Mathematical Analysis

1st Year Computer Science

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Contents

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❊ Real numbers

Let us start with some standard notation: $\emptyset$ is the empty set; $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers; $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\} = \{m - n \mid m, n \in \mathbb{N}\}$ the set of integers; $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ the set of rational numbers; $\mathbb{R}$ the set of real numbers.

You are very much used with the real numbers. However, it is not trivial to define them in a rigorous way, see for example [1] or [2] for different methods of constructing $\mathbb{R}$ from $\mathbb{Q}$. We will straightforwardly start working with the real numbers – for this reason some of their properties, e.g. definition 1.3, will simply be given as definitions.

Definition 1.1. Let A be a subset of $\mathbb{R}$, denoted as $A \subseteq \mathbb{R}$. We define $x \in \mathbb{R}$ to be

a *lower bound* for A if $x \leq a$, $\forall a \in A$; an *upper bound* for A if $x \geq a$, $\forall a \in A$.

We define

$$lb(A) := \{x \in \mathbb{R} \mid x \leq a, \forall a \in A\} \text{ the set of lower bounds of } A,$$

$$ub(A) := \{x \in \mathbb{R} \mid x \geq a, \forall a \in A\} \text{ the set of upper bounds of } A.$$

We define $x \in \mathbb{R}$ to be

the minimum of A if $x \in lb(A) \cap A$; the maximum of A if $x \in ub(A) \cap A$.

These are denoted by $\min(A)$ and $\max(A)$.

Note that there are sets which do no have minimum or maximum, e.g. $(0, 1)$.

Definition 1.2. A set $A \subseteq \mathbb{R}$ is defined to be

- bounded (from) below if $lb(A) \neq \emptyset$;
- bounded (from) above if $ub(A) \neq \emptyset$;
- bounded if it is both bounded below and above;
- unbounded if it is not bounded.

Definition 1.3 (Completeness Axiom)**.** Every set $A \subseteq \mathbb{R}$ that is bounded above has a *least upper bound*, called the *supremum* of A and denoted by $\sup(A)$. Similarly, every set $A \subseteq \mathbb{R}$ that is bounded below has a *greatest lower bound*, called the *infimum* of A and denoted by $\inf(A)$. In other words, we have

$$\sup(A) := \min(ub(A))$$

and

$$\inf(A) := \max(lb(A)).$$

Note that if A has a maximum, then $\sup(A) = \max(A)$. Similarly, if A has a minimum, then $\inf(A) = \min(A)$. Also, if $\sup(A) \in A$, then $\max(A) = \sup(A)$.

Example 1.4. (a) $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $\sup(A) = 1 = \max(A)$, $\inf(A) = 0$, $\nexists \min(A)$.

(b) $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$, $\sup(A) = \sqrt{2}$, $\nexists \max(A)$, $\inf(A) = -\sqrt{2}$, $\nexists \min(A)$.

Definition 1.5. Define the *extended real line* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, where ∞ and $-\infty$ are such that

$$\forall x \in \mathbb{R},\ -\infty < x < \infty.$$

If a set A is not bounded above, we define $\sup(A) := \infty$.
If a set A is not bounded below, we define $\inf(A) := -\infty$.

Note that the empty set $\emptyset$ is bounded by any real number and $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$.

Proposition 1.6. Let $A \subseteq B \subseteq \mathbb{R}$ be (nonempty) bounded sets. Then

$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$$

and

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\},$$
$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

Proof. TBA (left to the reader). □

Theorem 1.7. Let $A \subseteq \mathbb{R}$ be a bounded set. For $\sup(A)$ and $\inf(A)$ the following are true:

$$\forall \varepsilon > 0,\ \exists x \in A \text{ such that } \sup(A) - \varepsilon < x,$$
$$\forall \varepsilon > 0,\ \exists x \in A \text{ such that } x < \inf(A) + \varepsilon.$$

Proof. By definition, $\sup(A)$ is the least upper bound of A. In other words, $\sup(A)$ is an upper bound for A and any number less than $\sup(A)$ is not an upper bound for A. This means that for any $y < \sup(A)$ – say $y = \sup(A) - \varepsilon$, with $\varepsilon > 0$ – we have that $y \notin ub(A)$, hence there exists $x \in A$ such that $y = \sup(A) - \varepsilon < x$. The claim for $\inf(A)$ follows similarly. □

Definition 1.8. A set $V \subseteq \mathbb{R}$ is a *neighborhood (vecinity)* of $x \in \mathbb{R}$ if

$$\exists \varepsilon > 0 \text{ such that } (x - \varepsilon, x + \varepsilon) \subseteq V.$$

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of ∞ if

$$\exists a \in \mathbb{R} \text{ such that } (a, \infty) \subseteq V.$$

A set $V \subseteq \mathbb{R}$ is a *neighborhood* of $-\infty$ if

$$\exists a \in \mathbb{R} \text{ such that } (-\infty, a) \subseteq V.$$

We denote all the neighborhoods of x by $\mathcal{V}(x) := \{V \subseteq \mathbb{R} \mid V \text{ is a neighborhood of } x\}$.

Definition 1.9. Let $A \subseteq \mathbb{R}$. The following set is called the *interior* of A

$$\text{int}(A) := \{x \in \mathbb{R} \mid \exists V \in \mathcal{V}(x) \text{ such that } V \subseteq A\},$$

and the following set is called the *closure* of A

$$\text{cl}(A) := \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}.$$

Proposition 1.10. For any $A \subseteq \mathbb{R}$, it holds that $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$.

