第 13-14 周作业

2

解：

由题意知，$\boldsymbol{\alpha_1}=1,\boldsymbol{\alpha_2}=\sqrt{2},\boldsymbol{\alpha_3}=\sqrt{2}$

设$\boldsymbol{\alpha_1}=(1,0,0),\boldsymbol{\alpha_2}=\left(0,\sqrt{2},0\right),\boldsymbol{\alpha_3}=(-1,0,1)$

一组正交基为$\boldsymbol{\alpha_1},\frac{\boldsymbol{\alpha_2}}{\sqrt{2}},\boldsymbol{\alpha_1}+\boldsymbol{\alpha_3}$

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解：

(1)

$$|\boldsymbol{\alpha_1}|=\sqrt{1+4+1+1}=\sqrt{7}$$
$$|\boldsymbol{\alpha_2}|=\sqrt{4+9+1+1}=\sqrt{15}$$
$$|\boldsymbol{\alpha_3}|=\sqrt{1+1+4+4}=\sqrt{10}$$
$$<\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}>\ =2+6-1-1=6$$
$$<\boldsymbol{\alpha_1},\boldsymbol{\alpha_3}>\ =-1-2+2+2=1$$
$$<\boldsymbol{\alpha_2},\boldsymbol{\alpha_3}>\ =-2-3-2-2=-9$$

夹角分别为$\arccos\frac{6}{\sqrt{105}},\arccos\frac{1}{\sqrt{70}},\arccos-\frac{9}{5\sqrt{6}}$

(2)

设该向量$\boldsymbol{\beta}=(x_1,x_2,x_3,x_4)$

则$\begin{cases} x_1+2x_2-x_3+x_4=0 \\ 2x_1+3x_2+x_3-x_4=0 \\ -x_1-x_2-2x_3+2x_4=0 \end{cases}$

$$(x_1,x_2,x_3,x_4)=t_1(-5,3,1,0)+t_2(5,-3,0,1)$$

4(2)

解：

$$\boldsymbol{e_1}=(\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}},\frac{2}{\sqrt{7}})$$

$$\boldsymbol{\beta_2}=\boldsymbol{\alpha_2}-(\boldsymbol{e_1},\boldsymbol{\alpha_2})\boldsymbol{e_1}=(1,1,-5,3)-\frac{3}{\sqrt{7}}\left(\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}},\frac{2}{\sqrt{7}}\right)=\left(\frac{4}{7},\frac{4}{7},-\frac{38}{7},\frac{15}{7}\right)$$

$$\boldsymbol{e_2} = (0.097, 0.097, -0.921, 0.364)$$

$$\boldsymbol{\beta_3} = \boldsymbol{\alpha_3} - (\boldsymbol{e_1}, \boldsymbol{\alpha_3})\boldsymbol{e_1} - (\boldsymbol{e_2}, \boldsymbol{\alpha_3})\boldsymbol{e_2} = (4.058, 3.058, -0.547, -3.284)$$

$$\boldsymbol{e_3} = (0.668, 0.503, -0.090, -0.541)$$

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(1)证：

由于$\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$两两正交，则有

$$<\boldsymbol{\alpha_1}, \boldsymbol{\alpha_1}> = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

$$<\boldsymbol{\alpha_2}, \boldsymbol{\alpha_2}> = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

$$<\boldsymbol{\alpha_3}, \boldsymbol{\alpha_3}> = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

$$<\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}> = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

$$<\boldsymbol{\alpha_1}, \boldsymbol{\alpha_3}> = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

$$<\boldsymbol{\alpha_2}, \boldsymbol{\alpha_3}> = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

因此$\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\alpha_3}$也是一组标准正交基。

(2)解：

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

(3)解：

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

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解：

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

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解：$\frac{1}{\left(\frac{1}{4}\right)^2}=16$

