

Electrodynamics in 1 and 2 Spatial Dimensions

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(September 23, 2014; updated October 16, 2014)

1 Problem

In theoretical physics, consideration of phenomena in one or two spatial dimensions ($1 + 1$ or $2 + 1$ spacetime dimensions) is often regarded as a useful step toward understanding in three spatial dimensions ($3 + 1$ spacetime).¹ Why isn't classical electrodynamics more often considered in one or two spatial dimensions?^{2,3}

2 Solution

2.1 Electrodynamics in 1 Spatial Dimension

If we consider the case of a single spatial dimension as similar to that of three dimensions but with all charges and field lines somehow confined inside a narrow tube, we are led to suppose that the electric field strength is constant (in both space and time) away from charges, since a tube of field lines has constant field strength. Then, there are no waves of electric field away from charges, even if the charges are moving/accelerating.

Furthermore, the magnetic field (in 3 spatial dimensions) of moving charges does not affect other charges moving along the same axis. In effect, there is no magnetic field in $1 + 1$ electrodynamics (*i.e.*, in 1 spatial dimension, with only electric charges). As there are no waves and no magnetic field in 1 spatial dimension, the speed of light, c , plays no role in “electrodynamics” here.⁴

It is natural to choose (Gaussian) units in which the (Coulomb) force between two charges q and q' , which is independent of their separation, is simply $F = qq'$, with the implication that the magnitude of the electric field strength E of charge q is just $E = q$ for observers at $x > x_q$. Then, the electric field strength at a point x_0 is $E = q_+ - q_-$, where q_+ is the total charge at $x > x_0$ and q_- is the total charge at $x < x_0$, independent of the motion of these charges (unless they cross the point x_0 so as to change q_+ and q_-).

There is not much more content to classical electrodynamics in $1 + 1$ dimensions, which seems too trivial to provide useful “toy models” to help in understanding electrodynamics in $3 + 1$ dimensions.

¹A classic example is the Ising model of ferromagnetism [1].

²Examples of Schrödinger's equation in one and two spatial dimensions include the case of a “Coulomb” potential $V = -q/r$, but this potential does not correspond to the electric potential of a (point) charge q except in three spatial dimensions [2]. As can be confirmed from sec. 2, the electric scalar potential in one spatial dimension is $V_1 = q|x|$ [3], while in two spatial dimensions it is $V_2 = q \ln r$ [4].

³This problem should not be construed as a partial “explanation” as to why we live in $3 + 1$ spacetime. See [5] for general comments on such efforts.

⁴Electromagnetic plane waves in three spatial dimensions are often considered to be “one dimensional,” but such waves don't exist in “one dimensional” ($1 + 1$ spacetime) electrodynamics. For an example of how one-dimensional thinking can lead to misunderstandings as to three-dimensional electrodynamics, see [6].

2.2 Electrodynamics in 2 Spatial Dimensions

In two spatial dimensions, Coulomb's law for the force on electric charge q' due to charge q , both at rest, is $\mathbf{F}_{q'} = qq' \hat{\mathbf{r}}/r \equiv q' \mathbf{E}$, where \mathbf{r} is the vector from charge q to q' and we use Gaussian units, so the electric field \mathbf{E} of a point charge q (at rest) has the nontrivial form

$$\mathbf{E} = \frac{q}{r} \hat{\mathbf{r}}. \quad (1)$$

Moving charges lead to time-varying electric fields at a fixed observer, so wave phenomena are possible.

Gauss' law for the electric field $\mathbf{E} = (E_1, E_2)$ in 2 spatial dimensions is that the number of field lines crossing a closed loop is proportional to the total charge inside the loop,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 2\pi Q_{\text{in}} = 2\pi \int \rho d\text{Area}, \quad (2)$$

where ρ is the (surface) charge density, and the proportionality constant is 2π in Gaussian units. The differential form of Gauss' law in 2 spatial dimensions is

$$\nabla \cdot \mathbf{E} = 2\pi\rho, \quad \text{where} \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right). \quad (3)$$

