1.1 D Landau level eigenstates

In a magnetic $B\hat{\mathbf{z}}$, as you know, the non-interacting eigenstates of the Hamiltonian (1.1.8) of charged particles are no longer plane waves, but rather Landau level states.

From Hamiltonian to effective Hamiltonian

We have

$$\mathcal{H} = \frac{\hat{\mathbf{P}}^2}{2m_*} + V(\mathbf{r}); \tag{1.1.26}$$

we'll set aside $V(\mathbf{r})$ for now, and later on treat it as a perturbation. The canonical momentum is

$$\hat{\mathbf{P}} \equiv \frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}. \tag{1.1.27}$$

Assuming Landau gauge, with charge $q = -e$,the Scrödinger equation becomes

$$\mathcal{H}\psi(\mathbf{r}) = \frac{\hbar^2}{2m_*}\left[-\nabla_x^2 + \left(\frac{1}{i}\nabla_y - \frac{eB}{\hbar c}x\right)^2\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{1.1.28}$$

By our choice of gauge, the Hamltonian operator is independent of y, so we can separate the wavefunction $\psi(x, y) = e^{iky}\phi(x)$. Furthermore, the coefficent of x in (1.1.28) is just $1/\ell_B^2$, which manifestly has the right dimensions. Factoring out e^{iky}, the result is a one-dimensional Schrödinger equation $\mathcal{H}_x\phi(x) = E\phi_x$, with the effective Hamiltonian

$$\mathcal{H}_x \equiv \frac{\hbar^2}{2m_*}\left[-\nabla_x^2 + (x - x_k)^2\right] \tag{1.1.29}$$

where

$$x_k \equiv \ell_B^2 k. \tag{1.1.30}$$

Obviously, (1.1.29) describes a one-dimensional harmonic oscillator with its well centered at x_k and a spring constant $K_{\rm eff} = \hbar^2/m\ell_B^2$.

Solutions of simple harmonic oscillator (review)

The resonant frequency is $\sqrt{K_{\rm eff}/m_*} = \omega_c$. Thus, the energy levels are quantized as

$$E_n = (n + 1/2)\hbar\omega_c, \tag{1.1.31}$$

where $n = 0, 1, ...$; the states with index n constitute the *n-th Landau level*; the "lowest Landau level" ($n = 0$) will often be abbreviated "LLL". When one has a large magnetic field, it makes sense to start from a wavefunction made up *only* of orbitals from the *lowest* Landau level. So we shall generally assume we have restricted the electron state space to the LLL.

The ground state (LLL) wavefunction, of course, has Gaussian form, $\exp[-(x - x_k)^2/4\sigma^2]$. My trick to remember the variance σ^2 for any kind of harmonic oscillator is that, by the virial theorem, exactly half the zero-point energy is "kinetic" term, thus $\frac{1}{2}K\langle(\Delta x)^2\rangle \equiv \frac{1}{2}(\frac{1}{2}\hbar\omega_c)$, hence $\sigma^2 = \langle(\Delta x)^2\rangle = \hbar/2\sqrt{K_{\rm eff}m_*}$. In the present case this simplifies to $\ell_B^2/2$. Thus, our final answer for the LLL is

$$\psi_{0,k} \propto e^{iky}e^{-\frac{(x-x_k)^2}{2\ell_B^2}}. \tag{1.1.32}$$

For the n-th Landau level, the form is the same but multiplied by the Hermite polynomial $H_n([x - x_k]/\ell_B)$.

Recall that, classically, the equations of motion for a cyclotron motion are the same as for harmonic oscillation; thus Eq. (1.1.31) could be understood as quantizing the harmonic oscillator describing a cyclotron orbit, and the $\frac{1}{2}\omega_c$ is the oscillator's zero-point motion . The magnetic length is then the radius of the classical cyclotron orbit with the zero-point energy (the lowest Landau level is built from those states).

How many states?

