湖 南 大 学 理 工 类 必 修 课 程

大 学 数 学 AII

—— 多 元 微 分 学

2.5 多元复合函数的求导法则

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第二章　多元函数微分学

第五节　多元复合函数的求导法则

一. 链式法则

二. 全微分形式不变性

第五节 多元复合函数的求导法则

本节学习要求：

- 熟悉多元函数全导数的概念和计算方法。
- 熟练掌握复合函数的链式法则。
- 能熟练地、准确地计算二、三元复合函数的导数。
- 了解全微分形式不变性。

【例】 设 $z=x^2y^2$，$x=a\sin t$，$y=b\cos t$，求 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

【解】

$$z=x^2y^2=(a\sin t)^2(b\cos t)^2=\frac{1}{4}a^2b^2\sin^2 2t$$

故

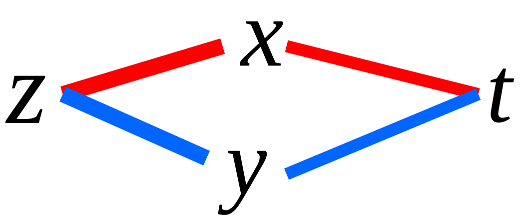
$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{1}{4}a^2b^2\cdot 2\sin 2t\cos 2t\cdot 2$$

$$=\frac{1}{2}a^2b^2\sin 4t$$

一. 链式法则

设$x=x(t)$，$y=y(t)$ 在点t处可导，$z=f(x,y)$ 在t对应

的点(x,y)处可微，则$z=f(x(t),y(t))$在t处可导，且



$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

z可微

x,y可导

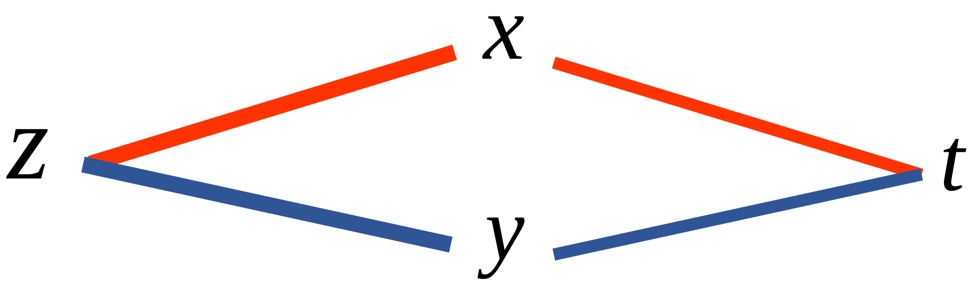
$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x++\frac{\partial z}{\partial y}\mathrm{d}y=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$$

一. 链式法则

【例】 设 $z=x^2y^2$，$x=a\sin t$，$y=b\cos t$，求 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

