Chapter 2
Solutions of the Dirac Equation in an External Electromagnetic Field

In this chapter, the solutions of the Dirac equation for a fermion in an external electromagnetic field are presented for the cases of a pure magnetic field of arbitrary strength, of a strong magnetic field when fermions occupy the ground Landau level, and of a crossed field. The density matrix of the plasma electron in a magnetic field with the fixed number of a Landau level is calculated. In this chapter, we use the notation for the 4-vectors and their components: $X^\mu = (t, x, y, z)$.

2.1 Magnetic Field

For calculation of the S matrix elements of quantum processes in external fields, the standard procedure is applied, which is based on the Feynman diagram technique using the field operators of charged fermions expanded over the solutions of the Dirac equation in an external magnetic field

$$\hat{\Psi}(X) = \sum_{\mathbf{p},s} \Big(\hat{a}_{\mathbf{p},s} \Psi^{(+)}_{\mathbf{p},s}(X) + \hat{b}^\dagger_{\mathbf{p},s} \Psi^{(-)}_{\mathbf{p},s}(X) \Big), \tag{2.1}$$

where $\hat{a}$ is the destruction operator for fermions, $\hat{b}^\dagger$ is the creation operator for antifermions, and $\Psi^{(+)}(X)$ and $\Psi^{(-)}(X)$ are the normalized solutions of the Dirac equation in a magnetic field with positive and negative energy, correspondingly.

There exist several methods of solving the Dirac equation in a magnetic field which are basically the similar but have some variations in details, see e.g. [1–5]. Here we present the basic points of the procedure which is the most simple and clear, in our opinion. The description is similar to the one of Ref. [5]. As a charged fermion, we consider an electron being the particle having the largest specific charge, i.e. being the most sensitive to the external field influence. More general case for an arbitrary charged fermion can be found e.g. in [1].

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The Dirac equation for an electron with the mass m_e and the charge $(-e)$ in an external electromagnetic field with the four-potential $A_\mu = A_\mu(X)$ has the form

$$\left(\mathrm{i}(\partial\gamma) + e(A\gamma) - m_e\right)\Psi(X) = 0, \tag{2.2}$$

where $(\partial\gamma) = \partial_\mu\gamma^\mu$ and $(A\gamma) = A_\mu\gamma^\mu$. For solving the Eq. (2.2) in a pure magnetic field $\mathbf{B}$, we take the frame where the field is directed along the z axis, and the Landau gauge where the four-potential is: $A^\mu = (0, 0, xB, 0)$.

To solve the Eq. (2.2), let us rewrite it in the Schrödinger form:

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(X) = \hat{H}\Psi(X), \tag{2.3}$$

with the Hamiltonian:

$$\hat{H} = \gamma_0\left[\boldsymbol{\gamma}\left(\hat{\mathbf{p}} + e\mathbf{A}\right)\right] + m_e\gamma_0. \tag{2.4}$$

Here, $\hat{\mathbf{p}} = -\mathrm{i}\nabla$ is the momentum operator.

Since the Hamiltonian does not depend explicitly on time, the problem reduces to finding the eigenvalues and eigenfunctions of the Schrödinger stationary equation:

$$\Psi(X) = \mathrm{e}^{-\mathrm{i}p_0 t}\psi(x, y, z), \quad \hat{H}\psi(x, y, z) = p_0\psi(x, y, z). \tag{2.5}$$

Consider the auxiliary operator, called the longitudinal polarization operator:

$$\hat{T}^0 = \frac{1}{m_e}\left[\boldsymbol{\Sigma}\left(\hat{\mathbf{p}} + e\mathbf{A}\right)\right], \tag{2.6}$$

where $\boldsymbol{\Sigma}$ is the 3-dimensional double spin operator:

$$\boldsymbol{\Sigma} = \gamma_0\boldsymbol{\gamma}\gamma_5 = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{2.7}$$

and $\boldsymbol{\sigma}$ are the Pauli matrices. It is easy to verify by direct calculation that the operator $\hat{T}^0$ commutes with the Hamiltonian (2.4).

First, we find the eigenvalues and the eigenfunctions of the operator $\hat{T}^0$,

$$\hat{T}^0\psi_T(x, y, z) = T^0\psi_T(x, y, z). \tag{2.8}$$

The functions $\psi_T(x, y, z)$ are also the eigenfunctions of the Hamiltonian (2.4), due to commutativity of $\hat{H}$ and $\hat{T}^0$.