Imagine a system of dimensions $L_x \times L_y$. With periodic boundary conditions in the y direction, the allowed k-vectors are

$$k = \frac{2\pi}{L_y} m_y \tag{1.1.33}$$

for any integer m. (It is not limited by the Brillouin zone.) Thus, the allowed values of x_k are separated by $\Delta x = \ell_B^2 \Delta k = 2\pi\ell_B^2/L_y$. Notice that if $L_y \gg \ell_B$ (and it is, or our system is effecively one-dimensional), then $\Delta x \ll \ell_B$ so the separation of successive eigenstates is much smaller than the width of each. (They are nevertheless orthogonal on account of the e^{iky} factors.)

So, the total number of independent states is $(L_x L_y)/2\pi\ell_B^2$ and each Landau level has

$$\frac{1}{2\pi\ell_B^2} \tag{1.1.34}$$

states per unit area. in a plot of the density of states these would be δ functions, which are broadened into the peaks of Fig. 1.1.2. Notice this comes out to exactly one state per flux quantum piercing the 2DEG. Note too that, over an energy interval with many Landau levels, this averages to a density of states $(\hbar\omega_c)^{-1}(2\pi\ell_B^2)^1 = m_*/2\pi\hbar^2$ per unit area, which is indeed the B-independent, constant density of states of a 2DEG (with one spin flavor).

Massive degeneracies provide rich opportunities for more elaborate electron correlations to develop. When any small perturbation is added to the Hamiltonian, the eigenstates will be some particular mixture of states from the *lowest* Landau level (LLL).

Adding a potential

Consider including a potential energy $V(x) = \frac{1}{2}Kx^2$. which we assume is slowly varying. The potential in our effective Hamiltonian now becomes

$$V_{\rm eff}(x) = \frac{1}{2}K_{\rm eff}(x - x_k)^2 + \frac{1}{2}Kx^2. \tag{1.1.35}$$

where, recall, $K_{\rm eff} = \hbar\omega_c/2\ell_B^2$. By the slowly varying assumption, $V(x)$ should change by much less than $\hbar\omega_c$ over the width ($\sim \ell_B$) of the ground state, or equivalently

$$K \ll K_{\rm eff}. \tag{1.1.36}$$

Rearranging the squares we get

$$V_{\rm eff}(x) = \frac{1}{2}K_{\rm eff}{}'(x - x'_k)^2 + \frac{1}{2}\left(1 - \frac{K}{K_{\rm eff}}\right)Kx^2. \tag{1.1.37}$$

where ${K_{\rm eff}}' = K_{\rm eff} + K \approx K_{\rm eff}$; also, ${x'}_k \equiv x_k - \delta x$, where

$$\delta x = {K_{\rm eff}}^{-1} \nabla V(x_k), \tag{1.1.38}$$

i.e. the centerline gets pushed a small distance by the force; the constant offset of the energy is practically the potential along the centerline,

$$V(x_k - \frac{1}{2}\delta x) \tag{1.1.39}$$

when linearized in δx.

Since the unperturbed wavefunction extended over a narrow range of x over which we could have linearized $V(x)$, it is not surprising that the same formulas hold for a linear potential. (Exercise: check it). It's also clear that they hold for any potential $V(x)$.

1.1 E Landau levels II: symmetric gauge

Why do we use two gauges? Technically, Landau gauge is a bit simpler, and perhaps more familiar, while the symmetric gauge is quite necessary for writing the wavefunctions in fractional QHE. More important, the respective eigenfunctions are useful for different physical pictures. The Landau level wavefunction, strung out along a line, was valid with little modification when a potential $V(x)$ is added, giving us the prototype of more general wavefunctions that hug the contour lines of a slowly varying potential. On the other hand, the first symmetric gauge wavefunction $\varphi_0(\mathbf{r})$ is localized, giving us an example of a coherent state wavepacket, which is particularly appropriate to handle external potentials or electron-electron interactions that are defined in real space.

