HW8.

(1). $EX = \int_0^{+\infty} x \cdot f(x)\, dx$

$= \int_0^{+\infty} \frac{x^2}{\sigma^2} \exp\{-\frac{x^2}{2\sigma^2}\}\, dx$

$= \int_0^{+\infty} -x \cdot d e^{-\frac{x^2}{2\sigma^2}}$

$= -x \cdot e^{-\frac{x^2}{2\sigma^2}} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-\frac{x^2}{2\sigma^2}}\, dx$

$= 0 + \frac{\sqrt{2\pi}\sigma}{2} = \frac{\sqrt{2\pi}\sigma}{2}$

$EX^2 = \int_0^{+\infty} x^2 f(x)\, dx$

$= \int_0^{+\infty} \frac{x^3}{\sigma^2} \exp\{-\frac{x^2}{2\sigma^2}\}\, dx$

$= \int_0^{+\infty} -x^2\, d e^{-\frac{x^2}{2\sigma^2}}$

$= -x^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_0^{+\infty} + \int_0^{+\infty} 2x\, e^{-\frac{x^2}{2\sigma^2}}\, dx$

$= -2\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_0^{+\infty} = 2\sigma^2$

$\therefore \operatorname{Var} X = EX^2 - (EX)^2 = 2\sigma^2 - \frac{\pi\sigma^2}{2} = (2-\frac{\pi}{2})\sigma^2$

(2) $EX = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha} (1-x)^{\beta-1}\, dx$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} B(\alpha+1, \beta)$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}$

同理. $EX^2 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$

$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$

$\therefore \operatorname{Var} X = EX^2 - (EX)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

注. $\Gamma(n) = (n-1)!$

$\Gamma(x+1) = x\Gamma(x)$.

$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$

$B(\alpha,\beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$

(3) $EX = \int_0^{+\infty} \frac{kx}{\lambda} (\frac{x}{\lambda})^{k-1} \exp\{-(\frac{x}{\lambda})^k\}\, dx$

$= \int_0^{+\infty} x \cdot \frac{kx^{k-1}}{\lambda^k} \exp\{-(\frac{x}{\lambda})^k\}\, dx$

$= \lambda \int_0^{+\infty} \frac{x}{\lambda} \exp\{-(\frac{x}{\lambda})^k\}\, d(\frac{x}{\lambda})^k$.

$y = (\frac{x}{\lambda})^k$

$= \lambda \int_0^{+\infty} y^{\frac{1}{k}} e^{-y}\, dy$

$= \lambda\Gamma(\frac{1}{k}+1) = \frac{\lambda}{k}\Gamma(\frac{1}{k})$.

$EX^2 = \int_0^{+\infty} x^2 \frac{kx^{k-1}}{\lambda^k} \exp\{-(\frac{x}{\lambda})^k\}\, dx$

$= \lambda^2 \int_0^{+\infty} (\frac{x}{\lambda})^2 \exp\{-(\frac{x}{\lambda})^k\}\, d(\frac{x}{\lambda})^k$

$y = (\frac{x}{\lambda})^k$

$= \lambda^2 \int_0^{+\infty} y^{\frac{2}{k}} e^{-y}\, dy$

$= \lambda^2 \Gamma(\frac{2}{k}+1) = \frac{2\lambda^2}{k}\Gamma(\frac{2}{k})$

$\operatorname{Var} X = EX^2 - (EX)^2 = \frac{2\lambda^2}{k}\Gamma(\frac{2}{k}) - \frac{\lambda^2}{k^2}\Gamma(\frac{1}{k})$

14. 求 pdf.

法1. 公式.

$Y = \ln X \sim N(\mu, \sigma^2)$.

$\therefore X = e^Y \triangleq g(Y) \qquad Y = h(X) = \ln X$.

