HW 10.

45. $A_i \sim \begin{pmatrix} 1 & 0 \\ 0.2 & 0.8 \end{pmatrix}$ $\quad S_n = \sum_{i=1}^{500} A_i.$ $\quad ES_n = n\cdot p = 500\cdot 0.2 = 100$

$Var S_n = n\cdot p\cdot(1-p) = 80.$

① $P(80 \le S_n \le 120) = P(|S_n - 100| \le 20) = 1 - P(|S_n - 100| > 20).$

$P(|S_n - 100| > 20) \le P(|S_n - ES_n| \ge 20) \le \frac{Var S_n}{20^2} = \frac{80}{20^2} = 0.2$

$\therefore P(80 \le S_n \le 120) \ge 0.8.$ Chebyshev 不等式只能给出一个下界. 而且一般都很"松".

② $P(80 \le S_n \le 120) = P\left(\frac{80 - ES_n}{\sqrt{Var S_n}} \le \frac{S_n - ES_n}{\sqrt{Var S_n}} \le \frac{120 - ES_n}{\sqrt{Var S_n}}\right)$ $\quad \frac{S_n - ES_n}{\sqrt{Var S_n}} \xrightarrow{d} N(0,1).$

$= P\left(\frac{-20}{\sqrt{80}} \le \frac{S_n - ES_n}{\sqrt{Var S_n}} \le \frac{20}{\sqrt{80}}\right) \overset{近似}{\approx} \Phi(\sqrt{5}) - \Phi(-\sqrt{5})$

$= 2\Phi(\sqrt{5}) - 1.$

$\approx 2\Phi(2.24) - 1 \approx 0.9748.$

48. (1) $X_i = I$(第i个部件起作用) 或用 $Y_i = I$(第i个部件损坏).

$X_i \sim \begin{pmatrix} 1 & 0 \\ 0.9 & 0.1 \end{pmatrix}$

$EX_i = 0.9 \quad Var X_i = 0.9 \times 0.1 = 0.09$

$P(整个系统起作用) = P\left(\sum_{i=1}^{100} X_i \ge 85\right)$

$= P\left(\frac{\sum_{i=1}^{100} X_i - 100\cdot EX_i}{\sqrt{100\cdot Var X_i}} \ge \frac{85 - 100\cdot 0.9}{\sqrt{100\cdot 0.09}}\right)$

$= P\left(\frac{\sum_{i=1}^{100} X_i - 100\cdot EX_i}{\sqrt{100\cdot Var X_i}} \ge -\frac{5}{3}\right) = 1 - P\left(\frac{\sum_{i=1}^{100} X_i - 100\, EX_i}{\sqrt{100\cdot Var X_i}} < -\frac{5}{3}\right)$

$\approx 1 - \Phi(-\frac{5}{3}) = \Phi(\frac{5}{3}) \approx \Phi(1.67) \approx 0.9525.$

(2) $S_n = \sum_{i=1}^{n} X_i$

$P(整个系统可靠) = P(S_n \ge 80\%\cdot n) = P\left(\frac{S_n - ES_n}{\sqrt{Var S_n}} \ge \frac{0.8n - 0.9n}{\sqrt{n\cdot 0.09}}\right)$

$\approx 1 - \Phi\left(-\frac{0.1n}{0.3\sqrt{n}}\right) = \Phi\left(\frac{\sqrt{n}}{3}\right) \ge 0.95 \Rightarrow \frac{\sqrt{n}}{3} \ge 1.65 \quad n \ge 24.5025.$

$\therefore$ n 至少为 25.

