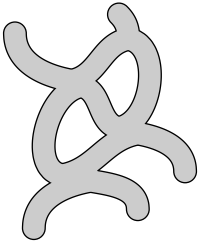
Reeb graph

Definition

Given:

- a manifold $\mathcal{M}$;

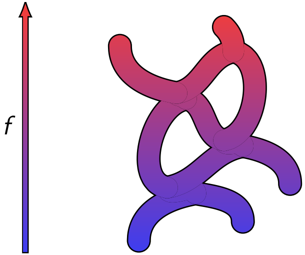


Reeb graph

Definition

Given:

- a manifold $\mathcal{M}$;
- a Morse function $f\colon \mathcal{M} \to \mathbb{R}$ with distinct critical values;



Reeb graph

Definition

Given:

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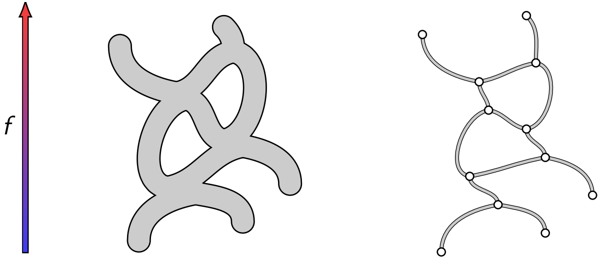
the **Reeb graph** of f is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M}/{\sim}.$$



Reeb graph

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the **Reeb graph** of f is the 1-dimensional simplicial complex

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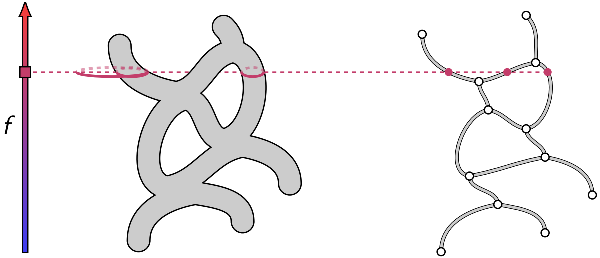
$x \sim y$ if $f(x) = f(y)$ and they belong to the same connected component of $f^{-1}(f(x))$



Reeb graph

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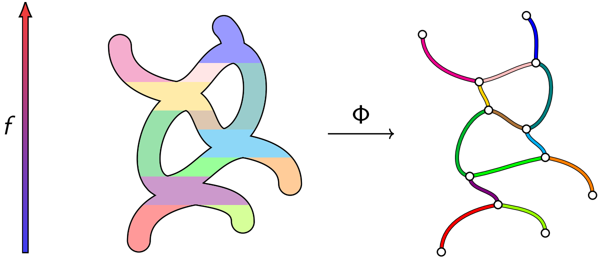
$$\mathcal{R}(f) = \mathcal{M}/\sim.$$

The **segmentation map** is the quotient map

$$\Phi\colon \mathcal{M} \longrightarrow \mathcal{R}(f).$$

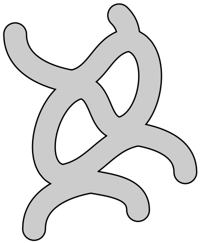


Reeb graph

Desired algorithm

Input:

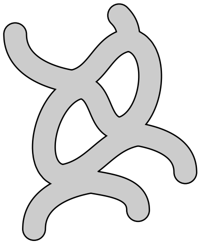
- a PL manifold $\mathcal{M}$



Reeb graph
Desired algorithm

Input:

- a PL manifold $\mathcal{M}$
  $\rightsquigarrow$ a triangulated mesh $\mathcal{M}$;

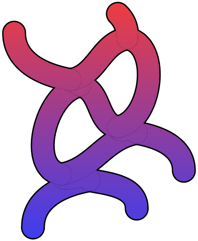


Reeb graph

Desired algorithm

Input:

- a PL manifold $\mathcal{M}$
  $\rightsquigarrow$ a triangulated mesh $\mathcal{M}$;
- a non-degenerate PL scalar field f on $\mathcal{M}$



Reeb graph

Desired algorithm

Input:

- a PL manifold $\mathcal{M}$
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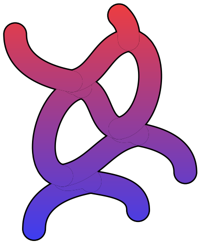
- a non-degenerate PL scalar field f on $\mathcal{M}$
  $\rightsquigarrow$ a scalar value $f(v)$ for each vertex v of $\mathcal{M}$.

pairwise different, in order to ensure non-degeneracy; this can be achieved by random perturbations



Reeb graph

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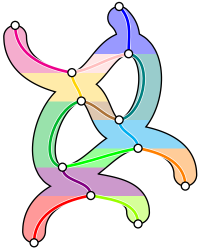

Output:

- the augmented Reeb graph $\mathcal{R}(f)$.

graph + segmentation map

Reeb graph

Desired algorithm

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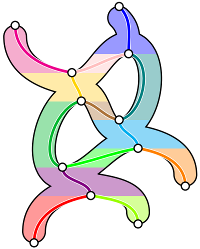

Output:

- the augmented Reeb graph $\mathcal{R}(f)$.

Time complexity:

- $O(m \cdot \log m)$, where m is the size of the 2-skeleton of $\mathcal{M}$.

#vertices + #edges + #triangles

Reeb graph

Desired algorithm

Input:

- a PL manifold $\mathcal{M}$
  $\rightsquigarrow$ a triangulated mesh $\mathcal{M}$;
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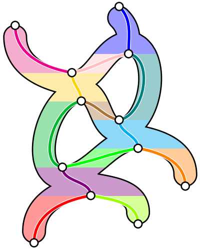

Output:

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Time complexity:

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Parallel.

Reeb graph

Geometry of critical points

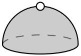
There are three kinds of critical points:

Reeb graph

Geometry of critical points

There are three kinds of critical points:

- (local) **maxima**



Reeb graph

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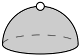
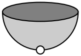


- (local) **minima**



Reeb graph

Geometry of critical points

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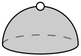
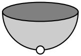
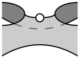


- (local) **minima**



- **saddles**



Reeb graph

Geometry of critical points

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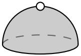

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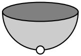

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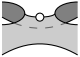
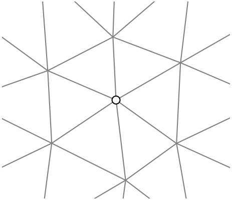


How to detect them on a PL manifold?



Reeb graph

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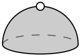

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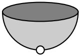

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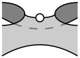
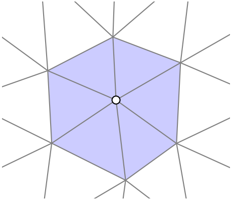


How to detect them on a PL manifold?

Given a vertex v, the **star** of v is the union of all simplices containing v.



Reeb graph

Geometry of critical points

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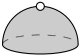

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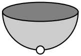

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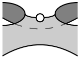

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Given a vertex v, the **star** of v is the union of all simplices containing v.
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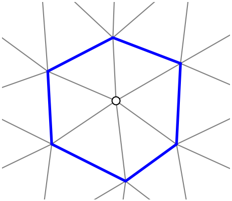

Reeb graph

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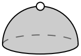

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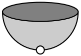
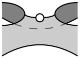


- **saddles**



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Given a vertex v, the **star** of v is the union of all simplices containing v.
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$$\mathrm{Link}^{+}(v) = \{x \in \mathrm{Link}(v) : f(x) > f(v)\}$$

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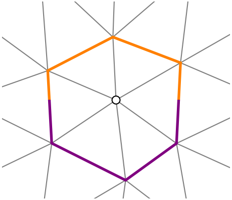
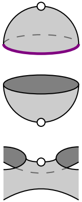


Reeb graph

Geometry of critical points

There are three kinds of critical points:

- (local) **maxima**
  $\rightsquigarrow$ Link$^+$ empty;
- (local) **minima**
- **saddles**



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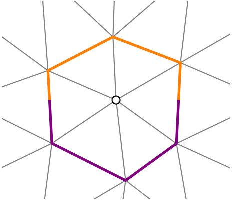

Reeb graph

Geometry of critical points

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- (local) **maxima** $\rightsquigarrow$ Link^+ empty;

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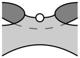

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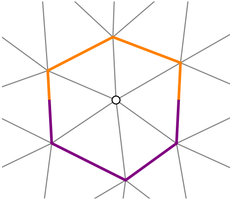

$$\text{Link}^+(v) = \{x \in \text{Link}(v) : f(x) > f(v)\}$$

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Reeb graph

Geometry of critical points

There are three kinds of critical points:

- (local) **maxima**
  $\rightsquigarrow$ Link$^+$ empty;

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  $\rightsquigarrow$ Link$^-$ empty;

- **saddles**
  $\rightsquigarrow$ Link$^+$ or Link$^-$ disconnected.