$h'(x) = \frac{1}{x}$

$\therefore p_X(x) = p_Y(h(x))|h'(x)|$

$= \frac{1}{\sqrt{2\pi}\sigma x} \exp\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\} \quad x>0$

$EX = \int_0^{+\infty} x \cdot p(x)\, dx$

$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(\ln x - u)^2}{2\sigma^2}\} dx$

令 $y = \frac{\ln x - u}{\sigma}$. $x = e^{\sigma y + u}$ $dx = \sigma e^{\sigma y + u} dy$

$\therefore EX = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \sigma e^{\sigma y + u} \cdot e^{-\frac{y^2}{2}} dy$

$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2 - 2\sigma y + \sigma^2}{2} + \frac{\sigma^2}{2} + u\} dy$

$= (\underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{(y-\sigma)^2}{2}\} dy}_{\text{正态密度 } N(\sigma,1)}) \exp\{\frac{\sigma^2}{2} + u\}$

$= e^{\frac{\sigma^2}{2} + u}$.

$EX^2 = \int_0^{+\infty} x^2 p(x)\, dx$

$= \int_0^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} \exp\{-\frac{(\ln x - u)^2}{2\sigma^2}\} dx$

$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{\sigma y + u + \sigma y + u - \frac{y^2}{2}\} dy$

$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2 - 4\sigma y + 4\sigma^2}{2} + 2\sigma^2 + 2u\} dy$

$= \exp\{2\sigma^2 + 2u\} \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{(y-2\sigma)^2}{2}\} dy$

$= e^{2\sigma^2 + 2u}$.

$\therefore VarX = EX^2 - (EX)^2 = e^{2\sigma^2 + 2u} - e^{\sigma^2 + 2u}$.

$= e^{\sigma^2 + 2u}(e^{\sigma^2} - 1)$.

18. (1) $0 < Y < 2$. $P(Y \le y) = P(Y \le y \mid X=1) P(X=1) + P(Y \le y \mid X=2) P(X=2)$.

① $0 < y \le 1$

$P(Y \le y) = y \cdot \frac{1}{2} + \frac{y}{2} \cdot \frac{1}{2} = \frac{3}{4} y$

易错: $F(y) = \begin{cases} y & X=1 \\ \frac{y}{2} & X=2. \end{cases}$ ×.

X是随机变量.

② $1 < y \le 2$

$P(Y \le y) = 1 \cdot \frac{1}{2} + \frac{y}{2} \cdot \frac{1}{2} = \frac{y}{4} + \frac{1}{2}$

$$F(y) = \begin{cases} 0 & y \le 0 \\ \frac{3}{4} y & 0 < y \le 1 \\ \frac{y}{4} + \frac{1}{2} & 1 < y \le 2 \\ 1 & y > 2. \end{cases}$$

(2) 法1. $p(y) = \begin{cases} \frac{3}{4} & 0 < y \le 1 \\ \frac{1}{4} & 1 < y \le 2 \\ 0 & \text{else}. \end{cases}$

$EY = \int_{-\infty}^{+\infty} y \cdot p(y)\, dy = \int_0^1 \frac{3}{4} y\, dy + \int_1^2 \frac{y}{4}\, dy$

$= \frac{3}{8} y^2 \Big|_0^1 + \frac{y^2}{8} \Big|_1^2 = \frac{3}{4}$.

法2. $EY = E[E[Y|X]]$

条件期望. $E[Y|X=1] = \frac{1}{2}$ $E[Y|X=2] = 1$

$EY = E[Y|X=1] P(X=1) + E[Y|X=2] P(X=2) = \frac{3}{4}$.

21. (1) $X \sim Poi(\lambda)$. $\Rightarrow P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$.

注. 有再生性的分布

① $B(n,p)$

② $Poi(\lambda)$

③ $N(u,\sigma^2)$.

$Y \sim Poi(u)$.