【解】

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$



$$=2xy^2\cdot a\cos t+2x^2y\cdot(-b\sin t)$$

$$=\frac{1}{2}a^2b^2\sin 4t$$

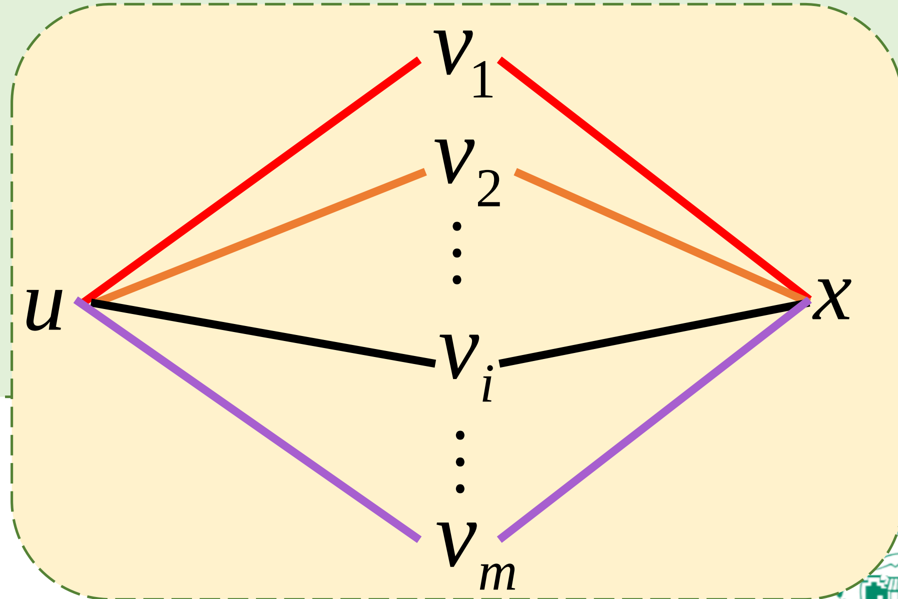
一. 链式法则

定理 设函数 $u=f(v_1,\cdots,v_m)$，$v_i=\phi_i(x)$ （$i=1,\cdots,m$）可复合为

$$u=f(\varphi_1(x),\cdots,\varphi_m(x)).$$

若 $\phi_i(x)$ 在点 x 处可导，函数 $f(v_1,\cdots,v_m)$ 在相应于 x 的点$(v_1,\cdots,v_m)$ 处可微，则复合函数 $u=f(\phi_1(x),\cdots,\phi_m(x))$ 在点 x 处可导，且

(全导数公式)
$$\frac{\mathrm{d}u}{\mathrm{d}x}=\sum_{i=1}^{m}\frac{\partial u}{\partial v_i}\frac{\mathrm{d}v_i}{\mathrm{d}x}.$$

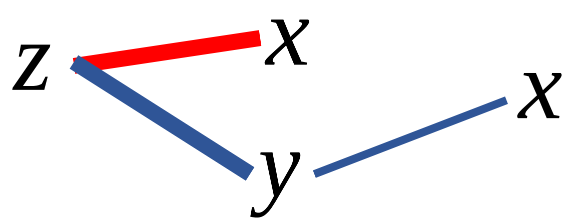


沿线相乘

分线相加

【例】 设$z=x^{\sin x}$，求$\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

【解】 令 $z=x^{y}$, $y=\sin x$, 则



$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$=yx^{y-1}+x^{y}\ln x\cdot\cos x$$

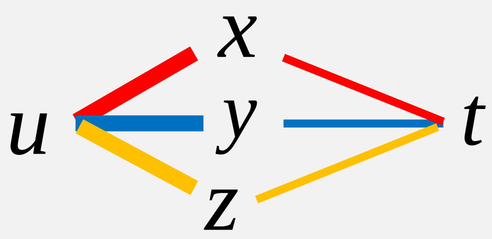
$$=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\ln x\right)$$

一. 链式法则

【例】 设以下函数满足定理的条件, 写出下列函数的全导数公式:

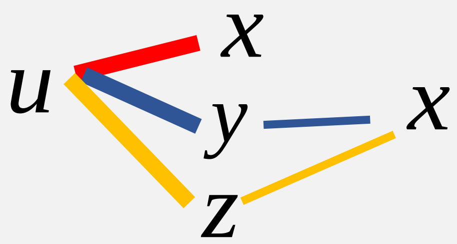
$$u = f(x,y,z),\ x = x(t),\ y = y(t),\ z = z(t);$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$



$$u = f(x,y,z),\ y = y(x),\ z = z(x).$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}$$



