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复习.

- 连续型 r.v.
  - def.
  - 性质(pdf) (P64)
    - $f(x)\geq 0$ ★
    - $\int_{-\infty}^{\infty} f(x)dx=1$
    - $P(X\in A)=\int_A f(x)\,dx$
    - 若 $f(x)$ 在 x_0 连续. $F'(x_0)=f(x_0)$
    - $\forall x\in\mathbb{R}$, $P(X=x)=0$ ★.
  - 常见分布
    - $U(a,b)$. uniform.
    - $Exp(\lambda)$.
    - $N(\mu,\sigma^2)$. normal　熟悉变换. 会查表.
  - $X\sim f(x)$. $Y=g(X)$
    - 分布 → 由定义
    - 密度变换公式 → 绝对值
    - 逐段单调时. $(a,b)=\bigcup_{i=1}^{n}(a_i,b_i)$　f在(a_i,b_i)单调.

      $f_Y(y)=\sum_j f(h_j(y))|h_j'(y)|I_j$ → 同样变换到y对应的上面.　示性变量((a_i,b_i)对应的)

      (书P74公式写的不太清楚)

      具体操作见2.46的解答

- 多维 r.v. $X=(X_1,X_2,\dots,X_n)$
  - 离散~　$P(X=x_i,Y=y_j)=P_{ij}$　$p_{ij}\geq 0$　$\sum p_{ij}=1$.
  - 连续~
    - def
    - 性质(4条)
  - 边际分布. (X,Y) $F(x,y)$　$F_1(x)=\lim_{y\to+\infty}F(x,y)$.

    $f_1(x)=\int_{\mathbb{R}} f(x,y)\,dy$.
  - 条件分布.
    - $f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_2(y)}$　$f_2(y)>0$
    - $(X,Y)\sim N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$　$(Y|X=x)\sim N\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),(1-\rho^2)\sigma_2^2\right)$
  - 独立性.
    - $X_1,\dots X_n$ 相互独立. $\Leftrightarrow$ $F(x_1,x_2,\dots,x_n)=F_1(x_1)F_2(x_2)\cdots F_n(x_n)$

- 随机向量的函数的分布　$Z=g(X,Y)$.　$Z=(Z_1,Z_2)=(g_1(X,Y),g_2(X,Y))$
    - 利用定义 $P(Z\in A)=\iint_{g(x,y)\in A} f(x,y)\,dx\,dy$　$P(Z\in A)=\iint_{((g_1,g_2)\in A)} f(x,y)\,dx\,dy$.
    - 公式. $X=(X_1,\dots X_n)\sim f(x_1,\dots x_n)$. $g:\mathbb{R}^n\to\mathbb{R}^n$ 双射. g^{-1} 存在一阶偏导
      $\Rightarrow g(X)=Y$ 连续 $p(y)=f(g^{-1}(y))|J|$　$J=\left|\frac{\partial g^{-1}}{\partial(y_1,\dots y_n)}\right|$

注. $\mathbb{R}^2\to\mathbb{R}^2$. $(X,Y)\sim f(x,y)$. $(U,V)=(g_1(X,Y),g_2(X,Y))=g(X,Y)$.

① $|J|=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}=\begin{vmatrix}\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y}\\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y}\end{vmatrix}^{-1}$

② 写出 $X=\varphi_1(U,V)$　$Y=\varphi_2(U,V)$.
$f(u,v)=f(\varphi_1,\varphi_2)\,|J|$

- 常考的
    - 和/差. $f_{X+Y}(z)=\int_{-\infty}^{+\infty} f(u,z-u)\,du$.
    - 商/积. $f_{\frac{X}{Y}}(z)=\int_{-\infty}^{+\infty}|v|f(zv,v)\,dv$.
    - 最大值/最小值 — 利用定义.
- 结论.
    - $X_1\dots X_n \overset{iid}{\sim} \mathrm{Exp}(\lambda)\Rightarrow Z=X_1+X_2+\dots+X_n\sim\Gamma(n,\lambda)/Ga(n,\lambda)$

      $X_1\dots X_n$ 独立.
    - $X_1,\dots X_n\sim N(\mu_i,\sigma_i^2)\Rightarrow Z=X_1+\dots+X_n\sim N(\sum_{i=1}^n\mu_i,\sum_{i=1}^n\sigma_i^2)$.