Proof. To prove that $\text{int}(A) \subseteq A$ we prove that if $x \in \text{int}(A)$, then $x \in A$. Let $x \in \text{int}(A)$, then $\exists \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq A$. Since $x \in (x-\varepsilon, x+\varepsilon)$, we have that $x \in A$.
To prove that $A \subseteq \text{cl}(A)$ we show that if $x \in A$, then $x \in \text{cl}(A)$. Let $x \in A$. Then for any $V \in \mathcal{V}(x)$ it holds that $x \in V$, giving that $x \in V \cap A$. Hence $x \in \text{cl}(A)$ since $V \cap A \neq \emptyset$. □

Definition 1.11. If $A = \text{int}(A)$, then A is called *open*. If $A = \text{cl}(A)$, then A is called *closed*.

Remark 1.12. To prove that a set A is open, it is sufficient to prove that $A \subseteq \text{int}(A)$.
To prove that a set A is closed, it is sufficient to prove that $\text{cl}(A) \subseteq A$.

Proposition 1.13. The following statements are true:

- The complement of an open set is closed.
- The complement of a closed set is open.

Proof. Let us prove the first statement, the other one being similar. Consider A an open set, i.e. $A = \text{int}(A)$, and denote by $A^c = \{x \in \mathbb{R} \mid x \notin A\}$ its complement. To prove that A^c is closed, we prove that $\text{cl}(A^c) \subseteq A^c$. Consider $x \in \text{cl}(A^c)$ and let's assume that $x \notin A^c$, i.e. $x \in A$, aiming to obtain a contradiction. Since A is open, there exists $V \in \mathcal{V}(x)$ such that $V \subseteq A$, giving that $V \cap A^c = \emptyset$: contradiction with $x \in \text{cl}(A^c)$. Hence the assumption $x \notin A^c$ is false, and we have that if $x \in \text{cl}(A^c)$, then $x \in A^c$. In other words, $\text{cl}(A^c) \subseteq A^c$. □

Proposition 1.14. Any union of open sets is open. Any finite intersection of closed sets is closed.

Proof. TBA (left to the reader). □

❊ Sequences

A set $\{x_n \mid n \in \mathbb{N}\}$ is called a sequence and is denoted by $(x_n)_{n\in\mathbb{N}}$ or simply (x_n). A sequence (x_n) is bounded above (or below) if the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded above (or below). A sequence (x_n) is increasing if $x_{n+1} \geq x_n$, $\forall n \in \mathbb{N}$, and decreasing if $x_{n+1} \leq x_n$, $\forall n \in \mathbb{N}$. A sequence is monotone if it either increasing or decreasing.

Definition 2.1. A sequence (x_n) has a limit $\ell \in \overline{\mathbb{R}}$, and we write $\lim\limits_{n\to\infty} x_n = \ell$ or $x_n \to \ell$, if

$$\forall V \in \mathcal{V}(\ell),\ \exists N_V \in \mathbb{N} \text{ such that } x_n \in V,\ \forall n \geq N_V.$$

If $\ell \in \mathbb{R}$, we say that (x_n) converges to ℓ: $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon$, $\forall n \geq N_\varepsilon$.

Proposition 2.2. A sequence (x_n) converges to ℓ if and only if $\lim\limits_{n\to\infty} |x_n - \ell| = 0$.

Proposition 2.3. Any convergent sequence is bounded.

Proof. TBA (left to the reader). □

Theorem 2.4 (Weierstrass)**.** Any monotone and bounded sequence is convergent.

Proof. Assume that the sequence is increasing, for example. Let $S = \{x_n \mid n \in \mathbb{N}\}$ and consider $\sup(S) \in \mathbb{R}$ (we know that S is bounded). From theorem 1.7 we have that

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } \sup(S) - \varepsilon < x_{N_\varepsilon}.$$

As (x_n) is increasing, $\sup(S) - \varepsilon < x_{N_\varepsilon} \leq x_n\ \forall n \geq N_\varepsilon$. Hence $\sup(S) - x_n \leq \varepsilon$, $\forall n \geq N_\varepsilon$. The sequence converges to its supremum. Similarly, a decreasing and bounded sequence converges to its infimum. □

Proposition 2.5. Any monotone sequence has a limit in $\overline{\mathbb{R}}$.

Theorem 2.6 (Squeeze theorem)**.** Let (x_n), (y_n), (z_n) be sequences for which there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \leq z_n,\ \forall n \geq n_0,$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n.$$

Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $\ell := \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$ and let $\varepsilon > 0$. Then

$$\exists N_1 \in \mathbb{N} \text{ such that } |x_n - \ell| < \varepsilon,\ \forall n \geq N_1$$

and

$$\exists N_2 \in \mathbb{N} \text{ such that } |z_n - \ell| < \varepsilon,\ \forall n \geq N_2.$$

Taking $N_\varepsilon := \max\{N_1, N_2\}$, we have that

$$|y_n - \ell| \leq \max\{|x_n - \ell|, |z_n - \ell|\} < \varepsilon,\ \forall n \geq N_\varepsilon.$$

□

Theorem 2.7 (Cantor's nested intervals)**.** Let (a_n) be increasing and (b_n) decreasing such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, $\forall n \in \mathbb{N}$. Consider the closed intervals $I_n := [a_n, b_n]$, with $I_{n+1} \subseteq I_n$. If $\lim_{n\to\infty}(b_n - a_n) = 0$, then there exists $x \in \mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty} I_n = \{x\}.$$

Proof. Consider the bounded sets $A := \{a_n \mid n \in \mathbb{N}\}$ and $B := \{b_n \mid n \in \mathbb{N}\}$. For any $k \in \mathbb{N}$, we have that

$$a_k \leq \sup(A) \leq b_k$$

and

$$b_k \geq \inf(B) \geq a_k.$$

Hence by the squeeze theorem we have that $\sup(A) = \inf(B)$ and $\bigcap_{n=1}^{\infty} I_n$ contains only the element $\sup(A)$. □