When both charges q and q' are in motion with uniform velocities $\mathbf{v} = (v_1, v_2)$ and $\mathbf{v}' = (v'_1, v'_2)$, charge q' experiences the (Lorentz) force

$$\mathbf{F}_{q'} = q' \left(\mathbf{E} + (v'_2, -v'_1) \frac{q(v_1 r_2 - v_2 r_1)}{c^2 r^3} \right), \quad (4)$$

where again vector \mathbf{r} points from q to q' . This might lead enthusiasts of vector analysis to invent the concept of “perpendicular vectors,” defined for any 2-vector $\mathbf{a} = (a_1, a_2)$ as⁵

$$\mathbf{a}_\perp \equiv (a_2, -a_1), \quad (\mathbf{a}_\perp)_\perp = -\mathbf{a}, \quad (5)$$

so that⁶

$$\mathbf{a}_\perp \cdot \mathbf{a}_\perp = \mathbf{a} \cdot \mathbf{a} = a^2 = a_1^2 + a_2^2, \quad \mathbf{a} \cdot \mathbf{a}_\perp = 0, \quad \mathbf{a} \cdot \mathbf{b}_\perp = -\mathbf{a}_\perp \cdot \mathbf{b} = a_1 b_2 - a_2 b_1. \quad (6)$$

Then, the Lorentz force law can be recast as⁷

$$\mathbf{F}_{q'} = q' \left(\mathbf{E} + \frac{\mathbf{v}'_\perp}{c} B \right), \quad (7)$$

⁵The vector $-\mathbf{a}_\perp$ is also perpendicular to \mathbf{a} , so the notion of a perpendicular vector has a two-fold ambiguity in 2 spatial dimensions. In $n \geq 3$ spatial dimensions there is an infinite set of perpendicular vectors (of same norm as an n -vector \mathbf{a}), such that perpendicular-vector algebra is not useful there.

⁶ $\mathbf{a} \cdot \mathbf{b}_\perp = -\mathbf{a}_\perp \cdot \mathbf{b}$ is the 2-vector equivalent of the third component of $\mathbf{a} \times \mathbf{b}$ when \mathbf{a} and \mathbf{b} are 3-vectors. Thus, perpendicular-vector algebra plays a special role in 2 spatial dimensions, akin to the special role of vector cross products in 3 spatial dimensions.

⁷It does not make sense to suppose that there could exist magnetic charges p in $2 + 1$ spacetime, as no 2-vector force \mathbf{F}_p can be constructed from p and B . As discussed in Appendix A, it seems that magnetic charges are physically plausible only in $3 + 1$ spacetime.

where the scalar B is the “magnetic” field in 2 spatial dimensions due to moving charge q ,

$$B = q \frac{\mathbf{v} \cdot \mathbf{r}_\perp}{cr^3} \left(= -q \frac{\mathbf{v}_\perp \cdot \mathbf{r}}{cr^3} \right). \quad (8)$$

For a continuous, steady (surface) current density \mathbf{J} of moving charges, the magnetic field has the (Biot-Savart) form

$$B = \int \frac{\mathbf{J} \cdot \mathbf{r}_\perp}{cr^3} d\text{Area}, \quad (9)$$

where “steady” means

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = 0. \quad (10)$$

The Ampère of $2 + 1$ spacetime might then introduce the 2-dimensional vector derivative operator

$$\nabla_\perp = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1} \right), \quad \nabla^2 = \nabla \cdot \nabla = \nabla_\perp \cdot \nabla_\perp = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad \nabla \cdot \nabla_\perp = 0, \quad (11)$$

to show that the scalar magnetic field (9) due to steady currents obeys the differential equation

$$\nabla_\perp B = \frac{2\pi}{c} \mathbf{J}. \quad (12)$$

The Faraday of $2 + 1$ spacetime would show that a time-varying magnetic field affects the electric field according to the forms

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \frac{d}{dt} \int B d\text{Area}, \quad \nabla \cdot \mathbf{E}_\perp = -\nabla_\perp \cdot \mathbf{E} = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad (13)$$

and finally the Maxwell of $2 + 1$ spacetime would generalize Ampère’s law (12) to time-varying situations by inventing the $2 + 1$ “displacement current,” $(1/2\pi)\partial\mathbf{E}/\partial t$, such that the microscopic Maxwell equations (for \mathbf{E} rather than \mathbf{E}_\perp) are

$$\nabla \cdot \mathbf{E} = 2\pi\rho, \quad \nabla_\perp \cdot \mathbf{E} = \frac{1}{c} \frac{\partial B}{\partial t}, \quad \nabla_\perp B = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{2\pi}{c} \mathbf{J}. \quad (14)$$

