# Reeb graph
## Definition

Given:

- a manifold $\mathcal{M}$;
- a Morse function $f\colon \mathcal{M} \to \mathbb{R}$ with distinct critical values;

the **Reeb graph** of $f$ is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M}/\sim.$$

$x \sim y$ if they belong to the same connected component of $f^{-1}(f(x))$

The **segmentation map** is the quotient map

$$\Phi\colon \mathcal{M} \longrightarrow \mathcal{R}(f).$$

# Reeb graph

## Desired algorithm

**Input:**

- a 2-dimensional PL manifold $\mathcal{M}$
  $\rightsquigarrow$ a triangulated mesh $\mathcal{M}$;
- a non-degenerate PL scalar field $f$ on $\mathcal{M}$
  $\rightsquigarrow$ a scalar value $f(v)$ for each vertex $v$ of $\mathcal{M}$.

  pairwise different, in order to ensure non-degeneracy; this can be achieved by random perturbations

**Output:**

- the augmented Reeb graph $\mathcal{R}(f)$.

  graph + segmentation map

**Time complexity:**

- $O(m \cdot \log m)$, where $m$ is the size of $\mathcal{M}$.

  #vertices + #edges + #triangles

**Parallelizable.**

# Reeb graph

## Geometry of critical points

There are three kinds of critical points:

- (local) maximums
  $\rightsquigarrow$ empty $\mathrm{Link}^+$;
- (local) minimums
  $\rightsquigarrow$ empty $\mathrm{Link}^-$;
- saddles
  $\rightsquigarrow$ disconnected $\mathrm{Link}^-$ and $\mathrm{Link}^+$.

### How to detect them on a PL manifold?

Given a vertex $v$, the **star** of $v$ is the union of all simplices containing $v$.
The **link** of $v$ is the boundary of its star.

$$\mathrm{Link}^+(v) = \{x \in \mathrm{Link}(v) : f(x) > f(v)\}$$

$$\mathrm{Link}^-(v) = \{x \in \mathrm{Link}(v) : f(x) < f(v)\}$$