Definition 2.8. For a sequence (x_n) we define the set of its *limit points* by

$$\operatorname{LIM}(x_n) := \{x \in \overline{\mathbb{R}} \mid \text{ there exists a subsequence } (x_{n_k}) \text{ s.t. } x_{n_k} \to x\},$$

and

$$\liminf_{n\to\infty} x_n := \inf\left(\operatorname{LIM}(x_n)\right),$$

$$\limsup_{n\to\infty} x_n := \sup\left(\operatorname{LIM}(x_n)\right).$$

Proposition 2.9. $\lim_{n\to\infty} x_n = \ell \in \overline{\mathbb{R}}$ if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell$.

Theorem 2.10 (Bolzano-Weierstrass)**.** Any bounded sequence has a convergent subsequence.

Proof. Consider the bounded set $A := \{x_n \mid n \in \mathbb{N}\}$. Let $a_1 := \inf(A)$ and $b_1 := \sup(A)$, and define $I_1 := [a_1, b_1]$. Bisect I_1 and notice that at least one of the two halves must contain infinitely many terms from the sequence. Take $I_2 := [a_2, b_2]$ to be the half that does. Continuing this procedure we obtain for each $k \in \mathbb{N}$ an interval $I_k := [a_k, b_k]$ containing (at least) a term $x_{n_k} \in A$, such that $I_{k+1} \subseteq I_k$ and $b_k - a_k \to 0$.

From Cantor's nested intervals theorem 2.7 we have that there exists $x \in \mathbb{R}$ such that $\bigcap_{n=1}^{\infty} I_n = \{x\}$, and hence the subsequence (x_{n_k}) converges to x. □

Definition 2.11 (Cauchy sequence)**.** A sequence (x_n) is called *Cauchy (or fundamental)* if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ such that } |x_m - x_n| < \varepsilon,\ \forall m, n \geq N_\varepsilon.$$

Proposition 2.12. Any Cauchy sequence is bounded.

Proof. For $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that $|x_m - x_n| < 1$, $\forall m, n \geq N_1$. In particular, $|x_n - x_{N_1}| < 1$, $\forall n \geq N_1$, hence the terms after index N_1 are bounded. The terms before index N_1 are also bounded since there is a finite number of them. We thus conclude that the entire sequence is bounded. □

Theorem 2.13. A sequence is convergent if and only if it is Cauchy.

Proof. Let's consider first a convergent sequence (x_n) with $x_n \to \ell$. For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n - \ell| < \frac{\varepsilon}{2}$, for any $n \geq N_\varepsilon$. Then $|x_m - x_n| \leq |x_m - \ell| + |x_n - \ell| < \varepsilon$, for any $n \geq N_\varepsilon$. Hence the sequence (x_n) is Cauchy.

Assume now that (x_n) is a Cauchy sequence. From the previous proposition we have that (x_n) must be bounded, and thus it has a convergent subsequence (x_{n_k}), $x_{n_k} \to x \in \mathbb{R}$. Let $\varepsilon > 0$. There exists thus $K_\varepsilon \in \mathbb{N}$ such that $|x_{n_k} - x| < \varepsilon$, $\forall k \geq K_\varepsilon$. Also, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$, $\forall m, n \geq N_\varepsilon$. In particular, $|x_{n_k} - x_n| < \varepsilon$, $\forall k, n \geq N_\varepsilon$. Hence $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < 2\varepsilon$, $\forall n \geq \max\{K_\varepsilon, N_\varepsilon\}$, meaning that $x_n \to x$. □

Example 2.14. The sequence defined by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is not convergent. Indeed, one can see, for example, that

$$x_{2n} - x_n = \frac{1}{n+1} + \ldots + \ldots + \frac{1}{2n} > \frac{n}{2n},$$

hence $x_{2n} - x_n > \frac{1}{2}$ for any $n \in \mathbb{N}$. Thus (x_n) is not convergent since it is not Cauchy.

❊ Series of real numbers

For a sequence (x_n), the sum $\sum_{n=1}^{\infty} x_n$ is called a *series* and $s_n := \sum_{k=1}^{n} x_k$ is called the *partial sum of the series*. The summation in a series can start from any index, not necessarily 0 or 1. A series is also often written as $\sum_{n\geq 1} x_n$.

Definition 3.1. The series $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of partial sums (s_n) converges.

Example 3.2. The *geometric series* $\sum_{n=0}^{\infty} q^n$ converges iff $|q| < 1$, with sum $\frac{1}{1-q}$.

Example 3.3. The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ diverges since (s_n) is not a Cauchy sequence.

Example 3.4 (Euler's number)**.** $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Proof. Let the partial sum $s_n = 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}$. Start from $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$ and expand

$$(1 + \frac{1}{n})^n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \ldots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdot \ldots \cdot (1 - \frac{n-1}{n}) \leq s_n.$$

We have that

$$(1 + \frac{1}{n})^n \leq s_n.$$

Consider now an index $k \geq n$. We have that

$$(1 + \frac{1}{k})^k \geq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{k}) + \ldots + \frac{1}{n!}(1 - \frac{1}{k})(1 - \frac{2}{k}) \cdot \ldots \cdot (1 - \frac{n-1}{k})$$

and taking $k \to \infty$ we obtain that $e \geq s_n$. We conclude with the squeeze theorem for

$$(1 + \frac{1}{n})^n \leq s_n \leq e,$$

obtaining that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges and its sum is e. □

Proposition 3.5. If the series $\sum_{n=1}^{\infty} x_n$ is convergent, then $\lim_{n\to\infty} x_n = 0$.