一. 链式法则

求函数$z=f(x(t),y(t))$的导数。

由$z=f(x,y),\begin{cases}x=x(t)\\y=y(t)\end{cases}$复合而成。

全导数问题 → **偏导数问题**

求函数$z=f(x(s,t),y(s,t))$的偏导数。

由$z=f(x,y),\begin{cases}x=x(s,t)\\y=y(s,t)\end{cases}$复合而成。

一般多元复合函数的求导法则

仅对其中一个自变量求偏导数时，

将其他自变量当成常数，按照全导数公式计算即可。

一. 链式法则

假设所有出现的函数求导运算均成立, 求下面函数的导数:

$$z = f(u(x,y), v(x,y), w(x,y))$$

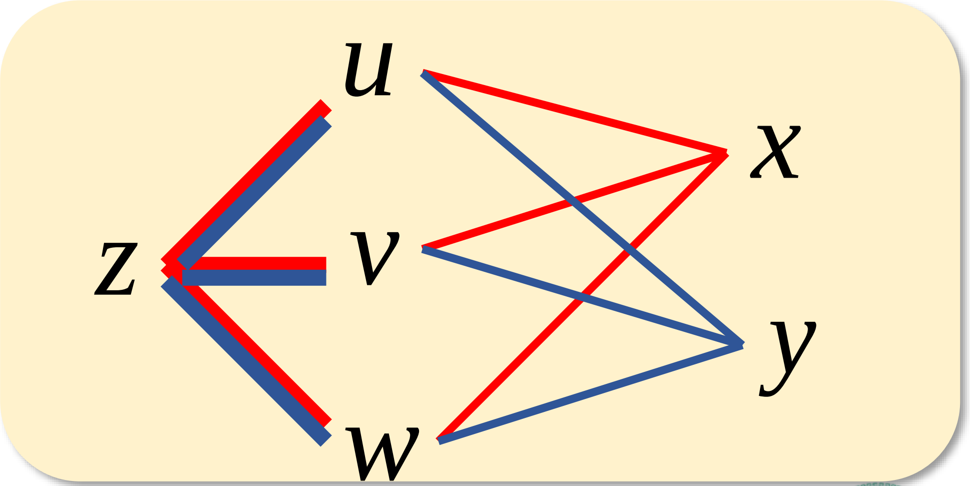
$$z = f(u,v,w), \quad \begin{cases} u = u(x,y) \\ v = v(x,y) \\ w = w(x,y) \end{cases}$$

