GHERASIM DELIA-CĂTĂLINA
923

# Bonus Problems - seminar 4

8). $X, Y$ independent random variables such that $X$ has $U(2)$, $Y$ has $Bern(\frac{1}{3})$

$U = X + Y$

$V = X - Y$

a). <u>joint pdf of $(U, V)$</u>

$X$ has a discrete Uniform distribution with parameter $2 \Rightarrow$ ~~X has~~ it's pdf is: $X\begin{pmatrix} k \\ \frac{1}{m} \end{pmatrix}_{k=\overline{1,m}} = X\begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$Y$ has a ~~dis~~ Bernoulli distribution with parameter $\frac{1}{3} \Rightarrow$ it's pdf is: $Y\begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} = Y\begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

The sum of $X\begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$, $Y\begin{pmatrix} y_j \\ q_j \end{pmatrix}_{j \in J}$ is the random variable with pdf given by $X+Y\begin{pmatrix} x_i + y_j \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}$

to find all possible values of $X+Y$ we take all the combinations of $x_i + y_j$, $i, j = 1, 2$

$x_1 + y_1 = 1 + 0 = 1$
$x_1 + y_2 = 1 + 1 = 2$
$x_2 + y_1 = 2 + 0 = 2$
$x_2 + y_2 = 2 + 1 = 3$

So, $X+Y$ can take the values $1, 2, 3$. Then we compute their corresponding probabilities:

$P(X+Y=1) = P(X=1, Y=0) \overset{indep.}{=} P(X=1)P(Y=0) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

$P(X+Y=2) = P((X=1, Y=1) \cup (X=2, Y=0)) = P(X=1, Y=1) + P(X=2, Y=0)$
$= P(X=1)P(Y=1) + P(X=2)P(Y=0) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}$

$P(X+Y=3) = P(X=2, Y=1) = P(X=2)P(Y=1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

Therefore, the pdf of $U$ is $X+Y\begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$

Now, we compute the pdf of $V = X - Y = X + (-1)Y$

The result of $Y\begin{pmatrix} y_j \\ q_j \end{pmatrix}_{j \in J}$ multiplied with a scalar $\alpha = -1$ is the random variable $\alpha Y$, having the pdf $\alpha Y\begin{pmatrix} \alpha y_j \\ q_j \end{pmatrix}$

$\Rightarrow (-1) \cdot Y\begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} = -Y\begin{pmatrix} 0 & -1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} = Y'\begin{pmatrix} 0 & -1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

For $X + (-1)Y$ we do the same thing as before:

$x_1 + y_1 = 1 + 0 = 1$

$x_1 + y_2 = 1 - 1 = 0$ $\Rightarrow$ the values of $X - Y$ are $0, 1, 2$

$x_2 + y_1 = 2 + 0 = 2$

$x_2 + y_2 = 2 - 1 = 1$

$P(X-Y=0) = P(X=1, Y'=-1) = P(X=1)P(Y'=-1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$P(X-Y=1) = P((X=1, Y'=0) \cup (X=2, Y'=-1)) = P(X=1, Y'=0) + P(X=2, Y'=-1)$

$= P(X=1)P(Y'=0) + P(X=2)P(Y'=-1) = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}$

$P(X-Y=2) = P(X=2, Y=0) = P(X=2)P(Y=0) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

So, the pdf of $V$ is $X-Y\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$

The joint pdf of $(U, V)$ is a 2-dimentional array of the form

$$\begin{array}{c|ccc|c} U/V & v_1 & v_2 & v_3 & \\ \hline u_1 & & \vdots & & p_1 \\ u_2 & \cdots & p_{ij} & \cdots & p_2 \\ u_3 & & \vdots & & p_3 \\ \hline & q_1 & q_2 & q_3 & \end{array}$$

, where $p_{ij} = P(U = u_i, V = v_j)$ is the probability that $(U, V)$ takes the values $(u_i, v_j)$