由Poisson分布再生性. $X+Y \sim Poi(\lambda+u)$.

$$P(X=k|X+Y=m) = \frac{P(X=k, X+Y=m)}{P(X+Y=m)} = \frac{P(X=k)P(Y=m-k)}{P(X+Y=m)}$$

$$= \frac{\frac{\lambda^k}{k!}e^{-\lambda} \cdot \frac{u^{m-k}}{(m-k)!}e^{-u}}{\frac{(\lambda+u)^m}{m!}e^{-(\lambda+u)}} = \frac{m!}{k!(m-k)!}\frac{\lambda^k u^{m-k}}{(\lambda+u)^m}$$

$$= \binom{m}{k}\left(\frac{\lambda}{\lambda+u}\right)^k\left(\frac{u}{\lambda+u}\right)^{m-k} \quad \sim B(m, \frac{\lambda}{\lambda+u})$$

$$\therefore E[X|X+Y=m] = m \cdot \frac{\lambda}{\lambda+u} = \frac{\lambda m}{\lambda+u}.$$

硬算.

$$E[X|X+Y=m] = \sum_{k=0}^{m} k \cdot P(X=k|X+Y=m) = \sum_{k=1}^{m} k P(X=k|X+Y=m)$$

$$= \sum_{k=1}^{m}\frac{k \cdot m!}{k!(m-k)!}\frac{\lambda^k u^{m-k}}{(\lambda+u)^m} = \frac{\lambda}{(\lambda+u)^m}\sum_{k=1}^{m}\frac{m(m-1)!}{(k-1)!(m-k)!}\lambda^{k-1}u^{m-k}$$

$$= \frac{\lambda m}{(\lambda+u)^m}\sum_{k=0}^{m-1}\frac{(m-1)!}{k!(m-1-k)!}\lambda^k u^{m-1-k} = \frac{\lambda m}{(\lambda+u)^m}(\lambda+u)^{m-1} = \frac{\lambda m}{\lambda+u}$$

(2) $X, Y \sim B(n,p)$

$X+Y \sim B(2n,p)$

$$\therefore P(X=k|X+Y=m) = \frac{P(X=k)P(Y=m-k)}{P(X+Y=m)}$$

$$= \frac{\binom{n}{k}p^k(1-p)^{n-k}\binom{n}{m-k}p^{m-k}(1-p)^{n-m+k}}{\binom{2n}{m}p^m(1-p)^{2n-m}} = \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

$\therefore X|X+Y=m \sim H(m,n,2n)$ 超几何分布 Hypergeometric. Distribution.

$$E[X|X+Y=m] = m \cdot \frac{n}{2n} = \frac{m}{2}$$

硬算. $$E[X|X+Y=m] = \sum_{k=0}^{m}\frac{k\cdot\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}} = \frac{1}{\binom{2n}{m}}\sum_{k=1}^{m} k \cdot \frac{n!}{k!(n-k)!}\frac{n!}{(m-k)!(n-m+k)!}$$

$$= \frac{n}{\binom{2n}{m}}\sum_{k=1}^{m}\frac{(n-1)!}{(k-1)!(n-k)!}\frac{n!}{(m-k)!(n-m+k)!} \overset{i=k-1}{=} \frac{n}{\binom{2n}{m}}\sum_{i=0}^{m-1}\frac{(n-1)!}{i!(n-i-1)!}\frac{n!}{(m-i-1)!(n-m+i+1)!}$$

$$= \frac{n}{\binom{2n}{m}}\sum_{i=0}^{m-1}\binom{n-1}{i}\binom{n}{m-i-1} = \frac{n}{\binom{2n}{m}}\binom{2n-1}{m-1} = \frac{m}{2}$$

22.

(1) $P(Y\le y)=\sum_{k=0}^{2}P(Y\le y\mid X=k)P(X=k)$

$=\frac{1}{3}(\Phi(y)+\Phi(y-1)+\Phi(y-2))$

$\therefore F(y)=\frac{1}{3}[\Phi(y)+\Phi(y-1)+\Phi(y-2)]$

$f_Y(y)=\frac{1}{3\sqrt{2\pi}}\left(e^{-\frac{y^2}{2}}+e^{-\frac{(y-1)^2}{2}}+e^{-\frac{(y-2)^2}{2}}\right)$

$EY=\int_{-\infty}^{+\infty}y\cdot f_Y(y)\,dy=\frac{1}{3}\left[\int_{-\infty}^{+\infty}y\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy+\int_{-\infty}^{+\infty}y\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-1)^2}{2}}dy+\int_{-\infty}^{+\infty}y\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-2)^2}{2}}\right]$

