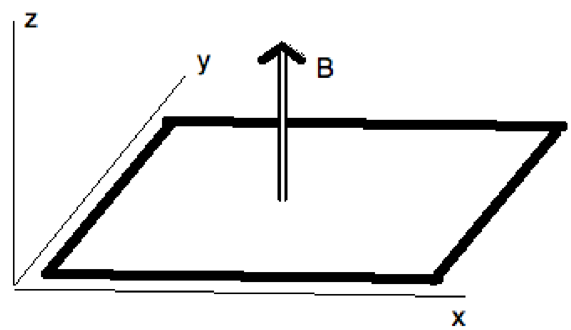
Problem Set 03

Landau Levels

In this problem we study the basics of Quantum Hall Effect physics. We want to describe the properties of a two dimensional gas of non-interacting electrons in a perpendicular magnetic field $\vec{B}$.



The description of a magnetic field via a vector potential $\vec{A}$ such that $\vec{\nabla} \times \vec{A} = \vec{B}$ gives us great freedom. Since the rotational (curl) of any gradient of a scalar function $\vec{\nabla} \times \left(\vec{\nabla} F(\mathbf{r})\right)$ is 0, any gradient of a scalar field can be added to the vector potential without affecting the physical properties of the system which only depend on the magnetic field itself. This is know as gauge invariance. Here we study the problem in two different gauges which while leaving the physical properties of the non-interacting system the same give rise to different representations of the wavefunctions having both advantages when trying to treat additional effects such as an external potential or the interaction between particles.

1. Landau Gauge

   First we study the system in Landau gauge defined by $\vec{A} = B(-y, 0, 0)$. Using the Hamiltonian for free particles in a magnetic field

   $$H = \frac{1}{2m}\left|\vec{P} + \frac{e}{c}\vec{A}\right|^2$$

   a) Write out explicitly the Hamiltonian in Landau gauge and the resulting differential form of the 2D Schrödinger equation for $\psi(x, y)$.

   Solution

$$H = \frac{1}{2m}\left(P_x - \frac{eB}{c}y\right)^2 + \frac{(P_y)^2}{2m}. \tag{1}$$

So that

$$\begin{aligned}
\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - \frac{eB}{c}y\right)^2\psi + \frac{1}{2m}\left(-i\hbar\right)^2\frac{\partial^2\psi}{\partial y^2} &= E\psi \\
-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \left(\frac{i\hbar}{m}\frac{eB}{c}y\right)\frac{\partial\psi}{\partial x} + \frac{1}{2m}\left(\frac{eB}{c}y\right)^2\psi - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2} &= E\psi
\end{aligned} \tag{2}$$

b) The equation being independent of x we can separate the wavefunctions as $\psi(x,y) = e^{ik_x x}\phi(y)$. Write the effective 1D Schrödinger equation for the $\phi(y)$ part of the wavefunction.

Solution

Making the replacement and simplifying $e^{ik_x x}$ factors, we find:

$$\begin{aligned}
\left(\frac{\hbar^2k_x^2}{2m}\right)\phi - \left(\frac{\hbar k_x}{m}\frac{eB}{c}y\right)\phi + \frac{1}{2m}\left(\frac{eB}{c}y\right)^2\phi - \left(\frac{\hbar^2}{2m}\right)\frac{\partial^2\phi}{\partial y^2} &= E\phi \\
-\left(\frac{\hbar}{2m}\right)^2\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2m}\left(\hbar k_x - \frac{eB}{c}y\right)^2\phi &= E\phi
\end{aligned} \tag{3}$$

which is just the replacement of P_x by $\hbar k_x$.

c) What are then the full wavefunctions and eigenenergies in Landau gauge ?

N.B.: Remember that the wavefunctions of a "shifted' 1D quantum harmonic oscillator

$$H = \frac{(\hat{P}_y)^2}{2m} + \frac{1}{2}m\omega^2(\hat{Y} - y_0)^2 \tag{4}$$

are given by:

$$\begin{aligned}
E_n &= \left(n + \frac{1}{2}\right)\hbar\omega \\
\psi_n &= \sqrt{\frac{1}{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega(y-y_0)^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}(y - y_0)\right),
\end{aligned} \tag{5}$$

where H_n are Hermite polynomials.