The symmetric gauge vector potential is

$$\mathbf{A} = \frac{1}{2} B(+y, -x) \tag{1.1.40}$$

We claim an eigenfunction in the lowest Landau level (unnormalized) is

$$\varphi_0(\mathbf{r}) \equiv e^{-|\mathbf{r}|^2/4\ell_B^2} \equiv e^{-|z|^2/4\ell_B^2} \tag{1.1.41}$$

It will be very handy to represent (x, y) as a complex number $z \equiv x + iy$. Let's see what we can get from transcribing everything into complex-variable language.

Note: After working this out, I discovered the text by P. Fazekas (see my book list) has a derivation for symmetric gauge in the same spirit, but done nicer.

First we define the operators

$$\frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \qquad \frac{\partial}{\partial z^*} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.1.42}$$

It is easy to check

$$\frac{\partial}{\partial z} z = \frac{\partial}{\partial z^*} z^* = 1; \qquad \frac{\partial}{\partial z} z^* = \frac{\partial}{\partial z^*} z = 0. \tag{1.1.43}$$

Of course, it follows that e.g. $(\partial/\partial z^*)^k f(z) \equiv 0$ for any power k and any function $f(z)$, and vice versa, so the behavior is as if z and z^* are *fully independent* variables – despite the fact that one can be obtained by simply conjugating the other! Note also the commutators:

$$[\frac{\partial}{\partial z}, z] = [\frac{\partial}{\partial z^*}, z^*] = 1. \tag{1.1.44}$$

The canonical momenta in this gauge are

$$(\hat{p}_x, \hat{p}_y) \equiv \hbar(\frac{1}{i}\nabla_x + \frac{y}{2\ell_B^2}, \frac{1}{i}\nabla_y - \frac{x}{2\ell_B^2}), \tag{1.1.45}$$

The Hamiltonian can now be arranged as

$$\hat{\mathcal{H}} = \frac{1}{2m_*}\left[(\hat{p}_x - i\hat{p}_y)(\hat{p}_x + i\hat{p}_y) - \frac{\hbar^2}{2\ell_B^2}\right]; \tag{1.1.46}$$

$$= -\frac{\hbar^2}{2m_*}\hat{D}_1\hat{D}_0 + \tfrac{1}{2}\hbar\omega_c, \tag{1.1.47}$$

where the extra constant came from the commutator $[\hat{p}_y, \hat{p}_x] = \hbar/i\ell_B^2$, and

$$\hat{D}_0 \equiv 2\frac{\partial}{\partial z^*} + \frac{z}{2\ell_B^2}; \qquad \hat{D}_1 \equiv 2\frac{\partial}{\partial z} - \frac{z^*}{2\ell_B^2}. \tag{1.1.48}$$

Because one of these is *not* the conjugate of the other, these operators will have different effects.

Although it is preferable to keep dimensions in most formulas, in this section the reader will profit by setting $\ell_B \to 1$ in every formula.

Firstly, it is trivial to check that $\hat{D}_0(\varphi_0(z)) = 0$ and hence $\mathcal{H}(\varphi_0(z)) = \frac{1}{2}\hbar\omega_x\varphi_0(z)$, so this is an eigenstate in the LLL. Furthermore,

$$[\hat{D}_0, z] = 0 \tag{1.1.49}$$

so $\hat{D}(z^m\varphi_0(z)) = 0$: all these functions are eigenfunctions in the LLL, too. Notice they have angular momentum component m.

Secondly, we have $[\hat{D}_1, \hat{D}_0] = 2/\ell_B^2$ and so

$$[\mathcal{H}, \hat{D}_1] = \hbar\omega_c\hat{D}_1. \tag{1.1.50}$$

Thus, $\hat{D}_1$ functions as a raising operator: if $\mathcal{H}\psi(z) = E\psi(z)$, then $\mathcal{H}(\hat{D}\psi(z)) = (E + \hbar\omega_c)\hat{D}\psi(z)$. Putting both results together, the eigenfunctions in symmetric gauge are

$$\psi_{n,m}(z) = \hat{D}_1^n(z^m e^{-|z|^2/4\ell_B^2}), \tag{1.1.51}$$

with energies

$$E_{n,m} \equiv (n + \frac{1}{2})\hbar\omega_c. \tag{1.1.52}$$

The states with index n are in the n-th Landau level, and the angular momentum is $m - n$.

