第八周作业

P155-20(2)

$$\begin{pmatrix} 3 & 6 & 1 & 5 \\ 1 & 4 & -1 & 3 \\ -1 & -10 & 5 & -7 \\ 4 & -2 & 8 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & -1 & 3 \\ 3 & 6 & 1 & 5 \\ -1 & -10 & 5 & -7 \\ 4 & -2 & 8 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & -6 & 4 & -4 \\ 0 & -6 & 4 & -4 \\ 0 & -18 & 12 & -12 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & -6 & 4 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

矩阵的秩为 2，行向量的一组基为$(1,4,-1,3),(0,-6,4,-4)$

P156-23

证：

设这 r 个向量为$\boldsymbol{\alpha}_{j_1},\boldsymbol{\alpha}_{j_2},\dots,\boldsymbol{\alpha}_{j_r}$，原向量组的一个极大无关组为$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_r}$，由题意得$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_r}$可以被$\boldsymbol{\alpha}_{j_1},\boldsymbol{\alpha}_{j_2},\dots,\boldsymbol{\alpha}_{j_r}$线性表示，则有

$$r = rank\{\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_r}\} \le rank\{\boldsymbol{\alpha}_{j_1},\boldsymbol{\alpha}_{j_2},\dots,\boldsymbol{\alpha}_{j_r}\}$$

因此$rank\{\boldsymbol{\alpha}_{j_1},\boldsymbol{\alpha}_{j_2},\dots,\boldsymbol{\alpha}_{j_r}\} = r$，$\boldsymbol{\alpha}_{j_1},\boldsymbol{\alpha}_{j_2},\dots,\boldsymbol{\alpha}_{j_r}$线性无关，则$\boldsymbol{\alpha}_{j_1},\boldsymbol{\alpha}_{j_2},\dots,\boldsymbol{\alpha}_{j_r}$是原向量组的一个极大无关组。

P156-24

证：

设$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_k}$为$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_r$的极大无关组，$\boldsymbol{\beta}_{j_1},\boldsymbol{\beta}_{j_2},\dots,\boldsymbol{\beta}_{j_l}$为$\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_s$的极大无关组，则$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_r,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_s$可以由$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_k},\boldsymbol{\beta}_{j_1},\boldsymbol{\beta}_{j_2},\dots,\boldsymbol{\beta}_{j_l}$线性表出

$$\begin{aligned} rank\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_r,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_s\} &\le rank\{\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_k},\boldsymbol{\beta}_{j_1},\boldsymbol{\beta}_{j_2},\dots,\boldsymbol{\beta}_{j_l}\} \le k+l \\ &= rank\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_r\} + rank\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_s\} \end{aligned}$$

P156-27

证：

设 A, B 的列向量组为$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_n$和$\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_n$，则 A+B 的列向量组为$\boldsymbol{\alpha}_1+\boldsymbol{\beta}_1,\boldsymbol{\alpha}_2+\boldsymbol{\beta}_2,\dots,\boldsymbol{\alpha}_n+\boldsymbol{\beta}_n$

设$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_k}$为$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_n$的极大无关组，$\boldsymbol{\beta}_{j_1},\boldsymbol{\beta}_{j_2},\dots,\boldsymbol{\beta}_{j_l}$为$\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_n$的极大无关组，则$\boldsymbol{\alpha}_1+\boldsymbol{\beta}_1,\boldsymbol{\alpha}_2+\boldsymbol{\beta}_2,\dots,\boldsymbol{\alpha}_n+\boldsymbol{\beta}_n$可以由$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_k},\boldsymbol{\beta}_{j_1},\boldsymbol{\beta}_{j_2},\dots,\boldsymbol{\beta}_{j_l}$线性表出

$$\begin{aligned} rank\{\boldsymbol{\alpha}_1+\boldsymbol{\beta}_1,\boldsymbol{\alpha}_2+\boldsymbol{\beta}_2,\dots,\boldsymbol{\alpha}_n+\boldsymbol{\beta}_n\} &\le rank\{\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_k},\boldsymbol{\beta}_{j_1},\boldsymbol{\beta}_{j_2},\dots,\boldsymbol{\beta}_{j_l}\} \le k+l \\ &= rank\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_n\} + rank\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_n\} = rank(A)+rank(B) \end{aligned}$$

P156-29

证：

$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_r}$为$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_m$的极大无关组$\Leftrightarrow\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_m$中的每个元素都可以被$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_r}$线性表示$\Leftrightarrow\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_m$中的每个元素都在$\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_r}$张成的子空间中$\Leftrightarrow$ $<\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_m>=<\boldsymbol{\alpha}_{i_1},\boldsymbol{\alpha}_{i_2},\dots,\boldsymbol{\alpha}_{i_r}>$