It is convenient to represent the operator $\hat{T}^0$ in the form

$$\hat{T}^0 = \begin{pmatrix} \hat{\tau}^0 & 0 \\ 0 & \hat{\tau}^0 \end{pmatrix}, \tag{2.9}$$

where

$$\hat{\tau}^0 = \frac{1}{m_e}\left[\boldsymbol{\sigma}\left(\hat{\mathbf{p}} + e\mathbf{A}\right)\right]. \tag{2.10}$$

By the structure of the operator $\hat{T}^0$, the system (2.8) of 4 equations splits into two exactly coinciding equations for the upper and lower spinors forming the bispinor $\psi_T(x, y, z)$.

In the chosen gauge, the operator $\hat{\tau}^0$ has the form:

$$\hat{\tau}^0 = \frac{1}{m_e}\left[\sigma_x\left(-\mathrm{i}\frac{\partial}{\partial x}\right) + \sigma_y\left(-\mathrm{i}\frac{\partial}{\partial y} + \beta x\right) + \sigma_z\left(-\mathrm{i}\frac{\partial}{\partial z}\right)\right], \tag{2.11}$$

where the notation is used: $\beta = eB$. Given the operator $\hat{T}^0$ not depending explicitly on the coordinates of y and z, one can write the bispinor $\psi_T(x, y, z)$ in the form:

$$\psi_T(x, y, z) = \mathrm{e}^{\mathrm{i}(p_y y + p_z z)}\begin{pmatrix} F(x) \\ \varkappa F(x) \end{pmatrix}, \quad F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \tag{2.12}$$

where $\varkappa$ is an arbitrary number. Introducing a new variable

$$\xi = \sqrt{\beta}\left(x + \frac{p_y}{\beta}\right), \tag{2.13}$$

one can transform the equation for the spinor $F(x)$ to the form:

$$\frac{1}{m_e}\begin{pmatrix} p_z & -\mathrm{i}\sqrt{2\beta}a^- \\ \mathrm{i}\sqrt{2\beta}a^+ & -p_z \end{pmatrix}\begin{pmatrix} f_1(\xi) \\ f_2(\xi) \end{pmatrix} = T^0\begin{pmatrix} f_1(\xi) \\ f_2(\xi) \end{pmatrix}, \tag{2.14}$$

where the raising and lowering operators of the problem of the quantum harmonic oscillator arise:

$$a^+ = \frac{1}{\sqrt{2}}\left(\xi - \frac{\mathrm{d}}{\mathrm{d}\xi}\right), \quad a^- = \frac{1}{\sqrt{2}}\left(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi}\right). \tag{2.15}$$

The expression (2.14) is a system of differential equations for the functions $f_1(\xi)$ and $f_2(\xi)$. We obtain:

$$\begin{gathered} f_1(\xi) = \frac{-\mathrm{i}\sqrt{2\beta}}{m_e T^0 - p_z}\, a^- f_2(\xi), \\ \left(a^+ a^- - \frac{m_e^2 (T^0)^2 - p_z^2}{2\beta}\right) f_2(\xi) = 0. \end{gathered} \tag{2.16}$$

Multiplying the operators (2.15), one can see that the equation for the function $f_2(\xi)$ is reduced to an equation for eigenfunctions of the quantum harmonic oscillator:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \xi^2 + 1 + \frac{m_e^2 (T^0)^2 - p_z^2}{\beta}\right) f_2(\xi) = 0. \tag{2.17}$$

Hence, we find the eigenvalues T^0 of the operator $\hat{T}^0$:

$$T^0 = \pm \frac{1}{m_e} \sqrt{p_z^2 + 2n\beta}. \tag{2.18}$$

Here, $n = 0, 1, 2, \ldots$. These numbers, as we shall see below, will determine the electron energy, i.e., will number the Landau levels. It should be noted that the eigenvalues T^0 are gauge invariant, being the eigenvalues of the Hermitian operator, i.e. the physically observable quantities.

The functions $f_1(\xi)$ and $f_2(\xi)$ are

$$f_1(\xi) = C \, \frac{-\mathrm{i}\sqrt{2n\beta}}{m_e T^0 - p_z} \, V_{n-1}(\xi), \qquad f_2(\xi) = C \, V_n(\xi), \tag{2.19}$$

where C is the normalization coefficient, and $V_n(\xi)$ $(n = 0, 1, 2, \ldots)$ are the normalized harmonic oscillator functions, which are expressed in terms of Hermite polynomials $H_n(\xi)$:

$$V_n(\xi) = \frac{\beta^{1/4}}{\sqrt{2^n \, n! \, \sqrt{\pi}}} \, \mathrm{e}^{-\xi^2/2} \, H_n(\xi), \quad H_n(\xi) = (-1)^n \, \mathrm{e}^{\xi^2} \, \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} \, \mathrm{e}^{-\xi^2},$$
$$\int\limits_{-\infty}^{+\infty} |V_n(\xi)|^2 \, \mathrm{d}x = 1, \tag{2.20}$$

and for negative values of the index n the function $V_n(\xi)$ is assumed to be zero.