The general eigenfunction in the LLL is any linear combination, $\sum c_m z^m \varphi_0(\mathbf{r})$. That prefactor, of course, is any analytic function. Thus,

$$\psi(\mathbf{r}) = f(z)e^{-\frac{|z|^2}{4\ell_B^2}} \tag{1.1.53}$$

is a general eigenfunction in the LLL, where $f(z)$ is any analytic function, (Similarly $\hat{D}_1^n[f(z)\exp(-|z|^2/4\ell_B^2)$ is a general eigenfunction in the n-th Landau level.) These wavefunctions have as many properties as analytic functions do; one amusing feature is that the locations of the zeroes of the wavefunction determine it completely.

Translations of wavepackets

If we translate the wavefunction $\varphi_0(\mathbf{r})$ along a straight line to another center $\bar{\mathbf{r}}$, we pick up additional (gauge-dependent) phase factors from the vector potential, as given in Appendix 1.1 X. The result is

$$\varphi_{\bar{\mathbf{r}}}(\mathbf{r}) = e^{-|\mathbf{r}-\bar{\mathbf{r}}|^2/4\ell_B^2 + i(\hat{z}\cdot\bar{\mathbf{r}}\times\mathbf{r})/2\ell_B^2} \tag{1.1.54}$$

In symmetric gauge, we could shift $\varphi_0(\mathbf{r})$ to a translation-equivalent state with a different center, and if the displacement more than $\sim \ell_B$ the overlap is small with the original state. Thus it is obvious the degenerate states have a massive degeneracy proportional of order ℓ_B^{-2} times the system area; the way to get the degeneracy is in (Ex. 1.1.2).

1.1 F Semiclassical wavefunctions

From our knowledge of the correspondence principle and our derivation of the classical motion, we can construct good approximations of the eigenstates (to be put to use in the next lecture). Recall that our classical motion implies $\hat{x}$ and $\hat{y}$ are canonically conjugate operators with $[\hat{x}, \hat{y}] = 1/\ell_B^2$. (Of course they aren't literally conjugate, but as projected into one Landau level – as we're assuming – they become conjugate.) Thus, within a Landau level, we cannot have an arbitarily definite x and y at the same time. Instead, let $\varphi_{\bar{\mathbf{r}}}(\mathbf{r})$ be a minimum uncertainty wavepacket centered at $\bar{\mathbf{r}}$ (in any gauge). It is an example of what is called a "coherent state".

As a digression, another situation where you have noncommuting variables that are related by a symmetry is the components of a spin (with large length S). In that case, too, we could like to think classically of having a particular direction, which forces us to use coherent states. And their centers follow the same sideways dynamics on the surface of a unit sphere, when a potential is created by an external field, crystal anisotropies, or interactions with other spins. (See P636 lectures, 5.1) The other standard spin wavefunction – having a completely definite S_z, and completely indefinite S_x or S_y – is similar to the ribbon-like electron eigenstates we found in either Landau or symmetric gauge.

In a nonzero potential, the guiding center drifts following the classical trajectory:

$$\Psi(\mathbf{r}, t) \approx e^{-i\Phi(t)}\varphi_{\bar{\mathbf{r}}(t)}(\mathbf{r}) \tag{1.1.55}$$

Here $\Theta(t)$ is the accumulated phase along that trajectory. Since the trajectory follows a contour of $V(\mathbf{r})$, we assume it is a closed loop with a period T. Here $\Theta(t)$ depends on gauge, but around the loop it will be gauge-invariant, just $2\pi(\Phi/\Phi_0)$ where $\Phi = AB$ is the net flux, assuming the loop encircles area A. Actually, in time the wavepacket spreads out , but this happens only to the degree that different parts of it experience differing forces ∇V; assuming the potential is slowly varying on the scale ℓ_B of the waveacket's width, this is a small (slow) effect.