How to detect them on a PL manifold?

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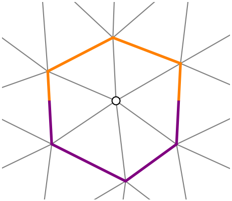

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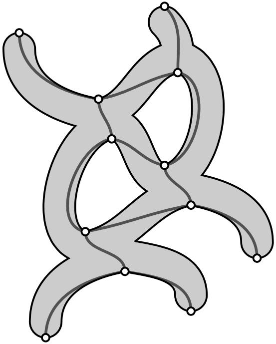
Reeb graph

Significance of critical points

The critical points of f are closely related to the topology of the Reeb graph $\mathcal{R}(f)$.

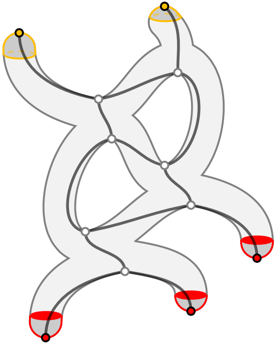


Reeb graph

Significance of critical points

The critical points of f are closely related to the topology of the Reeb graph $\mathcal{R}(f)$.

- **Maxima** and **minima** $\rightsquigarrow$ nodes of valence 1 (leaves).



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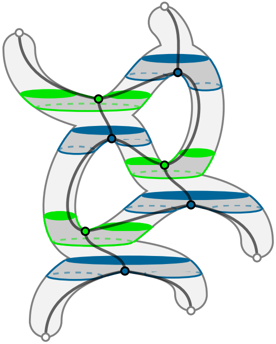

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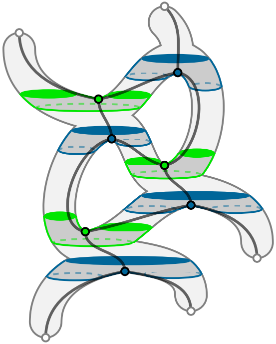
- **Saddles** $\rightsquigarrow$ nodes of valence ≥ 2.
  - **Join saddles**: multiple components below.
  - **Split saddles**: multiple components above.



Reeb graph

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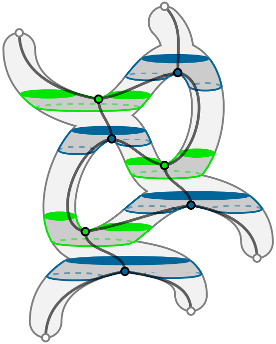
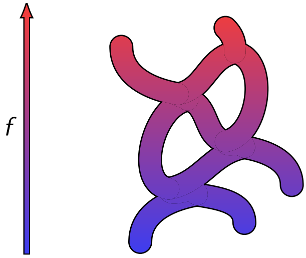
non-mutually exclusive in dimension ≥ 3



Sequential algorithm

Informal description

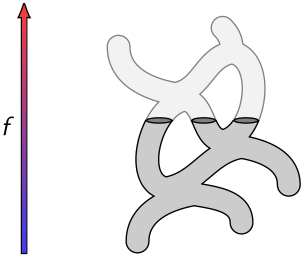
- Process the vertices of the mesh by **increasing** value of f.



Sequential algorithm

Informal description

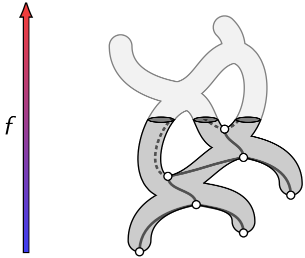
- Process the vertices of the mesh by **increasing** value of f.
- Construct the Reeb graph $\mathcal{R}(f)$ incrementally.



