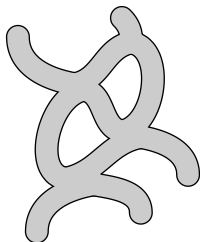


Reeb graph

Definition

Given:

- ▶ a manifold \mathcal{M} ;

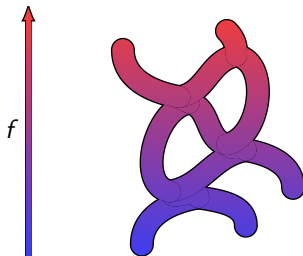


Reeb graph

Definition

Given:

- ▶ a manifold \mathcal{M} ;
- ▶ a Morse function $f: \mathcal{M} \rightarrow \mathbb{R}$ with distinct critical values;



Reeb graph

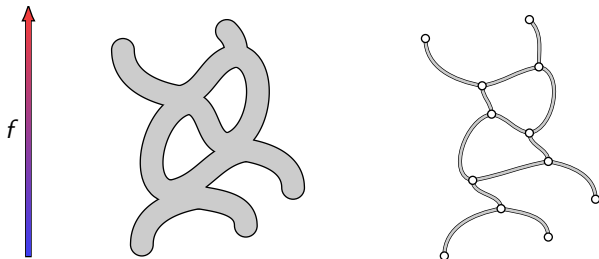
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the **Reeb graph** of f is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M}/\sim.$$



Reeb graph

Definition

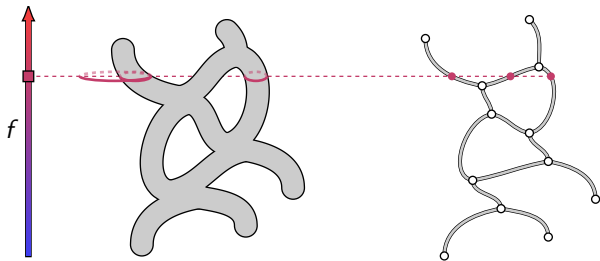
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$$\mathcal{R}(f) = \mathcal{M} / \sim,$$

$x \sim y$ if $f(x) = f(y)$ and they belong to the same connected component of $f^{-1}(f(x))$



Reeb graph

Definition

Given:

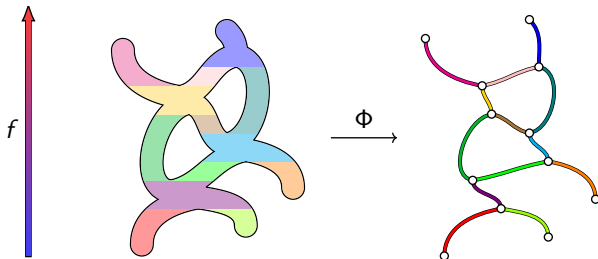
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the **Reeb graph** of f is the 1-dimensional simplicial complex

$$\mathcal{R}(f) = \mathcal{M}/\sim.$$

The **segmentation map** is the quotient map

$$\Phi: \mathcal{M} \longrightarrow \mathcal{R}(f).$$

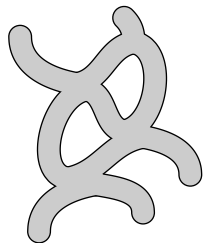


Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}

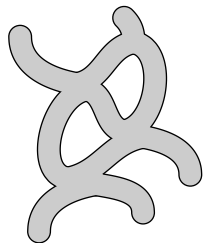


Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}
 \rightsquigarrow a triangulated mesh \mathcal{M} ;

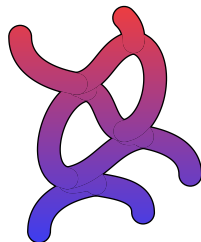


Reeb graph

Desired algorithm

Input:

- ▶ a PL manifold \mathcal{M}
 \rightsquigarrow a triangulated mesh \mathcal{M} ;
- ▶ a non-degenerate PL scalar field f on \mathcal{M}



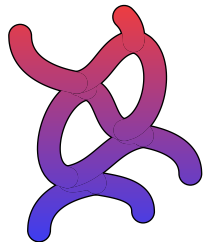
Reeb graph

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- ▶ a non-degenerate PL scalar field f on \mathcal{M}
 \rightsquigarrow a scalar value $f(v)$ for each vertex v of \mathcal{M} .

pairwise different, in order to ensure non-degeneracy; this can be achieved by random perturbations



Reeb graph

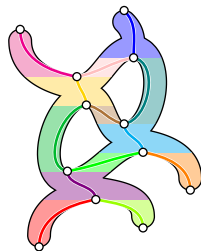
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Output:

- ▶ the augmented Reeb graph $\mathcal{R}(f)$.
 \searrow graph + segmentation map

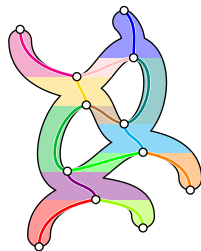


Reeb graph

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Output:

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Time complexity:

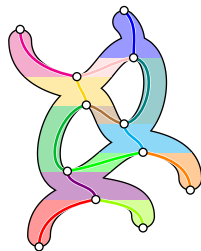
- ▶ $O(m \cdot \log m)$, where m is the **size** of the 2-skeleton of \mathcal{M} .
 #vertices + #edges + #triangles

Reeb graph

Desired algorithm

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- ▶ a PL manifold \mathcal{M}
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Parallel.

