湖 南 大 学 理 工 类 必 修 课 程

大 学 数 学 AII

—— 多 元 微 分 学

2.7 高阶偏导数

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第二章　多元函数微分学

第七节 高阶偏导数

一.高阶偏导数

二. 高阶微分

三. 泰勒公式

第七节 高阶偏导数

本节学习要求：

- 正确理解多元函数高阶偏导数的概念。
- 能熟练地计算二、三元函数的高阶偏导数($n \le 3$)。
- 熟悉求混合偏导数与求导顺序无关的条件。
- 了解高阶微分的概念及其算子表示法。
- 会求二、三元函数的二阶微分。
- 知道多元函数的泰勒公式。

1. 高阶偏导数

多元函数的高阶导数与一元函数的情形类似。

一般说来,在区域Ω内,函数$z = f(x, y)$的偏导数$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$仍是变量$x, y$的多元函数. 如果偏导数$\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$仍可偏导,则它们的偏导数就是原来函数的二阶偏导数

依此类推, 可定义多元函数的更高阶的导数.

1. 高阶偏导数

一般地, 若函数 $f(X)$ 的 $m-1$ 阶偏导数仍可偏导, 则称其偏导数为原来函数的 m 阶偏导数.

二阶和二阶以上的偏导数均称为高阶偏导数, 其中, 关于不同变量的高阶导数, 称为混合偏导数

1. 高阶偏导数

二元函数 $z = f(x, y)$ 的二阶偏导数：

二元函数的二阶偏导数共 $2^2 = 4$ 项

$$\frac{\partial z}{\partial x} \begin{cases} x \\ y \end{cases} \qquad \frac{\partial z}{\partial y} \begin{cases} x \\ y \end{cases}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx} = f''_{11}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f''_{xy} = f''_{12}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{yy} = f''_{22}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f''_{yx} = f''_{21}$$

1. 高阶偏导数

【例】 二元函数 $z=f(x,y)$ 的三阶偏导数：

1

$$\frac{\partial^2 z}{\partial y\partial x} \quad \frac{\partial^2 z}{\partial x\partial y} \quad \frac{\partial^2 z}{\partial y^2} \quad \frac{\partial^2 z}{\partial x^2} < \begin{matrix} x \\ y \end{matrix}$$

2

$$\frac{\partial^2 z}{\partial y\partial x} \quad \frac{\partial^2 z}{\partial x\partial y} \quad \frac{\partial^2 z}{\partial y^2} \quad \frac{\partial^2 z}{\partial x^2} < \begin{matrix} x \\ y \end{matrix}$$

$$\frac{\partial^2 z}{\partial x^2} \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 z}{\partial x^2}\right)=\dfrac{\partial^3 z}{\partial x^3} \\[2ex] \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 z}{\partial x^2}\right)=\dfrac{\partial^3 z}{\partial x^2\partial y} \end{cases}$$

$$\frac{\partial^2 z}{\partial y^2} \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 z}{\partial y^2}\right)=\dfrac{\partial^3 z}{\partial y^2\partial x} \\[2ex] \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 z}{\partial y^2}\right)=\dfrac{\partial^3 z}{\partial y^3} \end{cases}$$

1. 高阶偏导数

二元函数 $z=f(x,y)$ 的三阶偏导数：

共 $2^3=8$ 项.

【例】二元函数 $z=f(x,y)$ 的三阶偏导数：

3

$$\frac{\partial^2 z}{\partial y\partial x}\quad \frac{\partial^2 z}{\partial x\partial y}\quad \frac{\partial^2 z}{\partial y^2}\quad \frac{\partial^2 z}{\partial x^2}\quad \begin{matrix} x \\ y \end{matrix}$$

4

$$\frac{\partial^2 z}{\partial y\partial x}\quad \frac{\partial^2 z}{\partial x\partial y}\quad \frac{\partial^2 z}{\partial y^2}\quad \frac{\partial^2 z}{\partial x^2}\quad \begin{matrix} x \\ y \end{matrix}$$

$$\frac{\partial^2 z}{\partial x\partial y} \to \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 z}{\partial x\partial y}\right)=\dfrac{\partial^3 z}{\partial x\partial y\partial x} \\[2ex] \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 z}{\partial x\partial y}\right)=\dfrac{\partial^3 z}{\partial x\partial y^2} \end{cases}$$

$$\frac{\partial^2 z}{\partial y\partial x} \to \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 z}{\partial y\partial x}\right)=\dfrac{\partial^3 z}{\partial y\partial x^2} \\[2ex] \dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 z}{\partial y\partial x}\right)=\dfrac{\partial^3 z}{\partial y\partial x\partial y} \end{cases}$$

【例】 求 $z = x^3y^2 - 3xy^3 - xy + 1$ 的二阶偏导数.