Sequential algorithm

Informal description

- Process the vertices of the mesh by **increasing** value of f.
- Construct the Reeb graph $\mathcal{R}(f)$ incrementally.
- While sweeping upwards, keep:
  - the **partial Reeb graph** constructed so far;



Sequential algorithm

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- Process the vertices of the mesh by **increasing** value of f.
- Construct the Reeb graph $\mathcal{R}(f)$ incrementally.
- While sweeping upwards, keep:
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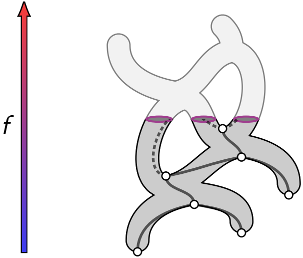
  - the current **level set** $f^{-1}(r)$.

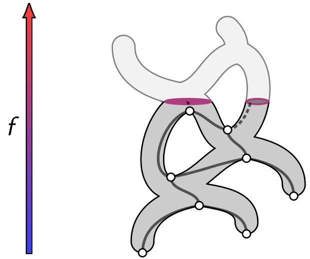
each connected component
corresponds to an open arc
of the partial Reeb graph



Sequential algorithm

Informal description

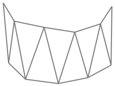
- Process the vertices of the mesh by **increasing** value of f.
- Construct the Reeb graph $\mathcal{R}(f)$ incrementally.
- While sweeping upwards, keep:
  - the **partial Reeb graph** constructed so far;
  - the current **level set** $f^{-1}(r)$.
- When processing a vertex, **update** the level set and the Reeb graph accordingly.



Sequential algorithm

The preimage graph

The level set $f^{-1}(r)$ can be represented by an abstract **graph** G_r:

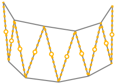


Sequential algorithm

The preimage graph

The level set $f^{-1}(r)$ can be represented by an abstract **graph** G_r:

- **nodes** $\rightsquigarrow$ edges of the mesh $\mathcal{M}$;



Sequential algorithm

The preimage graph

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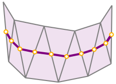


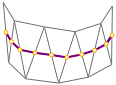
- **nodes** $\rightsquigarrow$ edges of the mesh $\mathcal{M}$;
- **arcs** $\rightsquigarrow$ triangles of $\mathcal{M}$ intersecting $f^{-1}(r)$.

a triangle connects its two sides intersecting $f^{-1}(r)$

Sequential algorithm

The preimage graph

The level set $f^{-1}(r)$ can be represented by an abstract **graph** G_r:



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Updating G_r

Sequential algorithm

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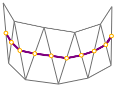

Updating G_r

- **Trigger**: update when processing a vertex v.

from $r = f(v) - \epsilon$ to $r = f(v) + \epsilon$

Sequential algorithm

The preimage graph

The level set $f^{-1}(r)$ can be represented by an abstract **graph** G_r:

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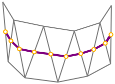
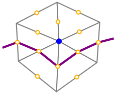


Updating G_r

- **Trigger**: update when processing a vertex v.
- **Action**: process each triangle $\mathcal{T}$ of $\mathrm{Star}(v)$ separately.



Sequential algorithm

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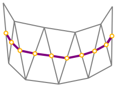

Updating G_r

- **Trigger**: update when processing a vertex v.
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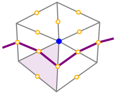
  1. v is the upper vertex of $\mathcal{T}$.



Sequential algorithm

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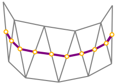

Updating G_r

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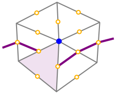

Sequential algorithm

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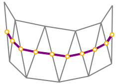

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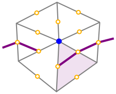
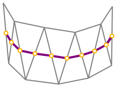


Sequential algorithm

The preimage graph

The level set $f^{-1}(r)$ can be represented by an abstract **graph** G_r:



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Updating G_r

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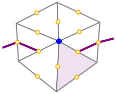

Sequential algorithm

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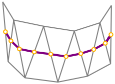

Updating G_r

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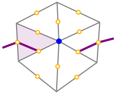
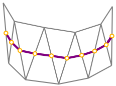


Sequential algorithm

The preimage graph

The level set $f^{-1}(r)$ can be represented by an abstract **graph** G_r:



