], [3,5])$.

(9) $\mathcal{D} = (\Sigma_1; 12; [2,2,2,2,2,2], [3,3,3,3], [5,7])$.

(10) $\mathcal{D} = (\Sigma_1; 16; [2,2,2,2,2,2,2,2], [1,3,3,3,3,3], [8,8])$.

(11) $\mathcal{D} = (\Sigma_1; 2k; [2,\ldots,2], [2,\ldots,2,3,5], [k,k])$ *with* $k \geq 5$.

(12) $\mathcal{D} = (\Sigma_2; 8; [2,2,2,2], [2,2,2,2], [2,2,2,2], [3,5])$.

(13) $\mathcal{D} = (\Sigma_{n-3}; 4; [2,2], \ldots, [2,2], [1,3])$.