1, 2, 4, 5, 6, 9, 11, 12　3.4

ex5.4

2022.11.28

5.4.1.(1) Ans: $\frac{1}{2}\int_0^{+\infty} e^{-x^2} dx^2 = -\frac{1}{2}e^{-x^2}\Big|_0^{+\infty} = 0+\frac{1}{2} = \frac{1}{2}$.

5.4.1.(2) Ans: $-\int_0^{+\infty} x\, d\cos x = \int_0^{+\infty} \cos x\, dx - x\cos x\Big|_0^{+\infty}$

其中 $\int_0^{+\infty} \cos x\, dx$ 不收敛，$x\cos x\Big|_0^{+\infty}$ 也不收敛

∴ Ans 不收敛.

5.4.1.(4) ~~Ans: $\int_1^{+\infty} \arctan x\, d\ln x = \arctan x \ln x\Big|_1^{+\infty} - \int_1^{+\infty} \ln x \cdot \frac{1}{1+x^2} dx$~~

Ans: $\int_1^{+\infty} \frac{\arctan x}{x} dx > \int_1^{+\infty} \frac{\pi/4}{x} dx = \frac{\pi}{4}\int_1^{+\infty} \frac{1}{x} dx$.

而 $\int_1^{+\infty} \frac{1}{x} dx$ 发散，∴ Ans 仍不收敛！

5.4.1.(5) Ans $= \int_0^{+\infty} -e^{-x} d\cos x = \int_0^{+\infty} \cos x\, de^{-x} - \cos x e^{-x}\Big|_0^{+\infty}$

$= \int_0^{+\infty} \cos x\, e^{-x}(-1) dx - (0-1)$

$= 1 - \int_0^{+\infty} e^{-x} d\sin x$

$= 1 - \left((e^{-x}\sin x)\Big|_0^{+\infty} - \int_0^{+\infty} \sin x\, de^{-x}\right)$

$= 1 + \int_0^{+\infty} \sin x\, de^{-x} = 1 - \int_0^{+\infty} \sin x\, e^{-x} dx. = 1 - \text{Ans}$.

综上 所有 Ans $= \frac{1}{2}$

5.4.1.(6) Ans: $\int_{-\infty}^{+\infty} \frac{1}{(x+1)^2+1} dx = \arctan(x+1)\Big|_{-\infty}^{+\infty} = \pi$

5.4.1.(9) 做不来啊 !!　$\int_0^1 \frac{x\ln x}{(1-x^2)^{\frac{3}{2}}} dx = (1-x^2)^{-\frac{1}{2}} \ln x\Big|_0^1 - \int_0^1 \frac{(1-x^2)^{-\frac{1}{2}}}{x} dx$

$= \frac{\ln x}{\sqrt{1-x^2}} - \ln\left|\frac{\sqrt{1-x^2}-1}{x}\right|\Bigg|_0^1$ 发散到 $+\infty$.

5.4.1.(11) Ans: $-\int_0^{+\infty} x^n de^{-x} = \int_0^{+\infty} e^{-x} dx^n - x^n e^{-x}\Big|_0^{+\infty} = \int_0^{+\infty} e^{-x} dx^n$

$= \int_0^{+\infty} e^{-x} n x^{n-1} dx = n\int x^{n-1} e^{-x} dx$

∴ $I_n = n \cdot I_{n-1} = n\cdot(n-1) I_{n-2} = \cdots = n!\, I_1$

而 $I_1 = \int_0^{+\infty} x e^{-x} dx = -\int_0^{+\infty} x\, de^{-x} = \int_0^{+\infty} e^{-x} dx - xe^{-x}\Big|_0^{+\infty}$

$= \int_0^{+\infty} e^{-x} dx = -e^{-x}\Big|_0^{+\infty} = 0+1 = 1$

综上，Ans $= n!$

5.4.1(12) Ans: $\ln^n x \cdot x\Big|_0^1 - \int_0^1 x\, d\ln^n x = 0 - \int_0^1 x\, n\ln^{n-1} x \cdot \frac{1}{x} dx$.

