Reeb graph

Definition

Given:

- a manifold $\mathcal{M}$;
- a Morse function $f\colon \mathcal{M} \to \mathbb{R}$ with distinct critical values;

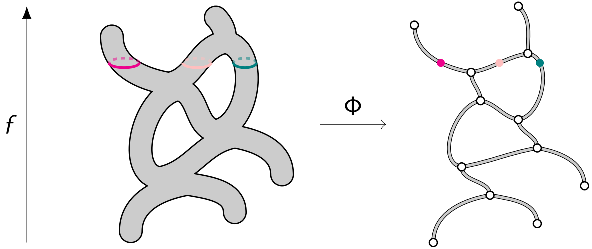
the **Reeb graph** of f is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M}/\sim.$$

$x \sim y$ if $f(x) = f(y)$ and they belong to the same connected component of $f^{-1}(f(x))$

The **segmentation map** is the quotient map

$$\Phi\colon \mathcal{M} \longrightarrow \mathcal{R}(f).$$



Reeb graph

Desired algorithm

Input:

- a PL manifold $\mathcal{M}$
  $\rightsquigarrow$ a triangulated mesh $\mathcal{M}$;
- a non-degenerate PL scalar field f on $\mathcal{M}$
  $\rightsquigarrow$ a scalar value $f(v)$ for each vertex v of $\mathcal{M}$.

pairwise different, in order to ensure non-degeneracy; this can be achieved by random perturbations

Output:

- the augmented Reeb graph $\mathcal{R}(f)$.

graph + segmentation map

Time complexity:

- $O(m \cdot \log m)$, where m is the size of the 2-skeleton of $\mathcal{M}$.

#vertices + #edges + #triangles

Parallel.

Reeb graph

Geometry of critical points

There are three kinds of critical points:

- (local) **maxima**
  $\rightsquigarrow$ Link^+ empty;
- (local) **minima**
  $\rightsquigarrow$ Link^- empty;
- **saddles**
  $\rightsquigarrow$ Link^- or Link^+ disconnected.

How to detect them on a PL manifold?

Given a vertex v, the **star** of v is the union of all simplices containing v.
The **link** of v is the boundary of its star.

$$\mathrm{Link}^+(v) = \{x \in \mathrm{Link}(v) : f(x) > f(v)\}$$

$$\mathrm{Link}^-(v) = \{x \in \mathrm{Link}(v) : f(x) < f(v)\}$$

Reeb graph

Significance of critical points

The critical points of f are closely related to the topology of the Reeb graph $\mathcal{R}(f)$.

- **Maxima** and **minima**
  $\rightsquigarrow$ nodes of valence 1 (leaves).
- **Saddles**
  $\rightsquigarrow$ nodes of valence ≥ 2.
  - **Join saddles**: multiple components below.
  - **Split saddles**: multiple components above.

non-mutually exclusive
in dimension ≥ 3