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4.1. 1.(2) 解：$\int\frac{e^{3x}+1}{e^x+1}dx=\int e^{2x}-e^x+1\,dx$

$=\int e^{2x}dx-\int e^x dx+\int 1dx=\frac{1}{2}e^{2x}-e^x+x+C$

(3) 解：$\int(2^x+3^x)^2dx=\int 4^x+2\cdot6^x+9^x dx$

$=\int 4^x dx+2\int 6^x dx+\int 9^x dx=\frac{4^x}{\ln4}+\frac{2\cdot6^x}{\ln6}+\frac{9^x}{\ln9}+C$

(5) 解：$\int\frac{x^2}{1+x^2}dx=\int 1-\frac{1}{1+x^2}dx$

$=\int 1dx-\int\frac{1}{1+x^2}dx=x-\arctan x+C$

2.(3) 解：$\int\frac{\cos x-\sin x}{1+\sin x+\cos x}dx=\int\frac{1}{1+\sin x+\cos x}d(\sin x+\cos x)=\ln|1+\sin x+\cos x|+C$

(4) 解：$\int\frac{\arctan x}{1+x^2}dx=\int\arctan x\, d(\arctan x)=\frac{1}{2}\arctan^2x+C$

(5) 解：$\int x\sqrt{1-x^2}dx=\int\sqrt{1-x^2}\,d(\frac{1}{2}x^2)=-\frac{1}{2}\int\sqrt{1-x^2}\,d(1-x^2)=-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$

(9) 解：$\int\sin^2x\,dx=\frac{1}{2}\int 1-\cos2x\,dx=\frac{1}{2}(\int dx-\frac{1}{2}\int\cos2x\,d(2x))$

$=\frac{1}{2}(x-\frac{1}{2}\sin2x)+C$

(10) 解：$\int\sin^5x\cos x\,dx=\int\sin^5x\,d(\sin x)=\frac{1}{6}\sin^6x+C$

Ch.3. 7. 证明：

不妨设 $f(a)>0$，$f(b)>0$.

$\because\lim\limits_{x\to a^+}\frac{f(x)-f(a)}{x-a}>0$　又$\because x>a\ (x\to a^+)$

$\therefore\exists\delta_1>0$，$\forall x\in(a,a+\delta_1)$ $f(x)-f(a)>0\Rightarrow f(x)>0$.

$\exists x_1\in(a,a\cdot\delta_1)$ $f(x_1)>0$.

$\because\lim\limits_{x\to b^-}\frac{f(x)-f(b)}{x-b}>0$.　又$\because x<b\ (x\to b^-)$

$\therefore\exists\delta_2>0$. $\forall x\in(b-\delta_2,b)$ $f(x)-f(b)<0\Rightarrow f(x)<0$

$\exists x_2\in(b-\delta_2,b)$ $f(x_2)<0$.

可使 $x_1<x_2$，$f(x)$ 在 $[x_1,x_2]$ 上连续

由零值性，$\exists\xi\in(x_1,x_2)\subseteq(a,b)$ $f(\xi)=0$.

18. 证明：$f(x)$ 在 $[-1,1]$ 上连续三阶导数

$f(x)$ 在 ~~[-1,1]~~ 0 处的二阶Taylor公式为

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(\xi)}{3!}x^3$$

取 $x=-1$ 和 $x=1$. 得

(既然来了两条一，就作(做)些也把这到的列出来

$$f(-1)=f(0)+f'(0)(-1)+\frac{f''(0)}{2}+\frac{f'''(\eta_1)}{6}(-1)^3 \qquad \eta_1\in (-1,0)$$

$$f(1)=f(0)+f'(0)+\frac{1}{2}f''(0)+\frac{1}{6}f'''(\eta_2) \qquad \eta_2\in(0,1)$$

$$\Rightarrow\begin{cases}0=f(0)+\frac{1}{2}f''(0)-\frac{1}{6}f'''(\eta_1) & ①\\ 1=f(0)+\frac{1}{2}f''(0)+\frac{1}{6}f'''(\eta_2) & ②\end{cases}$$

把未知量消去

② − ① 得 $\frac{1}{6}(f'''(\eta_1)+f'''(\eta_2))=1 \Rightarrow \frac{1}{2}(f'''(\eta_1)+f'''(\eta_2))=3.$

$\because f'''(x)$ 在 $[-1,1]$ 上连续. 则 $\exists m, M\in R$ ~~$m\le f'''(x)\le M$~~

$m=\min\limits_{-1\le x\le 1} f'''(x) \quad M=\max\limits_{-1\le x\le 1} f'''(x)$