所 $\int_0^1 (\ln x)^n dx = -n\int_0^1 \ln^{(n-1)} x\, dx = \cdots = (-1)^n n! \int_0^1 \ln x\, dx$

而 $\int_0^1 \ln x\, dx = (x\ln x - x)\Big|_0^1 = -1$　$\ln x + 1$

综上，Ans $= (-1)^n n!$

扫描全能王 创建

$\frac{\sin^2 x}{\cos^2 x}$ $\frac{1}{\cos^2 x}$ $\frac{1-\cos^2 x}{\cos^2 x}$

5.4.3 (1) 先求 $\int \frac{1}{x\sqrt{x-1}}dx$

令 $x=t^2$，$\therefore dx=2t\,dt$，$Ans=\int\frac{2t\,dt}{t^2\sqrt{t^2-1}}=2\int\frac{1}{t\sqrt{t^2-1}}dt$

令 $t=\sec k$ $\therefore dt=\frac{\sin x}{\cos^2 x}dk$，$Ans=2\int\frac{1}{\sec k\tan k}\cdot\frac{\sin k}{\cos^2 k}dk=2k$

$\therefore \int\frac{1}{x\sqrt{x-1}}dx=2\,\mathrm{arcsec}\sqrt{x}+C$

$\therefore Ans=2\,\mathrm{arcsec}\sqrt{x}\Big|_1^{+\infty}=2\left(\frac{\pi}{2}-0\right)=\pi$

5.4.3 (2). $\int_0^{+\infty}\frac{1}{x^\alpha}dx=\int_0^1\frac{1}{x^\alpha}dx+\int_1^{+\infty}\frac{1}{x^\alpha}dx$

接下来考查 $\int_0^1\frac{1}{x^\alpha}dx$ 的敛散性

① $\alpha=1$. $\int_0^1\frac{1}{x}dx=\ln|_0^1=+\infty$

② $\alpha\neq1$, $\int_0^1 x^{-\alpha}dx=\int_0^1\frac{x^{1-\alpha}}{1-\alpha}dx=\frac{1}{1-\alpha}-\infty$

$\therefore$ 对 $\forall\alpha\in R$, $\int_0^{+\infty}\frac{1}{x^\alpha}dx$ 发散.

5.4.4. (1) $Ans=\int_{-1}^4\frac{1}{(x+2)(x-1)}dx=\frac{1}{3}\int_{-1}^4\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx$

$\therefore Ans=\frac{1}{3}\int_{-1}^1\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx+\frac{1}{3}\int_1^4\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx.$

$=\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\Big|_{-1}^1+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|\Big|_1^4$

$=\frac{1}{3}(-\infty-\ln 2)+\cdots$

$\therefore$ 原积分发散

5.4.4. (2) $Ans=\int_{-1}^0 x^{-\frac{1}{3}}dx+\int_0^1 x^{-\frac{1}{3}}dx$

$=\frac{3}{2}x^{\frac{2}{3}}\Big|_{-1}^0+\frac{3}{2}x^{\frac{2}{3}}\Big|_0^1$

$=0$

原 $x^{-\frac{1}{3}}$ 为在 $[-1,1]$ 上的连续有界奇函数

扫描全能王 创建

5.1 5.3

1.3 ^{25}, 28, 30 $\frac{3}{2}$ 2.4 ^{5}

ex5.1+ex5.3

2022.11.30.

5.1.25.(1) $f(x)=x$, $f'(x)=1>0$

$[0,1]$: $\bar{f}_1=\frac{1}{1-0}\int_0^1 x\,dx=\frac{1}{2}x^2\Big|_0^1=\frac{1}{2}$. $max_1=1$ $min_1=0$.

$[0,10^5]$: $\bar{f}_2=\frac{1}{10^5-0}\int_0^{10^5}x\,dx=\frac{1}{10^5}\cdot\frac{1}{2}x^2\Big|_0^{10^5}=\frac{10^5}{2}$ $max_2=10^5$ $min_2=0$.

(2) $f(x)=xe^{-x}$, $\therefore f'(x)=e^{-x}+e^{-x}(-1)x=e^{-x}(1-x)$

$\int xe^{-x}dx=-xe^{-x}-e^{-x}+C$

$e^{-x}-xe^{-x}$

$-(1\cdot e^{-x}-e^{-x}x)-e^{-x}$

$-e^{-x}(x+1)$

$-(-e^{-x})(x+1)+e^{-x}$

$[0,1]$ $\bar{f}_1=\frac{1}{1-0}\int_0^1 f(x)dx=-xe^{-x}-e^{-x}\Big|_0^1=1-\frac{2}{e}$

$(-\frac{1}{e}-\frac{1}{e})+1$

$max_1=f(1)=\frac{1}{e}$ $min_1=f(0)=0$

$[0,10^5]$ $\bar{f}_2=\frac{1}{10^5}(-xe^{-x}-e^{-x})\Big|_0^{10^5}=\frac{1}{10^5}(1-\frac{10^5}{e^{10^5}}-\frac{1}{e^{10^5}})$

$max_2=f(1)=\frac{1}{e}$ $min_2=f(0)=0$

5.1.28.(1) 不妨令 $f(x)>0$, $f'(x)>0$.