将 y 看成常数

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

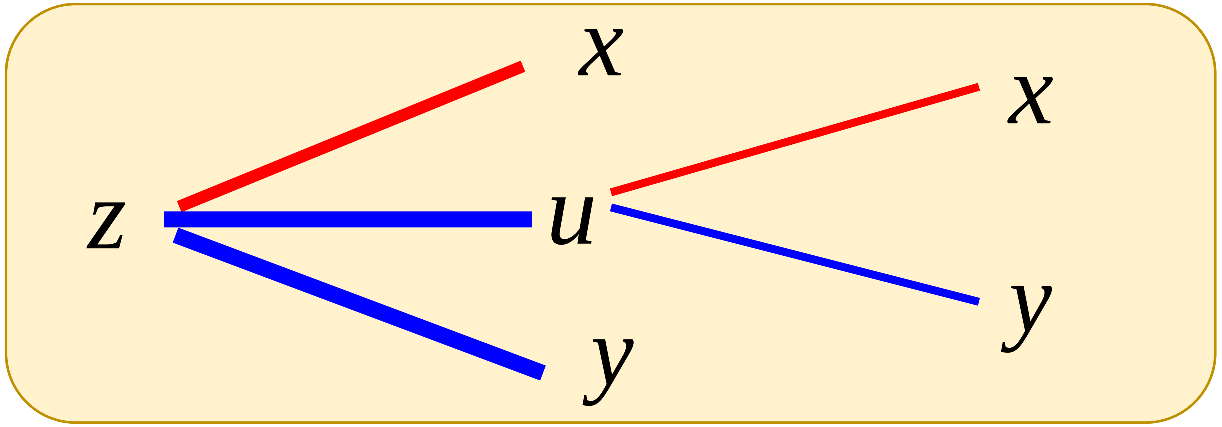
将 x 看成常数

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$



一. 链式法则

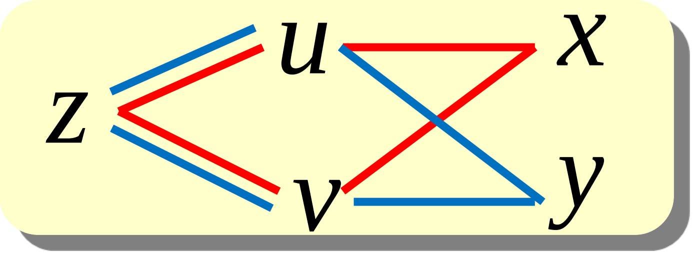
设$z=f(u,x,y)$,$u=u(x,y)$满足定理的条件,则有$z=f(u(x,y),x,y)$



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}$$

【例】 设$z=e^u\sin v, u=x^2y^2, v=x-y$,求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.



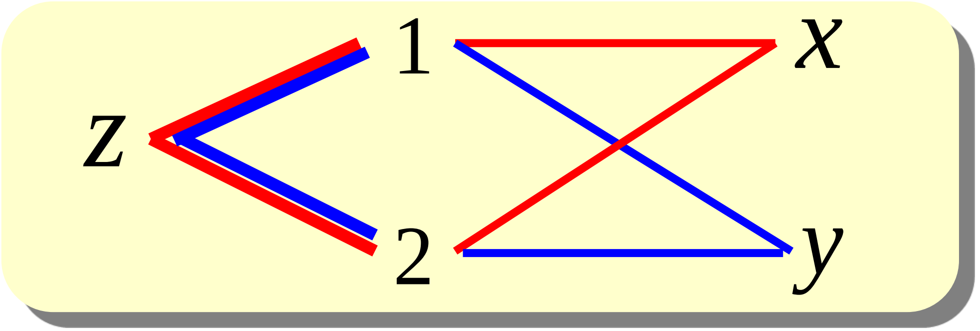
【解】 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^u\sin v\cdot 2xy^2+e^u\cos v\cdot 1$$

$$=e^{x^2y^2}(2xy^2\sin(x-y)+\cos(x-y))$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=e^u\sin v\cdot 2x^2y+e^u\cos v\cdot(-1)$$

$$=e^{x^2y^2}(2x^2y\sin(x-y)-\cos(x-y))$$

一. 链式法则



【例】 设$z = f(x^2 - y^2,\ e^{xy})$，求$\dfrac{\partial z}{\partial x}$。

【解】 $$\frac{\partial z}{\partial x} = f_1' \frac{\partial(x^2 - y^2)}{\partial x} + f_2' \frac{\partial(e^{xy})}{\partial x}$$

$$= 2x f_1' + y e^{xy} f_2'$$

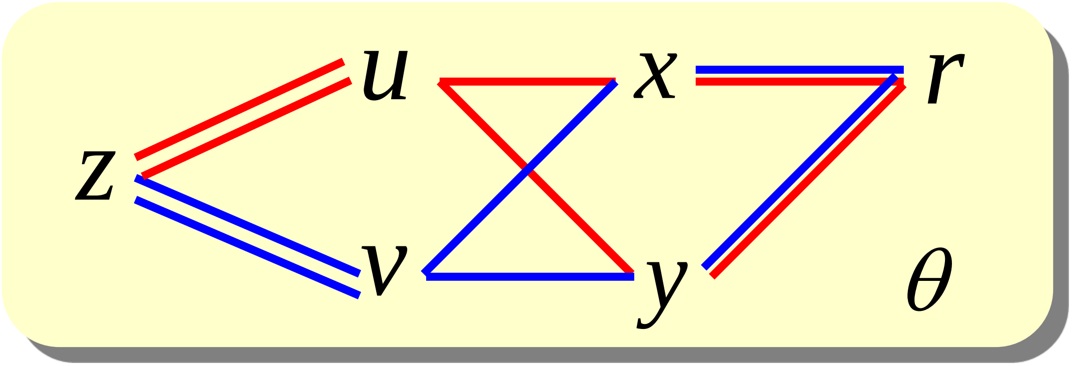
$$\frac{\partial z}{\partial y} = -2y f_1' + x e^{xy} f_2'$$