We want to make an eigenfunction that will be stationary in time. Now, (1.1.55) actually represents a whole family of time-dependent solutions depending on the shift t' of the initial time; a proper linear combination of all these lagged solutions is stationary. We equate $E/\hbar$ to the mean rate of phase change, $\Theta(T)/T$, and construct the desired linear combination by projecting out that Fourier component:

$$\Psi_E(\mathbf{r}, t) \propto \int dt' e^{-iE(t-t')/\hbar}\Psi(\mathbf{r}, t-t') = e^{-iEt/\hbar)}\Psi_E(\mathbf{r}), \tag{1.1.56}$$

where

$$\Psi_E(\mathbf{r}) = \int_0^T dt' e^{-i\Theta(t')+iEt'/\hbar}\varphi_{\bar{\mathbf{r}}(t)}(\mathbf{r}). \tag{1.1.57}$$

Thus the wavefunction is, in general, strung out along a contour. (Exercise: when the potential is $V(x)$, verify the semiclassical formula gives the Landau gauge eigenstates you already found.)

THis is just the utterly standard Bohr-Sommerfeld semiclassical construction. The quantization condition is

$$\Theta(T) = 2\pi m \tag{1.1.58}$$

with m an integer; otherwise there is a discontinuity in the integral. Thus, the shift between successive quantized orbits is such that the area between them has one flux quanta; their spacing is inversely proportional to the contour's length. (You already obtained that for the special case of the Landau level eigenstates: compare the formula for Δx.)

1.1 X Appendix: Review of gauges

To compare results in different gauges $\mathbf{A}(\mathbf{r})$ and $\mathbf{A}'(\mathbf{r})$, it's helpful to review how to convert between gauges. The respective vector potentials are related by the gauge transformation $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla\chi(\mathbf{r})$. If $\psi(\mathbf{r})$ is a wavefunction, the same wavefunction in the other gauge is

$$\psi'(\mathbf{r}) \equiv \exp\left(-\frac{ie}{\hbar c}\chi(\mathbf{r})\right)\psi(\mathbf{r}). \tag{1.1.59}$$

It's also handy to translate a wavefunction by a vector $\mathbf{u}$, in a particular gauge $\mathbf{A}(\mathbf{r})$. Define a path Γ in translation-space connecting the origin to $\mathbf{u}$. Then

$$\psi'(\mathbf{r}+\mathbf{u}) = \exp\left(-\frac{ie}{\hbar c}\right)\int_{\Gamma:0}^{\mathbf{u}} d\mathbf{u}'\cdot\mathbf{A}(\mathbf{r}+\mathbf{u}'). \tag{1.1.60}$$

The result is unavoidably different, depending what path you adopted in $\mathbf{u}$ space. But if the magnetic field is uniform The phase difference between the two paths is proportional to the integral of $\mathbf{A}(\mathbf{r}')$ around a closed loop attached to $\mathbf{r}$, which is the flux inside. Provided the field is uniform, this flux doesn't depend on $\mathbf{r}$, and therefore the difference is a trivial overall phase factor. Alternatively, say the field is nonuniform but the wavefunction is localized in an area much smaller than the scale over which $\mathbf{B}(\mathbf{r})$ varies. Then the same phase factor is given for every value $\mathbf{r}$ at which

Exercises

Ex. 1.1.1 Landau gauge wavepacket

(From Girvin Ex. 1.5). Work in Landau gauge where our eigenfunctions were proportional to e^{iky}, $k \equiv k_y$. With allowed k values being spaced by $2\pi/L_y$, the corresponding shift in $\bar{x}_k$ would be far smaller than the eigenfunction's width $\sim \ell_B$. That means we have a hope of combining many eigenfunctions together to make a wavepacket more compact in the y direction, but still $\sim \ell_B$ (though broader) in the x direction. Thus, we consider a wavefunction $\psi(x,y) = \int_{-\infty}^{+\infty} dk a(k)\psi_k(x,y)$.

