10.8

ex3.1 / 1. (4) $x=0$ 不连续，故不可导

(5) $f'_-(0)=-1$, $f'_+(0)=1$

(6) $f'_-(0)=f'_+(0)=0$

2.(1) 连续性 $\Rightarrow$ $a+b=1$. 可导 $\Rightarrow$ $a=2$ $\Rightarrow \begin{cases} a=2 \\ b=-1 \end{cases}$

3. $f'(a)=\lim\limits_{\Delta x\to 0}\dfrac{f(a+\Delta x)-f(a)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta x\, g(a+\Delta x)}{\Delta x}=g(a)$

$\downarrow$ 直得结果 $\rightarrow$ 用公式

5.(1) $|f(a)|>0$：不妨 $f(a)>0$. 由连续，$\exists\,\delta>0$ s.t. $\forall x\in U(a,\delta)$，$f(x)>0$. 故 $\forall x\in U(a,\delta)$，$|f(x)|=f(x)\Rightarrow(|f(x)|)'=f'(x)$，可导.

(2) $|x|$.

7. (4) $y'=3(\sin x+\cos x)^2(\cos x-\sin x)$ 一部分 部分求

(10) $y'=\sqrt{1+x^2}\sin x+\dfrac{x^2}{\sqrt{1+x^2}}\sin x+x\sqrt{1+x^2}\cos x$.

(16) $y'=10^x\ln 10\,(\sin x)^{\cos x}+10^x(\sin x)^{\cos x}(\cos x\cot x-\sin x\ln(\sin x))$

14. (1) $\dfrac{dy}{dx}=2x+2e^x$，$\dfrac{dx}{dy}=\dfrac{1}{2x+2e^x}$ / $\dfrac{dy}{dx}=(x+1)e^x$. $\dfrac{dx}{dy}=\dfrac{1}{(x+1)e^x}$.

(3) $\dfrac{dy}{dx}=-2x^{-x}(\ln x+1)+2e^{-2x}$

$\dfrac{dx}{dy}=\left(\dfrac{dy}{dx}\right)^{-1}=\cdots$

10.9

ch3 综/1. 令 $g=\prod_{i=1}^{n}(x+i)$. 则 $f(x)=xg(x)$ ☆

$f'(x)=g(x)+xg'(x)$. $f'(0)=g(0)=n!$.

引入 g(x)与 f(x) 配合 (可在求导时约掉 x, 而x为0, 化简求出)

ex3-1/ 1. (2) $f'_-(0)=1$, $f'_+(0)=0$

(3) $x=0$ 不连续.

8. $f'(x^2) = f'(t)\big|_{t=x^2} = 3x^4$. ⇒ 只为中间变量 (在结果中把中间变量代掉即可)

$(f(x^2))' = f'(x^2)(x^2)' = 6x^5$.

7. (18) $y' = \frac{2x-x^2}{x^2-2x+1}\sqrt{\frac{x+1}{x^2+x+1}} - \frac{1}{2}\frac{x^2}{1-x}\sqrt{\frac{x^2+x+1}{x+1}}\,\frac{x^2+2x}{(x^2+x+1)^2}$

10. (2) $\frac{dy}{dx} = 2\sin x\cos x\, f'(\sin^2 x) - 2\sin x\cos x\, f'(\cos^2 x)$

(6) $\frac{dy}{dx} = f'(e^x)\,e^x\,e^{f(x)} + f(e^x)\,e^{f(x)}f'(x)$

11 $y=\begin{cases}(1-2x)\sin x, & x\le\frac{1}{2}\\(2x-1)\sin x, & x>\frac{1}{2}\end{cases}$ ☆ 写成分段函数

$\Rightarrow y'=\begin{cases}-2\sin x+(1-2x)\cos x, & x<\frac{1}{2}\\2\sin x-(1-2x)\cos x, & x>\frac{1}{2}\end{cases}$

$x=\frac{1}{2}$ 处不可导.

10.10

7.(3) $y' = -\frac{2}{\sqrt{-4x^2+4x+2}}$

(12) $y' = \frac{1}{\ln^2(\ln^3 x)} \, 2\ln(\ln^3 x) \, \frac{1}{\ln^3 x} \, 3\ln^2 x \, \frac{1}{x}$

8.(1) $y' = -2xe^{-x^2}$, $y'' = 4x^2e^{-x^2} - 2e^{-x^2}$.