      $X_1\sim N(\mu_1,\sigma_1^2)\Rightarrow cX_1\sim N(c\mu_1,c^2\sigma_1^2)$

HW 4

20. 利用分布函数性质.

$P(1<X<2)=\int_1^2 f(x)\,dx=\frac{1}{2}ax^2\Big|_1^2=\frac{3}{2}a$

$P(2<X<3)=\int_2^3 f(x)\,dx=bx\Big|_2^3=b$

由题. $\frac{3}{2}a=b$.

又 $\because \int_{-\infty}^{+\infty} f(x)\,dx=1$.

$\therefore \frac{3}{2}a+b=1$

$\therefore a=\frac{1}{3} \quad b=\frac{1}{2}$

24. 设X密度函数为 $f(x)$

P(方程有实根) $=P(\Delta=X^2-4\geqslant 0)$

有同学,

$\Delta=X^2-4\geqslant 0 \Rightarrow X^2\leq 4$ why?

$=P(X\geqslant 2$ 或 $X\leq -2)$

$=\left(\int_{-\infty}^{-2}+\int_{2}^{+\infty}\right)f(x)\,dx$

$X\sim U(-5,5) \quad f(x)=\begin{cases}\frac{1}{10} & -5<X<5\\ 0 & \text{else}\end{cases}$

$\therefore$ 上式 $=\frac{3}{5}$

仔细用 $\Phi(x)+\Phi(-x)=1$ 表明只有 $x>0$ 的值.

31. $X\sim N(1,4)$

$Y=\frac{X-1}{2}\sim N(0,1)$

(1) $P(0\leq X\leq 4)=P(-\frac{1}{2}\leq Y\leq \frac{3}{2})$.

$=\Phi(\frac{3}{2})-\Phi(-\frac{1}{2})$.

$=\Phi(\frac{3}{2})-(1-\Phi(\frac{1}{2}))$

$=\Phi(\frac{3}{2})+\Phi(\frac{1}{2})-1$

$=0.9332+0.6915-1$

$P(X>2.4)=P(Y>0.7)$

$=1-\Phi(0.7)=1-0.7580$

$P(|X|>2)=P(X>2$ 或 $X<-2)$ $\neq 2(P(X\leq 2)-\frac{1}{2})$.

$=P(Y>\frac{1}{2})+P(Y<-\frac{3}{2})$ $\neq 2P(X>2)$.

$=1-\Phi(\frac{1}{2})+\Phi(-\frac{3}{2})$

$=2-\Phi(\frac{1}{2})-\Phi(\frac{3}{2})=2-0.6915-0.9332$.

(2) $P(X>c)=P(Y>\frac{c-1}{2})=1-\Phi(\frac{c-1}{2})$

$P(X\leq c)=P(Y<\frac{c-1}{2})=\Phi(\frac{c-1}{2})$.

$\because P(X>c)=2P(X\leq c)$

$\therefore 1=3\Phi(\frac{c-1}{2})$

$\therefore \Phi(\frac{c-1}{2})=\frac{1}{3}=0.333$

$1-\Phi(\frac{c-1}{2})=\Phi(\frac{1-c}{2})\approx 0.66667 \quad \Phi(0.43)=0.6664$

$\therefore \frac{1-c}{2}\approx 0.43$

$c\approx 0.14$.

36.

(1)

Y_1	3	1	-1	-3
P	0.2	0.3	0.1	0.4

(2)

Y_2	0	1	2
P	0.3	0.3	0.4

(3)

Y_3	0	1	4
P	0.1	0.7	0.2

37.