Proof. Consider the partial sum s_n. We have that $x_n = s_n - s_{n-1}$, hence the conclusion. □

It thus follows that if $\lim_{n\to\infty} x_n \neq 0$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Example 3.6. Series like $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) = 1$ are called *telescoping series*. The partial sum of a telescoping series can be easily computed since after cancellations the only remaining terms are the first one and the last one.

If the sequence (x_n) has only nonnegative terms $x_n \geq 0$, then the sequence of partial sums (s_n) is increasing. The series $\sum_{n=1}^{\infty} x_n$ then converges iff (s_n) are bounded.

Theorem 3.7 (Comparison test)**.** Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If there is an $n_0 \in \mathbb{N}$ such that

$$x_n \leq y_n,\ \forall n \geq n_0,\ \text{then}$$

(a) If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ also converges.

(b) If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ also diverges.

Proof. Consider the sequences of partial sums. In case (a), both sequences are bounded. In case (b), both sequences are unbounded. □

Example 3.8. If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ since $\frac{1}{n^p} \geq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. E.g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$.

Theorem 3.9. Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be series with nonnegative terms. If

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \ell,\ \text{then}$$

- if $\ell \in (0, \infty)$, then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ have the same nature.
- if $\ell = 0$, then if the series $\sum_{n=1}^{\infty} y_n$ converges, the series $\sum_{n=1}^{\infty} x_n$ also converges.
- if $\ell = \infty$, then if the series $\sum_{n=1}^{\infty} y_n$ diverges, the series $\sum_{n=1}^{\infty} x_n$ also diverges.

Theorem 3.10 (Ratio test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Proof. The idea is that $\sum_{n\geq 1} x_n$ behaves like a geometric series with ratio ℓ. We will only give a proof when $\ell < 1$, the other case being similar.

Take $\varepsilon > 0$ such that $q := \ell + \varepsilon < 1$. There exists $N_\varepsilon \in \mathbb{N}$ such that

$$\frac{x_{n+1}}{x_n} - \ell < \varepsilon, \ \forall n \geq N_\varepsilon,$$

giving that $x_{n+1} < x_n \cdot q$, $\forall n \geq N_\varepsilon$. Hence $x_n < q^{n-N_\varepsilon} x_{N_\varepsilon}$, that is $x_n < q^n \frac{x_{N_\varepsilon}}{q^{N_\varepsilon}}$. Since $q < 1$, the series converges by comparison with the geometric series $\sum_{n\geq 1} q^n$. □

Note that the Ratio test is *inconclusive* when $\ell = 1$.

Theorem 3.11 (Root test)**.** Let $\sum_{n=1}^{\infty} x_n$ be a series with nonnegative terms such that

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \ell.$$

- If $\ell < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $\ell > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Proof. The idea is that $\sum_{n\geq 1} x_n$ behaves like $\sum_{n\geq 1} \ell^n$. Like in the ratio test, the proof uses the comparison test with a geometric series. Left to the reader. □

Example 3.12. The series $\sum_{n\geq 0} \frac{x^n}{n!}$ converges for any $x \in \mathbb{R}$. We will see later that $\sum_{n\geq 0} \frac{x^n}{n!} = e^x$. We have that $\frac{x_{n+1}}{x_n} = \frac{x}{n+1} \to 0 < 1$, hence the series converges by the ratio test.

Theorem 3.13 (Cauchy condensation test)**.** Let (x_n) be a decreasing sequence with $x_n > 0$. Then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=0}^{\infty} 2^n \cdot x_{2^n}$ have the same nature.

Proof. TBA (given during the lectures). □

Example 3.14. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof. By the Cauchy condensation test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ has the same nature as $\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=1}^{\infty} \left(2^{1-p}\right)^n$, which converges if and only if $2^{1-p} < 1$, i.e for $p > 1$. □

Theorem 3.15 (Kummer's test)**.** Let (x_n) be a positive sequence and consider another positive sequence (c_n).

(a) If
$$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1}\right) > 0,$$
then $\sum_{n\geq 1} x_n$ is convergent.

(b) If $\sum_{n\geq 1} \frac{1}{c_n} = \infty$ and
$$\lim_{n\to\infty} \left(c_n \frac{x_n}{x_{n+1}} - c_{n+1}\right) < 0,$$
then $\sum_{n\geq 1} x_n$ is divergent.

Proof. TBA (given during the lectures). □

Theorem 3.16 (Raabe-Duhamel)**.** Let $\sum_{n\geq 1} x_n$ be a series with positive terms such that
$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = R.$$

- If $R > 1$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.
- If $R < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Proof. Take $c_n = n$ in Kummer's test (theorem 3.15). □

Example 3.17. Study the convergence of the series $\sum_{n\geq 0} \frac{n!}{a(a+1)\dots(a+n)}$, with $a > 0$.

Proof. The ratio test is inconclusive since $\frac{x_{n+1}}{x_n} = \frac{n+1}{a+n+1} \to 1$. Let us then try the Raabe-Duhamel test:

$$\lim_{n\to\infty} n\Big(\frac{x_n}{x_{n+1}} - 1\Big) = \lim_{n\to\infty} n\Big(\frac{a+n+1}{n+1} - 1\Big) = a.$$

Hence if $a > 1$ the series converges; and if $a < 1$ the series diverges. When $a = 1$ the series is $\sum_{n\geq 0} \frac{1}{n+1} = \infty$. □

A series $\sum_{n\geq 1} x_n$ is called an *alternating series* if $x_n x_{n+1} \leq 0$, $\forall n \in \mathbb{N}$. A fundamental class of alternating series are series of the form $\sum_{n\geq 1} (-1)^n a_n$ or $\sum_{n\geq 1} (-1)^{n+1} a_n$, with $a_n > 0$.