一. 链式法则

【例】 设$z=f(x^2-y^2,\ \cos xy)$, $x=r\cos\theta$, $y=r\sin\theta$, 其中$f\in C^1$, 求$\dfrac{\partial z}{\partial r}$。

【解】 令$u=x^2-y^2$, $v=\cos xy$,

则$z=f(u,v)$,



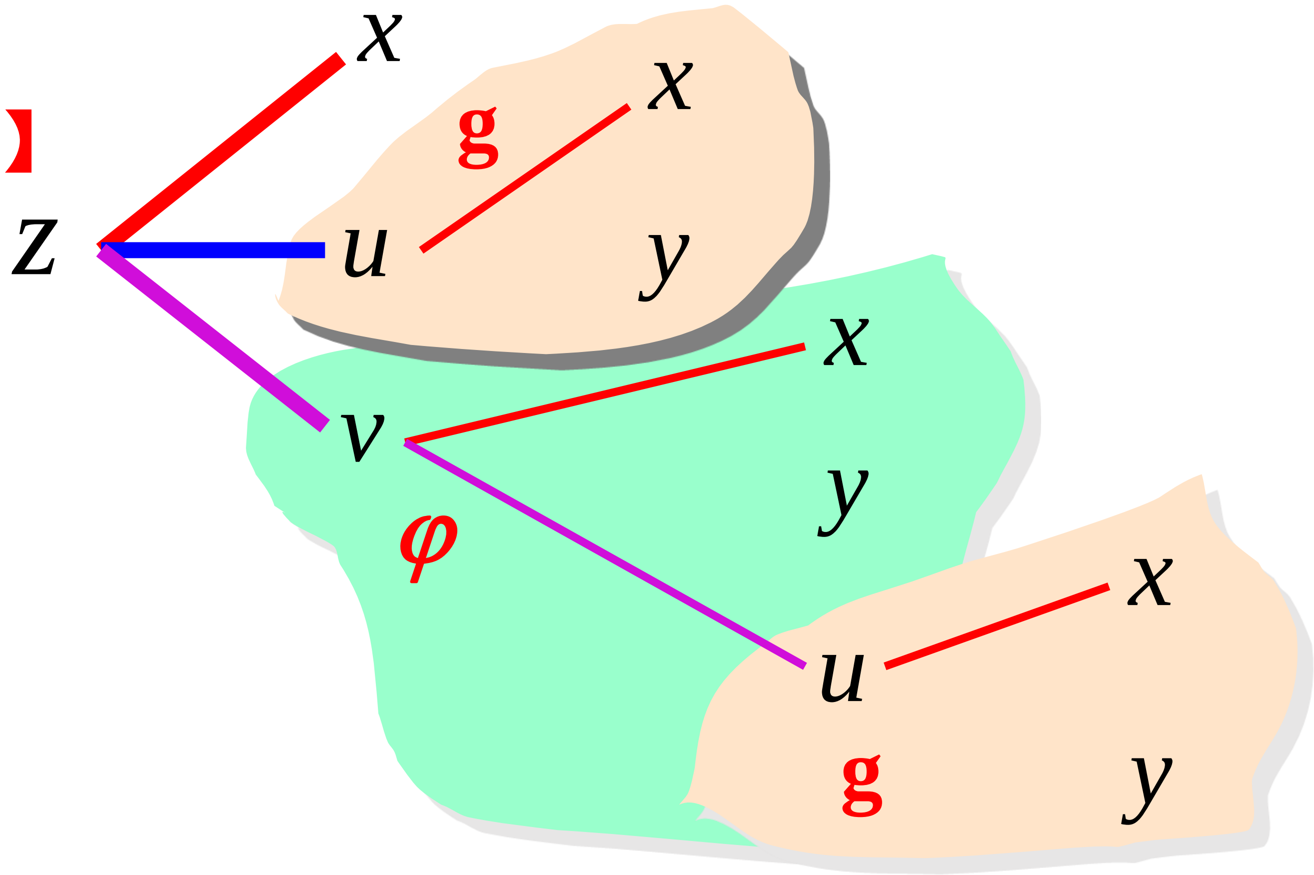
$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$=2(x\cos\theta-y\sin\theta)\frac{\partial z}{\partial u}-(y\cos\theta+x\sin\theta)\sin xy\frac{\partial z}{\partial v}$$