Reeb graph

Geometry of critical points

There are three kinds of critical points:

Reeb graph

Geometry of critical points

There are three kinds of critical points:

- ▶ (local) **maxima**



Reeb graph

Geometry of critical points

There are three kinds of critical points:

► (local) **maxima**



► (local) **minima**



Reeb graph

Geometry of critical points

There are three kinds of critical points:

▶ (local) **maxima**



▶ (local) **minima**



▶ **saddles**



Reeb graph

Geometry of critical points

There are three kinds of critical points:

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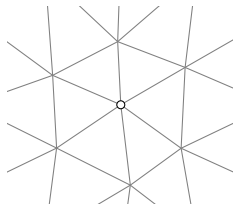
► (local) **minima**



► **saddles**



How to detect them on a PL manifold?



Reeb graph

Geometry of critical points

There are three kinds of critical points:

► (local) **maxima**



► (local) **minima**

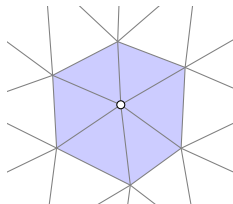


► **saddles**



How to detect them on a PL manifold?

Given a vertex v , the **star** of v is the union of all simplices containing v .



Reeb graph

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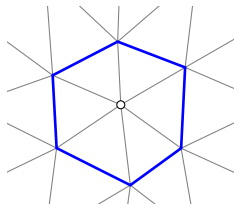
► **saddles**



How to detect them on a PL manifold?

Given a vertex v , the **star** of v is the union of all simplices containing v .

The **link** of v is the boundary of its star.



Reeb graph

Geometry of critical points

There are three kinds of critical points:

► (local) **maxima**



► (local) **minima**



► **saddles**



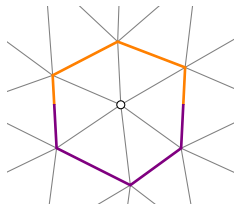
How to detect them on a PL manifold?

Given a vertex v , the **star** of v is the union of all simplices containing v .

The **link** of v is the boundary of its star.

$$\text{Link}^+(v) = \{x \in \text{Link}(v) : f(x) > f(v)\}$$

$$\text{Link}^-(v) = \{x \in \text{Link}(v) : f(x) < f(v)\}$$



Reeb graph

Geometry of critical points

There are three kinds of critical points:

- ▶ (local) **maxima**
 $\rightsquigarrow \text{Link}^+$ empty;
- ▶ (local) **minima**
- ▶ **saddles**



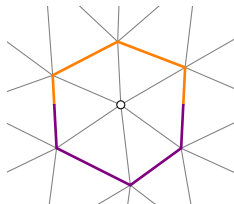
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Reeb graph

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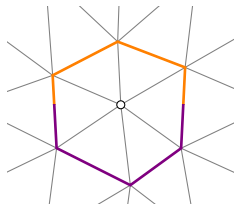
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Reeb graph

Geometry of critical points

There are three kinds of critical points:

- ▶ (local) **maxima**

\rightsquigarrow Link^+ empty;



- ▶ (local) **minima**

\rightsquigarrow Link^- empty;



- ▶ **saddles**

\rightsquigarrow Link^+ or Link^- disconnected.



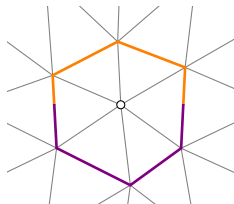
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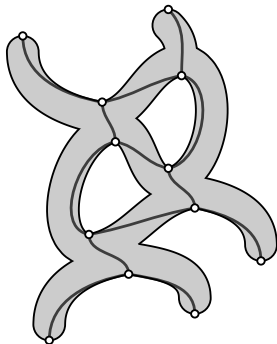
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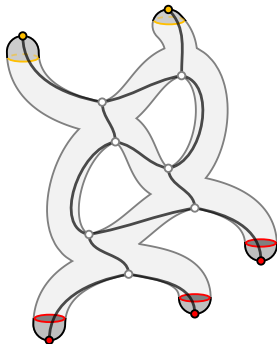


The critical points of f are closely related to the topology of the Reeb graph $\mathcal{R}(f)$.



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- **Maxima** and **minima** \rightsquigarrow nodes of valence 1 (leaves).



The critical points of f are closely related to the topology of the Reeb graph $\mathcal{R}(f)$.

- ▶ **Maxima** and **minima** \rightsquigarrow nodes of valence 1 (leaves).
 - ▶ **Saddles** \rightsquigarrow nodes of valence ≥ 2 .
 - ▶ **Join saddles**: multiple components below.
 - ▶ **Split saddles**: multiple components above.
- } non-mutually exclusive
in dimension ≥ 3