Example 3.18. The series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges to $\ln 2$.

Proof. Let us prove convergence by considering the partial sums $s_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Notice that $s_{2k+2} - s_{2k} = \frac{1}{2k+1} - \frac{1}{2k+2} > 0$ and that $s_{2k+3} - s_{2k+1} = \frac{1}{2k+3} - \frac{1}{2k+2} < 0$. This means that the subsequence (s_{2k}) is increasing, while the subsequence (s_{2k+1}) is decreasing. Notice also that $s_{2k+1} - s_{2k} = \frac{1}{2k+1}$ and $s_{2k} < s_{2k+1}$, so both subsequences are also bounded and converge to the same limit.

To find the sum of the alternating series, recall (from seminar) that

$$\lim_{n\to\infty} 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n = \gamma \in (0,1).$$

Hence

$$\begin{aligned} s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{2n} \\ &= 1 + \frac{1}{2} + \dots + \frac{1}{2n} - 2\Big(\frac{1}{2} + \dots + \frac{1}{2n}\Big) - \ln(2n) \\ &= \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{2n} - \ln(2n)}_{\to\gamma} - \underbrace{\Big(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\Big)}_{\to\gamma} + \ln 2 \to \ln 2. \end{aligned}$$

□

Theorem 3.19 (Leibniz test)**.** Let (x_n) be a decreasing sequence with $x_n \to 0$. Then the series $\sum_{n\geq 1} (-1)^n x_n$ is convergent.

Proof. Consider the partial sum $s_n = \sum_{k=1}^{n}(-1)^k x_k$. We will prove that (s_n) is convergent by showing that it is a Cauchy sequence. For $n, p \in \mathbb{N}$ consider

$$\begin{aligned}
|s_{n+p} - s_n| &= |(-1)^{n+1}x_{n+1} + \ldots + (-1)^{n+p}x_{n+p}| \\
&= |\underbrace{x_{n+1} - x_{n+2}}_{\geq 0} + \underbrace{x_{n+3} - x_{n+4}}_{\geq 0} + \ldots + (-1)^{p-2}x_{n+p-1} + (-1)^{p-1}x_{n+p}| \\
&= x_{n+1} \underbrace{- x_{n+2} + x_{n+3}}_{\leq 0} - x_{n+4} + \ldots + (-1)^{p-2}x_{n+p-1} + (-1)^{p-1}x_{n+p} \\
&\leq x_{n+1},
\end{aligned}$$

hence (s_n) is a Cauchy sequence since $|s_{n+p} - s_n|$ can be made arbitrarily small. □

Definition 3.20. A series $\sum_{n\geq 1} x_n$ is called *absolutely convergent* if $\sum_{n\geq 1}|x_n|$ is convergent.

Proposition 3.21. Any absolutely convergent series is also convergent.

Proof. If $\sum_{k=1}^{n}|x_k|$ gives a Cauchy sequence, then $\sum_{k=1}^{n} x_k$ also gives a Cauchy sequence. □

Theorem 3.22 (Cauchy)**.** Let $\sum_{n\geq 1} x_n$ be an *absolutely convergent series* and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n\geq 1} x_{\sigma(n)}$ is also absolutely convergent and $\sum_{n\geq 1} x_{\sigma(n)} = \sum_{n\geq 1} x_n$. In other words, any rearrangement of an absolutely convergent series has the same sum.

Proof. (Optional) See [2][Theorem 7.4.3]. □

Definition 3.23. A series $\sum_{n\geq 1} x_n$ is called *conditionally convergent* (or semi-convergent) if $\sum_{n\geq 1} x_n$ converges, but $\sum_{n\geq 1}|x_n|$ diverges.

Theorem 3.24 (Riemann)**.** Let $\sum_{n\geq 1} x_n$ be a *conditionally convergent series* and let $x \in \overline{\mathbb{R}}$. Then there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n\geq 1} x_{\sigma(n)} = x$. In other words, a conditionally convergent series can be rearranged to converge to any value or diverge to $\pm\infty$.

Proof. (Optional) See [2][Theorem 8.2.8]. □

※ Limits, continuity, differentiability

Definition 4.1. Let $A \subseteq \mathbb{R}$. We say that $x_0 \in \overline{\mathbb{R}}$ is an *accumulation point* (or cluster point) if

$$\forall V \in \mathcal{V}(x_0),\ V \cap \big(A \setminus \{x_0\}\big) \neq \emptyset.$$

We denote by A' the set of all the accumulation points of A.

We say that $x_0 \in A$ is an *isolated point* if $x_0 \in A \setminus A'$.

Proposition 4.2. Let $A \subseteq \mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$, then $x_0 \in A'$ if and only if there exists a sequence (x_n) in $A \setminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$.

Proof. Assume that $x_0 \in A'$, with $x_0 \in \mathbb{R}$, and consider the neighborhoods $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$. Then each neighborhood must contain an $x_n \in A \setminus \{x_0\}$ with $|x_n - x_0| < \frac{1}{n}$, hence $x_n \longrightarrow x_0$. If x_0 is infinite, the neighborhoods can be taken $(-\infty, -n)$ or (n, ∞), respectively.