(1) $F(-\infty) = a - b\frac{\pi}{2} = 0$

$F(+\infty) = a + b\cdot\frac{\pi}{2} = 1$

$\therefore a = \frac{1}{2} \quad b = \frac{1}{\pi}$

(2) $y = g(x) = 3 - \sqrt[3]{x}$ 单调(在 $(-\infty, +\infty)$)

$x = h(y) = g^{-1}(y) = (3-y)^3$

$h'(y) = -3(3-y)^2$

设 X, Y 的密度函数分别为 $f_X(x), f_Y(y)$

$f_X(x) = F'(x) = \frac{1}{\pi}\frac{1}{x^2+1}$

$\therefore f_Y(y) = f_X(h(y))\,|h'(y)|$ → 绝对值.

$= \frac{1}{\pi}\frac{1}{(y-3)^6+1}\,3(y-3)^2$

$= \frac{3}{\pi}\frac{(y-3)^2}{(y-3)^6+1}$

(3) 法1. $y = g_2(x) = \frac{1}{x}$ 在 $(-\infty, 0)$ 和 $(0, +\infty)$ 单减

$h_2(y) = \frac{1}{y} \qquad h_2'(y) = -\frac{1}{y^2}$

$\therefore f_{\frac{1}{X}}(y) = f_X(h_2(y))\,|h_2'(y)| + f_X(h_2(y))\,|h_2'(y)|$

$= \frac{2}{\pi}\frac{1}{(\frac{1}{y})^2+1}\frac{1}{y^2} = \frac{2}{\pi}\frac{1}{1+y^2}$

按定义

法2. 设 $\frac{1}{X}$ 分布函数为 $F_2(y)$

$F_2(y) = P(\frac{1}{X} \leq y)$

$y = 0$ 时.

$F_2(y) = P(X \leq 0) = F(0) = \frac{1}{2}$

$y > 0$ 时 $F_2(y) = P(X \leq 0 或 X \geq \frac{1}{y})$.

$= \frac{1}{2} + 1 - (\frac{1}{2} + \frac{1}{\pi}\arctan\frac{1}{y})$

$= 1 - \frac{1}{\pi}\arctan\frac{1}{y}$

$= 1 - \frac{1}{\pi}(\frac{\pi}{2} - \arctan y)$

40. $X \sim U(0,1)$. ① 不是 $N(0,1)$!

$f(x) = I(0<x<1)$.

② cdf. pdf 写清范围或用示性变量.

③ 代入时易错 $f(x)=1 \quad \forall\, 0<x<1$.

④ 绝对值! $f(x) \geq 0$.

(1) $X = \ln Y_1 \quad (1<Y_1<e)$

$h_1(y)=\ln y \quad h_1'(y) = \frac{1}{y}$

$\therefore f_{Y_1}(y) = f(\ln y)\left|\frac{1}{y}\right| = \frac{1}{y} I(1<y<e)$

(2) $X = \frac{1}{Y_2} \quad Y_2>1$

$h_2(y) = \frac{1}{y} \quad h_2'(y) = -\frac{1}{y^2}$

$f_{Y_2}(y) = f(\frac{1}{y})\frac{1}{y^2} = \frac{1}{y^2} I(y>1)$

(3) $Y_3 = -\frac{1}{\lambda}\ln X \quad (Y_3>0)$

$X = e^{-\lambda Y_3}$

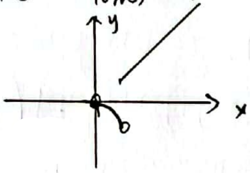
$h_3(y) = e^{-\lambda y} \quad h_3'(y) = -\lambda e^{-\lambda y}$

$\therefore f_{Y_3}(y) = \lambda e^{-\lambda y} \ I(y>0)$.

46. $X \sim Exp(\lambda)$. $f(x) = \lambda e^{-\lambda x} I_{[0,\infty)}(x)$.

$x>0$!

$Y = \begin{cases} X & X\geq 1 \\ -X^2 & X<1. \end{cases}$



先看Y的范围.
别急着算.

法1.

$(-\infty,+\infty)$ "=" $(-\infty,0)\cup(0,1)\cup(1,+\infty)$.