Returning to the stationary Schrödinger equation (2.5), let us substitute into it the found function $\psi_T(x, y, z)$ as an eigenfunction. The Hamiltonian (2.4) can be expressed in terms of the operator $\hat{\tau}^0$:

$$\hat{H} = m_e \begin{pmatrix} I & \hat{\tau}^0 \\ \hat{\tau}^0 & -I \end{pmatrix}. \tag{2.21}$$

In view of (2.14), we obtain the equation:

$$m_e \left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + T^0 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right] \begin{pmatrix} F(x) \\ \varkappa \, F(x) \end{pmatrix} = p_0 \begin{pmatrix} F(x) \\ \varkappa \, F(x) \end{pmatrix}, \tag{2.22}$$

which is transformed to a system of algebraic equations for p_0 and $\varkappa$ having two solutions. Two eigenvalues of the stationary Schrödinger equation (2.5) are:

$$(p_0)_{1,2} = \pm E_n, \tag{2.23}$$

where

$$E_n = m_e \sqrt{(T^0)^2 + 1} = \sqrt{p_z^2 + m_e^2 + 2n\beta}. \tag{2.24}$$

The values of $\varkappa$ corresponding to the two eigenvalues p_0 are:

$$\varkappa_1 = \mathrm{sign}(T^0)\sqrt{\frac{E_n - m_e}{E_n + m_e}}, \quad \varkappa_2 = -\mathrm{sign}(T^0)\sqrt{\frac{E_n + m_e}{E_n - m_e}}. \tag{2.25}$$

Thus, given the ambiguity of T^0 (2.18), there exist four independent solutions of the Eq. (2.2).

(i) The eigenvalue $p_0 = +E_n$.

The solutions corresponding to the positive eigenvalue $p_0 = +E_n$, called the solutions with positive energy, which differ in sign of T^0, can be written as:

$$\Psi^{(+\pm)}(X) = A^{(+\pm)}\, \mathrm{e}^{-\mathrm{i}(E_n t - p_y y - p_z z)}\, u^{(+\pm)}(\xi). \tag{2.26}$$

Here, the first of two signs in the superscript refers to p_0, while the second one refers to T^0. For the bispinors $u^{(+\pm)}(\xi)$ we obtain the expressions:

$$u^{(++)}(\xi) = \begin{pmatrix} \frac{-\mathrm{i}\sqrt{2n\beta}}{m_e|T^0| - p_z} V_{n-1}(\xi) \\ V_n(\xi) \\ \sqrt{\frac{E_n - m_e}{E_n + m_e}}\, \frac{-\mathrm{i}\sqrt{2n\beta}}{m_e|T^0| - p_z} V_{n-1}(\xi) \\ \sqrt{\frac{E_n - m_e}{E_n + m_e}}\, V_n(\xi) \end{pmatrix}, \tag{2.27}$$

$$u^{(+-)}(\xi) = \begin{pmatrix} \frac{-\mathrm{i}\sqrt{2n\beta}}{-m_e|T^0| - p_z} V_{n-1}(\xi) \\ V_n(\xi) \\ -\sqrt{\frac{E_n - m_e}{E_n + m_e}}\, \frac{-\mathrm{i}\sqrt{2n\beta}}{-m_e|T^0| - p_z} V_{n-1}(\xi) \\ -\sqrt{\frac{E_n - m_e}{E_n + m_e}}\, V_n(\xi) \end{pmatrix}. \tag{2.28}$$

The functions (2.26)–(2.28), as well as any of their linear combinations, are the solutions of the Dirac equation (2.2), corresponding to the eigenvalue $p_0 = +E_n$.

As in the analysis of solutions of the Dirac equation in vacuum, the solutions in a magnetic field are typically used in the form of linear combinations of the functions (2.26)–(2.28), in which the upper two components of the bispinor correspond to the states of the electron with the spin projections 1/2 and −1/2 on some direction, in this case, on the direction of the magnetic field.