既要用介值定理就找出(设出)上下界

☆ $\therefore\begin{cases}m\le f'''(\eta_1)\le M\\ m\le f'''(\eta_2)\le M\end{cases} \Rightarrow m\le\frac{f'''(\eta_1)+f'''(\eta_2)}{2}\le M$

分别代值相加 $\Rightarrow m\le 3\le M$

依介值性， $\exists\,\xi\in(-1,1)$ $f'''(\xi)=3$

求整体性质 ☆

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4.1 2(6) 解：$\int\frac{1}{\sqrt{x}(1+x)}dx=2\int\frac{1}{1+\sqrt{x}^2}d(\sqrt{x})=2\arctan\sqrt{x}+C$

(7) 解：$\int\frac{\arctan\frac{1}{x}}{1+x^2}dx=\int\frac{\frac{1}{x^2}\arctan\frac{1}{x}}{1+\frac{1}{x^2}}dx=-\int\frac{\arctan\frac{1}{x}}{1+\frac{1}{x^2}}d(\frac{1}{x})$

$=-\int\arctan\frac{1}{x}\,d(\arctan\frac{1}{x})=-\frac{1}{2}\arctan^2\frac{1}{x}+C$

(8) 解：$\because (x\ln x)'=1+\ln x$

$\therefore\int\frac{1+\ln x}{1+x\ln x}dx=\int\frac{1}{1+x\ln x}d(x\ln x)=\ln|x\ln x+1|+C$

3.(12) 解：$\int\frac{1}{x^8(x^2+1)}dx=\int\frac{1}{x^8}-\frac{1}{x^6}+\frac{1}{x^4}-\frac{1}{x^2}+\frac{1}{x^2+1}dx$

$= \int \frac{1}{x^8}dx - \int \frac{1}{x^6}dx + \int \frac{1}{x^4}dx - \int \frac{1}{x^2}dx + \int \frac{1}{x^2+1}dx$

$= \frac{1}{5x^5} - \frac{1}{7x^7} + \frac{1}{x} - \frac{1}{3x^3} + \arctan x + C$

4.(1) 解: $x \geq 0$ 时 $\int |x| dx = \int x dx = \frac{1}{2}x^2 + C$

$x < 0$ 时 $\int |x| dx = \int -x dx = -\frac{1}{2}x^2 + C$

7.(2) 解: $\int \frac{x^2-1}{x^4+x^2+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx = \int \frac{1}{x^2+\frac{1}{x^2}+1} d(x+\frac{1}{x})$

$= \frac{1}{2}(\int \frac{1}{x+\frac{1}{x}-1} - \frac{1}{x+\frac{1}{x}+1} dx) = \frac{1}{2}\ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + C$

(3) 解: $\int \frac{1}{x^4+x^6}dx = \int \frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{x^2+1} dx$

$= \frac{1}{x} - \frac{1}{3x^3} + \arctan x + C$

(22) 解: $\int \frac{1}{\sqrt{x-1}+\sqrt{x+1}}dx = \frac{1}{2}\int \sqrt{x+1} - \sqrt{x-1} dx$

$= \frac{1}{3}((x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}}) + C$

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ex4.1 3.(1) 解: 令 $x = \ln(t^2+2)$ $t \geq 0$ 则 $dx = \frac{2t}{t^2+2}dt$

原式 $= \int \frac{2t^2}{t^2+2}dt = \int 2 - \frac{4}{t^2+2}dt = \int 2dt - 4\int \frac{1}{t^2+2}dt$

$= 2t - \frac{4}{\sqrt{2}}\int \frac{1}{1+(\frac{t}{\sqrt{2}})^2} d(\frac{t}{\sqrt{2}}) = 2t - 2\sqrt{2}\arctan(\frac{\sqrt{2}t}{2}) + C$

$= 2\sqrt{e^x-2} - 2\sqrt{2}\arctan(\frac{\sqrt{2}\sqrt{e^x-2}}{2}) + C$

(2) 解: $\int \sqrt{x^2+a^2}dx = a\int \sqrt{1+(\frac{x}{a})^2}dx$

令 $x = a\tan t$, $t \in (0, \frac{\pi}{2})$ $\Rightarrow$ $dx = a\sec^2 t dt$

$\therefore$ 原式 $= a^2\int \sec^3 t dt = a^2\int \frac{\cos t}{\cos^4 t}dt = a^2\int \frac{1}{(1-\sin^2 x)^2} d(\sin x)$

$= \frac{a^2}{4}\int \frac{1}{1+\sin t} + \frac{1}{1-\sin t} + \frac{1}{(1+\sin t)^2} + \frac{1}{(1-\sin t)^2} d(\sin t)$

$= \frac{a^2}{4}(\ln|1+\sin t| - \ln|1-\sin t| - \frac{1}{1+\sin t} + \frac{1}{1-\sin t}) + C$