Assume now that there exists a sequence (x_n) in $A \setminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$. Then for any $V \in \mathcal{V}(x_0)$, there exists $N_V \in N$ such that $x_n \in V$, for any $n \geq N_V$. In particular, $x_{N_V} \in V \cap \big(A \setminus \{x_0\}\big)$, for any $V \in \mathcal{V}(x_0)$, hence $x_0 \in A'$. □

Example 4.3. For $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, each element $x \in A$ in an isolated point and $A' = \{0\}$.

Definition 4.4. Let $A \subseteq \mathbb{R}$, $f : A \longrightarrow \mathbb{R}$ and $x_0 \in A'$. We say that $\lim\limits_{x\to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ if

$$\forall V \in \mathcal{V}(\ell),\ \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V,\ \forall x \in U \cap \big(A \setminus \{x_0\}\big).$$

Remark 4.5 (ε–δ). Let $A \subseteq \mathbb{R}$, $f : A \longrightarrow \mathbb{R}$ and $x_0 \in A'$ finite. If $\lim\limits_{x\to x_0} f(x) = \ell \in \mathbb{R}$, then

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - \ell| < \varepsilon,\ \forall x \in A \text{ with } |x - x_0| < \delta.$$

Theorem 4.6. Let $A \subseteq \mathbb{R}$, $f : A \longrightarrow \mathbb{R}$ and $x_0 \in A'$. Then $\lim\limits_{x\to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ iff

$$\text{for any sequence } (x_n) \text{ in } A \setminus \{x_0\} \text{ with } \lim_{n\to\infty} x_n = x_0, \text{ we have that } \lim_{n\to\infty} f(x_n) = \ell.$$

Theorem 4.7. Let $A \subseteq \mathbb{R}$, $f : A \longrightarrow \mathbb{R}$, $x_0 \in \mathbb{R}$ s.t. $x_0 \in \big(A \cap (-\infty, x_0)\big)'$ and $x_0 \in \big(A \cap (x_0, \infty)\big)'$. Then

$$\lim_{x\to x_0} f(x) = \ell \text{ iff } \lim_{\substack{x\to x_0\\ x<x_0}} f(x) = \lim_{\substack{x\to x_0\\ x>x_0}} f(x) = \ell.$$

Example 4.8. (a) $\operatorname{sgn} : \mathbb{R} \longrightarrow \mathbb{R}$, $\operatorname{sgn}(x) = \begin{cases} +1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$ has no limit at 0.

(b) $f : \mathbb{R}^* \longrightarrow \mathbb{R}$, $f(x) = \sin(\frac{1}{x})$ has no limit at 0 since $f(\frac{1}{2n\pi}) = 0$, $f(\frac{1}{2n\pi + \pi/2}) = 1$.

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ has no limit at any $x \in R$.

Definition 4.9. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A$. We say that f is *continuous* at x_0 if

$$\forall V \in \mathcal{V}(f(x_0)),\ \exists U \in \mathcal{V}(x_0) \text{ s.t. } f(x) \in V,\ \forall x \in U \cap A.$$

Remark 4.10. If $x_0 \in A \cap A'$ is an accumulation point, then f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Remark 4.11. If x_0 is an isolated point of A, then $\exists U \in \mathcal{V}(x_0)$ with $U \cap A = \{x_0\}$, and since $f(x_0) \in V$, $\forall V \in \mathcal{V}(f(x_0))$, we have that f is continuous at x_0.

Definition 4.12. For $f : A \to \mathbb{R}$ denote by $f(A) := \{f(x) \mid x \in A\}$ the image of A. We say that f is *bounded* if $f(A)$ is *bounded*, i.e. $\inf\big(f(A)\big), \sup\big(f(A)\big)$ are finite.

Theorem 4.13. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A$. The following are equivalent:

(a) f is continuous at x_0.

(b) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x \in A$ with $|x - x_0| < \delta$.

(c) for any sequence (x_n) in A with $\lim_{n\to\infty} x_n = x_0$, we have that $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Remark 4.14. Elementary operations – e.g. sums, products or compositions – of continuous functions are continuous (when defined).

Theorem 4.15 (Weierstrass). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded and it attains its bounds, i.e. there exist $\min\big(f(A)\big), \max\big(f(A)\big)$.

Proof. Let us first prove that f is bounded. Assuming that this is not the case, we have that for any $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Since the sequence (x_n) is bounded, we have that it has a convergent subsequence (x_{n_k}), see theorem 2.10; denote its limit by x. We have that $x_{n_k} \to x$ and f is continuous, hence $f(x_{n_k}) \to f(x)$. But $|f(x_{n_k})| > n_k \to \infty$, contradiction. Hence f is bounded on $[a, b]$.

To prove that f attains its bounds, let's consider the upper bound and show that there exists $x_M \in [a, b]$ such that $f(x_M) = \sup\big(f(A)\big)$, i.e. $f(x_M) = \max\big(f(A)\big) = \sup\big(f(A)\big)$. From theorem 1.7, we obtain a sequence (x_n) in $[a, b]$ such that $f(x_n) \to \sup\big(f(A)\big)$. Since the sequence (x_n) is bounded, it has a convergent subsequence (x_{n_k}); let's call its limit $x_M \in [a, b]$. Since f is continuous, it follows that $f(x_{n_k}) \to f(x_M)$, but we know that $f(x_{n_k}) \to \sup\big(f(A)\big)$, hence $f(x_M) = \sup\big(f(A)\big)$ and f reaches its upper bound. □

Theorem 4.16 (Intermediate value property)**.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f has the intermediate value property, i.e. if $y \in \mathbb{R}$ is in between $f(a)$ and $f(b)$, there exists $c \in (a,b)$ such that $f(c) = y$.