There are three scalar, differential equations for the three field components E_1 , E_2 and B ; there is no equivalent to the 3-dimensional equation $\nabla \cdot \mathbf{B} = 0$.⁸

Applying the operator ∇_\perp to the third of eq. (14), and then using the second of these, leads to the wave equation for the magnetic field,

$$\nabla^2 B - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = \frac{2\pi}{c} \nabla_\perp \cdot \mathbf{J}. \quad (15)$$

⁸In $3 + 1$ spacetime the equation $\nabla \cdot \mathbf{B} = 0$ indicates that the 3-vector magnetic field \mathbf{B} is not due to magnetic charges. In $2 + 1$ spacetime the scalar character of the magnetic field B alerts us that this is not due to magnetic charges, which would lead to a 2-vector magnetic field \mathbf{B}_m . The Appendix considers electrodynamics with magnetic charges in $2 + 1$ spacetime.

Noting a vector calculus identity, $\nabla_{\perp}(\nabla_{\perp} \cdot \mathbf{E}) = \nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E})$, applying the operator ∇_{\perp} to the second of eq. (14), and then using the first and third of these, leads to the wave equation for the electric field,

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 2\pi \nabla \rho + \frac{2\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t}. \quad (16)$$

Thus, waves of both \mathbf{E} and B propagate at the speed c , which can be called the speed of light in $2 + 1$ electrodynamics.

2.2.1 Potentials

To write the fields \mathbf{E} and B in terms of potentials, we don't use a single scalar potential for B , but relate it to a two-component vector potential \mathbf{A} , according to

$$B = -\nabla \cdot \mathbf{A}_{\perp} = -\nabla_{\perp} \cdot \mathbf{A}, \quad (17)$$

so that the second Maxwell equation of (14) can be written as

$$\nabla_{\perp} \cdot \left(\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \quad (18)$$

From the definitions (11) we see that $\nabla \cdot \nabla_{\perp} = 0$, so that any vector field \mathbf{f} of the form $-\nabla V$ for a scalar potential V obeys $\nabla_{\perp} \cdot \mathbf{f} = 0$. Hence we can relate the field $\mathbf{f} = \mathbf{E} + \partial \mathbf{A} / \partial ct$ to a scalar potential V , and write

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \quad (19)$$

Wave equations for the potentials can be found by replacing the fields in the Maxwell equations (14) by their expressions (17) and (19) in terms of the potentials. The first of eq. (14) leads to

$$\nabla^2 V + \frac{1}{c} \frac{\partial \nabla \cdot \mathbf{A}}{\partial t} = -2\pi \rho, \quad (20)$$

and the third of eq. (14) leads to

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) - \frac{1}{c} \nabla \frac{\partial V}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -2\pi \mathbf{J}. \quad (21)$$

If we invoke the Lorenz gauge condition,

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial V}{\partial t} = 0, \quad (22)$$

then the wave equations of the potentials take the forms

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -2\pi \rho, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -2\pi \mathbf{J} \quad (\text{Lorenz gauge}). \quad (23)$$

Formal solutions to the wave equations (23) are the retarded potentials

$$V(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t' = t - r/c)}{r} d^2\mathbf{x}', \quad \mathbf{A}(\mathbf{x}, t) = \int \frac{\mathbf{J}(\mathbf{x}', t' = t - r/c)}{cr} d^2\mathbf{x}', \quad (24)$$

with $r = |\mathbf{x} - \mathbf{x}'|$.

Once electrodynamics in $2 + 1$ dimensions is related to potentials, we see that the formal structure is almost identical to that for $3 + 1$ dimensions, so that examples in the former case are essentially as complicated as in the latter. Hence, there is almost no literature on electrodynamics in $2 + 1$ dimensions as a simple path to understanding of that in $3 + 1$ dimensions.