Given the normalization

$$\int |\Psi(X)|^2\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = 1, \tag{2.29}$$

we obtain the final form of the exact solutions of the Dirac equation for an electron in an external magnetic field on the n-th Landau level:

$$\Psi^{(+)}_{n,\,p_y,\,p_z,\,s}(X) = \frac{\mathrm{e}^{-\mathrm{i}\,(E_n\,t - p_y\,y - p_z\,z)}}{\sqrt{2\,E_n\,(E_n + m_e)\,L_y\,L_z}}\;U^{(+)}_{n,\,p_y,\,p_z,\,s}(\xi), \tag{2.30}$$

where L_y and L_z are the normalizing sizes along the axes of y and z; the number $s = \pm 1$ is the eigenvalue of the double spin operator σ_z acting on the spinor composed of the upper two components of the bispinor.

The bispinor $U^{(+)}$ has different forms for the cases $s = +1$ and $s = -1$:

$$U^{(+)}_{n,\,p_y,\,p_z,\,s=+1}(\xi) = \begin{pmatrix} (E_n + m_e)\,V_{n-1}(\xi) \\ 0 \\ p_z\,V_{n-1}(\xi) \\ \mathrm{i}\,\sqrt{2n\beta}\,V_n(\xi) \end{pmatrix}, \tag{2.31}$$

$$U^{(+)}_{n,\,p_y,\,p_z,\,s=-1}(\xi) = \begin{pmatrix} 0 \\ (E_n + m_e)\,V_n(\xi) \\ -\,\mathrm{i}\,\sqrt{2n\beta}\,V_{n-1}(\xi) \\ -\,p_z\,V_n(\xi) \end{pmatrix}. \tag{2.32}$$

One can see that in each of the bispinors (2.31) and (2.32), the upper two components form a spinor being the eigenfunction of the operator σ_z. For the ground Landau level, $n = 0$, the solution exists only at $s = -1$.

Note that the value p_z in above expressions is a conserved component of the electron momentum along the z axis, i.e. along the field, while the value p_y is the generalized momentum, which determines the position of a center of the Gaussian packet along the x axis by the relation $x_0 = -p_y/\beta$ (see (2.13)).

(ii) The eigenvalue $p_0 = -E_n$.

The solutions $\Psi^{(-\pm)}(X)$ corresponding to this eigenvalue describe the states of an electron with negative energy in the Dirac sea. To obtain the functions corresponding to the states of a positron as a physical particle with the energy E_n and the momentum components p_y and p_z, one should construct the solutions $\Psi^{(-\pm)}(X)$ which are similar to the functions (2.26)–(2.28), in view of (2.25), and then change the signs of p_y and p_z. One should also remember that the projection of the spin of a positron, i.e. of a hole in the sea of negative energies, on any special direction is opposite to the spin projection of the electron, described by a bispinor.

There are two main variants of constructing the solutions with a negative energy, with using of different linear combinations of the functions $\Psi^{(-+)}(X)$ and $\Psi^{(--)}(X)$, which, of course, lead to identical results in calculations of observable quantities. In the first case, one can simply use the solutions (2.30)–(2.32) and change there the signs of E_n, p_y, and p_z. The second way is perhaps more physically justified. As in the analysis of the Dirac equation in vacuum, one can consider the solutions in which the upper two components of a bispinor are small, if the nonrelativistic limit,

$p_z^2 \ll m_e^2$, and the case of a weak field, $\beta \ll m_e^2$, are taken. In this case, the linear combinations of the functions $\Psi^{(-+)}(X)$ and $\Psi^{(--)}(X)$ should be used, in which the spinor composed of the two lower components of a bispinor, describes the states of the electron with the spin projections 1/2 and −1/2 on some direction, in this case, on the direction of the magnetic field.

The exact solutions of the Dirac equation corresponding to the positron states in an external magnetic field, on the nth Landau level have the form:

$$\Psi^{(-)}_{n,\, p_y,\, p_z,\, s}(X) = \frac{\mathrm{e}^{\mathrm{i}\,(E_n\, t - p_y\, y - p_z\, z)}}{\sqrt{2\, E_n\, (E_n + m_e)\, L_y\, L_z}}\, U^{(-)}_{n,\, p_y,\, p_z,\, s}(\xi^{(-)}), \tag{2.33}$$

where the number $s = \pm 1$ is the eigenvalue of the double electron spin operator σ_z acting on the spinor composed of the two lower components of the bispinor,

$$\xi^{(-)} = \sqrt{\beta}\left(x - \frac{p_y}{\beta}\right), \tag{2.34}$$

$$U^{(-)}_{n,\, p_y,\, p_z,\, s=+1}(\xi^{(-)}) = \begin{pmatrix} p_z\, V_{n-1}(\xi^{(-)}) \\ -\mathrm{i}\sqrt{2n\beta}\, V_n(\xi^{(-)}) \\ (E_n + m_e)\, V_{n-1}(\xi^{(-)}) \\ 0 \end{pmatrix}, \tag{2.35}$$

