5.3. Check whether the following formulas are theorems or not using predicate resolution

$$\mathcal{U}_3 = (\forall x)(\forall y)P(x,y) \leftrightarrow (\exists x)(\forall y)P(x,y)$$

A propositional formula $\mathcal{U}$ is a theorem iff and only if the empty clause can be derived from the clausal normal form of $\neg\mathcal{U}$, using the resolution algorithm.

$\mathcal{U}$ theorem iff $CNF(\neg\mathcal{U}) \vdash_{Res} \square$

A predicate formula $\mathcal{U}$ is in prenex normal form if it has the form $(Q_1x_1)\ldots(Q_nx_n)M$ where Q_i are quantifiers and M is quantifier free. The prenex normal form is obtained by applying transformations which preserve the logical equivalence, according to the following steps:

Step 1 The connectives $\rightarrow$ and $\leftrightarrow$ are replaced using the connectives $\neg, \wedge, \vee$ $\quad (A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

$\neg(\mathcal{U} \rightarrow V) \equiv \mathcal{U} \wedge \neg V)$

$$\mathcal{U}_3 = (\forall x)(\forall y)P(x,y) \rightarrow (\exists x)(\forall y)P(x,y) \wedge (\exists x)(\forall y)P(x,y) \rightarrow (\forall x)(\forall y)P(x,y)$$

$$\mathcal{U}_{3,1} = (\forall x)(\forall y)P(x,y) \rightarrow (\exists x)(\forall y)P(x,y)$$

$$\mathcal{U}_{3,2} = (\exists x)(\forall y)P(x,y) \rightarrow (\forall x)(\forall y)P(x,y)$$

$\mathcal{U}_3$ is a theorem if both $\mathcal{U}_{3,1}$ and $\mathcal{U}_{3,2}$ are theorems

Study $\neg U_{3,1}$ and apply DeMorgan's law:

$\neg U_{3,1} = (\forall x)(\forall y) P(x,y) \wedge \neg(\exists x)(\forall y) P(x,y)$

$= (\forall x)(\forall y) P(x,y) \wedge (\forall x)(\exists y) \neg P(x,y)$

Step 2 Rename the bound variables

$= (\forall x)(\forall y) P(x,y) \wedge (\forall u)\cancel{\forall}(\exists t) \neg P(u,t)$

Step 3 Extract quantifiers in front of the formula

$= (\forall u)(\exists t)(\forall x)(\forall y) P(x,y) \wedge \neg P(u,t)$

From the prenex form we obtain the Skolem form by applying transformations to the existential quantifiers.

$t = f(u) \rightarrow$ unary Skolem function

$U^S = (\forall u)(\forall x)(\forall y) P(x,y) \wedge \neg P(u, f(u))$

Obtain the clausal normal form U^C by deleting the prefix of U^S

$U^C = P(x,y) \wedge \neg P(u, f(u))$

$C_1 : P(x,y)$

$C_2 : \neg P(u, f(u))$

$\theta_1 = [x \leftarrow u, y \leftarrow f(u)]$

$C_3 : \vdash^{\theta_1} \square$

So $U_{3,1}$ is a theorem

We repeat the steps for $U_{3.2}$

$\neg U_{3.2} = (\exists x)(\forall y) P(x, y) \wedge \neg(\forall x)(\forall y) P(x, y)$
$= (\exists x)(\forall y) P(x, y) \wedge (\exists x)(\exists y) \neg P(x, y)$
$= (\exists x)(\forall y) P(x, y) \wedge (\exists u)(\exists t) \neg P(u, t)$

$U^P = (\exists u)(\exists t)(\exists x)(\forall y) P(x, y) \wedge \neg P(u, t)$

$u = a$
$t = b$ Skolem constants
$x = c$

$U^S = (\forall y) P(c, y) \wedge \neg P(a, b)$

$U^C = P(c, y) \wedge \neg P(a, b)$

$C_1 : P(c, y)$
$C_2 : \neg P(a, b)$

Because a, b, c are distinct constants we can't do any valid substitutions, so we can't obtain an empty clause. Therefore, $U_{3.2}$ is not a theorem.

$U_{3.1}$ theorem
$U_{3.2}$ not a theorem $\Rightarrow U_3$ not a theorem