2.2.2 Energy, Momentum and Stress

Poynting's argument [7] relates the rate of work done by electromagnetic fields on “free” electric and magnetic currents to both flow of energy and to rate of change of stored energy. The density of the time rate of change of work on electric currents follows from eq. (7),

$$\frac{dw}{dt} (= \mathbf{F} \cdot \mathbf{v}) = \mathbf{J} \cdot \mathbf{E} = \mathbf{E} \cdot \left(\nabla_{\perp} B - \frac{1}{2\pi} \frac{\partial \mathbf{E}}{\partial t} \right) = \frac{c}{2\pi} \mathbf{E} \cdot \nabla_{\perp} B - \frac{1}{4\pi} \frac{\partial E^2}{\partial t}. \quad (25)$$

Now,

$$\begin{aligned} \mathbf{E} \cdot \nabla_{\perp} B &= E_1 \frac{\partial B}{\partial x_2} - E_2 \frac{\partial B}{\partial x_1} = -\frac{\partial(-E_1 B)}{\partial x_2} - B \frac{\partial E_1}{\partial x_2} - \frac{\partial(E_2 B)}{\partial x_1} + B \frac{\partial E_2}{\partial x_1} \\ &= -\nabla \cdot \mathbf{E}_{\perp} B - B \nabla_{\perp} \cdot \mathbf{E} = -\nabla \cdot \mathbf{E}_{\perp} B - \frac{B}{c} \frac{\partial B}{\partial t} = -\nabla \cdot \mathbf{E}_{\perp} B - \frac{1}{2c} \frac{\partial B^2}{\partial t} \end{aligned} \quad (26)$$

so that eq. (25) can be written as

$$-\mathbf{J} \cdot \mathbf{E} = \nabla \cdot \left(\frac{c}{2\pi} \mathbf{E}_{\perp} B \right) + \frac{\partial}{\partial t} \frac{E^2 + B^2}{4\pi} \equiv \nabla \cdot \mathbf{S} + \frac{\partial u_{\text{EM}}}{\partial t}, \quad (27)$$

where

$$\mathbf{S} = \frac{c}{2\pi} \mathbf{E}_{\perp} B, \quad u_{\text{EM}} = \frac{E^2 + B^2}{4\pi}. \quad (28)$$

The Poynting vector \mathbf{S} described the flux of energy density in the electromagnetic field, and u_{EM} is the density of energy stored in it.

Related arguments (first given in $3 + 1$ spacetime by Abraham [8] and Minkowski [9]) start from the Lorentz force density as the time rate of change of mechanical momentum,

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = \mathbf{f} = \rho \mathbf{E} + \frac{\mathbf{J}_{\perp}}{c} B = \frac{\mathbf{E}(\nabla \cdot \mathbf{E})}{2\pi} + \frac{\mathbf{J}_{\perp}}{c} B \quad (29)$$

We can form the perpendicular-vector version of the third Maxwell equation of (14), finding

$$(\nabla_{\perp} B)_{\perp} = -\nabla B = \frac{1}{c} \frac{\partial \mathbf{E}_{\perp}}{\partial t} + \frac{2\pi}{c} \mathbf{J}_{\perp}, \quad (30)$$

$$\begin{aligned} \frac{\mathbf{J}_{\perp}}{c} B &= -\frac{B \nabla B}{2\pi} - \frac{B}{2\pi c} \frac{\partial \mathbf{E}_{\perp}}{\partial t} = -\frac{\partial}{\partial t} \frac{\mathbf{E}_{\perp} B}{2\pi c} + \frac{\mathbf{E}_{\perp}}{2\pi c} \frac{\partial B}{\partial t} - \frac{B \nabla B}{2\pi} \\ &= -\frac{\partial}{\partial t} \frac{\mathbf{E}_{\perp} B}{2\pi c} - \frac{\mathbf{E}_{\perp} (\nabla \cdot \mathbf{E}_{\perp})}{2\pi} - \frac{\nabla B^2}{4\pi}, \end{aligned} \quad (31)$$

such that eq. (29) can be written as

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\frac{\partial}{\partial t} \frac{\mathbf{E}_{\perp} B}{2\pi c} + \frac{\mathbf{E}(\nabla \cdot \mathbf{E})}{2\pi} - \frac{\mathbf{E}_{\perp}(\nabla \cdot \mathbf{E}_{\perp})}{2\pi} - \frac{\nabla B^2}{4\pi} = -\frac{d\mathbf{p}_{\text{EM}}}{dt} + \nabla \cdot \mathbf{T}, \quad (32)$$

where

$$\mathbf{p}_{\text{EM}} = \frac{\mathbf{E}_{\perp} B}{2\pi c} = \frac{\mathbf{S}}{c^2} \quad (33)$$

is the density of momentum stored in the electromagnetic field, and \mathbf{T} is the 2×2 Maxwell stress tensor,

$$\mathbf{T}_{ij} = \frac{E_i E_j}{2\pi} - \delta_{ij} \frac{E^2 + B^2}{4\pi}. \quad (34)$$