此结论是必然的吗？

解 先求一阶偏导数：$\dfrac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$, $\dfrac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x$,

再求二阶偏导数：

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2y^2 - 3y^3 - y) = 6xy^2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(2x^3y - 9xy^2 - x) = 2x^3 - 18xy$$

二阶混合偏导数：

两个混合偏导数相等！

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(3x^2y^2 - 3y^3 - y) = 6x^2y - 9y^2 - 1$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}(2x^3y - 9xy^2 - x) = 6x^2y - 9y^2 - 1$$

1. 高阶偏导数

两个混合偏导数相等取决于什么条件呢？！

【例】 设 $f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2\neq 0\end{cases}$ 求 $f''_{xy}(0,0)$，

不相等！！

解 需按定义求函数在点(0, 0) 处的偏导数:

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$$

$$f'_x(x,y)=\begin{cases}\dfrac{x^2y-y^3}{x^2+y^2}+\dfrac{4x^2y^3}{(x^2+y^2)^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

$$f'_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$$

$$f'_y(x,y)=\begin{cases}\dfrac{x^3-xy^2}{x^2+y^2}-\dfrac{4x^3y^2}{(x^2+y^2)^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

$$f''_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{f'_x(0,\Delta y)-f'_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1$$

$$f''_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f'_y(\Delta x,0)-f'_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

定理1

若 $z=f(x,y)$ 的二阶混合偏导数在 $U((x_0,y_0))$ 内存在且在点 (x_0,y_0) 处连续，则必有 $\dfrac{\partial^2 f(x_0,y_0)}{\partial x\partial y}=\dfrac{\partial^2 f(x_0,y_0)}{\partial y\partial x}$.

该定理的结论可推广到更高阶的混合偏导数的情形.

1. 高阶偏导数

引入记号： $f(X)$在 Ω内有直到 k 阶的连续偏导数，

记为 $f(X) \in C^k(\Omega)$，$k = 0,\ 1,\ 2, \cdots$。

二元函数的n 阶偏导数就有2^n 项，当 $f(x, y) \in C^n(\Omega)$ 时，则在求n 阶及n 阶以下的偏导数时，可大大减少运算次数。自变量的个数越多，求导与求导顺序无关的作用越明显。

【例】 求 $z=e^{x^2y}$ 的二阶偏导数.

【解】 $\dfrac{\partial z}{\partial x}=2xye^{x^2y}$ $\qquad$ $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial x}(2xye^{x^2y})=(2y+4x^2y^2)e^{x^2y}$

$\dfrac{\partial z}{\partial y}=x^2e^{x^2y}$ $\qquad$ $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}(x^2e^{x^2y})=x^4e^{x^2y}$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}(x^2e^{x^2y})=(2x+2x^3y)e^{x^2y}$$

【例】 设 $u=f(x+y+z,\ xyz)$，且 $f\in C^2$，求 $\dfrac{\partial^2 u}{\partial x\partial y}$.

【解】 这是求复合函数的高阶偏导数.

$$\frac{\partial u}{\partial x}=f_1'\cdot\frac{\partial(x+y+z)}{\partial x}+f_2'\cdot\frac{\partial(xyz)}{\partial x}=f_1'+yz\,f_2'$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}(f_1'+yz\,f_2')$$

此时，$\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial^2 u}{\partial x\partial y}$.