$h_1(y) = \ x=-\sqrt{-y} \qquad h_2(y) \ x=\sqrt{-y} \qquad h_3(y) \ x=y$.

$\frac{\partial x}{\partial y} = \frac{+1}{2\sqrt{-y}} \qquad \frac{\partial x}{\partial y} = \frac{-1}{2\sqrt{-y}} \qquad \frac{\partial x}{\partial y} = 1$

$f(x) = f_1(x) I_{(-\infty,0)}(x) + f_2(x) I_{(0,1)}(x) + f_2(x) I_{(1,\infty)}(x)$

$f_1(x) = 0 \qquad f_2(x) = \lambda e^{-\lambda x}$.

$\therefore p_Y(y) = f_1(h_1(y))\left|\frac{1}{2\sqrt{-y}}\right| I(x=-\sqrt{-y}<0) + f_2(h_2(y))\left|\frac{-1}{2\sqrt{-y}}\right| I(0<x=\sqrt{-y}<1) + f_2(h_3(y))\cdot 1 \cdot I(x=y>1)$.

$= 0 + \lambda e^{-\lambda\sqrt{-y}} \frac{1}{2\sqrt{-y}} I(-1<y<0) + \lambda e^{-\lambda y} I(y>1)$

$= \frac{\lambda}{2\sqrt{-y}} e^{-\lambda\sqrt{-y}} I(-1<y<0) + \lambda e^{-\lambda y} I(y>1)$.

法2. 先求 $F_Y(y)$. Y 取值为 $(-1,0)\cup(1,+\infty)$.

① $-1<y<0$ $\quad F_Y(y) = P(Y\leq y) = P(-X^2\leq y). = P(0<X\leq\sqrt{-y}) = \int_0^{\sqrt{-y}} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^{\sqrt{-y}}$
$= 1-e^{-\lambda\sqrt{-y}}$.

② $y\geq 1$ $\quad F_Y(y) = P(Y\leq y) = P(X<1)+P(1\leq X\leq y) = P(X\leq y) = \int_0^y \lambda e^{-\lambda x} dx = 1-e^{-\lambda x}$

$\therefore f_Y(y) = \begin{cases} \frac{\lambda}{2\sqrt{-y}} e^{-\lambda\sqrt{-y}} & (-1<y<0). \\ \lambda e^{-\lambda y} & (y\geq 1) \end{cases}$

HW6.

8. $\because \rho = 0$

$\therefore$ X、Y 独立.

$\therefore X \sim N(1,1) \quad Y \sim N(0,1)$

$P(XY-Y<0) = P((X-1)Y<0)$

$= P(X-1<0, Y>0) + P(X-1>0\ Y<0)$

$\overset{X、Y独立}{=} P(X-1<0)P(Y>0) + P(X-1>0)P(Y<0)$

$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

15. $(X,Y) \sim N(0,0,\sigma_1^2,\sigma_2^2,0)$

$X-Y \sim N(0,\sigma_1^2+\sigma_2^2)$. ＊结论

$\therefore P(Z=1) = P(X\le Y) = P(X-Y\le 0) = \frac{1}{2}$

$P(Z=0) = P(X>Y) = P(X-Y>0) = \frac{1}{2}$

$Z \sim B(1,\frac{1}{2}) \quad Z \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

26. (1) $P(Z\le\frac{1}{2}|X=0) = P(X+Y\le\frac{1}{2}|X=0)$

$= P(Y\le\frac{1}{2}|X=0)$

$\overset{X、Y独立}{=} P(Y\le\frac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f_Y(y) = \frac{1}{2}$

(2) $-1\le z\le 2$.

① $-1\le z\le 0$.

$F(z) = P(Z\le z) = P(X=-1)P(Y\le z+1|X=-1)$.

$= \frac{1}{3}\cdot(z+1)$

② $0\le z\le 1$

$F(z) = P(Z\le z) = F(0) + P(0\le Z\le z)$

$= \frac{1}{3} + P(0\le X+Y\le z\,|\,X=0)\,P(X=0)$

$= \frac{1}{3} + z\cdot\frac{1}{3} = \frac{1}{3}(z+1)$

③ $1\le z\le 2$.