$=\frac{1}{3}(0+1+2)=1$

或条件期望 $EY=E[E[Y|X]]$

(2). $F_{X+Y}(z)=P(X+Y\le z)=\sum_{k=0}^{2}P(X+Y\le z\mid X=k)P(X=k)$

$\stackrel{*}{=}\frac{1}{3}\sum_{k=0}^{2}P(Y\le z-k\mid X=k)\neq\frac{1}{3}\sum_{k=0}^{2}P(Y\le z-k)$. X与Y不独立

$Y|X\sim N(1,1)\sim\Phi(x-1)$

$=\frac{1}{3}[P(Y\le z|X=0)+P(\underset{\sim\sim}{Y\le z-1}|X=1)+P(Y\le z-2|X=2)]$

$=\frac{1}{3}[\Phi(z)+\Phi(z-1-1)+\Phi(z-2-2)]=\frac{1}{3}[\Phi(z)+\Phi(z-2)+\Phi(z-4)]$

(3) $cov(X,Y)=EXY-EX\cdot EY$

$EXY=E[E[XY|X]]$

$=E[XY|X=0]P(X=0)+E[XY|X=1]P(X=1)+E[XY|X=2]P(X=2)$

$=0\cdot\frac{1}{3}+1\cdot\frac{1}{3}+2\cdot2\cdot\frac{1}{3}=\frac{5}{3}$

$EX=1\quad EY=1$

$\therefore cov(X,Y)=\frac{2}{3}$.

25. $X_1.X_2$ iid $\sim Exp(\lambda)\quad \lambda=2$.

$f_1(x)=f_2(x)=\begin{cases}\lambda e^{-\lambda x} & x>0.\\ 0 & x\le 0\end{cases}$

$F_1(x)=F_2(x)=1-e^{-\lambda x}\quad x>0$.

记 $Y_1=\min\{X_1,X_2\}\quad Y_2=\max\{X_1,X_2\}$

$F_{Y_1}(z)=P(\min\{X_1,X_2\}\le z)$

$=1-P(\min\{X_1,X_2\}>z)$

$=1-e^{-\lambda x}\cdot e^{-\lambda x}$

$=1-e^{-2\lambda x}$

$f_{Y_1}(z)=2\lambda e^{-2\lambda x}\sim Exp(2\lambda)$.

$\therefore EY_1=\frac{1}{2\lambda}=\frac{1}{4}$

$$\min\{X_1, X_2\} + \max\{X_1, X_2\} = X_1 + X_2.$$

$$\therefore E\max\{X_1, X_2\} = EX_1 + EX_2 - EY_1$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$

或求 $\max\{X_1, X_2\}$ cdf $\to$ pdf.

$$P(Y_2 \le z) = P(X_1 \le z) \cdot P(X_2 \le z)$$
$$= (1 - e^{-\lambda x})^2$$
$$= e^{-2\lambda x} - 2e^{-\lambda x} + 1 = F_{Y_2}(z)$$

$$f_{Y_2}(z) = -2\lambda e^{-2\lambda z} + 2\lambda e^{-\lambda z}$$

$$EY_2 = \int_0^{+\infty} x(-2\lambda e^{-2\lambda x} + 2\lambda e^{-\lambda x})\,dx$$

$$= 2\int_0^{+\infty} x \cdot \lambda e^{-\lambda x}\,dx - \int_0^{+\infty} x 2\lambda e^{-2\lambda x}\,dx$$

$$= 2 \cdot \frac{1}{\lambda} - \frac{1}{2\lambda} = 1 - \frac{1}{4} = \frac{3}{4}.$$

法2. $$\min\{X_1, X_2\} = \frac{X_1 + X_2}{2} - \frac{|X_1 - X_2|}{2}$$

$$\max\{X_1, X_2\} = \frac{X_1 + X_2}{2} + \frac{|X_1 - X_2|}{2}$$