Solution

The equation found previously is obviously an harmonic oscillator. Rearranging the terms gives us

$$\frac{1}{2}m\omega^2(y-y_0)^2 = \frac{1}{2m}\left(\hbar k_x - \frac{eB}{c}y\right)^2 = \frac{1}{2}m\left(\frac{eB}{mc}\right)^2\left(y - \frac{\hbar k_x c}{eB}\right)^2 \tag{6}$$

and therefore we find a QHO with

$$\begin{aligned} \omega_c^2 &= \left(\frac{eB}{mc}\right)^2 \\ y_0 &= \frac{\hbar k_x c}{eB} \equiv \ell^2 k_x \end{aligned} \tag{7}$$

with the definition of the magnetic length:

$$\ell = \sqrt{\frac{\hbar c}{eB}} \tag{8}$$

The resulting wavefunctions are trivially found from the given information.

d) Schematically speaking what do these wavefunctions look like in the 2D plane.
Solution

Eigenstates in this gauge choice are then plane waves in x with wavevector k_x. For any given choice of k_x this will fix the position in y around which the wavefunction will be localized in y being given by the solutions of the QHO problem.

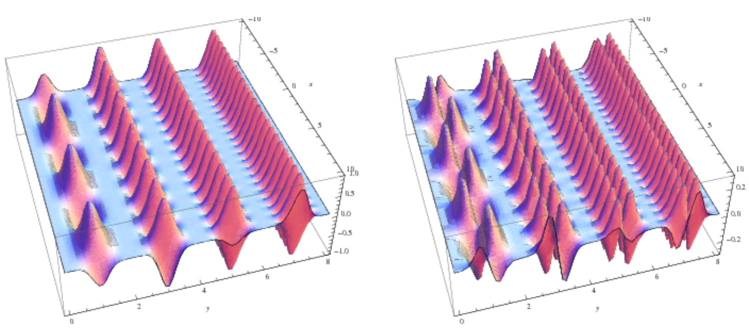


Figure 1: Schematic real part of the wavefunctions in Landau level n=0 (left) and n=1(right)

e) Assuming a finite sample of dimensions (L_x, L_y) how many orthogonal single particle states does a single Landau level (a given n) contain ?

This leads to the definition of the filling factor ν which tells us for a given density of electrons and a given magnetic field, the fraction of completely filled Landau levels. Find an expression for ν.

Solution

For a finite length L_x, periodic boundary conditions lead to a quantized momentum $k_x = \frac{2\pi}{L_x} n$ where n is any integer. Consequently, in y the distance between the center of two closeby wavefunctions in a given Landau level is given by $\Delta_y = \frac{2\pi\ell^2}{L_x}$. We can therefore in a sample of dimension L_y fit $\frac{L_y}{\Delta_y} = \frac{L_x L_y}{2\pi\ell^2}$ different wavefunctions. For a given number of electrons N_e we therefore populate $\frac{N_e}{L_x L_y} 2\pi\ell^2 = 2\pi\ell^2\rho = \rho\frac{\phi_0}{B}$ Landau levels.

2. Symmetric Gauge

We now turn to a second choice of gauge known as the symmetric gauge. The vector potential is now defined by $\vec{A} = \frac{B}{2}(-y, x, 0)$.

a) Write out the Hamiltonian in terms of the complex coordinates $z = \frac{x}{\ell} - i\frac{y}{\ell}$, $\bar{z} = \frac{x}{\ell} + i\frac{y}{\ell}$, where $\ell = \sqrt{\frac{\hbar c}{eB}}$.

