Submit on Crowdmark by Tuesday, July 7, 2020, 11pm

Upload one .pdf file with 2 pages: Page 1 is your typed report (your discussions, data and figures on a single page); Page 2 is a listing of your code(s). The assignment is due at 11:00pm. You will receive a Crowdmark link for the upload.

This computing assignment is about drawing pretty pictures of computed eigenvalues of some special matrices. In particular, we want to see how sensitive eigenvalues are or can be to perturbations. The focus is on setting up and conducting the experiments, and observing the outcome without going too deep into the analysis.

Your task. Set $\delta = 10^{-8}$. Run the experiment described in example Ex5 first. Then, for each of the four matrices A from examples Ex1, Ex2, Ex3, and Ex4, take $n = 42$, and compute the eigenvalues of $A + \delta Q$ for 100 random orthogonal matrices. Plot the eigenvalues of all 100 computations on one plot. If the vector `zeig` contains all your values, you might want to plot them with the command

```
plot(zeig,'.')
```

Investigate the condition number of the eigenvector matrix by computing it for **ten** random perturbations Q; report and comment on your results. The Matlab command `eig` will return eigenvalues and the eigenvector matrix.

Notes and examples.

```
B = 2*rand(n) - eye(n);
[Q,R] = qr(B);
```

creates an n by n random matrix B with elements uniformly distributed in $[-1, 1]$; you can use the QR-factorization to create a random **orthogonal** matrix Q. To build the matrices for your experiments you will also need the following:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

$$U = J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad LD = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

J is the n by n Jordan super block, $J_{i,i+1} = 1$ and all other elements zero. S is related to the discrete version of the second derivative operator, $S = J^T - 2I + J$; $U = J$ is the upper triangular part of S, whereas $LD = A - U = J^T - 2I$ is the lower triangular part and diagonal of S.

```
S =  -2*diag(ones(n,1)) + diag(ones(n-1,1),-1) + diag(ones(n-1,1),1);
LD = tril(S); U = triu(S,1);
```

Ex1 Take $A = 4J$. What are the exact eigenvalues and eigenvectors of A? Food for thought: If you have the equation $z^n = 0$, and perturb it slightly to $z^n = \delta$, what happens to the solutions?

Ex2 (Limaçon) Take $A = 4J + 4J^2$. What are the exact eigenvalues and eigenvectors of A?

Ex3 (Gauss-Seidel) This example is the Gauss-Seidel iteration matrix $A = -(LD)^{-1}U$ (use the Matlab command `inv` to compute the inverse). There is a formula for the exact eigenvalues of A; it is known that they are real, and are all in the interval $[0, 1]$. Is $\lambda = 0$ an eigenvalue of the matrix A?

Ex4 For this example we are starting out with the eigenvalues, so we definitely know what they are – or ought to be. We take n equally spaced points on the interval $[-2, 2]$,

$$t_j = 4(j-1)/(n-1) - 2, \qquad j = 1, 2, \ldots n.$$

We then compute the polynomial

$$p(z) = c_0 + c_1 z + \cdots c_{n-1} z^{n-1} + z^n = \prod_{j=1}^{n} (z - t_j)$$

whose roots are precisely the t_j. Its companion matrix is a matrix that has $p(z)$ as its characteristic polynomial, and therefore the t_j as its eigenvalues:

$$A = \begin{pmatrix} -c_{n-1} & -c_{n-2} & \cdots & -c_1 & -c_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

```
t=4*[0:n-1]/(n-1) - 2;
p=poly(t);   A=compan(p);
```

Ex5 Taking the same t_j from Ex4, create the symmetric matrix A with t_j, $j = 1, 2, \ldots, n$ as its eigenvalues and with random (orthogonal) eigenvectors $w_1, \ldots, w_n$:

```
B = 2*rand(n) - eye(n);   [W,R]=qr(B);   A = W*diag(t)*W';
```

Compute the eigenvalues of A as well as those of $A + \delta Q$ and $A + \delta(Q + Q^T)$, for ten different random orthogonal matrices Q. The second perturbation is symmetric, so the eigenvalues will remain real. **Do not plot your results in this case;** instead, report by how much the eigenvalues move under these perturbations. Although Ex4 and Ex5 have the same eigenvalues, what do you believe to be the reason for obtaining very different results with perturbations?