A Microscopic Electrodynamics via $n + 1$ Spacetime Vectors

To generalize $3 + 1$ spacetime electrodynamics to $n + 1$ dimensions, it is useful cast $3 + 1$ electrodynamics into $(3 + 1)$ -vector notation. We also consider the possibility of magnetic charges in addition to electric charges.

When Heaviside first presented Maxwell's equations in vector notation [10] he assumed that in addition to electric charge and current densities, ρ_e and \mathbf{J}_e , there existed magnetic charge and current densities, ρ_m and \mathbf{J}_m , although there remains no experimental evidence for the latter.⁹ Maxwell's equations for microscopic electrodynamics are then (in Gaussian units)¹⁰

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \quad \nabla \cdot \mathbf{B} = 4\pi\rho_m, \quad -c\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + 4\pi\mathbf{J}_m, \quad c\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J}_e, \quad (35)$$

where c is the speed of light in vacuum.

The factors of 4π in eq. (35) for $3 + 1$ electrodynamics are associated with the surface area of a unit 3-sphere. We anticipate that in $n + 1$ spacetime, these factors become

$$S_n = \frac{n\pi^{n/2}}{\Gamma(n/2 + 1)}, \quad (36)$$

such that $S_1 = 2$, $S_2 = 2\pi$, $S_3 = 4\pi$, $S_4 = 2\pi^2$, \dots ¹¹

The fields \mathbf{E} and \mathbf{B} can be deduced from potentials according to

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m, \quad \mathbf{B} = \mathbf{B}_e + \mathbf{B}_m, \quad (37)$$

$$\mathbf{E}_e = -\nabla V_e - \frac{1}{c} \frac{\partial \mathbf{A}_e}{\partial t}, \quad \mathbf{B}_e = \nabla \times \mathbf{A}_e, \quad (38)$$

$$\mathbf{B}_m = -\nabla V_m - \frac{1}{c} \frac{\partial \mathbf{A}_m}{\partial t}, \quad \mathbf{E}_m = -\nabla \times \mathbf{A}_m, \quad (39)$$

⁹Heaviside seems to have regarded magnetic charges as “fictitious,” as indicated on p. 25 of [11].

¹⁰See [12] for Maxwell's equations in SI units, including a discussion in footnote 7 there about the ambiguous placement of the permeability μ_0 for terms involving magnetic charges.

¹¹While many people prefer to use units such that the factors of 4π do not appear in Maxwell's equations in $3 + 1$ electrodynamics, these units obscure the generalization to $n + 1$ spacetime.

where in the Lorenz gauge,

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial V}{\partial t} = 0, \quad (40)$$

the (retarded) potentials are related to the source charge/current densities by¹²

$$V_{e,m}(\mathbf{x}, t) = \int \frac{\rho_{e,m}(\mathbf{x}', t' = t - r/c)}{r} d^3\mathbf{x}', \quad \mathbf{A}_{e,m}(\mathbf{x}, t) = \int \frac{\mathbf{J}_{e,m}(\mathbf{x}', t' = t - r/c)}{cr} d^3\mathbf{x}', \quad (41)$$

with $r = |\mathbf{x} - \mathbf{x}'|$.

