553.430/630, Spring 2021: Homework 5

The homework is due on **Monday, March 30, 2021, by 11:59 p.m. Submission is via Gradescope.** Please note: all Rice problems refer to the 3rd (US) edition. If this is not the edition you are using—for instance, if you have the international edition—please check to make sure the problems in your version of the text match those in the 3rd US edition.

You are free to talk with each other and get help. However, you should write up your own answers and understand everything you write. You must show ALL YOUR WORK in order to receive credit. ANSWERS WITHOUT APPROPRIATE JUSTIFICATION WILL RECEIVE NO CREDIT. Please feel free to stop by office hours if you need additional help!

Problem 1. Rice, Chapter 8, Exercise 7, parts (a) through (c) only, page 314.

Problem 2. Rice, Chapter 8, Problem 13.

Problem 3 (Is the MLE for i.i.d exponential data asymptotically normal?)**.** Let $X_i, 1 \leq i \leq n$, be i.i.d exponential with parameter $\lambda > 0$.

(a) Does the support of this distribution depend on λ?

(b) Compute the maximum likelihood estimate for λ, $\hat{\lambda}$.

(c) Consider the function $g(x) = 1/x$. Construct a second order Taylor expansion of this function around the value $1/\lambda$ (why?), similar to the more general case you considered in problem on Taylor expansions from the previous homework.

(d) Suppose the true value of λ is $\lambda = \lambda_0$. Use this Taylor expansion to determine the asymptotic distribution of
$$\sqrt{n}(\hat{\lambda} - \lambda_0)$$

(e) Compute the Fisher information $I(\lambda_0)$ and determine whether your answers to the previous part agree with the asymptotic normality results we described in class.

Problem 4 (MLE of uniform distribution)**.** Rice, Exercise 53, page 324: we worked through the essential ideas in lecture!

Problem 5 (More on MLE of a uniform distribution)**.** Let X_i be i.i.d uniform on $[0, \theta]$. Let $\hat{\theta}_n$ be the MLE for θ that you obtained from the previous exercise.

a) Show that $P[\hat{\theta}_n - \theta > \epsilon] = 0$ for any $\epsilon > 0$.

b) For any $\epsilon > 0$, determine an explicit expression for the probability
$$P[|\hat{\theta} - \theta| > \epsilon]$$

c) Compute, for any $\epsilon > 0$, the limit

$$P[|\sqrt{n}(\hat{\theta}_n - \theta)| > \epsilon]$$

as $n \to \infty$.

d) What do your previous answers suggest about the asymptotic distribution of

$$\sqrt{n}(\hat{\theta}_n - \theta)?$$

In particular, does this still look approximately normal?

e) What is the Method-of-Moments estimate for θ, $\hat{\theta}_{MOM}$? What is the limiting distribution of

$$\sqrt{n}(\hat{\theta}_{MOM} - \theta_0)$$

where θ_0 is the true value of the parameter?

Problem 6. Choose the correct answer and justify your choice completely.
Suppose X_n, $n \geq 1$, is a sequence of random variables on a probablity space Ω. Suppose all the random variables X_n have common mean $E[X_n] = \mu$. Suppose X is also a random variable defined on Ω. Suppose that for all n, the variance $V(X_n) \leq M$, where M is a fixed constant. Which of the following is true?

(a) Because the variances of the X_n random variables are all bounded by a constant M, this guarantees, by Chebyshev's inequality, that X_n converges in probability to the common mean μ.

(b) Because the variances of X_n random variables are all bounded by a constant M, this guarantees, by Chebyshev's inequality, that X_n converges to μ in L^2.

(c) The only way to guarantee either of the options in (a) and (b) is to require that the random variables X_n be independent.

(d) If there exists a single sample point $\omega \in \Omega$ such that the sequence $X_n(\omega)$ converges to μ, then we know that X_n converges to μ with probability one.

(e) Both (a) and (b) are true.

(f) All of (a), (b), and (d) are true.

(g) None of the above.

Problem 7. Choose the correct answer, and justify your response completely. Suppose $\Omega = [0, 1]$, and let $\omega \in \Omega$ be drawn uniformly at random from the unit interval. Define the sequence of random variables X_n, $n \geq 1$, on Ω by

$$X_n(\omega) = n^\alpha I_{[0, \frac{1}{n^2}]}(\omega) + n^\alpha I_{[1-\frac{1}{n}, 1]}(\omega)$$

Here $\alpha \in \mathbb{R}$ is a fixed, finite real number, and the notation I_A above denotes the indicator function of the set A. Let X be the random variable defined by $X(\omega) = 0$ for all $\omega \in [0, 1]$. Which of the following is true?

(a) For $\alpha = 1$, the expected value of X_n is $\mathbb{E}[X_n] = 1 + 1/n$.

(b) Regardless of the value of α, X_n converges to X in probability.

(c) Regardless of the value of α, X_n cannot converge to X with probability one because for any $\delta > 0$, the set $\{\omega : |X_n(\omega) - X(\omega)| > \delta\}$ has positive probability.

(d) If $\alpha = 1$, then for any $n > 51$, we are guaranteed that

$$|X_n(\omega) - X(\omega)| < 1/50$$

no matter what the value of $\omega \in [0, 1]$ happens to be.

(e) Both (a) and (b) are true.

(f) Both (a) and (c) are true.

(g) Both (b) and (c) are true.

(h) All three of (a), (b), and (c) are true.

(i) All three of (a), (b), and (d) are true.

(j) All four of (a), (b), (c), and (d) are true.

(k) None of the above.