$$=f_{11}''\cdot\frac{\partial(x+y+z)}{\partial y}+\underline{f_{12}''}\cdot\frac{\partial(xyz)}{\partial y}+z\,f_2'+yz\left[\underline{f_{21}''}\cdot\frac{\partial(x+y+z)}{\partial y}+f_{22}''\cdot\frac{\partial(xyz)}{\partial y}\right]$$

$$=f_{11}''+(x+y)z\,f_{12}''+xyz^2\,f_{22}''+z\,f_2'$$

1. 高阶偏导数

【练】 设 $u = f(x^2 + y^2 - z^2)$，其中 $f \in C^2$，求 $\dfrac{\partial^2 u}{\partial x^2}$，$\dfrac{\partial^2 u}{\partial x \partial y}$．

【解】

$$\frac{\partial u}{\partial x} = f'(x^2 + y^2 - z^2) \cdot \frac{\partial (x^2 + y^2 - z^2)}{\partial x} = 2x f'(x^2 + y^2 - z^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2x f'(x^2 + y^2 - z^2)) = 2 f'(x^2 + y^2 - z^2) + 4x^2 f''(x^2 + y^2 - z^2)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}(2x f'(x^2 + y^2 - z^2)) = 4xy f''(x^2 + y^2 - z^2)$$

【例】 设 $z^3-3xyz=a^3$，求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

【解】令 $F(x,y,z)=z^3-3xyz\text{-}a^3$，则

$$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}=-\frac{-3yz}{3z^2-3xy}.\qquad \frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}=-\frac{-3xz}{3z^2-3xy}.$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{3yz}{3z^2-3xy}\right)=\frac{(3yz)'_y(3z^2-3xy)-3xz(3z^2-3xy)'_y}{(3z^2-3xy)^2}$$

$$=\frac{(3z+3yz'_y)(3z^2-3xy)-3xz(6zz'_y-3x)}{(3z^2-3xy)^2}$$

$$=\frac{(3z+3y\dfrac{3xz}{3z^2-3xy})(3z^2-3xy)-3xz(6z\dfrac{3xz}{3z^2-3xy}-3x)}{(3z^2-3xy)^2}=\frac{9z(x^2+z^2)}{(3z^2-3xy)^2}-\frac{54x^2z^3}{(3z^2-3xy)^3}.$$

【练】 设 $e^z - xyz = 0$，求 $\dfrac{\partial^2 z}{\partial x^2}$.

【解】 这是求**隐函数的高阶偏导数**.

令 $F(x,y,z) = e^z - xyz$，则 $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = -\dfrac{-yz}{e^z - xy} = \dfrac{yz}{e^z - xy}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{yz}{e^z - xy}\right) = \frac{y\dfrac{\partial z}{\partial x}(e^z - xy) - yz\left(e^z\dfrac{\partial z}{\partial x} - y\right)}{(e^z - xy)^2}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}}$$

$$= \frac{y^2 z(e^z - xy) - yz\left(e^z yz - y(e^z - xy)\right)}{(e^z - xy)^3} = \frac{2y^2 z e^z - 2xy^3 z - y^2 z^2 e^z}{(e^z - xy)^3}$$

【例】 设 $u = yf(\frac{x}{y}) + xg(\frac{y}{x})$，其中$f, g$具有二阶连续导数，试求 $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$

【解】 $\frac{\partial u}{\partial x} = y \cdot f'(\frac{x}{y}) \cdot \frac{1}{y} + g(\frac{y}{x}) + x \cdot g'(\frac{y}{x}) \cdot (-\frac{y}{x^2}) = f'(\frac{x}{y}) + g(\frac{y}{x}) - \frac{y}{x} g'(\frac{y}{x}).$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(f'(\frac{x}{y}) + g(\frac{y}{x}) - \frac{y}{x} g'(\frac{y}{x}))$$

$$= f''(\frac{x}{y})\frac{1}{y} + g'(\frac{y}{x})(-\frac{y}{x^2}) - (-\frac{y}{x^2})g'(\frac{y}{x}) - \frac{y}{x} g''(\frac{y}{x})(-\frac{y}{x^2}) = \frac{1}{y} f''(\frac{x}{y}) + \frac{y^2}{x^3} g''(\frac{y}{x}).$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}(f'(\frac{x}{y}) + g(\frac{y}{x}) - \frac{y}{x} g'(\frac{y}{x}))$$