$$U^{(-)}_{n,\, p_y,\, p_z,\, s=-1}(\xi^{(-)}) = \begin{pmatrix} \mathrm{i}\sqrt{2n\beta}\, V_{n-1}(\xi^{(-)}) \\ -\, p_z\, V_n(\xi^{(-)}) \\ 0 \\ (E_n + m_e)\, V_n(\xi^{(-)}) \end{pmatrix}. \tag{2.36}$$

For the ground Landau level, $n = 0$, the solution exists only for the value of the double spin $s = -1$ of an electron with negative energy. This corresponds to the positron state with a value of the double spin $s = +1$.

2.2 The Ground Landau Level

If some physical process with electrons/positrons, with a typical energy E is realised in a strong magnetic field, where the field induction B determines the maximum energy scale of a problem, namely, $eB > E^2, m_e^2$, electrons/positrons can occupy only the states that correspond to the ground Landau level, $n = 0$. Contrary to other Landau levels with $n \geqslant 1$, which are doubly degenerate with respect to spin, the ground level is not degenerate, i.e. the electron/positron spin is fixed, $s = -1/+1$.

The solution of the Dirac equation for the electron with energy E and momentum components p_y and p_z can be presented in this case in the following form

$$\Psi^{(+)}_{0,\,p_y,\,p_z,\,s=-1}(X) \;=\; \frac{\beta^{1/4}\,\mathrm{e}^{-\mathrm{i}(Et-p_y y-p_z z)}}{(\sqrt{\pi}\,2\,E\,(E+m_e)L_y L_z)^{1/2}}\;\mathrm{e}^{-\xi^2/2}\,u(p_\parallel)\,, \tag{2.37}$$

where $p_\parallel$ is the energy-momentum vector of an electron in the Minkowski $\{0{,}3\}$ plane. Here, $E = \sqrt{p_z{}^2 + m_e^2}$, and ξ is defined by (2.13) and describes the motion along the x axis.

The bispinor amplitude is given by

$$u(p_\parallel) \;=\; \begin{pmatrix} 0 \\ E+m_e \\ 0 \\ -p_z \end{pmatrix}. \tag{2.38}$$

It is interesting to note that the bispinor amplitude (2.38) is exactly the same as the solution of the free Dirac equation for an electron having a momentum directed along the z axis. This separation of a bispinor amplitude that does not depend on the spatial coordinate x is typical for the ground Landau level only.

The calculation technique of electroweak processes in a strong magnetic field, where electrons occupy the ground Landau level, the so-called two-dimensional electrodynamics, was developed by Loskutov and Skobelev [6, 7]; for details and a complete list of references see e.g. [8]. That technique was essentially improved, with a covariant extension, in our papers; see e.g. [9–14]. For example, the antisymmetric tensor $\varepsilon_{\alpha\beta}$ ($\varepsilon_{30} = -\varepsilon_{03} = 1$) in the subspace $\{0, 3\}$, used in that technique, appears to be not a mathematical abstraction, but has a clear physical meaning of the dimensionless dual magnetic field tensor, $\varepsilon_{\alpha\beta} = -\tilde{\varphi}_{\alpha\beta}$. Similarly, all the formulae can be written in a covariant form with obvious rules of transformation to any frame.

2.3 Crossed Field

There exists a special case of external electromagnetic field, in which the analysis of quantum processes is essentially simplified. It is the case of a crossed field, where the vectors of the electric field $\boldsymbol{\mathcal{E}}$ and the magnetic field $\mathbf{B}$ are orthogonal and their values are equal, $\boldsymbol{\mathcal{E}} \perp \mathbf{B}$, $\mathcal{E} = B$. The calculation technique of electromagnetic processes in the crossed field was developed by Nikishov and Ritus; for details and the list of references see e.g. [15, 16].

