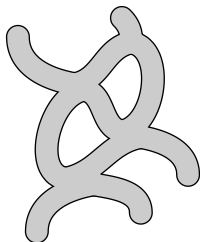


Reeb graph

Definition

Given:

- ▶ a manifold \mathcal{M} ;

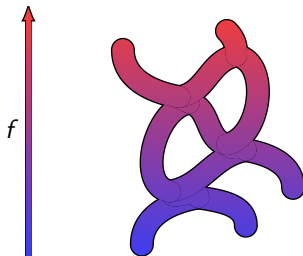


Reeb graph

Definition

Given:

- ▶ a manifold \mathcal{M} ;
- ▶ a Morse function $f: \mathcal{M} \rightarrow \mathbb{R}$ with distinct critical values;



Reeb graph

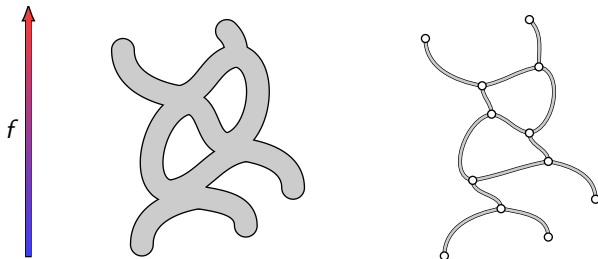
Definition

Given:

- ▶ a manifold \mathcal{M} ;
- ▶ a Morse function $f: \mathcal{M} \rightarrow \mathbb{R}$ with distinct critical values;

the **Reeb graph** of f is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M}/\sim.$$



Reeb graph

Definition

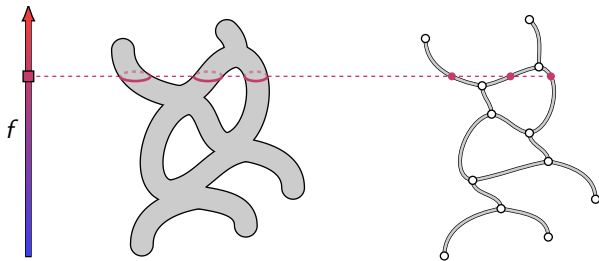
Given:

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- ▶ a Morse function $f: \mathcal{M} \rightarrow \mathbb{R}$ with distinct critical values;

the **Reeb graph** of f is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M} / \sim,$$

$x \sim y$ if $f(x) = f(y)$ and they belong to the same connected component of $f^{-1}(f(x))$



Reeb graph

Definition

Given:

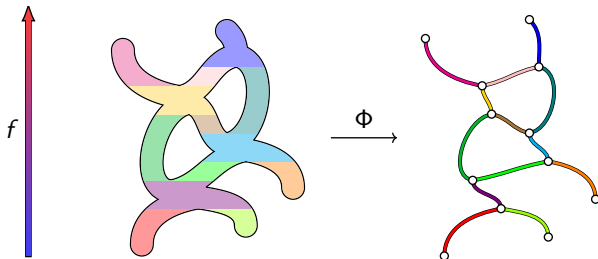
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- ▶ a Morse function $f: \mathcal{M} \rightarrow \mathbb{R}$ with distinct critical values;

the **Reeb graph** of f is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M}/\sim.$$

The **segmentation map** is the quotient map

$$\Phi: \mathcal{M} \longrightarrow \mathcal{R}(f).$$

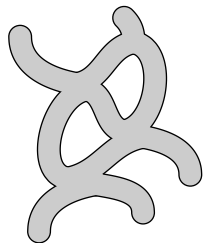


Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}

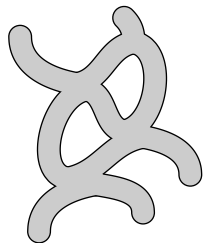


Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}
 \rightsquigarrow a triangulated mesh \mathcal{M} ;

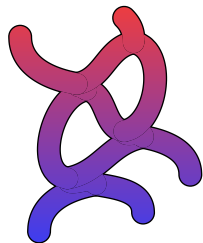


Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}
 \rightsquigarrow a triangulated mesh \mathcal{M} ;
- ▶ a non-degenerate PL scalar field f on \mathcal{M}



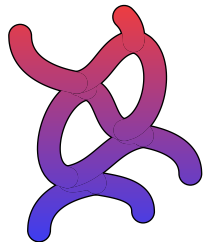
Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}
 \rightsquigarrow a triangulated mesh \mathcal{M} ;
- ▶ a non-degenerate PL scalar field f on \mathcal{M}
 \rightsquigarrow a scalar value $f(v)$ for each vertex v of \mathcal{M} .

→ pairwise different, in order to ensure non-degeneracy; this can be achieved by random perturbations




Reeb graph

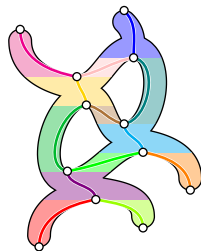
Desired algorithm

Input:

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 \rightsquigarrow a scalar value $f(v)$ for each vertex v of \mathcal{M} .

Output:

- ▶ the augmented Reeb graph $\mathcal{R}(f)$.


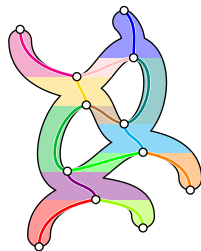


Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}
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 \rightsquigarrow a scalar value $f(v)$ for each vertex v of \mathcal{M} .



Output:

- ▶ the augmented Reeb graph $\mathcal{R}(f)$.

Time complexity:

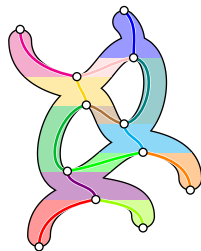
- ▶ $O(m \cdot \log m)$, where m is the **size** of the 2-skeleton of \mathcal{M} .
 #vertices + #edges + #triangles

Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}
 \rightsquigarrow a triangulated mesh \mathcal{M} ;
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 \rightsquigarrow a scalar value $f(v)$ for each vertex v of \mathcal{M} .



Output:

- ▶ the augmented Reeb graph $\mathcal{R}(f)$.

Time complexity:

- ▶ $O(m \cdot \log m)$, where m is the size of the 2-skeleton of \mathcal{M} .

Parallel.

Reeb graph

Geometry of critical points

There are three kinds of critical points:

- ▶ (local) **maxima**
 $\rightsquigarrow \text{Link}^+$ empty;
- ▶ (local) **minima**
 $\rightsquigarrow \text{Link}^-$ empty;
- ▶ **saddles**
 $\rightsquigarrow \text{Link}^-$ or Link^+ disconnected.

How to detect them on a PL manifold?

Given a vertex v , the **star** of v is the union of all simplices containing v .

The **link** of v is the boundary of its star.

$$\text{Link}^+(v) = \{x \in \text{Link}(v) : f(x) > f(v)\}$$

$$\text{Link}^-(v) = \{x \in \text{Link}(v) : f(x) < f(v)\}$$

Reeb graph

Significance of critical points

The critical points of f are closely related to the topology of the Reeb graph $\mathcal{R}(f)$.

- ▶ **Maxima and minima**

\rightsquigarrow nodes of valence 1 (leaves).

- ▶ **Saddles**

\rightsquigarrow nodes of valence ≥ 2 .

- ▶ **Join saddles:** multiple components below.
 - ▶ **Split saddles:** multiple components above.
- } non-mutually exclusive
in dimension ≥ 3