$F(z) = P(Z\le z) = F(1) + P(1\le Z\le z)$

$= \frac{2}{3} + P(1\le X+Y\le z\,|\,X=1)\,P(X=1)$

$= \frac{2}{3} + (z-1)\frac{1}{3} = \frac{1}{3}(z+1)$

$\therefore F(z) = \frac{1}{3}(z+1) \quad (-1\le z\le 2)$

$f(z) = \begin{cases} \frac{1}{3} & (-1\le z\le 2) \\ 0 & (\text{else}) \end{cases}$

29. $P(X\cdot Y \le z)$

① $z<0$.

$P(X\cdot Y\le z) = P(Y=1)\,P(X\le z)$.

② $z \geqslant 0$.

$P(X\cdot Y\le z) = P(Y=0)\cdot 1 + P(Y=1)\,P(X\le z)$.

$X \sim N(u, \sigma^2)$.

$\frac{X-u}{\sigma} \sim N(0,1)$

$\therefore P(X\le z) = P\left(\frac{X-u}{\sigma} \le \frac{z-u}{\sigma}\right) = \Phi\left(\frac{z-u}{\sigma}\right)$

$\therefore F_{XY}(z) = \begin{cases} p\cdot\Phi\left(\frac{z-u}{\sigma}\right) & (z<0) \\ 1-p+p\,\Phi\left(\frac{z-u}{\sigma}\right). & (z\geqslant 0.) \end{cases}$ 没有pdf.

r.v $X \sim F(x)$, 若存在 $f(x)\geqslant 0$.
s.t $\forall x\in\mathbb{R}$ $F(x)=\int_{-\infty}^{x} f(t)\,dt$.
则称 X 为连续型 r.v. $f(x)$ 称为分布函数的pdf.

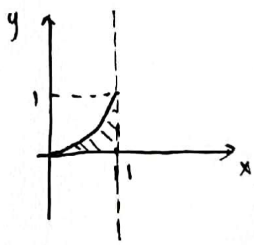
30、(1) $\because$ X. Y 独立

$\therefore f(x,y) = f_X(x)\cdot f_Y(y)$

$= I_{(0,1)}(x)\cdot \frac{1}{2}e^{-\frac{y}{2}}\, I_{(0,+\infty)}(y)$

$\therefore f(x,y) = \begin{cases} \frac{1}{2}e^{-\frac{y}{2}} & 0<x<1,\ y>0 \\ 0 & \text{else}. \end{cases}$

(2) $\Delta = 4X^2 - 4Y$



$P(\text{方程有实根}) = P(\Delta\geqslant 0) = P(X^2 - Y \geqslant 0)$.

$= \iint\limits_{x^2\geqslant y} \frac{1}{2}e^{-\frac{y}{2}}\, I_{(0,1)}(x)\, I_{(0,+\infty)}(y)\, dx\, dy$

$= \int_0^1 \left(\int_0^{x^2} \frac{1}{2}e^{-\frac{y}{2}}\, dy\right) dx$

$= \int_0^1 -e^{-\frac{y}{2}}\Big|_0^{x^2}\, dx$

$= \int_0^1 1 - e^{-\frac{x^2}{2}}\, dx$

$= 1 - \int_0^1 e^{-\frac{x^2}{2}}\, dx$

$\int_0^1 e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}\left(\Phi(1) - \Phi(0)\right) = \sqrt{2\pi}\,(0.8413 - 0.5)$

≈ 0.1445

32. (1) 下表中值为 $P(X\cdot Y=\ )$

X\Y	-1	0	1	
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

← 由题干可得.