49. 设每次取款额为 X_i $(i=1,2,\cdots,200)$

$$EX_i=\sum_{k=1}^{10}k\cdot P(X_i=k)=\frac{1}{10}\sum_{k=1}^{10}k=\frac{11}{2}$$

$$Var X_i=EX_i^2-(EX_i)^2$$

$$EX_i^2=\sum_{k=1}^{10}k^2P(X_i=k)=\frac{1}{10}\sum_{k=1}^{10}k^2=38.5$$

$$\therefore Var X_i=38.5-(5.5)^2=8.25$$

$$S_n=\sum_{i=1}^{n=200}X_i$$ 设至少要存入 x 百元.

$$P(S_n\le x)=P\left(\frac{S_n-ES_n}{\sqrt{Var S_n}}\le\frac{x-200\cdot5.5}{\sqrt{200\cdot38.5}}\right)\ge0.95.$$

$$\approx\Phi\left(\frac{x-1100}{5\sqrt{66}}\right)\ge0.95$$

$$\therefore\frac{x-1100}{5\sqrt{66}}\ge1.65\qquad x\ge1167.023$$

$\therefore$ 至少存 1168 百元

50. (1) 记第 i 次运算的误差为 X_i $(i=1,\cdots,1500)$　　审题：绝对值

$$X_i\sim U(-0.5,0.5)\qquad EX_i=0.\qquad Var X_i=\frac{(0.5+0.5)^2}{12}=\frac{1}{12}$$

$$S_n=\sum_{i=1}^{n=1500}X_i$$

$$P(|S_n|>15)=1-P(|S_n|\le15)$$

$$=1-P(-15\le S_n\le15)$$

$$=1-P\left(\frac{-15-0}{\sqrt{1500\cdot\frac{1}{12}}}\le\frac{S_n-ES_n}{\sqrt{Var S_n}}\le\frac{15-0}{\sqrt{1500\cdot\frac{1}{12}}}\right)$$

$$\approx1-\left[\Phi\left(\frac{3}{\sqrt5}\right)-\Phi\left(-\frac{3}{\sqrt5}\right)\right]$$

$$=1-\Phi\left(\frac{3}{\sqrt5}\right)+1-\Phi\left(\frac{3}{\sqrt5}\right)=2\left(1-\Phi\left(\frac{3}{\sqrt5}\right)\right)\approx2(1-\Phi(1.34))$$

$$\approx2(1-0.9099)=0.1802$$

(2) 设至多可进行 n 次加法

$$P(|S_n|\le10)=P(-10\le S_n\le10)\ge0.9$$

$$=P\left(\frac{-10-0}{\sqrt{n\cdot\frac{1}{12}}}\le\frac{S_n-ES_n}{\sqrt{Var S_n}}\le\frac{10-0}{\sqrt{n\cdot\frac{1}{12}}}\right)\approx\Phi\left(\frac{10\sqrt{12}}{\sqrt n}\right)-\Phi\left(-\frac{10\sqrt{12}}{\sqrt n}\right)$$

$$=2\Phi\left(\frac{10\sqrt{12}}{\sqrt n}\right)-1\ge0.9$$

$\therefore\Phi\left(\frac{10\sqrt{12}}{\sqrt n}\right)\ge0.95$　　$\frac{10\sqrt{12}}{\sqrt n}\ge1.65\Rightarrow n\le\frac{100\cdot12}{(1.65)^2}\approx440.7713$　$\therefore$ 至多进行 440 次

52. $n=2400$. 为简化计算，均以"万元"为单位

收入 $=2400\times5000=1200$ 万元

盈利 200 万元 $\Leftrightarrow$ 赔偿总额 ≤ 1000 万元.

设第 i 辆车的事故数为 $X_i\sim Poi(2)$ $(i=1,\dots,2400)$

第 i 辆车第 j 次(事故)索赔额度为 $Y_{ij}\sim U(0.1,0.5)$ $j=1,2,\dots,X_i$.

第 i 辆车赔款总额为 $Y_i=\sum_{j=1}^{X_i}Y_{ij}$ $(Y_1, Y_2\dots Y_{2400}$ iid$)$

保险公司一年赔款总额为 $S=\sum_{i=1}^{n=2400}Y_i$.

想利用 CLT 进行估计: $P(S\le1000)=P\left(\frac{S-ES}{\sqrt{Var S}}\le\frac{1000-ES}{\sqrt{Var S}}\right)\approx\Phi\left(\frac{1000-ES}{\sqrt{Var S}}\right)$.