$= \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + \frac{x\sqrt{x^2+a^2}}{2} + C$

(3) 解：令 $x = a\sec t$，$t \in (0, \frac{\pi}{2})$． $dx = a\sec t\tan t\,dt$

$\int \frac{dx}{(x^2-a^2)^{\frac{3}{2}}} = \frac{1}{a^3}\int \frac{a\sec t\tan t}{\tan^3 t}dt = \frac{1}{a^2}\int \csc t\cot t\,dt = -\frac{\csc t}{a^2}+C$

$= -\frac{1}{a^2\sqrt{1-\frac{a^2}{x^2}}}+C$

(4) 解：令 $x = a\sin t$ $t \in (0, \frac{\pi}{2})$． $dx = a\cos t\,dt$

$\therefore \int \frac{x^2}{\sqrt{a^2-x^2}}dx = a^2\int \sin^2 t\,dt = a^2\int \frac{1-\cos 2t}{2}dt = \frac{a^2}{2}\int dt - \frac{a^2}{2}\int \cos 2t\,dt$

$= \frac{a^2}{2}t - \frac{a^2}{4}\sin 2t + C = \frac{a^2}{2}\arcsin(\frac{x}{a}) - \frac{ax}{2}\sqrt{1-\frac{x^2}{a^2}} + C$

(5) 解：令 $x = t^2-1$，$t \geq 0$． $dx = 2t\,dt$

$\therefore \int \frac{1}{1+\sqrt{x+1}}dx = \int \frac{2t}{1+t}dt = \int 2-\frac{2}{1+t}dt = 2t-2\ln|1+t|+C$

$= 2\sqrt{x+1}-2\ln|1+\sqrt{x+1}|+C$

(9) 解：令 $x = \frac{t-1}{2}$ $dx = \frac{1}{2}dt$

$\int \frac{x+2}{\sqrt[3]{2x+1}}dx = \frac{1}{2}\int \frac{\frac{t-1}{2}+2}{\sqrt[3]{t}}dt = \frac{1}{4}(\int t^{\frac{2}{3}}dt + 3\int t^{-\frac{1}{3}}dt)$

$= \frac{3}{20}t^{\frac{5}{3}} + \frac{9}{8}t^{\frac{2}{3}} + C = \frac{3}{20}(2x+1)^{\frac{5}{3}} + \frac{9}{8}(2x+1)^{\frac{2}{3}} + C$

(10) 解：令 $x = t^{14}$ $(t>0)$ $dx = 14t^{13}dt$

$\therefore \int \frac{x^{\frac{1}{7}}+x^{\frac{1}{2}}}{x^{\frac{3}{7}}+x^{\frac{1}{14}}}dx = 14\int \frac{t^{12}(t^2+t^7)}{t^5+1}dt = 14\int \frac{t^{14}}{t^{10}-t^5+1}dt$

$= 14\int t^4 + \frac{t^9-\frac{1}{2}t^4}{t^{10}-t^5+1} - \frac{1}{2}\cdot\frac{t^4}{(t^5-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dt$

$= \frac{14}{5}t^5 + \frac{14}{10}\ln|t^{10}-t^5+1| - 7\int \frac{t^4}{(t^5-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$= \frac{14}{5}t^5 + \frac{7}{5}\ln|t^{10}-t^5+1| - \frac{14\sqrt{3}}{15}\arctan\frac{2t^5-1}{\sqrt{3}} + C$

$= \frac{14}{5}x^{\frac{5}{14}} + \frac{7}{5}\ln|x^{\frac{5}{7}}-x^{\frac{5}{14}}+1| - \frac{14\sqrt{3}}{15}\arctan(\frac{\sqrt{3}(2x^{\frac{5}{14}}-1)}{3}) + C$

7.(1) 解：令 $x = \ln(t-1)$ $dx = \frac{1}{t-1}dt$

$\int \frac{1}{1+e^x}dx = \int \frac{1}{t(t-1)}dt = \int \frac{1}{t-1}-\frac{1}{t}\,dt = \ln|\frac{t-1}{t}|+C$

$= \ln|\frac{e^x}{e^x+1}|+C$

↓ 将e^x用 ln t代换，而使用中一t e) 中x

(4) 解：令 $x = t+2$ $dx = dt$

$$\int x\sqrt{x-2}\,dx = \int (t+2)\sqrt{t}\,dt = \int t^{\frac{3}{2}}dt + 2\int \sqrt{t}\,dt$$

$$= \frac{2}{5}t^{\frac{5}{2}} + \frac{4}{3}t^{\frac{3}{2}} + C = \frac{2}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}} + C$$


ESTHER