Solution

With $x = \frac{\ell}{2}(z + \bar{z})$, $y = i\frac{\ell}{2}(z - \bar{z})$, we find the differential operators

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{1}{\ell}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = -i\frac{1}{\ell}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right).
\end{aligned} \tag{9}
$$

so that we can write

$$
\begin{aligned}
P_x &= -i\hbar\frac{\partial}{\partial x} = -i\hbar\frac{1}{\ell}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) \\
P_x^2 &= -\hbar^2\frac{\partial^2}{\partial x^2} = -\hbar^2\frac{1}{\ell^2}\left(\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z\partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right) \\
P_y &= -i\hbar\frac{\partial}{\partial y} = -\hbar\frac{1}{\ell}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \\
P_y^2 &= -\hbar^2\frac{\partial^2}{\partial y^2} = \hbar^2\frac{1}{\ell^2}\left(\frac{\partial^2}{\partial z^2} - 2\frac{\partial^2}{\partial z\partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right)
\end{aligned} \tag{10}
$$

Here we used the fact that $z, \bar{z}$ can be treated as two independant variables (orthogonal superpositions of independant variables x, y) and that the wavefunctions will have continuous second derivatives at every point in space therefore allowing us to use Schwarz's theorem to exchange the order of the two partial derivatives.

Finally we have in this gauge,

$$
\begin{aligned}
H &= \frac{1}{2m}\left(P_x - \frac{eB}{c}\frac{y}{2}\right)^2 + \frac{1}{2m}\left(P_y + \frac{eB}{c}\frac{x}{2}\right)^2 \\
&= \frac{1}{2m}\left(P_x^2 - \frac{eB}{c}yP_x + \left(\frac{eB}{c}\frac{y}{2}\right)^2\right) + \frac{1}{2m}\left(P_y^2 + \frac{eB}{c}xP_y + \left(\frac{eB}{c}\frac{x}{2}\right)^2\right) \\
&= \frac{1}{2m}\left(-4\frac{\hbar^2}{\ell^2}\frac{\partial^2}{\partial z\partial\bar{z}} + i\frac{\hbar eB}{c\ell}y\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}\right) - \frac{\hbar eB}{c\ell}x\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right) + \frac{1}{4}\left(\frac{eB}{c}\right)^2\left(x^2+y^2\right)\right) \\
&= \frac{1}{2m}\left(-4\frac{\hbar^2}{\ell^2}\frac{\partial^2}{\partial z\partial\bar{z}} + i\frac{\hbar^2}{\ell^2}\frac{1}{\ell}y\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}\right) - \frac{\hbar^2}{\ell^2}\frac{1}{\ell}x\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right) + \frac{1}{4}\left(\frac{\hbar}{\ell^2}\right)^2\left(x^2+y^2\right)\right) \\
&= \frac{1}{2m}\left(\frac{\hbar}{\ell}\right)^2\left(-4\frac{\partial^2}{\partial z\partial\bar{z}} - z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial\bar{z}} + \frac{1}{4}z\bar{z}\right) \\
&= \hbar\omega_c\left(-2\frac{\partial^2}{\partial z\partial\bar{z}} - \frac{1}{2}z\frac{\partial}{\partial z} + \frac{1}{2}\bar{z}\frac{\partial}{\partial\bar{z}} + \frac{1}{8}z\bar{z}\right) \quad \text{with} \quad \omega_c = \frac{eB}{mc}
\end{aligned} \tag{11}
$$

b) Defining the ladder operators

$$
\begin{aligned}
b &= \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right) \\
b^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial\bar{z}}\right) \\
a &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial\bar{z}}\right) \\
a^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right)
\end{aligned} \tag{12}
$$

write the Hamiltonian in terms of this new set of operators.

Solution one can immediately see that since the only second derivative term in H is the crossed $\frac{\partial^2}{\partial z\partial\bar{z}}$, only terms $a^\dagger a$ or $b^\dagger b$ can occur. Computing the action of $a^\dagger a$ on any function $\psi(z,\bar{z})$

$$
a^\dagger a\psi = \frac{1}{2}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right)\left(\frac{z}{2} + 2\frac{\partial}{\partial\bar{z}}\right)\psi = \frac{\bar{z}z}{8}\psi + \frac{\bar{z}}{2}\left(\frac{\partial}{\partial\bar{z}}\right)\psi - \frac{1}{2}\psi - \frac{z}{2}\frac{\partial}{\partial z}\psi - 2\frac{\partial^2}{\partial z\partial\bar{z}}\psi \tag{13}
$$

which can be compared with the expression of H to conclude that

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2}\right) \quad \text{with} \quad \omega_c = \frac{eB}{mc} \tag{14}$$

c) Defining $\psi_{0,0}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{4}z\bar{z}}$, show that it is an eigenstate of both $a^\dagger a$ and $b^\dagger b$ with eigenvalue 0. What is then the eigenenergy of this state.