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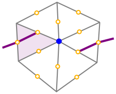

Sequential algorithm

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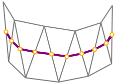

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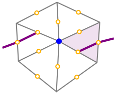

Sequential algorithm

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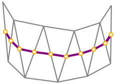

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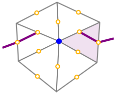

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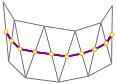

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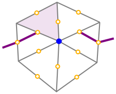

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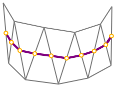

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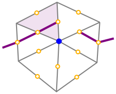

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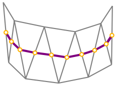

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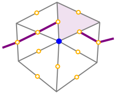

Sequential algorithm

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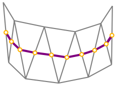

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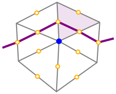

Sequential algorithm

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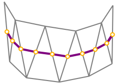

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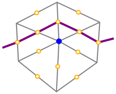

- **Data structure**: the following operations are required;

Sequential algorithm

The preimage graph

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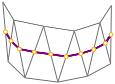

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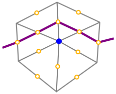

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Sequential algorithm

The preimage graph

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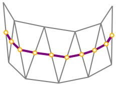

Updating G_r

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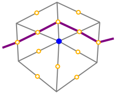

- **Data structure**: the following operations are required;
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    - `insert` a new arc between nodes e_1, e_2;

Sequential algorithm

The preimage graph

The level set $f^{-1}(r)$ can be represented by an abstract **graph** G_r:

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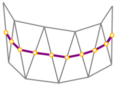

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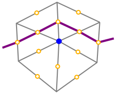

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Sequential algorithm

The preimage graph

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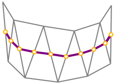

Updating G_r

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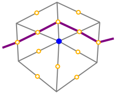

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$\rightsquigarrow$ offline dynamic connectivity problem

Sequential algorithm

The preimage graph

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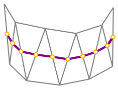

- **nodes** $\rightsquigarrow$ edges of the mesh $\mathcal{M}$;
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Updating G_r

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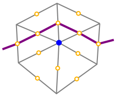


- **Data structure**: the following operations are required;
  - `find` the connected component of a node e;
  - `insert` a new arc between nodes e_1, e_2;
  - `delete` the arc between nodes e_1, e_2;

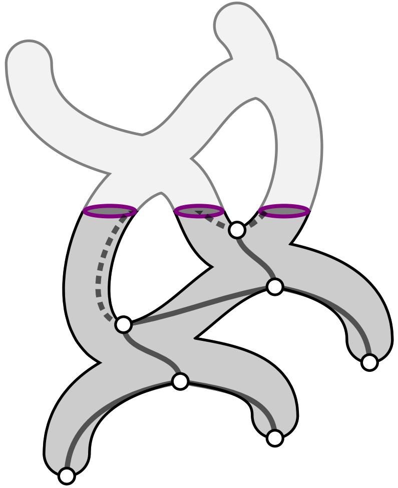
$\rightsquigarrow$ offline dynamic connectivity problem $\rightsquigarrow$ **ST-trees**

support all the operations in $O(\log m)$

Sequential algorithm

The augmented Reeb graph

The partial augmented Reeb graph is represented by a pair $(\mathcal{R}, \Phi)$.



Sequential algorithm

The augmented Reeb graph

Reeb graph constructed so far;
has one open arc for each component of G_r

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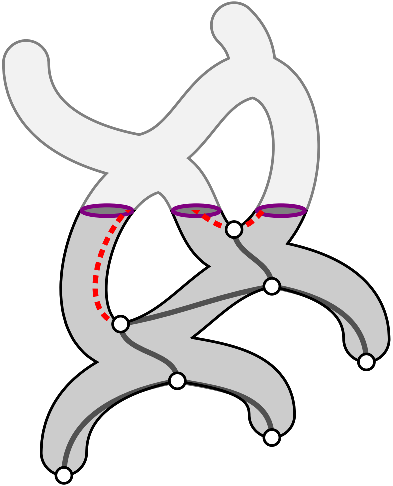