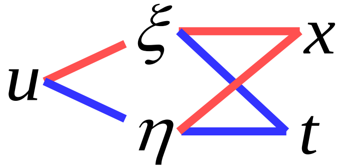
$$= f''(\frac{x}{y})(-\frac{x}{y^2}) + g'(\frac{y}{x})\frac{1}{x} - \frac{1}{x} g'(\frac{y}{x}) - \frac{y}{x} g''(\frac{y}{x})\frac{1}{x} = -\frac{x}{y^2} f''(\frac{x}{y}) - \frac{y}{x^2} g''(\frac{y}{x}).$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = x(\frac{1}{y} f''(\frac{x}{y}) + \frac{y^2}{x^3} g''(\frac{y}{x})) + y(-\frac{x}{y^2} f''(\frac{x}{y}) - \frac{y}{x^2} g''(\frac{y}{x})) = 0.$$

1. 高阶偏导数

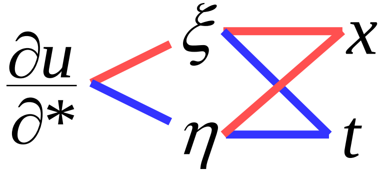
【例】 利用变量代换$\xi = x - at$，$\eta = x + at$将方程$\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$

化为关于变量ξ,η的方程.$(u \in C^2)$



【解】 令 $u = u(\xi,\eta)$，$\xi = x - at$， $\eta = x + at$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -a\frac{\partial u}{\partial \xi} + a\frac{\partial u}{\partial \eta}$$



$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-a\frac{\partial u}{\partial \xi} + a\frac{\partial u}{\partial \eta}\right) = -a\left[\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial t}\right] + a\left[\frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial t}\right]$$

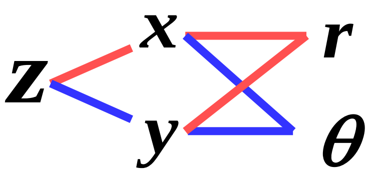
即 $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial \xi^2} - 2a^2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + a^2\dfrac{\partial^2 u}{\partial \eta^2}$ 同理可得 $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2}$

将上述偏导数带入原方程，得到 $\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0$.

【例】 设$z=f(x,y)\in C^2$,记$x=r\cos\theta, y=r\sin\theta$,试证明:

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$

$\frac{\partial z}{\partial *}$



【证】 $$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=(-r\sin\theta)\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial \theta^2}=\frac{\partial}{\partial \theta}[(-r\sin\theta)\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}]$$

$$=(-r\cos\theta)\frac{\partial z}{\partial x}+(-r\sin\theta)[(-r\sin\theta)\frac{\partial^2 z}{\partial x^2}+r\cos\theta\frac{\partial^2 z}{\partial x\partial y}]+(-r\sin\theta)\frac{\partial z}{\partial y}+r\cos\theta[(-r\sin\theta)\frac{\partial^2 z}{\partial y\partial x}+r\cos\theta\frac{\partial^2 z}{\partial y^2}]$$

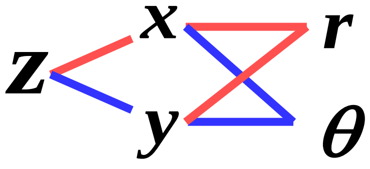
$$=(-r\cos\theta)\frac{\partial z}{\partial x}+(-r\sin\theta)\frac{\partial z}{\partial y}+r^2\sin^2\theta\frac{\partial^2 z}{\partial x^2}+r^2\cos^2\theta\frac{\partial^2 z}{\partial y^2}-2r^2\sin\theta\cos\theta\frac{\partial^2 z}{\partial y\partial x}.$$

1. 高阶偏导数

【例】 设$z=f(x,y)\in C^2$，记$x=r\cos\theta, y=r\sin\theta$，试证明：

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$

$$\frac{\partial z}{\partial *}$$



$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial r^2}=\frac{\partial}{\partial r}(\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y})=\cos\theta(\cos\theta\frac{\partial^2 z}{\partial x^2}+\sin\theta\frac{\partial^2 z}{\partial x\partial y})+\sin\theta(\cos\theta\frac{\partial^2 z}{\partial y\partial x}+\sin\theta\frac{\partial^2 z}{\partial y^2})$$

$$=\cos^2\theta\frac{\partial^2 z}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 z}{\partial y\partial x}+\sin^2\theta\frac{\partial^2 z}{\partial y^2}.$$