Solution

By computing

$$\begin{aligned} a\psi_{0,0} &= \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial\bar{z}}\right)\frac{1}{2\pi}e^{-\frac{1}{4}z\bar{z}} \propto \left(\frac{z}{2} + (-\frac{2}{4}z)\right)\psi_{0,0} = 0 \\ b\psi_{0,0} &= \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2} + 2\frac{\partial}{\partial z}\right)\frac{1}{2\pi}e^{-\frac{1}{4}z\bar{z}} \propto \left(\frac{\bar{z}}{2} + (-\frac{2}{4}\bar{z})\right)\psi_{0,0} = 0 \end{aligned} \tag{15}$$

The energy is then simply given by $E = \frac{1}{2}\omega_c$, which obviously puts this state in the lowest Landau level.

d) Compute for any integer m, the function $\psi_{0,m}(x,y) \equiv (b^\dagger)^m \psi_{0,0}(x,y)$ and show that it is also an eigenstate of $a^\dagger a$ and $b^\dagger b$ and give the eigenvalue. What is its energy?

Plot the schematic density profile ψ^2 for these eigenfunctions.

N.B. Together, these $\psi_{0,m}(x,y)$ form a basis for the lowest Landau level ($n = 0$). In principle, one can use any linear combinations within an energy degenerate subspace of an Hamitonian to form eigenstates of an Hamiltonian.

Using the form of the $\psi_{0,m}(x,y)$ we can conclude that for any analytical function $f(z)$, the state

$$\psi(x,y) \propto f(z) e^{-\frac{1}{4}z\bar{z}}, \tag{16}$$

is a correct eigenstate.

Solution

Applying $b^\dagger$ on this state gives:

$$\psi_{0,1}(x,y) = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial\bar{z}}\right)\frac{1}{2\pi}e^{-\frac{1}{4}z\bar{z}} \propto \left(z - 4\frac{\partial}{\partial\bar{z}}\right)e^{-\frac{1}{4}z\bar{z}} = 2ze^{-\frac{1}{4}z\bar{z}} \propto ze^{-\frac{1}{4}z\bar{z}} \tag{17}$$

Repeating the procedure a second time gives

$$\psi_{0,2}(x,y) \propto \left(z - 4\frac{\partial}{\partial \bar{z}}\right) z e^{-\frac{1}{4}z\bar{z}} = z^2 e^{-\frac{1}{4}z\bar{z}} - 4\frac{\partial}{\partial \bar{z}} z e^{-\frac{1}{4}z\bar{z}} = 2z^2 e^{-\frac{1}{4}z\bar{z}} \propto z^2 e^{-\frac{1}{4}z\bar{z}} \quad (18)$$

It should be clear that recursively we simply have

$$\psi_{0,m}(x,y) \propto z^m e^{-\frac{1}{4}z\bar{z}} \equiv z^m \psi_{0,0}(x,y). \quad (19)$$

Since a commutes with z^m because it only contains a derivative with respect to $\bar{z}$, the application of $a^\dagger a$ on this state leads again to

$$a^\dagger a \psi_{0,m}(x,y) = a^\dagger 0 = 0 \quad (20)$$

and all of these states are also eigenstates of the Hamitonian with the same eigenenergy found previously, i.e. $E = \frac{\omega_c}{2}$.