Sequential algorithm

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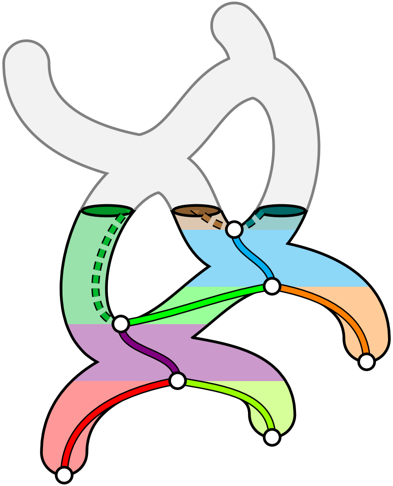
partial segmentation map;
maps each vertex of the mesh to a node or an arc of $\mathcal{R}$



Sequential algorithm

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Updating $(\mathcal{R}, \Phi)$

When processing a vertex v:

Sequential algorithm

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Updating $(\mathcal{R}, \Phi)$

When processing a vertex v:

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lower components

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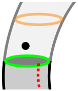
3. **If** $|$Lc$| = |$Uc$| = 1$ **then**:



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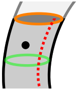
   - $\mathcal{R}$ is unchanged;



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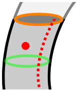

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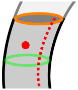

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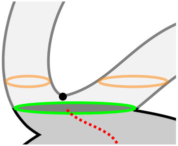

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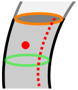

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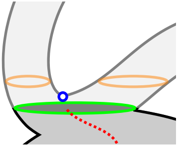

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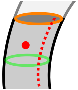

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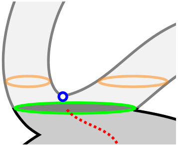

Sequential algorithm

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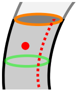

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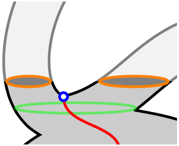

Sequential algorithm

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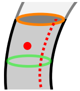
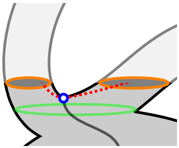


4. **Otherwise**:
   - create a new node w in $\mathcal{R}$;
   - $\Phi(v) = w$;
   - all the open arcs associated to the lower components end at w;
   - open a new arc in $\mathcal{R}$ starting at w for each upper component.



Sequential algorithm

Full implementation

input : a triangulated mesh $\mathcal{M}$
a scalar field f on $\mathcal{M}$
output : the augmented Reeb graph $(\mathcal{R}, \Phi)$

begin
  $\mathcal{R}$, $\Phi \leftarrow \varnothing$ [graph], $\varnothing$ [function]
  $G_r \leftarrow \varnothing$ [ST-tree]
  sort the vertices of $\mathcal{M}$ by increasing value of f
  foreach *v vertex of $\mathcal{M}$* **do**
    Lc $\leftarrow$ `GetLowerComponents(`v`)`
    `UpdatePreimageGraph()`
    Uc $\leftarrow$ `GetUpperComponents(`v`)`
    if $|\,\mathrm{Lc}\,| = |\,\mathrm{Uc}\,| = 1$ **then** update $\Phi(v)$
    else `UpdateReebGraph(`v, Lc, Uc`)`
  end
  return $(\mathcal{R}, \Phi)$
end

Parallel algorithm

Core ideas

- **Sequential**: single procedure sweeping all the vertices sequentially.

Parallel algorithm

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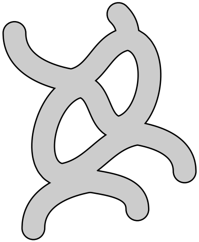

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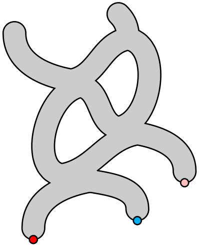

Parallel algorithm

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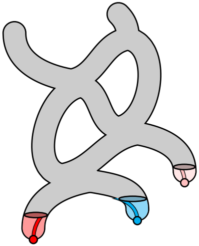

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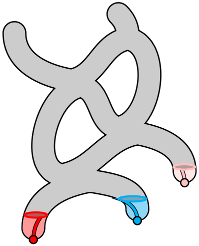