(2) $y' = 2x\cdot 2^x + x^2\cdot 2^x\ln 2$

$y'' = 2^{x+1} + 4x\cdot 2^x\ln 2 + x^2\cdot 2^x(\ln 2)^2$

(3) $y' = 2x\arctan x + 1$

$y'' = 2\arctan x + \frac{2x}{x^2+1}$

(4) $y' = |2x|$, $y'' = 2\operatorname{sgn}(x)$, $x \neq 0$ $\rightarrow$ 成立区间.

(注意细节)

19.(1) $y' = 2xf'(x^2)$. $y'' = 4x^2f''(x^2) + 2f'(x^2)$,

$y''' = 12x f''(x^2) + 8x^3 f'''(x^2)$.

(2) $y' = f'(e^x+x)(e^x+1)$ $y'' = f''(e^x+x)(e^x+1)^2 + f'(e^x+x)e^x$

$y''' = f'''(e^x+x)(e^x+1)^3 + 3f''(e^x+x)(e^x+x)e^x + f'(e^x+x)e^x$

20. pf: $f(x) = \begin{cases} x^{n+1}, & x \geq 0 \\ -x^{n+1}, & x < 0. \end{cases} \Rightarrow f^{(n)}(x) = \begin{cases} (n+1)!\,x, & x > 0 \\ -(n+1)!\,x, & x < 0. \end{cases}$

$f^{(n+1)}(x) = \begin{cases} (n+1)!, & x > 0 \\ -(n+1)!, & x < 0. \end{cases}$ 由于 $f^{(n)}(x)$ C, 但 $f^{(n+1)}(x)$ 间断.

故在 $x=0$ 处, $f^{(n)}(0) = 0$ 但 $f^{(n+1)}(0)$ 不存在.

21. (1) $(x^2e^x)^{(n)} = x^2 \cdot e^x + 2nxe^x + n(n-1)e^x$

(2) $((x^2+1)\sin x)^{(n)} = (x^2+1)\sin(x+\frac{n}{2}\pi) + 2nx\sin(x+\frac{n-1}{2}\pi)$
$+ n(n-1)\sin(x+\frac{n-2}{2}\pi)$

(3) $(\frac{1}{x^2-3x+2})^{(n)} = (\frac{1}{x-2}-\frac{1}{x-1})^{(n)} = \frac{n!(-1)^n}{(x-2)^{n+1}} - \frac{n!(-1)^n}{(x-1)^{n+1}}$

↓ 同阶分解 ☆

(4) $(\sin x\cos x)^{(n)} = \frac{1}{2}(\sin 2x)^{(n)} = 2^{n-1}\sin(2x+\frac{n}{2}\pi)$

三角函数建议 化为公式 ☆

22. $x=\frac{\pi}{4}$ 时，$y=\frac{\sqrt{2}}{2}$，$y'=-\frac{\sqrt{2}}{2} \Rightarrow y=-\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})+\frac{\sqrt{2}}{2}$

10.12

2.(2) $dy = x\sin x\, dx$ ☆

$=\frac{|x|}{x}$

(3) $dy = \operatorname{sgn}(x)\sqrt{\frac{x^2}{x^2-1}}\cdot\frac{1}{x^2}dx$,　$x \in \mathring{U}(0,1)$

小邻域

(4) $dy = (\frac{1}{x-1}-\frac{1}{x+1})dx$　　$x \neq \pm 1$.

(5) $dy = 5^{\sqrt{\tan^{-1}x^2}} \cdot \ln 5 \cdot \frac{1}{2\sqrt{\tan^{-1}x^2}} \cdot \frac{1}{x^4+1} \cdot 2x\, dx$

(6) $dy = 8\tan(1+2x^2)\sec^2(1+2x^2)x\, dx$

☆ 中间变量

3. (1) $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot(\frac{dx}{dt})^{-1} = \frac{t}{2}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx})\cdot\frac{dt}{dx} = \frac{d}{dt}(\frac{dy}{dx})\cdot(\frac{dx}{dt})^{-1} = \frac{t^2+1}{4t}$ ☆

$\frac{d(\frac{dy}{dx})}{dx}$

(3) $\frac{dy}{dx}=\frac{\sin\varphi+\varphi\cos\varphi}{\cos\varphi-\varphi\sin\varphi}$ $\frac{d^2y}{dx^2}=\frac{\varphi^2+2}{(\cos\varphi-\varphi\sin\varphi)^3}$

r.m.k：无需考虑 y关于x是否为多值函数，因为对每个参数t，相当于将 x与y限制在某一邻域中，在该邻域内y关于x单值。

4. (1) $x+y=\sqrt{2}$ (2) $y=-\frac{4}{3}x+4$

$-f(x)=-f(-x) \Rightarrow f'(x)=f'(-x) \Rightarrow -f''(x)=f''(x)$

综(1)(2)由 $f(x)$ 奇知 $f(0)=f''(0)=0$，$f'(x)$ 偶。

$\lim\limits_{x\to 0} g(x)=\lim\frac{f(x)-f(0)}{x-0}=f'(0)=a$

不一定有值

合→结

(2) $x\neq 0$：$g'(x)=\frac{xf'(x)-f(x)}{x^2}$.