$$=2(x\cos\theta-y\sin\theta)\cdot f_1'-(y\cos\theta+x\sin\theta)\sin xy\cdot f_2'$$

【例】设函数$z=f(x,u,v), v=\phi(x,y,u), u=g(x,y)$均可微,求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

【解】



$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial v}\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}\frac{\partial g}{\partial x}\right)$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial v}\left(\frac{\partial \varphi}{\partial y}+\frac{\partial \varphi}{\partial u}\frac{\partial g}{\partial y}\right)$$

【练】 设$z=xy+xf\left(\frac{y}{x}\right), f\in C^1$, 证明$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z$.

【证】

$$\frac{\partial z}{\partial x}=y+f+xf'\cdot\left(-\frac{y}{x^2}\right)=y+f-\frac{y}{x}f'$$

$$\frac{\partial z}{\partial y}=x+xf'\cdot\frac{1}{x}=x+f'$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=(xy+xf-yf')+(xy+yf')$$

$$=2xy+xf=xy+(xy+xf)=xy+z.$$

二. 全微分形式不变性

一元函数的微分有一个重要性质：

一阶微分形式不变性

对函数 $y = f(u)$ 不论 u 是自变量

还是中间变量，在可微的条件下，均有

$$\mathrm{d}y = f'(u)\,\mathrm{d}u$$

对二元函数 $z = f(x, y)$ 来说，

全微分形式不变性

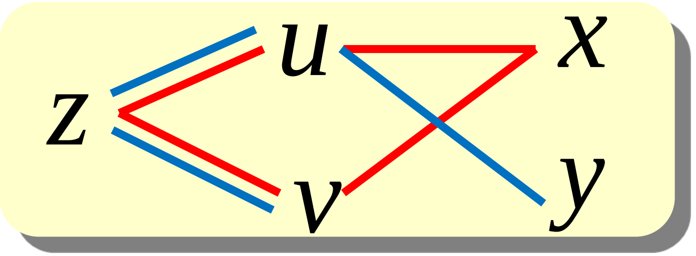
不论 x 和 y 是自变量还是中间变量，

在可微的条件下，f 的全微分总可写为：

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

二. 全微分形式不变性

设二元函数 $z = f(u, v), u = u(x, y), v = v(x, y)$,



则 $dz = \boxed{\dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dy}$

$$= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) dx + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right) dy$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right)$$

$$= \boxed{\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv}$$

x 和 y 是自变量时的全微分公式与为中间变量时的全微分公式相同！

二. 全微分形式不变性

一般说来,设$u=f(x_1,\cdots,x_n)$,不论x_i是自变量还是中间变量,

在可微的条件下,全微分均有如下形式:

$$\mathrm{d}u=\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\mathrm{d}x_i$$

二. 全微分形式不变性

【例】 设$z=f(xy,\dfrac{y}{x}),f\in C^1$,求$\mathrm{d}z$

【解】 设$u=xy,v=\dfrac{y}{x}$.