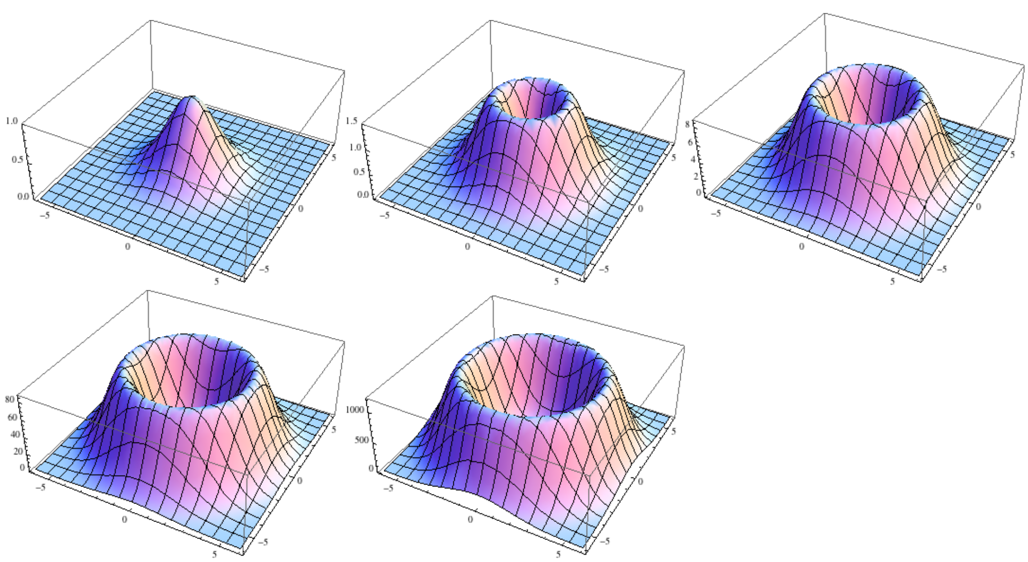


Figure 2: Density profile in the lowest Landau level for states $\psi_{0,0}, \psi_{0,1}, \psi_{0,2}, \psi_{0,3}, \psi_{0,4}$

e) Find expressions for $\psi_{n,0}(x,y) \equiv (a^\dagger)^n \psi_{0,0}(x,y)$ and for its eigenenergy.

Applying $b^\dagger$ to $\psi_{n,0}(x,y)$ in a similar fashion as in c) and d) we could show that $\psi_{n,m}(x,y) \equiv (b^\dagger)^m \psi_{n,0}(x,y)$ are degenerate as well and form a basis for Landau level n.

Solution

Since $a^\dagger$ is equivalent to $b^\dagger$ by simply switching the roles of z and $\bar{z}$ and since $\psi_{0,0}(x,y)$ is invariant under this transformation, it should be immediately clear that

$$\psi_{n,0}(x,y) \propto \bar{z}^n e^{-\frac{1}{4}z\bar{z}} \tag{21}$$

With knowledge of ladder operator, one should know that since $a^\dagger$ raises the eigenvalue of $a^\dagger a$ by one, the resulting eigenenergy is given by $E_n = \hbar\omega_c(n+\frac{1}{2})$.

One can however show it explicitely by applying the operator $a^\dagger a$ to the state, giving us

$$\begin{aligned}
\frac{1}{2}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right)\left(\frac{z}{2} + 2\frac{\partial}{\partial \bar{z}}\right)\bar{z}^n e^{-\frac{1}{4}z\bar{z}} &= \frac{1}{2}\left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right)\left(\frac{1}{2}z\bar{z}^n e^{-\frac{1}{4}z\bar{z}} + 2n\bar{z}^{n-1}e^{-\frac{1}{4}z\bar{z}} - \frac{1}{2}z\bar{z}^n e^{-\frac{1}{4}z\bar{z}}\right) \\
&= \left(\frac{\bar{z}}{2} - 2\frac{\partial}{\partial z}\right) n\bar{z}^{n-1}e^{-\frac{1}{4}z\bar{z}} = n\bar{z}^n e^{-\frac{1}{4}z\bar{z}} = n\psi_{n,0}(x,y),
\end{aligned} \tag{22}$$

giving us the previously stated result.