P156-34

解：

$$\begin{pmatrix}3&6&1\\1&3&3\\0&2&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$$

$$\begin{pmatrix}3&6&1&1&0&0\\1&3&3&0&1&0\\0&2&5&0&0&1\end{pmatrix}\to\begin{pmatrix}1&3&3&0&1&0\\3&6&1&1&0&0\\0&2&5&0&0&1\end{pmatrix}\to\begin{pmatrix}1&3&3&0&1&0\\0&-3&-8&1&-3&0\\0&2&5&0&0&1\end{pmatrix}$$

$$\to\begin{pmatrix}1&3&3&0&1&0\\0&1&\frac{8}{3}&-\frac{1}{3}&1&0\\0&2&5&0&0&1\end{pmatrix}\to\begin{pmatrix}1&3&3&0&1&0\\0&1&\frac{8}{3}&-\frac{1}{3}&1&0\\0&0&-1/3&2/3&-2&1\end{pmatrix}$$

$$\to\begin{pmatrix}1&3&3&0&1&0\\0&1&\frac{8}{3}&-\frac{1}{3}&1&0\\0&0&1&-2&6&-3\end{pmatrix}\to\begin{pmatrix}1&0&0&-9&28&-15\\0&1&0&5&-15&8\\0&0&1&-2&6&-3\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3&6&1\\1&3&3\\0&2&5\end{pmatrix}^{-1}\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}-9&28&-15\\5&-15&8\\-2&6&-3\end{pmatrix}\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}-76\\41\\-16\end{pmatrix}$$

P156-35

（答案不唯一）

解：

(1)

$$\boldsymbol{\alpha}_1=(3,2,-1,4),\boldsymbol{\alpha}_2=(2,3,0,-1),\boldsymbol{\alpha}_3=(1,0,0,0),\boldsymbol{\alpha}_4=(0,1,0,0)$$

（取一组线性无关的基均可。）

(2)

$$\begin{pmatrix}3&2&1&0\\2&3&0&1\\-1&0&0&0\\4&-1&0&0\end{pmatrix}^{-1}=\begin{pmatrix}0&0&-1&0\\0&0&-4&-1\\1&0&11&2\\0&1&14&3\end{pmatrix}$$

(3)

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -14 \\ 41 \\ 53 \end{pmatrix}$$

P156-37

解：

设 A 的前 n-1 列为 A1，各列为$a_1, a_2, \dots, a_n$，去掉第 i 列得到的矩阵的行列式为d_i

假设$x_n = 0$，$rank(A) = n-1$→A 的列向量的极大无关组的元素为 n-1→不妨设极大无关组为$\{a_1, a_2, \dots, a_{n-1}\}$，则 A1 为$(n-1)\times(n-1)$满秩矩阵，$A_1\boldsymbol{x} = 0$的解为 0

$x_n \neq 0$时，设$y_i = \frac{x_i}{x_n}$，则有$a_1y_1 + a_2y_2 + \cdots + a_{n-1}y_{n-1} + a_n = 0$

由 Cramer 法则，$y_i = \frac{\det(a_1,\dots,a_{i-1},-a_{n+1},a_{i+1},\dots,a_{n-1})}{\det(a_1,a_2,\dots,a_{n-1})} = \frac{(-1)^{n-i}d_i}{d_n} = \frac{x_i}{x_n}$

则有$\frac{x_i}{(-1)^i d_i} = \frac{x_n}{(-1)^n d_n} = c$，即$x_i = (-1)^i c d_i$

因此可以得到方程组的一组基础解系为$(-d_1, d_2, \dots, (-1)^{n-1}d_{n-1})$

P157-40(2)

解：

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -4 \\ 1 & -2 & 3 & -4 & 2 \\ -1 & 3 & -5 & 7 & -4 \\ 1 & 2 & -1 & 4 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & -4 \\ 0 & -3 & 2 & -5 & 6 \\ 0 & 4 & -4 & 8 & -8 \\ 0 & 1 & -2 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & -4 \\ 0 & 1 & -2 & 3 & -2 \\ 0 & 4 & -4 & 8 & -8 \\ 0 & -3 & 2 & -5 & 6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 & -4 \\ 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & -4 \\ 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

r=3，解空间的维数为 5-3=2

设$x_4 = t_1, x_5 = t_2$，则$x_3 = t_1, x_2 = -t_1 + 2t_2, x_1 = 2t_2 - t_1$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} t_2$$

$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$和$\begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$为原方程组的基础解系。

P157-41

（答案不唯一）

解：

设方程的系数为 a，b，c，d，e

则有$\begin{cases}a+2b+3c+2d+e=0\\a+3b+2c+d+2e=0\end{cases}$

$$b=c+d-e, a=-5c-4d+e$$

因此可以构造方程组

$$\begin{cases}-5x_1+x_2+x_3=0\\-4x_1+x_2+x_4=0\\x_1-x_2+x_5=0\end{cases}$$

给出的方程组应满足的条件：5 个未知数，系数矩阵的秩为 3，基础解系的两个解均满足原方程组。