$\because f'(x)\le M$, 所 $f'(x)-M\le 0$. 所 $f(x)-M(x-a)\le f(a)-M(a-a)=0$.

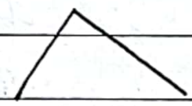
$\therefore \int_a^b|f(x)|dx=\int_a^b f(x)dx\le\int_a^b M(x-a)\,dx=M\frac{1}{2}(x-a)^2\Big|_a^b=\frac{M}{2}(b-a)^2$

所有 $\int_a^b|f(x)|dx\le\frac{M}{2}(b-a)^2$

删去

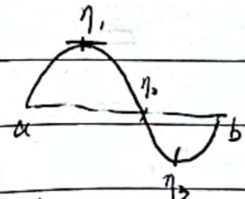
(2) 由积分中值定理. $\exists\,\xi\in(a,b)$ s.t.:

$\int_a^b|f(x)|dx=f(\xi)(b-a)$.



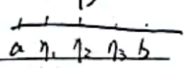
① $f(x)\equiv 0$. 显然, 成立

② $f(x)$ 有极值点与零点. 从a到b放出这些点



$\therefore\int_a^b|f(x)|dx=\int_a^{\eta_1}|f(x)|dx+\int_{\eta_1}^{\eta_2}|f(x)|dx+\cdots+\int_{\eta_n}^b|f(x)|dx$

由(1), R.H.S. $\le\frac{M}{2}\left((\eta_1-a)^2+(\eta_2-\eta_1)^2+\cdots+(b-\eta_n)^2\right)$

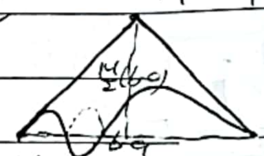


$\therefore$ 只需证 $\left((\eta_1-a)^2+(\eta_2-\eta_1)^2+\cdots+(b-\eta_n)^2\right)\le\frac{1}{2}(b-a)^2$ $n\ge 1$.

而 L.H.S $\le(\eta_1-a)^2+(b-\eta_1)^2<(\eta_1-a+b-\eta_1)^2$

不会!

构造 $g(x)=\begin{cases}M(x-a), & x\in[a,\frac{b-a}{2}]\\ -M(x-b) & x\in[\frac{b-a}{2},b]\end{cases}$ $\therefore g(x)\in C$



且 $\int_a^b g(x)dx=(b-a)(\frac{M}{2}(b-a))(\frac{1}{2})=\frac{M}{4}(b-a)^2$

因此, 只需证 $|f(x)|<g(x)$ 所有 $\int_a^b|f(x)|dx<\frac{M}{4}(b-a)^2$

~~而 $(g(x)-|f(x)|)'_x$ 在 $f(x)\ne 0$ 处 $\ge M-M=0$.~~

$f(x)=f(a)+f'(a)(x-a)+o(x-a)^2<f(a)+M(x-a)$

$f(x)=f(b)+f'(b)(x-b)+o(x-b)^2<f(b)-M(b-x)$.

见第二次习题课表梳习题了. 一样的思路.

扫描全能王 创建

5.1.30 已知 $f(x)-f(a)=\int_a^x f'(t)\,dt=\int_a^x f'(t)\,d(t-x)$

$$=f'(t)(t-x)\Big|_a^x-\int_a^x (t-x)\,df'(t)$$

$$=f'(a)(x-a)-\int_a^x (t-x)f''(t)\,dx = P_1(x)-\frac{1}{2!}\int_a^x f''(t)\,d(t-x)^2$$

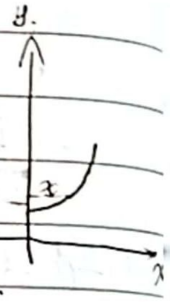
$$=P_1(x)-\frac{1}{2!}\left(f''(t)(t-x)^2\Big|_a^x-\int_a^x (t-x)^2\,df''(t)\right)$$

$$=P_1(x)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{2!}\cdot\frac{1}{3}\int_a^x f'''(t)\,d(t-x)^3$$

$$=P_2(x)+\frac{1}{3!}f'''(a)(x-a)^3-\frac{1}{3!}\int_a^x (t-x)^3\,df'''(t)\cdots\cdots$$

$$=P_n(x)+\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$$

5.3.3.(2) ∴ $x=\sqrt{\ln y}$.