$\because P(XY=0)=1$

$\therefore P(XY=1)=0 \quad P(X\cdot Y=-1)=0.$

① $P(Y=-1)=P(X=0,Y=-1)+P(X=1,Y=-1)$

$\therefore \quad \frac{1}{4} = P(X=0,Y=-1)+0.$

② $P(Y=1)=P(X=0,Y=1)+P(X=1,Y=-1)$

$\frac{1}{4} = P(X=0,Y=1) + 0$

③ $P(X=1)=P(X=1,Y=-1)+P(X=1,Y=0)+P(X=1,Y=1)$

$\frac{1}{2} = \quad 0 \quad + P(X=1,Y=0) + 0$

④ $P(Y=0)=P(X=0,Y=0)+P(X=1,Y=0)$

$\frac{1}{2} = P(X=0,Y=0)+\frac{1}{2}.$

X.Y分布律.

X	Y		
	-1	0	1
0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	0	$\frac{1}{2}$	0

(2) $P(X\cdot Y=0)=1 \neq P(X=0)P(Y=0)$

$\therefore$ X, Y 不独立.

41. (1) $F_X(x)=\lim_{y\to\infty}F(x,y)=\lim_{y\to\infty}\dfrac{1-(x+1)e^{-x}}{\frac{1}{y}+1} = 1-(x+1)e^{-x}$

$F_Y(y)=\lim_{x\to\infty}F(x,y)=\lim_{x\to\infty}\dfrac{y}{1+y}-(x+1)e^{-x}\dfrac{y}{1+y} = \dfrac{y}{1+y}$

(2) $\dfrac{\partial F}{\partial x} = -\dfrac{y}{1+y}\left(e^{-x}-(x+1)e^{-x}\right) = \dfrac{y}{1+y}x\cdot e^{-x} = \left(1-\dfrac{1}{1+y}\right)x\cdot e^{-x}$

$\dfrac{\partial^2 F}{\partial x\partial y} = xe^{-x}\dfrac{1}{(1+y)^2} \qquad \therefore f(x,y)=\begin{cases}\dfrac{xe^{-x}}{(1+y)^2} & (x>0,y>0).\\ 0 & (\text{else})\end{cases}$

$f_X(x)=F_X'(x)=(x+1)e^{-x}-e^{-x}=x\cdot e^{-x}\ (x>0)$

$f_Y(y)=F_Y'(y)=\dfrac{1}{(1+y)^2} \quad y>0.$

(3) $f(x,y)=f_X(x)\cdot f_Y(y)$ X, Y 相互独立

42. 证明: $f(x,y)=\int_{-\infty}^{+\infty} f(x,y,z)\,dz \qquad 0\le x,y\le 2\pi.$

$$=\int_0^{2\pi}\frac{1}{8\pi^3}(1-\sin x\sin y\sin z)\,dz.$$

$$=\frac{2\pi-0}{8\pi^3}=\frac{1}{4\pi^2}$$

同理 $f(y,z)=f(x,z)=\frac{1}{4\pi^2}$

$f_X(x)=\int_0^{2\pi} f(x,y)\,dy=\int_0^{2\pi}\frac{1}{4\pi^2}=\frac{1}{2\pi}.$

同理 $f_Y(y)=f_Z(z)=\frac{1}{2\pi}.$

$\therefore\ f(x,y)=f_X(x)\cdot f_Y(y) \qquad f(x,z)=f_X(x)\cdot f_Z(z) \qquad f(y,z)=f_Y(y)\cdot f_Z(z).$

X,Y,Z 两两独立.

但 $f(x,y,z)\neq f(x)\cdot f(y)\cdot f(z)$

故 X,Y,Z 不相互独立.

34. 把 Z_n 限制在 $\{0,1,\dots,k-1\}$ 上考虑. $Z_n=X+nY$ 模 k 的余数.

$P(Z_n=z)=\sum_{i=0}^{k-1}P(Z_n=z\mid Y=i)\,P(Y=i)$

$P(Z_n=z\mid Y=i)=\frac{1}{k}.$ (固定 Y 时, $X\equiv z-ni\pmod k$. X 有解且解唯一)

$\therefore\ P(Z_n=z)=\sum_{i=0}^{k-1}\frac{1}{k}\cdot\frac{1}{k}=\frac{1}{k}$

$\therefore$ Z_n 服从 $\{0,1,\dots,k-1\}$ 上均匀分布.