Proof. Assume that $f(a) < y < f(b)$ and consider the set $S := \{x \in [a,b] \mid f(x) \leq y\}$. Take

$$c := \sup(S)$$

Let $\varepsilon > 0$, then $\exists \delta > 0$ such that $|f(x) - f(c)| < \varepsilon$, whenever $|x - c| < \delta$. Since $c = \sup(S)$, we have from theorem 1.7 that there exists $x_1 \in S$ such that $c - \delta < x_1 \leq c$. From continuity we have that $f(c) < f(x_1) + \varepsilon \leq y + \varepsilon$. Also, for $x_2 \in (c, c+\delta)$, we have from continuity that $f(c) > f(x_2) - \varepsilon$. From the definition of the supremum, $x_2 \notin S$ hence $f(x_2) > y$ and $f(c) > y - \varepsilon$. We conclude that $y - \varepsilon < f(c) < y + \varepsilon$, for any $\varepsilon > 0$. Hence $f(c) = y$. □

Definition 4.17. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. The *derivative* of f at x_0 is

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \overline{\mathbb{R}}$$

If $f'(x_0) \in \mathbb{R}$ (finite) we say that f is *differentiable* at x_0.

Remark 4.18. $f'(x_0)$ represents the gradient of the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. The equation of the tangent is $f(x) - f(x_0) = f'(x_0)(x - x_0)$.

Theorem 4.19. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. If f is differentiable at x_0, then f is continuous at x_0.

Proof. Since $f(x) = f(x_0) + \frac{f(x)-f(x_0)}{x-x_0}(x - x_0)$, we have that $\lim\limits_{x \to x_0} f(x) = f(x_0) + 0 = f(x_0)$. □

Example 4.20. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is not differentiable in 0 since $\nexists \lim\limits_{x \to 0} \dfrac{|x|}{x}$.

Theorem 4.21 (Calculus Rules)**.**

- $(cf)'(x) = cf'(x)$, for any constant $c \in \mathbb{R}$.
- $(f + g)'(x) = f'(x) + g'(x)$.
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. (Product Rule)
- $\left(fg\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$. (Quotient Rule)
- $(f \circ g)'(x) = f'\big(g(x)\big)g'(x)$. (Chain Rule)

Definition 4.22. $f : A \to \mathbb{R}$ has a local extremum (minimum/maximum) at $x_0 \in A$ if

$$\exists V \in \mathcal{V}(x_0) \text{ s.t. } f(x_0) \leq f(x)/f(x_0) \geq f(x),\ \forall x \in V \cap A.$$

Theorem 4.23 (Fermat)**.** Let $f : (a,b) \longrightarrow \mathbb{R}$ and $x_0 \in (a,b)$. If f is differentiable at x_0 and x_0 is a local extremum, then $f'(x_0) = 0$.

Proof. The lateral derivatives at x_0 are equal. Since x_0 is a local extremum, one of them is ≥ 0, the other ≤ 0. Hence $f'(x_0) = 0$. □

Theorem 4.24 (Rolle)**.** Let $f : (a,b) \longrightarrow \mathbb{R}$ with $f(a) = f(b)$. If is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ s.t. $f'(c) = 0$.

Proof. Since f is continuous, it is bounded and it attains its bounds. Denote by x_m and x_M the minimum and maximum points of f on $[a,b]$. If at least one of x_m and x_M belongs to (a,b), then $f'(x_m) = 0$ or $f'(x_M) = 0$. Otherwise, $x_m, x_M \in \{a,b\}$ and $f(x_m) = f(x_M)$, hence the function is constant and its derivative is zero on (a,b). □

Theorem 4.25 (Mean value theorem)**.** Let $f : (a,b) \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider the function $g : (a,b) \longrightarrow \mathbb{R}$, $g(x) := f(x) - x\frac{f(b)-f(a)}{b-a}$. Since $g(a) = g(b)$, the conclusion follows from Rolle's theorem. □

Theorem 4.26. Let $f : (a,b) \longrightarrow \mathbb{R}$ be differentiable on (a,b). Then

$$f \text{ is increasing iff } f' \geq 0,$$

$$f \text{ is decreasing iff } f' \leq 0.$$

Proof. $\Rightarrow$ follows from the definition of the derivative; $\Leftarrow$ from the mean value theorem. □

Proposition 4.27 (l'Hôpital's rule)**.** Let I be an open interval, $x_0 \in \overline{\mathbb{R}}$ and $f, g : I \setminus \{x_0\} \longrightarrow \mathbb{R}$ differentiable. If $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$ or $\lim\limits_{x \to x_0} g(x) = \pm\infty$, and $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

※ Taylor series and power series

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times ($n \in \mathbb{N}$). Does there exist a polynomial $P : \mathbb{R} \to \mathbb{R}$ that matches the function f and all its derivatives up to order n at the point x_0? That is

$$\begin{aligned} P(x_0) &= f(x_0) \\ P'(x_0) &= f'(x_0) \\ P''(x_0) &= f''(x_0) \\ &\vdots \\ P^{(n)}(x_0) &= f^{(n)}(x_0). \end{aligned}$$

Let us look for P of degree at most n of the form

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n.$$

By imposing the conditions at x_0 and differentiating P we have that

$$P(x_0) = a_0 = f(x_0),\ P'(x_0) = a_1 = f'(x_0),\ P''(x_0) = 2a_2 = f''(x_0),\ \ldots,\ P^{(n)}(x_0) = n!a_n = f^{(n)}(x_0).$$

We thus see that there exists a unique such polynomial P of degree at most n given by

$$P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

that matches the function f and all its derivatives up to order n at the point x_0.

Definition 5.1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times. The polynomial $T_n : \mathbb{R} \to \mathbb{R}$,

$$T_n(x) := f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor polynomial* of degree n centered around x_0.