故有：$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$

利用**算子**可以方便地表示

高阶微分

泰勒公式

二. 高阶微分

若 $z=f(x,y)\in C^2$, 则它的全微分存在,且

$$\mathrm{d}z=f_x'(x,y)\,\mathrm{d}x+f_y'(x,y)\,\mathrm{d}y$$

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

$$=\left(\frac{\partial}{\partial x}\mathrm{d}x+\frac{\partial}{\partial y}\mathrm{d}y\right)z$$

$$\mathrm{d}(\mathrm{d}z)=\mathrm{d}(f_x'(x,y)\,\mathrm{d}x+f_y'(x,y)\,\mathrm{d}y)$$

$$=\mathrm{d}f_x'(x,y)\cdot\mathrm{d}x+\mathrm{d}f_y'(x,y)\cdot\mathrm{d}y$$

$$=(f_{xx}''(x,y)\,\mathrm{d}x+f_{xy}''(x,y)\,\mathrm{d}y)\,\mathrm{d}x$$

$$+(f_{yx}''(x,y)\,\mathrm{d}x+f_{yy}''(x,y)\,\mathrm{d}y)\,\mathrm{d}y$$

$$\mathrm{d}^2z=\left(\frac{\partial}{\partial x}\mathrm{d}x+\frac{\partial}{\partial y}\mathrm{d}y\right)^2z$$

$$=\frac{\partial^2 z}{\partial x^2}\mathrm{d}x^2+2\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x\,\mathrm{d}y+\frac{\partial^2 z}{\partial y^2}\mathrm{d}y^2$$

$$=f_{xx}''(x,y)\,\mathrm{d}x^2+2f_{xy}''(x,y)\,\mathrm{d}x\,\mathrm{d}y+f_{yy}''(x,y)\,\mathrm{d}y^2$$

二. 高阶微分

若$u=f(x_1,\cdots,x_n)\in C^2$,则$\mathrm{d}u\in C^1$，故 $\mathrm{d}u$ 的全微分 $\mathrm{d}(\mathrm{d}u)$ 存在，

称为f 的二阶微分，记为 d^2u，且

$$\mathbf{d}^2 \boldsymbol{u}=\left(\frac{\partial}{\partial x_1}\mathbf{d}\,x_1+\frac{\partial}{\partial x_2}\mathbf{d}\,x_2+\cdots+\frac{\partial}{\partial x_n}\mathbf{d}\,x_n\right)^2 \boldsymbol{u}.$$

一般地，若 $u=f(x_1,\cdots,x_n)\in C^k$，则$f$ 有直到k 阶的微分：

$$\mathbf{d}^k \boldsymbol{u}=\mathbf{d}(\mathbf{d}^{k-1}\boldsymbol{u}).$$

$$\boxed{\mathbf{d}^k \boldsymbol{u}=\left(\frac{\partial}{\partial x_1}\mathbf{d}\,x_1+\frac{\partial}{\partial x_2}\mathbf{d}\,x_2+\cdots+\frac{\partial}{\partial x_n}\mathbf{d}\,x_n\right)^k \boldsymbol{u}}$$

二. 高阶微分

【例】 设 $u=x^3+y^3-3xy(x-y)$，求 d^3u .

【解】 $$\mathrm{d}^3u=\left(\frac{\partial}{\partial x}\mathrm{d}x+\frac{\partial}{\partial y}\mathrm{d}y\right)^3u$$

$$=\frac{\partial^3u}{\partial x^3}\mathrm{d}x^3+3\frac{\partial^3u}{\partial x^2\partial y}\mathrm{d}x^2\mathrm{d}y+3\frac{\partial^3u}{\partial x\partial y^2}\mathrm{d}x\mathrm{d}y^2+\frac{\partial^3u}{\partial y^3}\mathrm{d}y^3$$

又 $$\frac{\partial^3u}{\partial x^3}=6,\quad \frac{\partial^3u}{\partial x^2\partial y}=-6,\quad \frac{\partial^3u}{\partial x\partial y^2}=6,\quad \frac{\partial^3u}{\partial y^3}=6$$

故 $$\mathrm{d}^3u=6(\mathrm{d}x^3-3\mathrm{d}x^2\mathrm{d}y+3\mathrm{d}x\mathrm{d}y^2+\mathrm{d}y^3)$$

与求多元函数的偏导数的方法类似,我们想借助一元函数的泰勒公式来建立多元函数的泰勒公式.