The fields \mathbf{E} and \mathbf{B} obey the duality relations that

$$(c\rho_e, \mathbf{J}_e) \rightarrow (c\rho_m, \mathbf{J}_m), \quad (c\rho_m, \mathbf{J}_m) \rightarrow -(c\rho_e, \mathbf{J}_e) \quad \Rightarrow \quad \mathbf{E} \rightarrow \mathbf{B}, \quad \mathbf{B} \rightarrow -\mathbf{E}. \quad (42)$$

Expressions (35)-(42) can be taken over to the case of n spatial dimensions, except for eqs. (35), (38)-(39) which involve the vector cross product, which is defined only for three spatial dimensions. Hence, we recast these equations into $(n+1)$ -dimensional vector/tensor form where a charge-current vector is written $J_\mu = (c\rho, \mathbf{J})$, $\mu = 0, 1, \dots, n$, a potential is written $A_\mu = (V, \mathbf{A})$, the derivative operator is written $\partial_\mu = (\partial/\partial ct, -\partial/\partial \mathbf{x})$, and the scalar product of two vectors is written $a_\mu b_\mu = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$.

Conservation of electric and magnetic charge can now be expressed as

$$\partial_\mu J_\mu = 0 \quad \left(= \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right). \quad (43)$$

We introduce the antisymmetric field tensors F_e and F_m with components.

$$F_{e,\mu\nu} = \partial_\mu A_{e,\nu} - \partial_\nu A_{e,\mu}, \quad F_{m,\mu\nu} = \partial_\mu A_{m,\nu} - \partial_\nu A_{m,\mu}, \quad (44)$$

In 3 spatial dimensions ($3+1$ electrodynamics) these tensors have components

$$F_e = \begin{pmatrix} 0 & -E_{e,1} & -E_{e,2} & -E_{e,3} \\ E_{e,1} & 0 & -B_{e,3} & B_{e,2} \\ E_{e,2} & B_{e,3} & 0 & -B_{e,1} \\ E_{e,3} & -B_{e,2} & B_{e,1} & 0 \end{pmatrix}, \quad F_m = \begin{pmatrix} 0 & -B_{m,1} & -B_{m,2} & -B_{m,3} \\ B_{m,1} & 0 & E_{m,3} & -E_{m,2} \\ B_{m,2} & -E_{m,3} & 0 & E_{m,1} \\ B_{m,3} & E_{m,2} & -E_{m,1} & 0 \end{pmatrix}. \quad (45)$$

Then, the tensor relations, where S_n is defined in eq. (36),

$$\partial_\mu F_{e,\mu\nu} = \frac{S_n}{c} J_{e,\nu}, \quad \partial_\mu F_{m,\mu\nu} = \frac{S_n}{c} J_{m,\nu}, \quad (46)$$

lead to the Maxwell equations in $n=3$ spatial dimensions,

$$\nabla \cdot \mathbf{E}_e = 4\pi\rho_e, \quad c\nabla \times \mathbf{B}_e = \frac{\partial \mathbf{E}_e}{\partial t} + 4\pi\mathbf{J}_e, \quad \nabla \cdot \mathbf{B}_m = 4\pi\rho_m, \quad -c\nabla \times \mathbf{E}_m = -\frac{\partial \mathbf{B}_m}{\partial t} + 4\pi\mathbf{J}_m. \quad (47)$$

¹² \mathbf{A}_e is often called the magnetic vector potential, but as its source is the electrical current \mathbf{J}_e , it is better called the electric vector potential.

The remaining Maxwell equations,

$$\nabla \cdot \mathbf{B}_e = 0, \quad -c \nabla \times \mathbf{E}_e = \frac{\partial B_e}{\partial t}, \quad \nabla \cdot \mathbf{E}_m = 0, \quad c \nabla \times \mathbf{B}_m = \frac{\partial E_m}{\partial t}, \quad (48)$$

can be obtained from

$$\partial_\lambda F_{e,\mu\nu} + \partial_\mu F_{e,\nu\lambda} + \partial_\nu F_{e,\lambda\mu} = 0, \quad \partial_\lambda F_{m,\mu\nu} + \partial_\mu F_{m,\nu\lambda} + \partial_\nu F_{m,\lambda\mu} = 0, \quad (49)$$

which are true for any values of the indices $\{\lambda, \mu, \nu\}$, but which are nontrivial only if all three indices are distinct, leading to only four different relations for indices $\{1, 2, 3\}$ (which corresponds to the divergence equations in (48)), and $\{0, 1, 2\}$, $\{0, 2, 3\}$, $\{0, 3, 1\}$ (which correspond to the curl equations).