What form for $a(k)$? You have a big hint in the fact that x and y behave as conjugate operators. You are probably familiar with the construction of wavepackets minimum-uncertainty in x and p_x in ordinary QM. (What is their shape?)

We want it to be centered around $(\bar{x},\bar{y})$. To ensure x is near $\bar{x}$, we want to concentrate around k such that $x_k = \bar{x}$, namely $\bar{k} = \bar{x}/\ell_B^2$. To ensure y is near $\bar{y}$, we want to project thatout by convolving with $e^{-i\bar{y}k}$.

So assume $a(k) \propto \exp(-i\bar{y}k)\exp(-\frac{1}{2}\sigma^2[k-\bar{k}]^2)$. Here σ is a parameter with length units that controls the Gaussian's width.

(a). Do the integral over k and write the whole wavefunction you get, as a function of x and y. (Would you be surprised if it is a Gaussian? Or are there other factors?)

(b). Write $|\psi(x,y)|^2$. Is it centered on $(\bar{x},\bar{y})$ as we guessed?

(c). Compute $\langle\Delta x^2\rangle + \langle\Delta y^2\rangle$. What value of σ^2 minimizes the total uncertainty? What does that give for $|\psi(x,y)|$? (Hint: the problem has a complete translational and rotational symmetry, so – apart from the gauge-dependent phase factrs, this ought to be symmetric under rotations in the xy plane.)

How does this compare with $|\psi_0(x,y)|$ as found in symmetric gauge?

Ex. 1.1.2 Symmetric gauge eigenstates

This problem is not intended to very lengthy algebraically.

Work in symmetric gauge! Consider the (single-particle) eigenstate $\psi_m(x,y) \propto z^m \exp(-|z|^2/4\ell_B^2)$.

(a). Show that $|\psi_m()|$ is concentrated near a circle of radius R_m, and also show its profile (transverse to the circle) is approximately a Gaussian with variance σ_m^2. is a multiple of ℓ_B^2. Assume $m \gg 1$.

Hint (i). Write $|\psi_m(x,y)| = \exp(-F(r)$; Taylor expand $F(r)$ around its minimum, to second order. (Does the expansion get more valid as m gets large.)

Hint (ii). When the radius gets large, the circle is locally nearly a straight line. We should suspect, then, that this circular eigenfunction looks locally just like the linear eigenfunctions we got in Landau gauge; in particular, the angular momentum number m here is exactly analogous to k (the y direction wavenumber) in the Landau gauge case. (If you try to compare the wavefunctions directly, of course, you should be careful about the additional phase factors due to the gauge change.)

(b). In light of your result for R_m, how many eigenstates are there in a disk of radius R? Derive the number of eigenstates per unit area (for the lowest Landau level) from this: does it agree with the result we got in Landau gauge?

Ex. 1.1.3 Landau level eigenstates

The point of this exercise is to demonstrate how two apparently different looking eigenfunctions can be the same.

Remember our conventions: Landau gauge $\mathbf{A} = (0, -Bx)$, or symmetric gauge $\mathbf{A} = (By/2, -Bx/2)$, corresponding to $\mathbf{B} = -B\hat{\mathbf{z}}$.

(a). Find how to write Eq. 1.1.54 in the form (1.1.53).

(b). In the Landau gauge $\mathbf{A} = (0, -Bx)$, a Landau level wavefunction had

$$\psi_k^L(\mathbf{r}) = \exp(iky) e^{-(x-k\ell_B^2)^2/2\ell_B^2}. \tag{1.1.61}$$

Concoct a wavefunction $\psi(\mathbf{r})$ of form (1.1.53) such that $|\psi(\mathbf{r})|^2 = |\psi_k^L(\mathbf{r})|^2$; in other words, so the wavefunctions differ by at most a phase.

(c). Using Appendix 1.1 X, work out the phase factor in $\psi_k^L(\mathbf{r})$ induced by the transformation to symmetric gauge. Show the result is indeed the wavefunction you constructed in (b).