首先, 将一元函数的泰勒公式写成微分形式:

$$f(x)=f(x_0)+\boxed{f'(x_0)\Delta x}+\cdots+\frac{1}{n!}\boxed{f^{(n)}(x_0)\Delta x^n}+R_n(x)$$

$$=f(x_0)+\boxed{\mathrm{d}f(x_0)}+\cdots+\frac{1}{n!}\boxed{\mathrm{d}^n f(x_0)}+R_n(x)$$

$$=f(x_0)+\sum_{k=1}^{n}\frac{1}{k!}\boxed{\mathrm{d}^k f(x_0)}+R_n(x)$$

$$R_n(x)=\frac{1}{(n+1)!}\mathrm{d}^{n+1} f(x_0+\theta\Delta x)$$

运用点函数进行推广

$$(0<\theta<1)$$

定理

在 R^n 中，设 $f(X)\in C^{m+1}(\mathrm{U}(X_0))$，则在 $\mathrm{U}(X_0)$ 内有

$$f(X)=f(X_0)+\sum_{k=1}^{m}\frac{1}{k!}\boxed{\mathrm{d}^k f(X_0)}+R_m(X)$$

其中 $R_m(X)=\dfrac{1}{(m+1)!}\mathrm{d}^{m+1}f(X_0+\theta\Delta X)$，$(0<\theta<1)$ 称为拉格朗日余项.

该公式称为多元函数泰勒公式的微分形式

三. 泰勒公式

由多元函数高阶微分式：

$(\mathrm{d}\,x_i = \Delta x_i)$

$$\mathrm{d}^k u = \left(\mathrm{d}\,x_1 \frac{\partial}{\partial x_1} + \mathrm{d}\,x_2 \frac{\partial}{\partial x_2} + \cdots + \mathrm{d}\,x_n \frac{\partial}{\partial x_n}\right)^k u = \left(\sum_{i=1}^{n} \mathrm{d}\,x_i \frac{\partial}{\partial x_i}\right)^k u$$

多元函数的泰勒公式可写成一般形式：

$$f(X) = f(X_0) + \sum_{k=1}^{m} \frac{1}{k!}\left(\sum_{i=1}^{n} \Delta x_i \frac{\partial}{\partial x_i}\right)^k f(X_0) + \ldots + \frac{1}{(m+1)!}\left(\sum_{i=1}^{n} \Delta x_i \frac{\partial}{\partial x_i}\right)^{m+1} f(X_0 + \theta \Delta X),$$

$\boxed{R_m(X)}$ $\quad (0 < \theta < 1)$

三. 泰勒公式

设二元函数$z=f(x,y)\in C^{n+1}$, 点$(x_0+\Delta x, y_0+\Delta y)$为点$X_0(x_0,y_0)$邻域$U(X_0)$内的任意一点, 则$z=f(x,y)$点$(x_0,y_0)$的泰勒公式为

$$f(x_0+\Delta x, y_0+\Delta y)=f(x_0,y_0)+(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})f(x_0,y_0)+\frac{1}{2!}(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^2 f(x_0,y_0)$$

$$+\dots+\frac{1}{(n+1)!}(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^{n+1} f(x_0+\theta\Delta x, y_0+\theta\Delta y)\ . \quad (0<\theta<1)$$

$$f(x,y)=f(0,0)+(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})f(0,0)+\frac{1}{2!}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^2 f(0,0)$$

$$+\dots+\frac{1}{(n+1)!}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^{n+1} f(\theta x,\theta y)\ . \quad (0<\theta<1)$$

$z=f(x,y)$点(x_0,y_0)的麦克劳林公式

【例】求函数$f(x,y)=\sin x\sin y$在点$(\frac{\pi}{4},\frac{\pi}{4})$的二阶泰勒公式，并写出余项$R_2$.