Turning to the Maxwell stress-energy-momentum tensor, we note that in $3 + 1$ spacetime this can be written as

$$T_{\mu\nu} = \frac{1}{4\pi} F_{\mu\lambda} F_{\lambda\nu} + \frac{1}{16\pi} \eta_{\mu,\nu} F_{\kappa\lambda} F_{\kappa\lambda}, \quad (50)$$

where

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \quad (51)$$

and $\eta_{\mu\nu}$ is the “metric” tensor with $\eta_{00} = 1$, $\eta_{ii} = -1$, and all other components 0.

However, $3 + 1$ spacetime is a special case in this regard, as number of distinct field components in the electric and magnetic field tensors F_e or F_m , eq. (44), in $n + 1$ spacetime is n components with indices $0i$ and $n(n - 1)/2$ components with indices ij for a total of $N_n = n(n + 1)/2$ in $n + 1$ electrodynamics; $N_1 = 1$, $N_2 = 3$, $N_3 = 6$, $N_4 = 10$, *etc.* The only case with the same number of components of the two types, *i.e.*, with $n = n(n - 1)/2$, is $n = 3$; only in $3 + 1$ spacetime do the components of F_e or F_m combine into a single 4-tensor (51). In any other spatial dimension than 3, the components of the electric and magnetic field tensors (44) are different physical entities, and the stress-energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{4\pi} F_{e,\mu\lambda} F_{e,\lambda\nu} + \frac{1}{16\pi} \eta_{\mu,\nu} F_{e,\kappa\lambda} F_{e,\kappa\lambda} + \frac{1}{4\pi} F_{m,\mu\lambda} F_{m,\lambda\nu} + \frac{1}{16\pi} \eta_{\mu,\nu} F_{m,\kappa\lambda} F_{m,\kappa\lambda}. \quad (52)$$

Finally (in the present survey) the Lorentz force (density) law in $3 + 1$ spacetime is

$$\mathbf{f} = \mathbf{f}_e + \mathbf{f}_m = \rho_e \mathbf{E} + \frac{\mathbf{J}_e}{c} \times \mathbf{B} + \rho_m \mathbf{B} - \frac{\mathbf{J}_m}{c} \times \mathbf{E}, \quad (53)$$

assuming that the electromagnetic fields in eq.(53) are the total fields $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ and $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_m$. To cast the Lorentz force into tensor notation, it is useful to introduce the

dual tensors $F_{\mu\nu}^* = \frac{1}{2}\epsilon_{\mu\mu\kappa\lambda}F_{\mu\nu}$,

$$F_e^* = \begin{pmatrix} 0 & -B_{e,1} & -B_{e,2} & -B_{e,3} \\ B_{e,1} & 0 & E_{e,3} & -E_{e,2} \\ B_{e,2} & -E_{e,3} & 0 & E_{e,1} \\ B_{e,3} & E_{e,2} & -E_{e,1} & 0 \end{pmatrix}, \quad F_m^* = \begin{pmatrix} 0 & -E_{m,1} & -E_{m,2} & -E_{m,3} \\ E_{m,1} & 0 & -B_{m,3} & B_{m,2} \\ E_{m,2} & B_{m,3} & 0 & -B_{m,1} \\ E_{m,3} & -B_{m,2} & B_{m,1} & 0 \end{pmatrix}, \quad (54)$$

Then, we can define the total field tensors

$$F = F_e + F_m^*, \quad F^* = F_e^* + F_m, \quad (55)$$

and the Lorentz force law can be written as

$$f_\mu = F_{\mu\nu} \frac{J_{e,\nu}}{c} + F_{\mu\nu}^* \frac{J_{e,\nu}}{c} \quad (3 + 1 \text{ spacetime}), \quad (56)$$

However, as seen above, only in $3 + 1$ spacetime can the components of the field tensors $F_{e,\mu\nu}$ and $F_{m,\mu\nu}$ be combined into a single field tensor $F_{\mu\nu}$ such that total 3-vector electric and magnetic fields can be defined as the sums of the 3-vector fields due to electric and magnetic charges, $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ and $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_m$. For other spatial dimensions we can only have

$$f_\mu = F_{e,\mu\nu} \frac{J_{e,\nu}}{c} + F_{m,\mu\nu} \frac{J_{m,\nu}}{c} \quad (n \neq 3). \quad (57)$$

which means that magnetic charges, if they exist, do not interact electromagnetically with electric charges (and so would be very hard to detect in apparatus made of electric charges). There is no distinction between the behavior of “electric” and “magnetic” charges when eq. (57) applies, so if both types of charges exist, it as if there were two types of electric charges with no interactions between the two types.