$p_{12} = P(U=1, V=0) = P(U=1)P(V=0) = \frac{1}{3} \cdot \frac{1}{6}$

$p_{12} = P(U=1, V=1) = P(U=1)P(V=1) = \frac{1}{3} \cdot \frac{1}{2}$

$p_{13} = P(U=1, V=2) = P(U=1)P(V=2) = \frac{1}{3} \cdot \frac{1}{3}$

$\cdots$

$p_{33} = P(U=3, V=2) = P(U=3)P(V=2) = \frac{1}{6} \cdot \frac{1}{3}$

The joint pdf of $U, V$ is:

| $U \backslash V$ | 0 | 1 | 2 |
|---|---|---|---|
| 1 | $\frac{1}{18}$ | $\frac{1}{6}$ | $\frac{1}{9}$ |
| 2 | $\frac{1}{12}$ | $\frac{1}{4}$ | $\frac{1}{6}$ |
| 3 | $\frac{1}{36}$ | $\frac{1}{12}$ | $\frac{1}{18}$ |

## b). marginal pdf of U and V

The marginal pdf's are the probabilities $p_i$ and $q_j$, where

$p_i = \sum_{j \in J} p_{ij}$, $p_i = P(U = u_i)$, $i \in I$ and

$q_j = \sum_{i \in I} p_{ij}$, $q_j = P(V = v_j)$, $j \in J$.

For $U$, the marginal pdf's are: $\begin{cases} p_1 = P(U=1) = \sum_{j=0,2} p_{1j} \\ p_2 = P(U=2) = \sum p_{2j} \\ p_3 = P(U=3) = \sum p_{3j} \end{cases}$

$$\Rightarrow \begin{cases} p_1 = \frac{1}{18} + \frac{1}{6} + \frac{1}{9} \\ p_2 = \frac{1}{12} + \frac{1}{4} + \frac{1}{6} \\ p_3 = \frac{1}{36} + \frac{1}{12} + \frac{1}{18} \end{cases} \Rightarrow \begin{cases} p_1 = \frac{1}{3} \\ p_2 = \frac{1}{2} \\ p_3 = \frac{1}{6} \end{cases}$$

For $V$, the marginal pdf's are $\begin{cases} q_1 = \sum_{i=1,3} p_{i1} \\ q_2 = \sum p_{i2} \\ q_3 = \sum p_{i3} \end{cases} \Rightarrow \begin{cases} q_1 = \frac{1}{18} + \frac{1}{12} + \frac{1}{36} \\ q_2 = \frac{1}{6} + \frac{1}{4} + \frac{1}{12} \\ q_3 = \frac{1}{9} + \frac{1}{6} + \frac{1}{18} \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} q_1 = \frac{1}{6} \\ q_2 = \frac{1}{2} \\ q_3 = \frac{1}{3} \end{cases}$$

## c). Are U and V independent?

2 discrete random variables $U$ and $V$ with pdf: $U \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$ and $V \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$ are independent iff $p_{ij} = P(U = u_i, V = v_j) = P(U = u_i) P(V = v_j) = p_i q_j$ for all $(i, j) \in I \times J$

$p_1 q_1 = P(U = u_1) P(V = v_1) = P(U=1) P(V=0) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18} = p_{11}$

$p_1 q_2 = P(U=1) P(V=1) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} = p_{12}$

$p_1 q_3 = P(U=1) P(V=3) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} = p_{13}$

$p_3 q_3 = P(U = u_3) P(V=2) = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18} = p_{33}$

Therefore $U$ and $V$ are independent.

9). coordinates $(x, y)$

$x, y$ determined by rolling a dice

$P((x,y) \in \mathcal{C}(x^2+y^2=10)) = ?$

If the values of $x$ and $y$ are determined by rolling a dice, $x$ and $y$ are random variables having the pdf:

$x\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$, $y\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$, meaning they each represent a discrete uniform distribution.

For the ~~variab~~ coordinates to belong to the circle $\mathcal{C}(x^2+y^2=10)$, the possible values for each coordinate need to respect the formula $x^2+y^2=10$.

The only natural number satisfying the condition are:

$(x, y) \in \{(1, 3), (3, 1)\}$. So the probability that the point is on the circle is the same as the probability that:

$P((X=1, Y=3) \cup (X=3, Y=1)) = P(X=1, Y=3) + P(X=3, Y=1) \overset{ind.}{=}$

$= P(X=1)P(Y=3) + P(X=3)P(Y=1) = \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{2}{36}$