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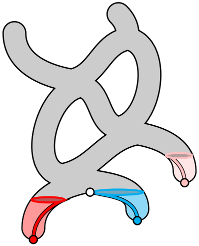

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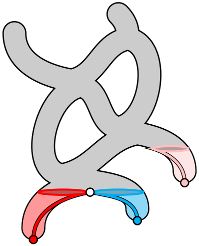

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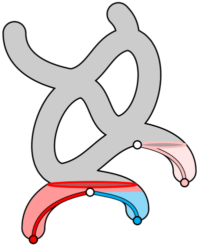

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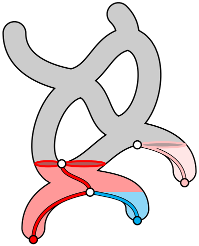

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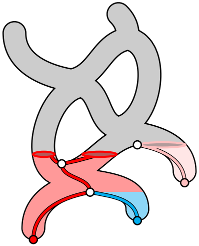

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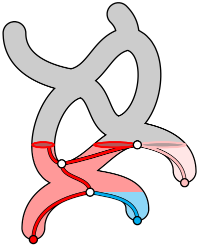

Parallel algorithm

Local growths

Data structures

Each local growth keeps:

Parallel algorithm

Local growths

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- a **Fibonacci heap** θ to store candidates for the ordered BFS;

Parallel algorithm

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Parallel algorithm

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  can be merged in $O(1)$

Parallel algorithm

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Each local growth keeps:

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- an **ST-tree** G_r to store the preimage graph.

Parallel algorithm

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What if a saddle joins components from different local growths?

Parallel algorithm

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What if a saddle joins components from different local growths?

- **Detection**: before processing a vertex v, check whether all the vertices in $\mathrm{Link}^{-}(v)$ have already been visited.

Parallel algorithm

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What if a saddle joins components from different local growths?

- **Detection**: before processing a vertex v, check whether all the vertices in $\mathrm{Link}^-(v)$ have already been visited.

  concretely, update an atomic counter visitedLower[v] and check whether visitedLower[v] $= |\,\mathrm{Link}^-(v)|$

Parallel algorithm

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Join saddles

What if a saddle joins components from different local growths?

- **Detection**: before processing a vertex v, check whether all the vertices in $\mathrm{Link}^{-}(v)$ have already been visited.
- **Stopping**: if not, terminate this local growth.

Parallel algorithm

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Data structures

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Parallel algorithm

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  - join the priority queues (θ) and the preimage graphs (G_r) of all local growths terminated at v;

Parallel algorithm

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  - join the priority queues (θ) and the preimage graphs (G_r) of all local growths terminated at v;
  - process v as usual.

Parallel algorithm

Local growth implementation

procedure $\texttt{LocalGrowth}(v_0, \mathcal{R}, \Phi)$
  θ, $G_r \leftarrow \{v_0\}$ [Fibonacci heap], $\varnothing$ [ST-tree]
  while $\theta \neq \varnothing$ **do**   add $|\{w \in \mathrm{Link}^-(v) : w \text{ visited by this local growth}\}|$
    $v \leftarrow$ vertex in θ with minimal f value
    update visitedLower[v]
    if visitedLower[v] $< |\mathrm{Link}^-(v)|$ **then**
      append (θ, G_r) to pending[v]
      terminate
    end
critical section
    foreach $(\theta', G_r') \in$ pending[v] **do**
      $\theta.\texttt{join}(\theta')$; $G_r.\texttt{join}(G_r')$
    end
    process v, updating G_r, $\mathcal{R}$ and Φ
    add vertices in $\mathrm{Link}^+(v)$ to θ
  end   just as in the sequential algorithm
end

Parallel algorithm

Full implementation

input : a triangulated mesh $\mathcal{M}$
a scalar field f on $\mathcal{M}$
output : the augmented Reeb graph $(\mathcal{R}, \Phi)$

begin
    $\mathcal{R}$, $\Phi \leftarrow \varnothing$ [graph], $\varnothing$ [function]
    $V \leftarrow$ `FindMinima`$(\mathcal{M}, f)$ → easy to run in parallel
    foreach $v_0 \in V$ **in parallel do**
        start procedure `LocalGrowth`$(v_0, \mathcal{R}, \Phi)$
    end
    return $(\mathcal{R}, \Phi)$
end