Note also that it is a logical possibility that the form (57) holds in $3 + 1$ spacetime as well, which could be why magnetic charges have not been detected in apparatus based on the electromagnetic interaction of electric charges.¹³

A.1 Electrodynamics in 1 Spatial Dimension

In $1 + 1$ electrodynamics, with one spatial dimension, x , the field tensors (44) have components

$$F_e = \begin{pmatrix} 0 & -E_e \\ E_e & 0 \end{pmatrix}, \quad F_m = \begin{pmatrix} 0 & -B_m \\ B_m & 0 \end{pmatrix}. \quad (58)$$

There is no magnetic field associated with moving electric charge, and no electric field due to moving magnetic charge. That is, the electric field is only due to electric charge, and the magnetic field is only due to magnetic charge.

¹³Magnetic charges that do not couple to electric charges, but have their own “magnetodynamics,” are not candidates for the dark matter of galactic haloes, as the magnetodynamic interaction would permit clumping of magnetic-charge matter similar to the clumping of electric-charge matter via electrodynamics.

The relations (49) lead to only trivial equations, as there are now only two value for the indices $\{\lambda, \mu, \nu\}$ which must be all distinct to have a nontrivial relation. The relations (46) lead to the “Maxwell” equations for $S_1 = 2$.

$$\frac{\partial E_e}{\partial x} = 2\rho_e, \quad \frac{\partial E_e}{\partial t} = -2J_e, \quad \frac{\partial B_m}{\partial x} = 2\rho_m, \quad \frac{\partial B_m}{\partial t} = -2J_m. \quad (59)$$

Only two of these four equations are independent, in view of charge conservation,

$$\frac{\partial \rho_e}{\partial t} = -\frac{\partial J_e}{\partial x}, \quad \frac{\partial \rho_m}{\partial t} = -\frac{\partial J_m}{\partial x}. \quad (60)$$

Equations (59) indicate that the fields E_e and B_m are constant in both x and t in charge-free regions. Hence, there are no electromagnetic waves in charge-free regions in $1 + 1$ electrodynamics, for which the constant c does not have the significance of the speed of electromagnetic waves.

If electric charge q is distributed uniformly between on the interval $[-dx/2, dx/2]$ then the charge density is $\rho = q/dx$, and the constant fields $\pm E_e$ outside the charge distribution are related by the first of eq. (59) by

$$\frac{dE_e}{dx} = \frac{2E_e}{dx} = 2\rho_e = \frac{2q}{dx}, \quad (61)$$

so $E_e = q$, as previously argued in sec. 2.1.

The force density on electric and magnetic charge distributions is

$$f = \rho_e E_e + \rho_m B_m. \quad (62)$$

There is no coupling between “electric” and “magnetic” charges.

A.2 Electrodynamics in 2 Spatial Dimensions

In $2 + 1$ electrodynamics the field tensors (44) have components

$$F_e = \begin{pmatrix} 0 & -E_{e,1} & -E_{e,2} \\ E_{e,1} & 0 & -B_e \\ E_{e,2} & B_e & 0 \end{pmatrix}, \quad F_m = \begin{pmatrix} 0 & -B_{m,1} & -B_{m,2} \\ B_{m,1} & 0 & -E_m \\ B_{m,2} & E_m & 0 \end{pmatrix}. \quad (63)$$

The fields \mathbf{E}_e and \mathbf{B}_m are 2-vectors, while the fields B_e and E_m are scalars.

The force density on electric and magnetic charge distributions is

$$\mathbf{f} = \rho_e \mathbf{E}_e + \frac{\mathbf{J}_{e,\perp} B_e}{c} + \rho_m \mathbf{B}_m + \frac{\mathbf{J}_{m,\perp} B_m}{c}. \quad (64)$$

As in any $n+1$ spacetime except $3 + 1$, there is no coupling between “electric” and “magnetic” charges, and no conceptual distinction between them.

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