1：偏导数法

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=f_1'\cdot y+f_2'\cdot(-\frac{y}{x^2})$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=f_1'\cdot x+f_2'\cdot\frac{1}{x}$$

$$dz=(yf_1'-\frac{y}{x^2}f_2')dx+(xf_1'+\frac{1}{x}f_2')dy.$$

法2：全微分法

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

$$=\frac{\partial z}{\partial u}\mathrm{d}(xy)+\frac{\partial z}{\partial v}\mathrm{d}(\frac{y}{x})$$

$$=f_1'(xdy+ydx)+f_2'(\frac{xdy-ydx}{x^2})$$

$$=(yf_1'-\frac{y}{x^2}f_2')dx+(xf_1'+\frac{1}{x}f_2')dy.$$

【练】设$z=e^u\sin v, u=xy, v=x+y$,应用全微分形式不变性求$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$。

【解】 $$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

与 $\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$ 比较, 得

$$=e^u\sin v(y\,\mathrm{d}x+x\,\mathrm{d}y)+e^u\cos v(\mathrm{d}x+\mathrm{d}y)$$

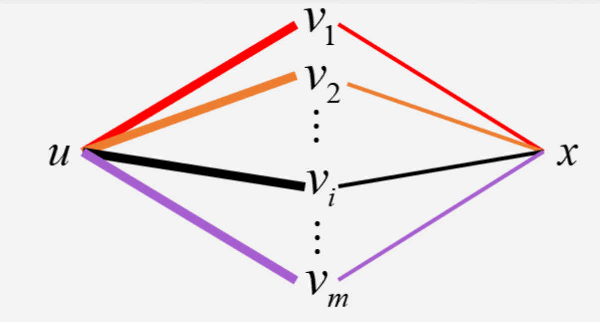
$$=e^{xy}[y\sin(x+y)+\cos(x+y)]\mathrm{d}x$$

$$+e^{xy}[x\sin(x+y)+\cos(x+y)]\mathrm{d}y$$

$$\frac{\partial z}{\partial x}=e^{xy}[y\sin(x+y)+\cos(x+y)] \qquad \frac{\partial z}{\partial y}=e^{xy}[x\sin(x+y)+\cos(x+y)]$$

本节小结

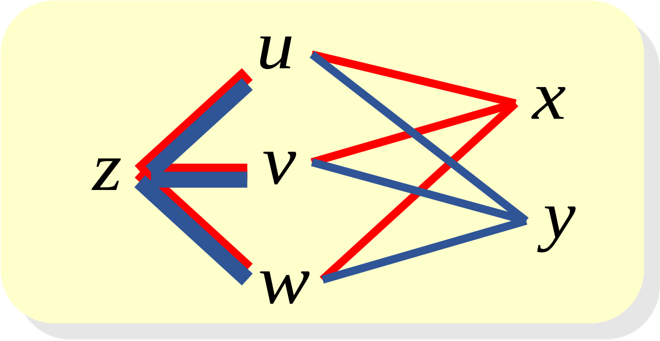
全导数公式



$$\frac{\mathrm{d}u}{\mathrm{d}x}=\sum_{i=1}^{m}\frac{\partial u}{\partial v_i}\frac{\mathrm{d}v_i}{\mathrm{d}x}$$

沿线相乘
分线相加

链导法则



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

全微分的形式不变性

$$\mathrm{d}u=\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\mathrm{d}x_i$$

无论x_i是自变量还是中间变量，全微分形式一样。

设$u = xyf\left(\frac{x+y}{xy}\right)$, $f(t) \in C^1$,且满足$x^2\frac{\partial u}{\partial x} - y^2\frac{\partial u}{\partial y} = G(x, y)u$,

证明：$G(x, y) = x - y$

思考题2

利用变量代换$u=x, v=\dfrac{y}{x}$把方程$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$ 化为新的方程

(z关于u, v的方程).

$$u\frac{\partial z}{\partial u}=z.$$