$$dV=\pi x^2\,dy=\pi\ln y\,dy$$

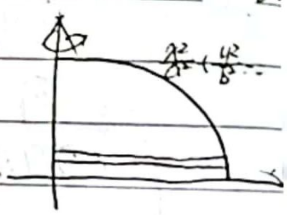
∴ $V=\int_1^e dV=\int_1^e \pi\ln y\,dy$ ；注意看算的是哪一块的面积.

$$=\pi(x\ln x-x)\Big|_1^e$$

$$=\pi[(e-e)-(0-1)]=\pi \quad (e-1)\pi.$$

$x^2=a^2-\frac{a}{b}$

5.3.5.(2) $dS=\pi(x(y)+x(y)+\Delta x)\,dl=2\pi x\,dl.$



$x=\frac{a}{b}\sqrt{b^2-y^2}$，$x_y'=\frac{a}{b}\frac{1}{2\sqrt{b^2-y^2}}\cdot(-2y)=\frac{a}{b}\frac{y}{\sqrt{b^2-y^2}}$

$$dl=\sqrt{1+x_y'^2}\,dy=\sqrt{1+\frac{a^2}{b^2}\frac{y^2}{b^2-y^2}}\,dy$$

$$\therefore dS=2\pi\frac{a}{b}\sqrt{b^2-y^2}\sqrt{1+\frac{a^2}{b^2}\frac{y^2}{b^2-y^2}}\,dy$$

$$=2\pi\frac{a}{b}\sqrt{b^2-y^2+\frac{a^2}{b^2}y^2}\,dy$$

$$=2\frac{a\pi}{b}\sqrt{b^2+\frac{c^2}{a^2}y^2}\,dy$$

$$=2\frac{a}{b}\pi\frac{a}{c}\sqrt{b^2+\frac{c^2}{a^2}y^2}\,d\frac{c}{a}y$$

$$\therefore S=\int_{-b}^{b}dS$$

$$=\int_{-b}^{b}\frac{2\pi a^2}{bc}\sqrt{b^2+\left(\frac{c}{a}y\right)^2}\,d\frac{c}{a}y$$

$$=\frac{2\pi a^2}{bc}\ln\left(\frac{c}{a}y+\sqrt{b^2+\left(\frac{c}{a}y\right)^2}\right)\Big|_{-b}^{b}$$

$$=\frac{4\pi a^2}{bc}\ln\frac{\frac{c}{a}b+b\sqrt{1+\frac{c^2}{a^2}}}{-\frac{c}{a}b+b\sqrt{1+\frac{c^2}{a^2}}}$$

$$=\frac{4\pi a^2}{bc}\ln\frac{c+\sqrt{a^2+c^2}}{-c+\sqrt{a^2+c^2}}$$

5.3.5(4) 上次做过了. $\dfrac{\pi a^2 b\arcsin\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}+2\pi b^2$

扫描全能王 创建

6.1

1 1.4 2 2.4 4 1.4 5 1.2

ex 6.1

2022.12.2

6.1.1.(1) $(1+x^2)dy = y\,dx$

SOl: $\therefore \int \frac{1}{y}dy = \int \frac{1}{1+x^2}dx \Rightarrow \ln y + \ln C' = \arctan x$

$\Rightarrow y = Ce^{\arctan x}$ $C=\frac{1}{C'}$

6.1.1.(4) $yy' = \frac{1-2x}{y}$ $\therefore y\neq 0$.

SOl: $\therefore y^2 dy = (1-2x)dx \Rightarrow \frac{1}{3}y^3 + C' = x - x^2$

$\Rightarrow y = \sqrt[3]{-3x^2+3x-C}$

6.1.2.(2) $y' = \frac{y^2}{x^2} - 2$, $\therefore x\neq 0$

SOl. 令 $\frac{y}{x} = u$, $\therefore y = ux$, $\therefore y'_x = u + u'x$

1° $\therefore y'_x = u + u'x = u^2 - 2 \Rightarrow \frac{du}{dx}\cdot x = u^2 - u - 2$

即 $\int \frac{1}{(u-2)(u+1)}du = \int \frac{1}{x}dx$, L.H.S. $= \int \frac{1}{3}(\frac{1}{u-2} - \frac{1}{u+1})du$

$\therefore \frac{1}{3}\ln\frac{u-2}{u+1} = 3\ln x\ C' \Rightarrow 1 - \frac{3}{u+1} = Cx^3$ $C=(C')^3$

即 $\frac{3}{u+1} = 1 - Cx^3 \Rightarrow u = \frac{3}{1-Cx^3} - 1 = \frac{y}{x}$

$\Rightarrow y = \frac{3x}{1-Cx^3} - x$

2° $u^2 - u - 2 = 0 \Rightarrow u = 2$ 或 $u = -1$ $Y = \pm\sqrt{x^2(2\ln x + 2C_1)}$