The particular case of a crossed field is in fact more general than it may seem at first glance. Really, the situation is possible when the so-called field dynamical parameter χ of the relativistic particle propagating in a relatively weak electromagnetic field, $F < B_e$ ($F = \mathcal{E}$ and/or B), could appear rather high. The definition of the dynamical parameter χ is

$$\chi = \frac{e(pFFp)^{1/2}}{m_e^3}, \tag{2.39}$$

where p^α is the particle four-momentum, and $F^{\alpha\beta}$ is the electromagnetic field tensor. In this case the field in the particle rest frame can exceed essentially the critical value and is very close to the crossed field. Even in a magnetic field whose strength is much greater than the critical value, the result obtained in a crossed field will correctly describe the leading contribution to the probability of a process in a pure magnetic field, provided that $\chi \gg B/B_e$. If, in addition, the invariant $|e^2(pFFp)|^{1/3}$ for a particle moving in an arbitrary electromagnetic field considerably exceeds the pure field invariants $|e^2(FF)|^{1/2}$ and $|e^2(\tilde{F}F)|^{1/2}$, the problem is reducible to a still simpler calculation, that in a crossed field for which one has $(FF) = 0$ and $(\tilde{F}F) = 0$. Thus, the calculation in a constant crossed field is the relativistic limit of the calculation in an arbitrary relatively weak smooth field. Consequently, the results obtained in a crossed field possess a great extent of generality, and acquire interest by itself.

The crossed field is described by the 4-vector potential $A^\mu = a^\mu \varphi$, where $\varphi = (kX)$, and a^μ and k^μ are the constant 4-vectors, $(kk) = 0,\ (ak) = 0$.

The field tensor in this case is $F^{\mu\nu} = k^\mu a^\nu - k^\nu a^\mu$, and the contraction of the two tensors over one index is $(FF)^{\mu\nu} = -k^\mu k^\nu (aa)$.

The solution of the Dirac equation for an electron in the crossed field can be found as a particular case of the Dirac equation solution in the field of a plane electromagnetic wave obtained by Volkov [17, 18], where the above-mentioned linear dependence of the field vector potential on the phase φ, $A^\mu = a^\mu \varphi$, should be taken. The solution has the form

$$\Psi_p(X) = \left(1 - \frac{e(k\gamma)(a\gamma)}{2(kp)}\,\varphi\right) \frac{u(p)}{\sqrt{2EV}} \times \exp\left[-\mathrm{i}\left((pX) - \frac{e(ap)}{2(kp)}\,\varphi^2 - \frac{e^2(aa)}{6(kp)}\,\varphi^3\right)\right]. \tag{2.40}$$

where $u(p)$ is the bispinor amplitude of a free electron with the 4-momentum $p^\mu = (E, \mathbf{p})$.

The solution with negative energy corresponding to an antiparticle can be obtained from (2.40) by the change of sign of all the components of the 4-momentum p^μ.

The directions of the coordinate frame axes can be taken as follows, without loss of generality:

$$k^\mu = (k_0, k_0, 0, 0),\ a^\mu = (0, 0, -a, 0). \tag{2.41}$$

In this case

$$\varphi = (kX) = k_0(t - x),\ \boldsymbol{\mathcal{E}} = (0, \mathcal{E}, 0),\ \mathbf{B} = (0, 0, B),\ \mathcal{E} = B = k_0 a.$$

It is worthwhile to introduce also the vector $b^\mu = (0, 0, 0, -a)$, which can be used for representing the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ by the following form $\tilde{F}^{\mu\nu} = k^\mu b^\nu - k^\nu b^\mu$.

2.4 Density Matrix of the Plasma Electron in a Magnetic Field with the Fixed Number of a Landau Level

When quantum processes in a magnetized plasma are investigated, it is occasionally necessary to calculate the plasma electron density matrix in the coordinate space, summed over all quantum states, except the Landau level number. In this Section, we present the calculations of this matrix which can be defined by the formula:

$$R_n(X, X') = \sum_s \int \frac{\mathrm{d}p_y \mathrm{d}p_z}{(2\pi)^2} L_y\, L_z\, f(E_n)\, \Psi^{(+)}_{n,\, p_y,\, p_z,\, s}(X)\, \bar{\Psi}^{(+)}_{n,\, p_y,\, p_z,\, s}(X'). \quad (2.42)$$

Here, $\Psi^{(+)}_{n,\, p_y,\, p_z,\, s}(X)$ are the solutions (2.30)–(2.32) of the Dirac equation for an electron in an external magnetic field, $E_n = \sqrt{p_z^2 + m_e^2 + 2n\beta}$ is the energy of the electron at the nth Landau level, $\beta = eB$, and $f(E_n)$ is the electron distribution function that allows for the presence of a plasma. In the plasma rest frame, it is

$$f(E) = [\mathrm{e}^{(E-\mu)/T} + 1]^{-1},$$

where μ is the chemical potential of plasma and T is its temperature.