【解】

$$f_x'=\cos x\sin y$$

$$f_y'=\sin x\cos y$$

$$f_{xx}''=-\sin x\sin y$$

$$f_{yy}''=-\sin x\sin y$$

$$f_{xy}''=\cos x\cos y$$

$$f_{xxx}'''=-\cos x\sin y$$

$$f_{xxy}'''=-\sin x\cos y$$

$$f_{yyx}'''=-\cos x\sin y$$

$$f_{yyy}'''=-\sin x\cos y$$

$$f(x,y)=\sin x\sin y=f(\frac{\pi}{4}+(x-\frac{\pi}{4}),\frac{\pi}{4}+(y-\frac{\pi}{4}))$$

$$=f(\frac{\pi}{4},\frac{\pi}{4})+[(x-\frac{\pi}{4})\frac{\partial}{\partial x}+(y-\frac{\pi}{4})\frac{\partial}{\partial y}]f(\frac{\pi}{4},\frac{\pi}{4})+\frac{1}{2!}[(x-\frac{\pi}{4})\frac{\partial}{\partial x}+(y-\frac{\pi}{4})\frac{\partial}{\partial y}]^2f(\frac{\pi}{4},\frac{\pi}{4})+R_2$$

三. 泰勒公式

$$= f(\frac{\pi}{4},\frac{\pi}{4}) + [(x-\frac{\pi}{4}) f_x'(\frac{\pi}{4},\frac{\pi}{4}) + (y-\frac{\pi}{4}) f_y'(\frac{\pi}{4},\frac{\pi}{4})] +$$

$$\frac{1}{2!}[(x-\frac{\pi}{4})^2 f_{xx}''(\frac{\pi}{4},\frac{\pi}{4}) + 2(x-\frac{\pi}{4})(y-\frac{\pi}{4}) f_{xy}''(\frac{\pi}{4},\frac{\pi}{4}) + (y-\frac{\pi}{4})^2 f_{yy}''(\frac{\pi}{4},\frac{\pi}{4})] + R_2$$

$$= \frac{1}{2} + \frac{1}{2}[(x-\frac{\pi}{4}) + (y-\frac{\pi}{4})] - \frac{1}{4}[(x-\frac{\pi}{4})^2 - 2(x-\frac{\pi}{4})(y-\frac{\pi}{4}) + (y-\frac{\pi}{4})^2] + R_2$$

$$R_2 = \frac{1}{3!}[(x-\frac{\pi}{4})\frac{\partial}{\partial x} + (y-\frac{\pi}{4})\frac{\partial}{\partial y}]^3 f(\xi,\eta)$$

$$= -\frac{1}{6}[\cos\xi\sin\eta(x-\frac{\pi}{4})^3 + 3\sin\xi\cos\eta(x-\frac{\pi}{4})^2(y-\frac{\pi}{4})$$

$$+ 3\cos\xi\sin\eta(x-\frac{\pi}{4})(y-\frac{\pi}{4})^2 + \sin\xi\cos\eta(y-\frac{\pi}{4})^3]$$

$$\xi = \frac{\pi}{4} + \theta(x-\frac{\pi}{4})$$

$$\eta = \frac{\pi}{4} + \theta(y-\frac{\pi}{4})$$

本节小结

混合偏导数与求导顺序无关的条件

对复合函数和隐函数方程：

计算二、三元函数的高阶偏导数

了解多元函数的高阶微分和泰勒公式

思考题1

设函数$z = f(xy, yg(x))$,其中$f \in C^2$,函数$g(x)$可导,

且在$x=1$处取得极值$g(1)=1$,求$\dfrac{\partial^2 z}{\partial x \partial y}\Big|_{\substack{x=1\\y=1}}$.

思考题2

设$x+y-z=e^{z}, xe^{y}=\tan t, y=\cos t$，求$\left.\dfrac{\mathrm{d}^2 z}{\mathrm{d}t^2}\right|_{t=0}$.

思考题3

已知$z = xf(\frac{y}{x}) + 2y\varphi(\frac{x}{y})$,其中$f,\varphi$均为二次可微函数。

（1）求$\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x \partial y}$;

（2）* 当$f=\varphi$且$\frac{\partial^2 z}{\partial x \partial y}\Big|_{x=a} = -by^2$时,求$f(y)$.