(通过洛必达法则可知，$g'_+(0)=g'_-(0)=g'(0+0)=g'(0-0)=\frac{1}{2}f''(0)=0$.

故 $g(x)$ 可导且导连续)

另一法：$g'(0+0)=\lim\limits_{\Delta x\to 0^+}\frac{f(\Delta x)}{\Delta x}\overset{奇}{=}\lim\limits_{\Delta x\to 0^+}\frac{f(-\Delta x)}{-\Delta x}=\lim\limits_{\Delta x\to 0^-}\frac{f(\Delta x)}{\Delta x}=g'(0-0)$.

从而 $g(x)$ 在 $x=0$ 处可导 ……

10.14

1. 利用Rolle中值定理即可.

3个根. $x_1\in(1,2)$，$x_2\in(2,3)$，$x_3\in(3,4)$

2. $F'(x)=(2x-2)f(x)+(x^2-2x+1)f'(x) \Rightarrow F'(1)=0$.

由于 $F(1)=0=F(2)$，由Rolle $\exists x_0\in(1,2)$ s.t. $F'(x_0)=0$.

再由Rolle，$\exists\xi\in(1,x_0)\subseteq(1,2)$ s.t. $F''(\xi)=0$.

不能取等

4.(1) $x^n \in C^\infty[b,a]$

由 Lagrange，$\exists\, x_0 \in (a,b)$ s.t. $(x_0^n)' = \frac{a^n-b^n}{a-b}$.

即 $n x_0^{n-1} = \frac{a^n-b^n}{a-b}$. 由于 nx^{n-1} 在 $[b,a]$ 上 $\uparrow$

故 $nb^{n-1} < f(x_0) < na^{n-1}$. □

5.(1) 令 $F(x) = LHS - RHS$. 则 $F'(x) \equiv 0 \Rightarrow F(x) \equiv C$

又 $F(0)=0$. 故 $F(x) \equiv 0$. 即 $LHS = RHS$.

表格记住即可.

15. 即证 $xf'(x) - f(x) > 0$.

由 Lagrange，$\exists\, \xi \in (0,x)$ s.t. $f(x) - f(0) = f'(\xi)x$.

从而 $xf'(x) - f(x) = x(f'(x) - f'(\xi)) > 0$. $\forall x \in \mathbb{R}^+$.

R.m.k. 没说 $f(x)$ 二阶可导.

19.(2) $f(\pm\frac{\pi}{2}) = \mp\frac{\pi}{2}$. 驻点 $x = \pm\frac{\pi}{6}$. $f(\pm\frac{\pi}{6}) = \pm(\frac{\sqrt{3}}{2} - \frac{\pi}{6})$.

$\Rightarrow f_{max} = \frac{\pi}{2}$. $f_{min} = -\frac{\pi}{2}$ 端点+极值点

20.(1) $f'(x) = p[x^{p-1} - (1-x)^{p-1}]$. 令 $f'(x) = 0 \Rightarrow x_0 = \frac{1}{2}$.

由于 $f''(x) \geq 0$. 故 $f'(x) \uparrow$. 故 $x \in [0,\frac{1}{2})$ 时，$f(x) \downarrow$

$\downarrow (1-x)$ 对 x 求导 -1 拖出来 $x \in (\frac{1}{2},1]$ 时，$f(x) \uparrow$.

$\Rightarrow f_{min} = f(\frac{1}{2}) = 2^{1-p} \Rightarrow 2^{1-p} \leq x^p + (1-x)^p \leq 1$. $\forall x \in [0,1]$.

让 arctan 独立

(4) 即证 $(1+x)\ln(1+x) - \arctan x \triangleq f(x) > 0$.

$f'(x) = \ln(1+x) + \dfrac{x^2}{1+x^2} > 0 \Rightarrow f(x)\uparrow$. 又 $f(0)=0$.

故 $\ln(1+x) > \dfrac{\arctan x}{1+x}$.

- Remark：我写答案可以偷懒，你们写作业或者考试不行！