Substituting the explicit form of the electron wave functions (2.30)–(2.32) into Eq. (2.42) for the density matrix, we can reduce it to the form

$$R_n(X, X') = \mathrm{e}^{\mathrm{i}\Phi(X,X')} \sum_s R_{n\parallel}((X - X')_\parallel)\, R_{ns\perp}((X - X')_\perp). \quad (2.43)$$

Here, the following functions are introduced:

$$\Phi(X, X') = -\frac{\beta}{2}\,(x + x')\,(y - y'), \quad (2.44)$$

$$R_{n\parallel}(X_\parallel) = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{E_n(E_n + m_e)}\, f(E_n)\, \mathrm{e}^{-\mathrm{i}(pX)_\parallel}, \quad (2.45)$$

$$R_{ns\perp}(X_\perp) = \frac{\sqrt{\beta}}{8\pi^2} \int\limits_{-\infty}^{+\infty} \mathrm{d}\xi\, U_s(\xi)\, \bar{U}_s(\xi - \sqrt{\beta}x)\, \mathrm{e}^{-\mathrm{i}\sqrt{\beta}(\sqrt{\beta}xy/2 - \xi y)}, \quad (2.46)$$

where we changed the integration variable from p_y to ξ, see (2.13). In Eq. (2.46), we have omitted all the indexes except s of the bispinors $U^{(+)}_{n,\, p_y,\, p_z,\, s}$. However, one should keep in mind that there is not a simple product of the functions $R_{n\parallel}$ and $R_{ns\perp}$ stands in Eq. (2.43), because $R_{ns\perp}$ depends on p_z.

The function $R_{ns\perp}(X_\perp)$ as a function of two variables x and y can be expanded into a Fourier integral:

$$R_{ns\perp}(X_\perp) = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}(pX)_\perp}\, R_{ns\perp}(p_\perp), \tag{2.47}$$

$$R_{ns\perp}(p_\perp) = \int \mathrm{d}^2 X_\perp\, \mathrm{e}^{-\mathrm{i}(pX)_\perp}\, R_{ns\perp}(X_\perp)\,. \tag{2.48}$$

Integrating the function $R_{ns\perp}(p_\perp)$ over the coordinates x and y and substituting the result into (2.43) yields

$$\begin{aligned} R_n(X, X') = {} & \frac{\mathrm{e}^{\mathrm{i}\Phi(X,X')}}{(2\pi)^3\sqrt{\beta}} \int \frac{\mathrm{d}^3 p\, f(E_n)}{E_n(E_n + m_e)}\, \mathrm{e}^{-\mathrm{i}(p(X-X'))}\, \mathrm{e}^{2\mathrm{i}p_x p_y/\beta} \\ & \times \int\limits_{-\infty}^{+\infty} \mathrm{d}\xi\, \mathrm{e}^{-2\mathrm{i}p_x\xi/\sqrt{\beta}} \sum_s U_s(\xi)\, \bar{U}_s(\xi'), \end{aligned} \tag{2.49}$$

where $\xi' = 2p_y/\sqrt{\beta} - \xi$.

After simple but slightly cumbersome calculations, including the summation over the spin states of the initial and final electrons that occupy the same Landau level n, the product of the bispinor amplitudes can be reduced to

$$\begin{aligned} & \sum_s U_s(\xi)\, \bar{U}_s(\xi') = (E_n + m_e) \\ & \quad \times \{((p\gamma)_\parallel + m_e)\ [\Pi_+ V_{n-1}(\xi) V_{n-1}(\xi') + \Pi_- V_n(\xi) V_n(\xi')] \\ & \quad - \sqrt{2n\beta}\ [\Pi_+ \gamma^2 V_{n-1}(\xi) V_n(\xi') + \Pi_- \gamma^2 V_n(\xi) V_{n-1}(\xi')]\}. \end{aligned} \tag{2.50}$$

Here, the projection operators are introduced:

$$\Pi_\pm = \frac{1}{2}\,(I \pm \mathrm{i}\,\gamma^1\,\gamma^2)\,, \quad \Pi_\pm\,\Pi_\pm = \Pi_\pm\,, \quad \Pi_\pm\,\Pi_\mp = 0. \tag{2.51}$$