The Taylor polynomial T_n gives a good approximation of f around x_0, i.e. when $x \approx x_0$,

$$f(x) \approx T_n(x).$$

The simplest approximations are: the *linear approximation* of f around x_0 given by T_1, i.e.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

and the *quadratic approximation* of f around x_0 given by T_2, i.e.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

The closer x is to x_0 and the higher the degree of T_n is, the better $T_n(x)$ approximates $f(x)$.

Example 5.2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ and $x_0 = 0$. Then $f(0) = f'(0) = \ldots = f^{(n)}(0) = 1$ and

$$T_n(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}.$$

Definition 5.3. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times. We define $R_n : \mathbb{R} \to \mathbb{R}$,

$$R_n(x) := f(x) - T_n(x)$$

to be the remainder when approximating f by T_n around x_0. Note that (Taylor's formula)

$$f(x) = T_n(x) + R_n(x).$$

Theorem 5.4 (Taylor-Lagrange)**.** Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ differentiable $n+1$ times. Then for any $x, x_0 \in I$, there exists $c \in (x_0, x)$ or $c \in (x, x_0)$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

In other words, $f(x) = T_n(x) + R_n(x)$ with

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

being called *the remainder in Lagrange's form*.

There are several other forms of the remainder, but we will mostly only use this one. Its main advantage is that assuming that the $(n+1)^{th}$ derivative of f is bounded by $M > 0$,

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1},$$

and we see that if this holds for any n, then $|R_n(x)| \to 0$ as $n \to \infty$.

Corollary 5.5 (Local optimality conditions)**.** Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ and $x_0 \in I$ a point where f is differentiable n times and

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \text{ and } f^{(n)}(x_0) \neq 0.$$

1. If n is even and $f^{(n)}(x_0) > 0$, then x_0 is a *local minimum* of f.
2. If n is even and $f^{(n)}(x_0) < 0$, then x_0 is a *local maximum* of f.
3. If n is odd, then x_0 is not a local extremum point of f.

Example 5.6 (Convex/concave)**.** Let $f : I \to R$ be two times differentiable, with a critical point at x_0, i.e. $f'(x_0) = 0$. Then from Taylor's formula we have that

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + R_2(x).$$

When x is very close to x_0, the quadratic approximation is very accurate and the remainder $R_2(x)$ is very small. Thus the behaviour of $f(x)$ around x_0 is dictated by the quadratic term $f''(x_0)(x - x_0)^2$ and we see that:

- If $f''(x_0) > 0$ (convexity), then $f(x) > f(x_0)$ and x_0 is a local minimum.
- If $f''(x_0) < 0$ (concavity), then $f(x) < f(x_0)$ and x_0 is a local maximum.

Definition 5.7. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be infinitely differentiable. For $x_0 \in I$ and $x \in \mathbb{R}$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor series* of f around x_0. If

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

then f can be expanded as a Taylor series around x_0 (also called a Taylor expansion).

Note that the partial sum of a Taylor series is simply the Taylor polynomial $T_n(x)$, and that a Taylor series converges to $f(x)$ if and only if the remainder $R_n(x) \to 0$ as $n \to \infty$.

Example 5.8. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ and $x_0 = 0$. Then $f(0) = f'(0) = \ldots = f^{(n)}(0) = 1$ and

$$T_n(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}.$$

Consider Taylor's formula $f(x) = T_n(x) + R_n(x)$ with the Lagrange remainder, for which there exists c in between 0 and x such that

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}.$$

Since $\frac{|x|^n}{n!} \to 0$ as $n \to \infty$, it follows that e^x can be expanded as a Taylor series around 0:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x}{2} + \ldots + \frac{x^n}{n!} + \ldots, \forall x \in \mathbb{R}.$$

Example 5.9. The functions sin and cos can be expanded in a Taylor series around 0.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

Definition 5.10. Let (a_n) be a sequence of real numbers and let $c \in \mathbb{R}$. The series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is called a *power series* centered at c.

Definition 5.11. The convergence set of a power series is

$$C := \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n (x-c)^n \text{ converges}\}.$$

Theorem 5.12. There exists a unique $R \in [0, \infty]$, called the *radius of convergence* of the power series, such that

- the power series converges absolutely when $|x - c| < R$.
- the power series diverges when $|x - c| > R$.

Remark 5.13. The convergence set C contains the open interval $(c - R, c + R)$ and possibly the endpoints $\{c - R, c + R\}$.

Example 5.14. The power series $\sum_{n \geq 1} \frac{x^n}{n}$ converges absolutely for $|x| < 1$ and diverges when $|x| > 1$ (by the ratio test), hence its radius of convergence is $R = 1$. Moreover, the series converges for $x = -1$ (alternating harmonic series) and diverges for $x = 1$ (harmonic series), hence its convergence set is $C = [-1, 1)$.

Theorem 5.15. Consider a power series with radius of convergence R, given by

$$s(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Then for any $x \in (c - R, c + R)$, the power series can be differentiated term by term and

$$s'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1},$$

and for any $t \in (c-R, c+R)$ the power series can be integrated term by term

$$\int_c^t s(x)\mathrm{dx} = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(t-c)^{n+1}.$$

Theorem 5.16. If the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \in [0,\infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0,\infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

Proof. It follows from the root test for series with positive terms. □

Corollary 5.17. If the limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \in [0,\infty]$$

exists, then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has the radius of convergence

$$R = \begin{cases} \frac{1}{L}, & \text{if } L \in (0,\infty) \\ 0, & \text{if } L = \infty \\ \infty, & \text{if } L = 0. \end{cases}$$

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