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证：

$$Q^TQ=(I-2\boldsymbol{\alpha\alpha^T})^T(I-2\boldsymbol{\alpha\alpha^T})=(I-2\boldsymbol{\alpha\alpha^T})(I-2\boldsymbol{\alpha\alpha^T})=I-4\boldsymbol{\alpha\alpha^T}+4\boldsymbol{\alpha\alpha^T\alpha\alpha^T}$$
$$=I-4\boldsymbol{\alpha\alpha^T}+4\boldsymbol{\alpha\alpha^T}=I$$

即 Q 为正交矩阵。

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

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(1)证：

由于 A 为正交矩阵，则 A 可逆，且|A|=±1

$$(A^*)^TA^*=(|A|A^{-1})^T|A|A^{-1}=|A|^2(A^T)^{-1}A^{-1}=(AA^T)^{-1}=I$$

A^*为正交矩阵。

(2)证：

$$(AB)^TAB=B^TA^TAB=B^TB=I$$

AB为正交矩阵。

(3)证：

$$(A^{-1})^TA^{-1}=(A^T)^{-1}A^{-1}=(AA^T)^{-1}=I$$

A^{-1}为正交矩阵。

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证：

$$A^TA = I \to A^T = A^{-1} \to \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

由于 A 为正交阵，则|A|只可能为 1 或-1。

$|A| = 1$时，$a = d, b = -c, a^2 + b^2 = 1$，令$a = \cos\theta\,, b = \sin\theta$

此时的正交矩阵为$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$|A| = -1$时，$a = -d, b = c, a^2 + b^2 = 1$，令$a = \cos\theta\,, b = \sin\theta$

此时的正交矩阵为$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

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解：

先求出 A 的特征值和特征向量。

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 2 & 0 \\ 2 & \lambda - 2 & 2 \\ 0 & 2 & \lambda - 3 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 3\lambda + 10 = (\lambda - 2)(\lambda^2 - 4\lambda - 5)$$

$$= (\lambda + 1)(\lambda - 2)(\lambda - 5)$$

A 的特征值为-1,2,5，

对于$\lambda = -1$，求解$(-I - A)x = 0$得$x = (2,2,1)^T$

对于$\lambda = 2$，求解$(2I - A)x = 0$得$x = (-2,1,2)^T$

对于$\lambda = 5$，求解$(5I - A)x = 0$得$x = (1,-2,2)^T$

单位化并正交化得

$$P = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$

$$A^k = P\begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}^k P^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} (-1)^k & & \\ & 2^k & \\ & & 5^k \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3}(-1)^k & -\frac{2}{3}2^k & \frac{1}{3}5^k \\ \frac{2}{3}(-1)^k & \frac{1}{3}2^k & -\frac{2}{3}5^k \\ \frac{1}{3}(-1)^k & \frac{2}{3}2^k & \frac{2}{3}5^k \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

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证：

12 推 3：$A = A^T, A^T A = I \to A^2 = I$

13 推 2：$A = A^T, A^2 = I \to A^T A = I$

23 推 1：$A^2 = I, A^T A = I \to A^{-1} = A = A^T$

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由于$e_1, \dots, e_n$是 n 维空间的一组标准正交基，则对任意x_i均可以被唯一表示为$e_1, \dots, e_n$的线性组合。

设$x_i = t_{i1}e_1 + \dots + t_{in}e_n (i = 1, \dots, k)$

必要性：

$$\left(x_i, x_j\right) = 0 \to t_{i1}t_{j1} + \dots + t_{in}t_{jn} = 0$$

$$\sum_{s=1}^{n}(x_i, e_s)(x_j, e_s) = \sum_{s=1}^{n} t_{is}t_{js} = 0$$

必要性成立。

充分性：

$$\sum_{s=1}^{n}(x_i, e_s)(x_j, e_s) = \sum_{s=1}^{n} t_{is}t_{js} = 0 \to \left(x_i, x_j\right) = 0$$

充分性成立。