讨论 k

$k=1$ 时. $P(X=0)=P(Y=0)=1 \qquad P(Z_n=0)=1 \qquad Z_n$ 退化为单点分布.

$\therefore$ Z_n 相互独立且两两独立.

$k\ge 2$ 时. 先考虑 Z_n. $n=0$ …… 的独立性.

$\forall n.\quad P(Z_n=0,\ Z_{n+1}=0,\ Z_{n+2}=1)=0$

$$\left.\begin{cases} X+nY\equiv 0\pmod k \\ X+(n+1)Y\equiv 0\pmod k \\ X+(n+2)Y\equiv 1\pmod k\end{cases}\right. \Rightarrow Y\equiv 0\pmod k,\quad \Rightarrow Y\equiv 1\pmod k.$$

$\therefore$ 方程组无解 $Z_n=0.\ Z_{n+1}=0\ Z_{n+2}=1$ 不可能成立

$\therefore P(Z_n=0,\ Z_{n+1}=0,\ Z_{n+2}=1)=0\neq P(Z_n=0)\,P(Z_{n+1}=0)\,P(Z_{n+2}=1)$

$\therefore$ Z_n 不可能相互独立.

下面考虑 Z_n 的两两独立性.

假设 Z_a 与 Z_b 独立. $(a \neq b)$

则 $P(Z_a=c, Z_b=d) = P(Z_a=c)\cdot P(Z_b=d). = \frac{1}{k^2}$

$$\begin{cases} Z_a = c \\ Z_b = d \end{cases} \Leftrightarrow \begin{cases} X + aY \equiv c \pmod{k} \\ X + bY \equiv d \pmod{k} \end{cases} \Rightarrow (a-b)Y \equiv (c-d) \pmod{k}$$

注. $ax \equiv b \pmod{k}$ $(k \nmid a)$ 称为一次同余方程.

greatest common divisor ①若 $\gcd(a,k)=1$ (即. a 与 k 互素) 则方程有且只有一个解.

$a \mid b$: b 能被 a 整除. ②若 $\gcd(a,k)=d>1$ (即 $a.k$ 的最大公约数为 d)

1° $d \nmid b$ 则方程无解

2° $d \mid b$. 则方程有 d 个解.

∴ 回到题目中.

$a-b$ 与 k 互素时. Y 有唯一解. 记为 y_0.

此时 $X + aY \equiv c \pmod{k} \Rightarrow X \equiv c - ay_0 \pmod{k}$ 也有唯一解. 记为 x_0.

验证 $x_0. y_0$ 是 $X + bY \equiv d \pmod{k}$ 的解.

$$\begin{cases} x_0 + ay_0 \equiv c \pmod{k} \\ (a-b)y_0 \equiv c-d \pmod{k} \end{cases} \Rightarrow x_0 + by_0 \equiv d \pmod{k}$$

∴ (x_0, y_0) 是方程组 $\begin{cases} Z_a = c \\ Z_b = d \end{cases}$ 的解. 且为唯一解.

∴ $P(Z_a=c, Z_b=d) = P(X=x_0)P(Y=y_0) = \frac{1}{k^2} = P(Z_a=c)P(Z_b=d)$

Z_a 与 Z_b 独立.

若 $a-b$ 不与 k 互素. Y 有多解或无解. 都会使 $P(Z_a=c, Z_b=d) \neq \frac{1}{k^2}$

∴ Z_a 与 Z_b 不独立.

若(a–b)与k互素，则Za与Zb独立
但总是存在Zn与Znk不独立，所以Zn也不是两两独立的

综上. $k=1$ 时. Z_n 两两独立且相互独立.

$k \geq 2$ 时. Zn不两两独立也不相互独立

注. 两两独立 $\Leftrightarrow$ 任两个相互独立.

感觉题目应该加上条件，讨论Z0,Z1…Zk–1的独立性。
这时，若k为素数，则Z0,Z1…Zk–1两两独立