$\Rightarrow y = 2x$ 或 $y = -x$

6.1.2.(4) $(x^2+3y^2)dx = 2xy\,dy$

SOl: 令 $\frac{y}{x} = u$, $\therefore y = ux$, $y'_x = u'x + u = \frac{dy}{dx} = \frac{x^2+3y^2}{2xy} = \frac{1+3u^2}{2u}$

$\therefore u'x = \frac{1+u^2}{2u} = \frac{du}{dx}x \Rightarrow \frac{2u}{1+u^2}du = \frac{1}{x}dx$

即 $\int \frac{du^2}{1+u^2} = \int \frac{1}{x}dx \Rightarrow \ln(1+u^2) = \ln x \cdot C$

$\Rightarrow u = \pm\sqrt{Cx-1} = \frac{y}{x}$

$\Rightarrow y = \pm x\sqrt{Cx-1}$.

6.1.4.(1) $(1+x^2)y' - 2xy = (1+x^2)^2$

SOl: $y' - \frac{2x}{1+x^2}y = (1+x^2)$, $P(x) = -\frac{2x}{1+x^2}$, $Q(x) = 1+x^2$

而 $y = e^{-\int p(x)dx}\left[\int Q(x)e^{\int p(x)dx}dx + C\right]$

$\therefore y = e^{\int\frac{2x}{1+x^2}dx}\left[\int(1+x^2)\cdot e^{-\int\frac{2x}{1+x^2}dx}dx + C\right]$. $x+C+x^3+Cx^2$

$\int\frac{2x}{1+x^2}dx = \int\frac{dx^2}{1+x^2} = \ln(1+x^2)$

$= (1+x^2)\cdot(x+C)$.

$\therefore y = (1+x^2)(x+C)$

扫描全能王 创建

6.1.4 (4) $y' + \frac{y}{x} = y^2 \ln x$

sol: ∴ $\frac{1}{y^2}y' + \frac{1}{y}\frac{1}{x} = \ln x$

令 $\frac{1}{y}=u$, ∴ $-\frac{1}{y^2}y'_x = u'_x$

故 $-u' + \frac{1}{x}u = \ln x$, 即 $u' - \frac{1}{x}u = -\ln x$

∴ $u = e^{-\int -\frac{1}{x}dx}\left[\int(-\ln x)e^{\int -\frac{1}{x}dx} + C\right]$

$= x\cdot\left[\int \frac{\ln x}{x}dx + C\right]$

$= x\left(-\frac{1}{2}\ln^2 x + C\right)$

∴ $y = \frac{1}{x(-\frac{1}{2}\ln^2 x+C)}$

6.1.5.(1) $y' = \frac{y}{x}\ln\frac{y}{x}$, $y(1)=1$

sol: 令 $\frac{y}{x}=u$, ∴ $y=ux$, $y'_x = u'x + u = u\ln u$

即 $\frac{du}{dx}x = u(\ln u - 1) \Rightarrow \frac{1}{u(\ln u-1)}du = \frac{1}{x}dx$

故 $\int\frac{1}{\ln u-1}d\ln u = \ln Cx \Rightarrow \ln(\ln u-1) = \ln Cx$.

$\Rightarrow \ln u - 1 = Cx$, $u = e^{Cx+1} = \frac{y}{x}$

∴ $y = xe^{Cx+1}$ [illegible] $y = xe^{-x+1}$

6.1.5.(2) $y' + \frac{y}{x} = \frac{\sin x}{x}$, $y(\pi)=1$ 注:求特解.

sol: $P(x)=\frac{1}{x}$, $Q(x)=\frac{\sin x}{x}$

∴ $y = e^{-\int_\pi^x \frac{1}{x}dx}\left[\int_\pi^x Q(x)e^{\int_\pi^x\frac{1}{x}dx}dx + 1\right]$

$= e^{-\ln x|_\pi^x}\cdot\left[\int_\pi^x \frac{\sin x}{x}e^{\int_\pi^x \frac{1}{x}dx}dx+1\right]$

$= e^{\ln\pi - \ln x}\cdot\left[\int_\pi^x\frac{\sin x}{x}e^{\ln x-\ln\pi}dx+1\right]$

$= \frac{\pi}{x}\cdot\left[\frac{1}{\pi}\int_\pi^x \sin x\,dx + 1\right]$

$= \frac{1}{x}(-1-\cos x + \pi)$

∴ $y = \frac{-1-\cos x+\pi}{x}$

9

扫描全能王 创建