下面求 ES, $Var S$.

$ES=\sum_{i=1}^{2400}EY_i$ $\quad Var S=\sum_{i=1}^{2400}Var Y_i$.

$EY_i=E[E[Y_i|X_i]]=E[E[\sum_{j=1}^{X_i}Y_{ij}|X_i]]\overset{\star}{=}E[X_i\cdot EY_{ij}]=EX_i\cdot EY_{ij}=2\cdot0.3=0.6$

$Var Y_i$ 有两种求法.

① 条件方差公式. $Var Y_i=Var(E[Y_i|X_i])+E[Var(Y_i|X_i)]$

$=Var(X_i\cdot EY_i)+E[X_i(Var Y_i)]$

$=(0.3)^2 Var X_i+EX_i\cdot Var Y_i=(0.3)^2\cdot2+\frac{(0.4)^2}{12}\cdot2=0.18+\frac{0.08}{3}=\frac{0.62}{3}$

② $Var Y_i=EY_i^2-(EY_i)^2$

$EY_i^2=E[E(Y_i^2|X_i)]$

$E[Y_i^2|X_i=k]=E[(\sum_{j=1}^{X_i}Y_{ij})^2|X_i=k]=E[(\sum_{j=1}^{k}Y_{ij})^2|X_i=k]\overset{Y_{ij}\text{与}X_i\text{无关}}{=}E[(\sum_{j=1}^{k}Y_{ij})^2]$

$=E[\sum_{j=1}^{k}Y_{ij}^2+\sum_{\substack{p\ne q\\p,q=1}}^{k}Y_{ip}\cdot Y_{iq}]=\sum_{j=1}^{k}EY_{ij}^2+k(k-1)(EY_{ij})^2$ $\quad Y_{ip}$与Y_{iq}独立$\Rightarrow EY_{ip}Y_{iq}=EY_{ip}EY_{iq}$

$=k\cdot[Var Y_{ij}+(EY_{ij})^2]+k(k-1)(EY_{ij})^2=k\cdot[\frac{(0.4)^2}{12}+(0.3)^2]+k(k-1)(0.3)^2$

$=k\cdot(\frac{0.04}{3}+0.09)+0.09k^2-0.09k$.

$=0.09k^2+\frac{0.04}{3}k$.

$\therefore E[Y_i^2|X_i]=0.09X_i^2+\frac{0.04}{3}X_i$

$EY_i^2=E[0.09X_i^2+\frac{0.04}{3}X_i]=0.09(Var X_i+(EX_i)^2)+\frac{0.04}{3}EX_i=0.09\cdot6+\frac{0.08}{3}=\frac{1.7}{3}$

$\therefore \mathrm{Var}\, Y_i = EY_i^2 - (EY_i)^2 = \frac{1.7}{3} - (0.6)^2 = \frac{0.62}{3}$.

$\therefore P(S \le 1000) = \Phi\left(\frac{1000 - 2400 \cdot 0.6}{\sqrt{2400 \cdot \frac{0.62}{3}}}\right) = \Phi\left(\frac{1000-1440}{\sqrt{496}}\right) \approx \Phi(-19.76) \approx 0$.

① 这道题目的难点在于如何求方差.

② 题干中说的平均车辆数为2400. 这里平均是指可以当作2400辆车来计算.
相当于, 可以把背景转化为数学语言: 有随机变量第i辆车的赔偿 $Y_1 \dots Y_{2400}$.

但不可以看作有4800次赔偿, 这里每辆车赔偿的次数 $X_i \sim Poi(2)$ 是一个随机变量.
而非一个确定的数

③ 不少同学想写了 $P(\text{赔偿} \le 1000\text{万元}) = P\left(Y_{ij} \le \frac{1000\text{万元}}{2400 \cdot 2}\right)$.

但这个概率仅是一次赔偿小于 $\frac{1000\text{万元}}{4800}$ 的概率