The integral over the variable ξ in Eq. (2.49) can be calculated using the formula

$$\begin{aligned} J_{n,n'} &= \frac{\mathrm{e}^{\mathrm{i}ab/2}}{\sqrt{\beta}} \int\limits_{-\infty}^{+\infty} \mathrm{d}\xi\, \mathrm{e}^{-\mathrm{i}a\,\xi}\, V_n(\xi) V_{n'}(b-\xi) \\ &= (-1)^{n'}\, \mathrm{e}^{-\mathrm{i}(n-n')\varphi}\, F_{n',n}(u), \qquad n \geqslant n'\,, \end{aligned} \tag{2.52}$$

where

$$\tan\varphi = \frac{a}{b}\,, \quad u = \frac{a^2+b^2}{4},$$

$$F_{n',n}(u) = \sqrt{\frac{n'!}{n!}}\ (2u)^{(n-n')/2}\ \mathrm{e}^{-u}\ L_{n'}^{n-n'}(2u),$$

and the associated Laguerre polynomials $L_k^s(x)$ are defined as

$$L_k^s(x) = \frac{1}{k!}\, \mathrm{e}^x\, x^{-s}\, \frac{\mathrm{d}^k}{\mathrm{d}x^k}\, (\mathrm{e}^{-x}\, x^{k+s})\,. \tag{2.53}$$

Finally, the electron density matrix can be reduced to a triple integral convenient for the subsequent use:

$$\begin{aligned} R_n(X, X') = {} & \mathrm{e}^{\mathrm{i}\Phi(X,X')}\,(-1)^n \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, \frac{f(E_n)}{E_n}\, \mathrm{e}^{-u}\, \mathrm{e}^{-\mathrm{i}p(X-X')} \\ & \times \{((p\gamma)_{\|} + m_e)[L_n(2u)\Pi_- - L_{n-1}(2u)\Pi_+] + 2(p\gamma)_\perp L^1_{n-1}(2u)\}, \end{aligned} \tag{2.54}$$

where $u = p_\perp^2/\beta$. Equation (2.54) can be used to investigate quantum processes in a plasma in the presence of a magnetic field with an arbitrary strength.

References

1. A.V. Kuznetsov, N.V. Mikheev, *Electroweak Processes in External Electromagnetic Fields* (Springer, New York, 2003)
2. A.I. Akhiezer, V.B. Berestetskii, *Quantum Electrodynamics*, 2nd edn. (Wiley, New York, 1965)
3. A.A. Sokolov, I.M. Ternov, *Radiation from Relativistic Electrons* (American Institute of Physics, New York, 1986)
4. K. Bhattacharya, P.B. Pal, Pramana. J. Phys. **62**, 1041 (2004)
5. I.A. Balantsev, Yu.V. Popov, A.I. Studenikin, J. Phys. A **44**, 255301 (2011)
6. V.V. Skobelev, Izv. Vyssh. Uchebn. Zaved., Fiz. No. 10, 142 (1975)
7. Yu.M. Loskutov, V.V. Skobelev, Phys. Lett. A **56**, 151 (1976)
8. V.V. Skobelev, Zh. Eksp. Teor. Fiz. **120**, 786 (2001) [JETP **93**, 685 (2001)]
9. M.V. Chistyakov, A.V. Kuznetsov, N.V. Mikheev, Phys. Lett. B **434**, 67 (1998)
10. A.V. Kuznetsov, N.V. Mikheev, Yad. Fiz. **60**, 2038 (1997) [Phys. At. Nucl. **60**, 1865 (1997)]
11. A.A. Gvozdev, N.V. Mikheev, L.A. Vassilevskaya, Phys. Rev. D **54**, 5674 (1996)
12. M.Y. Borovkov, A.V. Kuznetsov, N.V. Mikheev, Yad. Fiz. **62**, 1714 (1999) [Phys. At. Nucl. **62**, 1601 (1999)]
13. A.V. Kuznetsov, N.V. Mikheev, G.G. Raffelt, L.A. Vassilevskaya, Phys. Rev. D **73**, 023001 (2006)
14. A.V. Kuznetsov, N.V. Mikheev, Yad. Fiz. **70**, 1299 (2007) [Phys. At. Nucl. **70**, 1258 (2007)]
15. V.I. Ritus, Quantum effects of the interaction of elementary particles with an intense electromagnetic field (in Russian). in Quantum Electrodynamics of Phenomena in an Intense Field Proceeding, P.N. Lebedev Physical Institute, vol. 111 (Nauka, Moscow, 1979) pp. 5–151
16. V.I. Ritus, Radiative Effects and Their Amplification in an Intense Electromagnetic Field, in *Issues in Intense-Field Quantum Electrodynamics*, ed. by V.L. Ginzburg (Nova Science Publishers, New York, 1989), pp. 180–199
17. D.M. Volkov, Z. Phys. **94**, 250 (1935)
18. D.M. Volkov, Zh Eksp, Teor. Fiz. **7**, 1286 (1937)

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